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SINGULARITY AND ABSOLUTE CONTINUITY WITH RESPECT TO STRATEGIC MEASURES

BY

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Abstract

Extending the result of Prikry and Sudderth that a reverse strategic product measure on $N \times N$ with diffuse marginal measures is singular to all strategic measures (i.e. purely non-strategic) we show in Section 1 that any reverse strategic product measure an $X \times Y$ (where X and Y are arbitrary sets) is purely non-strategic if it has purely finitely additive marginal measures. If there are no real-valued measurable cardinals so all countably additive measures are discrete the converse is true. In Section 2, we introduce the language of split faces of probability measures as a convenient tool for discussing decompositions of probability measures. In this section we characterize which nearly strategic measures are absolutely continuous with respect to a given strategic measure. In Section 3, atomicity and non-atomicity of strategic measures are characterized. In Section 4, we deal with κ-additivity of strategic measures for an infinite cardinal κ. In Section 5, κ-uniformity of strategic measures is discussed. In Section 6, we give examples of reverse strategic product measures with diffuse marginals, one of which is countably additive, which are strategic. We also examine when a reverse strategic product measure with diffuse marginals, one of which is countably additive, may be purely non-strategic.

1. Introduction

Gambling Theory has as a central notion the concept of a strategy, [15]. A strategy σ is, essentially, a finitely additive Markov process on a discrete space F which is termed the *fortune space* (although state space is occasionally used in analogy with the terminology of the countably additive theory of Markov processes where F would be a locally compact Hausdorff space with a countable base.) The strategy σ describes the random movement of a particle (or player) through F in time. There is an initial distribution $\sigma_0(df)$ after one step from a given fortune f_0. σ_0 is an element of $P(F)$ the finitely

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additive probability measures defined on all subsets of F. There is, conditional on being at f_1 after step 1, a distribution $\sigma_1(f_1, df)$ of fortunes after step 2. Thus, σ_1 is a Markov kernel and is a function from F into $P(F)$. The pair (σ_0, σ_1) give rise to a probability distribution σ^2 in $F \times F$ describing the distribution of fortunes occurring in the first two steps of the Markov process. If s is a bounded function on $F \times F$ then

$$\int_{F\times F} s(f_1, f_2)\sigma^2(df_1, df_2) = \int\left[\int s(f_1, f_2)\sigma_1(f_1, df_2)\right]\sigma_0(df_1).$$

In general one is interested not just in the distribution of the first two steps of the Markov process but rather in the distribution of all possible sequences or histories $h = (f_1, f_2, \ldots, f_n, \ldots)$ of fortunes. For this, one needs, for any n and any $(f_1, \ldots, f_n) \in F^n$, a conditional distribution $\sigma_n(f_1, \ldots, f_n, df_{n+1})$ of f_{n+1} given that the first n fortunes occurring were $(f_1, \ldots, f_n)$. Thus, σ_n maps F^n into $P(F)$ and $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ gives rise to a probability distribution on F^{n+1} given by the inductively defined integration formula

$$\int s(f_1, \ldots, f_n, f_{n+1})\sigma^{n+1}(df_1, \ldots, df_n, df_{n+1}) \\ = \int\left[\int s(f_1, \ldots, f_n, f_{n+1})\sigma_n(f_1, \ldots, f_n, df_{n+1})\right]\sigma^n(df_1, \ldots, df_n).$$

The entire sequence $(\sigma_0, \ldots, \sigma_n, \ldots)$ is termed a *strategy* and is denoted by σ. The strategy σ gives rise to a probability distribution defined on the clopen algebra of the history space

$$H = \{h = (f_1, \ldots, f_n, \ldots): f_i \in F \text{ all } i\} = F^\infty.$$

The details appear in Dubins and Savage [15]. The measure on H is called the *strategic measure* on H associated with the strategy σ and is also denoted by σ.

Of central importance to the construction of strategic measures on H is the situation where one has two discrete spaces X and Y. One has an *initial distribution* $\sigma_0 \in P(X)$, which may be thought of as the distribution, of the initial step in X of a finitely additive Markov process. Conditional on $x \in X$ one has a probability distribution $\sigma_1(x, dy) \in P(Y)$ which may be thought of as the distribution of the second step in Y. As before the pair $\sigma = (\sigma_0, \sigma_1)$ gives rise to a probability distribution on $X \times Y$ again denoted by σ. We call the pair σ a *strategy* (actually a two step strategy) and the measure σ a *strategic measure* on $X \times Y$. Let Σ denote the set of all strategic measures on $X \times Y$. An example is $X = F^n$ and $Y = F$. Here σ_0 describes the distribution of the first n-steps of a finitely additive Markov process and σ_1 describes the distribution of the $(n + 1)$-*st* step conditional on the first n steps. More generally, Y could be F^m and σ_1 would describe the distribution of steps $n + 1$ through $n + m$ conditional on steps 1 through n.

It is natural to ask which measures in $P(X \times Y)$ arise as strategic measures.

If we were dealing with countably additive Markov processes and with X and Y locally compact Hansdorff spaces with countable bases we would have the result that all probability Radon measures on $X \times Y$ are strategic measures corresponding to strategies (σ_0, σ_1) where σ_0 is a probability Radon measure on X and σ_1 is a (suitably measurable) Markov kernel from X to $\mathcal{M}_1^+(Y)$, the probability Radon measures on Y. This is a standard consequence of the theory of disintegration of measures [16], [29].

In contrast to the situation for Radon measures it is almost never the case that $P(X \times Y) = \Sigma$. In fact, Dubins found in [15] that if X and Y are countably infinite there exists a measure $\gamma \in P(X \times Y)$ so that $\gamma \perp \Sigma$; that is, γ is singular to all strategic measures. Such measures will be called *purely non-strategic* and we will denote their totality by $\Sigma^\perp$. It was shown by Armstrong and Sudderth in [9] that every measure $\gamma \in P(X \times Y)$ may be expressed uniquely as a convex combination $\lambda\gamma_1 + (1 - \lambda)\gamma_2$ where γ_1 (unique if $\lambda \neq 0$) is in $\Sigma^\perp$ and γ_2 (unique if $\lambda \neq 1$) is in the closure $\bar{\Sigma}$ of Σ for the variation norm. Elements of $\bar{\Sigma}$ are called *nearly strategic* measures. Thus, it follows that $\bar{\Sigma} = \Sigma^{\perp\perp}$. It is also shown in [9] that Σ need not be convex hence need not equal $\bar{\Sigma}$.

Decompositions of finitely additive probabilities similar to the decomposition into purely non-strategic and nearly strategic measures are the *Hewitt-Yosida* [19] *decomposition* ($\gamma = \alpha\gamma_{ca} + \beta\gamma_{pfa}$ where γ_{ca} is countably additive and γ_{pfa} is purely finitely additive in that γ_{pfa} is singular to all countably additive probabilities); the *Sobczyk-Hammer decomposition* [26] ($\gamma = \alpha\gamma_{at} + \beta\gamma_{na}$ where γ_{at} is atomic so it is a countable convex combination of $\{0, 1\}$-valued measures and γ_{na} is non-atomic in that for all $\varepsilon > 0$ there is a finite partition into sets of measure at most ε); the *diffuse-discrete decomposition* ($\gamma = \alpha\gamma_{\text{diff}} + \beta\gamma_{\text{disc}}$ where γ_{diff} is diffuse in that γ assigns 0 measure to singletons and γ_{disc} is *discrete* in that it is a countable convex combination of point masses); and the *Lebesque decomposition* [10] ($\gamma = \alpha\gamma_s + \beta\gamma_\alpha$ where $\gamma_s \perp \mu_0$ and $\gamma_\alpha \ll \mu_0$ where μ_0 is a fixed measure). We shall discuss these types of decompositions at length in Section 2 as split face decompositions.

One type of strategic measure $\sigma = (\sigma_0, \sigma_1)$ is of special importance. This is the *strategic product measure* where $\sigma_0 = \alpha \in P(X)$ and for all (or α almost all) $x \in X$, $\sigma_1(x, \cdot) = \beta$ where β is a fixed element of $P(Y)$. This measure σ has the property that $\sigma(A \times B) = \alpha(A)\beta(B)$ if $A \subset X$ and $B \subset y$. This measure σ will be denoted by $\sigma(\alpha, \beta)$ and is an extension of the product measure $\alpha \otimes \beta$ from the product algebra $2^X \otimes 2^Y$ to $2^{X \times Y}$.

When the roles of X and Y are interchanged, one obtains *reverse strategies* $\tau = (\tau_0, \tau_1)$ where $\tau_0 \in P(Y)$ and, for $y \in Y$, $\tau_1(y, \cdot) \in P(X)$. Corresponding to a reverse strategy τ is a *reverse strategic measure,* also denoted by τ, in $P(X \times Y)$ defined by the integration formula

$$\int f(x, y)d\tau = \int\left[\int f(x, y)\tau_1(y, dx)\right]\tau_0(dy).$$

Corresponding to an $(\alpha, \beta) \in P(X) \times P(Y)$ there is a *reverse strategic product measure* $\tau(\alpha, \beta) = (\tau_0, \tau_1)$ with $\tau_0 = \beta$ and $\tau_1(y, dx) = \alpha$ for all $y \in Y$.

If μ is an element of $P(X \times Y)$ then the *X-margin* of μ, $\mu_X \in P(X)$ is defined by $\mu_X(A) = \mu(A \times Y)$ for $A \subset X$ and the *Y-margin* $\mu^Y \in P(Y)$ is defined by $\mu^Y(B) = \mu(X \times B)$ for $B \subset X$. Thus, strategic and reverse strategic product measures $\sigma(\alpha, \beta)$ and $\tau(\alpha, \beta)$ are uniquely specified by their X-margins α and their Y-margins β. Dubins established in [14] that if Y is finite then $P(X \times Y)$ consists entirely of reverse strategic measures. Prikry and Sudderth noted in [25] that, in this case $P(X \times Y) = \overline{\Sigma}$. From this it follows that if X and Y are arbitrary and if $\gamma \in P(X \times Y)$ either has a discrete X-margin or a discrete Y-margin then γ is both nearly strategic and nearly reverse strategic (so is approximable in variation norm by reverse strategic measures).

In [14], Dubins established that if $X = Y = N$ and if α and β are diffuse $\{0, 1\}$-valued elements of $P(X)$ and $P(Y)$ respectively then $\tau(\alpha, \beta) \in \Sigma^{\perp}$. This was the first example of a purely non-strategic measure.

In [25], Prikry and Sudderth showed that reverse strategic product measure τ associated with arbitrary diffuse α and β on $X = N = Y$ belongs to $\Sigma^{\perp}$. This is the present state of the question of existence of elements of $\Sigma^{\perp}$. Of course, when X and Y are countable it is immediate that a reverse strategic product measure $\tau(\alpha, \beta)$ is in $\Sigma^{\perp}$ only if both α and β are diffuse. If this weren't the case and $\alpha = \lambda\alpha_{\text{diff}} + (1 - \lambda)\alpha_{\text{disc}}$ with $\lambda < 1$ then

$$\tau = \lambda\tau^1 + (1 - \lambda)\tau^2 \quad \text{where } \tau^1 = \tau(\alpha_{\text{disc}}, \beta)$$

is nearly strategic. This reasoning works for general X and Y and allows us to consider only $\tau(\alpha, \beta)$ where α and β are diffuse.

We are interested in extending the known results to the cases where X and Y are uncountable. For instance, $X = Y = [0, 1]$ is a case of interest to many probabilists and statisticians. Our first result is nearly a corollary of the result of Prikry and Sudderth.

We recall from [6] that a $p \in P(X)$ is *strongly finitely additive* if there is a countable partition $\{X_n : n \in \omega\}$ of X so that $p(X_n) = 0$ for all $n \in \omega$. We also recall that $p \in P(X)$ is purely finitely additive if it can be written as a countable convex combination $\Sigma_{n=1}^{\infty}\lambda_n p_n$ where each p_n is strongly finitely additive. (Actually, in [6], a bounded positive p was shown to be purely finitely additive if it could be written as a countable sum of strongly finitely positive measures.) Furthermore one may choose for any $\varepsilon > 0$ such a countable convex combination with $\lambda_1 > 1 - \varepsilon$.

THEOREM 1.1. *Let $\alpha \in P(X)$ and $\beta \in P(Y)$ be purely finitely additive. Then $\tau(\alpha, \beta) \in \Sigma^{\perp}$.*

Proof. If it is shown that when α and β are strongly finitely additive then $\tau(\alpha, \beta) \in \Sigma^{\perp}$ the theorem will follow in general. To see this, write

$$\alpha = (1 - \lambda)\alpha_1 + \lambda\alpha_2 \quad \text{and} \quad \beta = (1 - \lambda)\beta_1 + \lambda\beta_2$$

where $0 < \lambda < \varepsilon$ and where both α_1 and β_1 are strongly finitely additive. We have

$$\tau = (1 - \lambda)^2\tau^1 + [1 - (1 - \lambda)^2]\tau^2$$

where $\tau^1 = \tau(\alpha_1, \beta_1)$ and τ^2 is some other element of $P(X \times Y)$. Since $\tau^1 \in \Sigma^\perp$ and $\varepsilon > 0$ is arbitrary it follows that τ is in the norm closure of $\Sigma^\perp$ which is $\Sigma^\perp$.

For the remainder of the proof we assume that both α and β are strongly finitely additive. We first note that Prikry and Sudderth in the proof of the Theorem of [25] actually give a proof that when $Y = N$ and X is arbitrary then $\tau(\alpha, \beta) \in \Sigma^\perp$. The only modification necessary in their proof is in the demonstration that if $\sigma = (\sigma_0, \sigma_1)$ is a strategic measure with $\sigma_1(x, \cdot)$ diffuse for all X then $\tau(\alpha, \beta) \perp \sigma$. To establish this use the strong finite additivity of α to find a decreasing sequence $\{X_n: n \in N\}$ of subsets of X with empty intersection with $\alpha(X_n) = 1$ for all $n \in N$. Set

$$S = (X_n \times \{n\}) \subset X \times N.$$

Note that for all $x \in X$, S_x is finite so $\sigma(S) = 0$ by diffusivity of σ_1 and that $\tau(\alpha, \beta)(S) = 1$. Thus, $\sigma \perp \tau(\alpha, \beta)$.

It only remains to establish the result when Y is uncountable. Since β is strongly finitely additive there is a $\Phi : Y \to N$ so that the image β' of β under Φ (defined by $\beta'(A) = \beta(\Phi^{-1}(A))$ for $A \subset N$ or equivalently by $\int_N f\, d\beta' = \int f(\Phi(y))\beta(dy)$ for bounded f on N), is diffuse. Define

$$\tau: X \times Y \to X \times N$$

by $\tau(x, y) = (x, \Phi(y))$. If $\sigma = (\sigma_0, \sigma_1)$ is a strategic measure on $X \times Y$ the image σ' of σ under τ is the strategic measure (σ_0, σ'_1) on $X \times N$ where $\sigma'_1(x, \cdot)$ is the image of $\sigma_1(x, \cdot)$ on N for all $x \in X$. To see this, calculate as follows for a bounded f on $X \times N$:

$$\begin{aligned} \int f\, d\sigma' &= \int\left[\int f(x, \Phi(y))\sigma_1(x, dy)\right]\sigma_0(dx) \\ &= \int\left[\int f(x, n)\sigma'_1(x, dn)\right]\sigma_0(dx). \end{aligned}$$

A similar verification shows that the image of $\tau(\alpha, \beta)$ under τ is the reverse strategic product measure $\tau(\alpha, \beta')$ on $X \times N$ where β' is the image of β under Φ. Since α and β' are strongly finitely additive $\tau(\alpha, \beta')$ is purely non-strategic on $X \times N$. If σ is a strategic measure on $X \times Y$ let σ' be the image of σ on $X \times N$ under τ. For any $\varepsilon > 0$ there is an $A'_\varepsilon \subset X \times N$ so that

$$\sigma'(A'_\varepsilon) < \varepsilon \quad \text{and} \quad \tau(\alpha, \beta')(A'_\varepsilon) > 1 - \varepsilon.$$

If $A_\varepsilon = \tau^{-1}(A'_\varepsilon)$ then $\sigma(A_\varepsilon) < \varepsilon$ and $\tau(\alpha, \beta)(A_\varepsilon) > 1 - \varepsilon$. Since ε and σ are arbitrary τ is purely non-strategic. ■

COROLLARY 1.1.1. *Let* $(\alpha, \beta) \in P(X) \times P(Y)$ *be such that* $\tau(\alpha, \beta) \notin \Sigma^\perp$. *One of* α *or* β *fails to be purely finitely additive.*

In effect, the question now facing us is whether $\tau(\alpha, \beta)$ may be in $\Sigma^\perp$ if α or β is countably additive and diffuse. The existence of such an α or β is equivalent to the cardinality of X or Y being *real-valued measurable*. It is consistent with the axioms, ZFC, of set theory that no real-valued measurable cardinals exist and it is consistent that $2^{\aleph_0} = c = \text{card}[0, 1]$ be real-valued measurable [6], [28]. If real-valued measurable cardinals don't exist then $\tau(\alpha, \beta)$ is in $\Sigma^\perp$ if α and β are diffuse. Further investigations of the question will, of necessity, be more set theoretic and be based in large part upon material in [6]. Although only partial results will be obtained these give considerable insight into the problem.

To facilitate discussion in later sections we introduce in the next section the notion of split faces of the simplex of probability measures on a set. This notion deals with convex direct sum decompositions. In particular the notation $\Sigma^\perp$, which should denote the ideal in the Banach lattice of finitely additive signed measures of bounded variation on $2^{X \times Y}$ which are singular to elements of Σ, will be replaced by $\Sigma' = \Sigma^\perp \cap P(X \times Y)$, the split face of $P(X \times Y)$ complementary to the split face $\overline{\Sigma}$.

2. Split faces of $\overline{\Sigma}$

A subset A of a convex set F is said to be a *split face* [1] of F if A is convex and there exists another convex set B so that F is the convex direct sum $A \oplus B$ so every $f \in F$ is representable uniquely as a convex combination $\lambda f_A + (1 - \lambda) f_B$ with $f_A \in A$ and $f_B \in B$. Here, λ is unique, f_A is unique if $\lambda \neq 0$ and f_B is unique if $1 - \lambda \neq 0$. If A is a split face of F it is a *face* [1], so that if $\{f_1, f_2\} \subset F$ and $0 < \lambda < 1$ is such that $\lambda f_1 + (1 - \lambda) f_2 \in A$ then $\{f_1, f_2\} \subset F$. If A is a split face of F then B consists of those points $f \in F$ so that if $f' \in F$ and $0 \leq \lambda \leq 1$ is such that $f = \lambda a + (1 - \lambda) f'$ for some $a \in A$ then $\lambda = 0$ and $f' = f$. B is uniquely determined by the requirement that $F = A \oplus B$ and is a split face of F called the *complementary split face* to A and is denoted by A'. When A is a split face then $A = (A')'$. The intersection of two split faces of F is again a split face of F as is the convex hull of the union of two split faces or a split face of a split face of F. Split faces form a Boolean algebra with F as supremum, and $\emptyset$ as infimum. The infimum of a finite family of split faces is their intersection and the supremum the convex hull of their union, [1].

In the simplex $P(X)$ of finitely additive probabilities on X (or for Choquet simplexes, or K-simplexes as in [2], [5], in general) the Boolean algebra of split faces is a complete Boolean algebra. The infimum of an arbitrary family of split faces is a split face and the supremum of an arbitrary family is the σ-convex hull, σ conv(E), of the union E where for a σ-convex hull one allows countable convex combinations [2], [5], [18]. Given any $E \subset P(X)$, there is a smallest split face of $P(X)$ containing E which will be denoted by sface(E). If E is the singleton $\{\mu\}$ then sface(E) will be denoted by sface(μ). One has

$$\text{sface}(\mu) = \{\nu \in P(X) : \nu << \mu\}$$

and

$$[\text{sface}(\mu)]' = \{\nu \in P(X) : \nu \perp \mu\} = \{\mu\}^{\perp} \cap P(X) \quad [2], [5], [10], [18], [24].$$

For any $E \subset P(X)$,

$$\text{sface}(E) = \cup \{\text{sface}(\mu) : \mu \in \sigma \text{ conv}(E)\} \quad \text{and} \quad [\text{sface}(E)]' = E^{\perp} \cap P(X).$$

If $E \subset P(X)$ we will denote $[\text{sface}(E)]'$ by E'.

There are several characterizations of which convex sets $A \subset P(X)$ are split faces due to Lima [24], and Goodearl [18]. If A is a face then is a split face iff it is σ-convex [18], iff it is norm closed [18]. A convex set $A \subset P(X)$ is a face if $\nu \in A$ whenever $\nu < \lambda\mu$ for some $\mu \in A$ and $\lambda \in (0, \infty]$.

A face F of $P(X)$ is split iff its linear span S in $BA(X)$ (the signed finitely additive measures of bounded variation on 2^X) is a norm-closed ideal in the Banach lattice $BA(X)$. In this case F' has linear span $F^{\perp} = S^{\perp}$ and $S^{\perp\perp} = S$ so $F = F^{\perp\perp} \cap P(X)$. Furthermore, $BA(X)$ is the l^1-direct sum $S \oplus S^{\perp}$ so that if $\mu_F \in S$ and $\mu_{F'} \in S^{\perp}$ then

$$\|\mu_F + \mu_{F'}\| = \|\mu_F\| + \|\mu_{F'}\| \quad [1], [2], [3], [5].$$

Most decomposition theorems for measures are split face decompositions in that they assert the existence of complementary pairs of split faces of the simplex $P(X)$. The Lebesque decomposition is the prime example. The Hewitt-Yosida decomposition states that purely finitely additive probability measures form a split face complementary to the countably additive probability measures. The Sobczyk-Hammer decomposition theorem says that atomic and non-atomic probability measures form complementary split faces. The diffuse and discrete probability measures form complementary split faces.

The main content of [9] is that the nearly strategic measures $\overline{\Sigma}$ form a split face of $P(X \times Y)$ whose complementary split face is the purely non-strategic measures $\overline{\Sigma}'$ $(= \Sigma^{\perp} \cap P(X))$. It is of interest to see how split faces of $P(X)$ and $P(Y)$ give rise to split faces of $\overline{\Sigma}$ (hence of $P(X \times Y)$).

LEMMA 2.1. *Let $\sigma = (\sigma_0, \sigma_1)$ be a strategy, and, for each $x \in X$, let*

$$\sigma_1(x, \cdot) = \lambda(x)\sigma_{11}(x, \cdot) + (1 - \lambda(x))\sigma_{12}(x, \cdot)$$

be a convex combination of probabilities. Let

$$\lambda = \int \lambda(x)\sigma_0(dx),$$

$$f_1(x) = \lambda(x)\lambda^{-1}, \sigma_{01} = f_1(x)\sigma_0,$$

$$f_2(x) = [1 - \lambda(x)](1 - \lambda)^{-1}, \quad \sigma_{02} = f_2(x)\sigma_0,$$

σ^1 be the strategy $(\sigma_{01}, \sigma_{11})$ and σ^2 be the strategy $(\sigma_{02}, \sigma_{12})$. As strategic measures, $\sigma = \lambda\sigma^1 + (1 - \lambda)\sigma^2$.

Proof. For any bounded $g(x, y)$ we calculate as follows.

$$\begin{aligned}
\int g(x, y)d\sigma &= \int\left[\int g(x, y)\sigma_1(x, dy)\right]\sigma_0(dx) \\
&= \int\left[\int g(x, y)\sigma_{11}(x, dy)\right]\lambda(x)\sigma_0(dx) \\
&\quad + \int\left[\int g(x, y)\sigma_{12}(x, dy)\right][1 - \lambda(x)]\sigma_0(dx) \\
&= \lambda\int\left[\int g(x, y)\sigma_{11}(x, dy)\right]f_1(x)\sigma_0(dx) \\
&\quad + \Big(1 - \lambda \int\left[\int g(x, y)\sigma_{12}(x, dy)\right]f_2(x)\sigma_0(dx) \\
&= \lambda\int g(x, y)d\sigma^1 + (1 - \lambda)\int g(x, y)d\sigma^2. \quad \blacksquare
\end{aligned}$$

Remark (1) Strictly speaking Lemma 2.1 is valid only if $0 < \lambda < 1$.

(2) This lemma will be used extensively not only in this section but throughout.

(3) If we were dealing with countably additive Markov kernels on a measurable space care would have to be taken in this lemma to ensure the measurability of $x \to \lambda(x)$ and $x \to \sigma_{11}(x, \cdot)$. See [21].

PROPOSITION 2.2. *Let S_X be a split face of $P(X)$, and, for all $x \in X$, let $S_Y(x)$ be a split face of $P(Y)$. Let $\mathscr{C}$ be all strategic measures σ with the strategy $\sigma = (\sigma_0, \sigma_1)$ satisfying $\sigma_0 \in S_X$, and $\sigma_1(x, \cdot) \in S_Y(x)$ for all $x \in X$. The norm closure of $\mathscr{C}$ is a split face of Σ.*

Proof. It is necessary to show that if $\nu << \mu \in \overline{\mathscr{C}}$ then $\nu \in \overline{\mathscr{C}}$. It may be assumed that $\mu \in \mathscr{C}$. To see this let $\{\mu_n: n \in \omega\} \subset \mathscr{C}$ converge to μ. For $n \in \omega$, let ν_n be the part of ν absolutely continuous with respect to μ_n. It is easily seen that $\{\nu_n: n \in \omega\}$ converges to ν. If it is known that $\{\nu_n: n \in \omega\} \subset \overline{\mathscr{C}}$ then it follows that $\nu \in \overline{\mathscr{C}}$.

Since $\nu << \mu \in \overline{\Sigma}$ it follows that $\nu \in \overline{\Sigma}$. Let $\{\nu_n: n \in \omega\} \subset \Sigma$ converge to ν. If, for $n \in \omega$, ν^n is the part of ν_n absolutely continuous with respect to μ then $\{\nu^n: n \in \omega\}$ converges to ν. For $n \in \omega$, let (ν_0^n, ν_1^n) be a strategy with strategic measure ν^n. Decompose $\nu_1^n(\chi, \cdot)$ as

$$\lambda_n(\chi)\nu_{11}^n(\chi, \cdot) + (1 - \lambda_n(x))\, \nu_{12}^n(\chi, \cdot)$$

with $\nu_{11}^n(x, \cdot) \in S_{\overline{Y}}(x)$ and $\nu_{12}^n(x, \cdot) \in S_Y^{\perp}(x)$ for all n. If $\lambda_n = \int \lambda_n(x)\, \nu_0^n(dx) = 1$ for infinitely many n then ν is the limit of a subsequence of (ν_0^n, ν_{11}^n). If this is not the case, $\lambda_n \neq 1$ if n is large. We assume that $\lambda_n \neq 1$ for all n. Write ν^n as $\lambda_n(\nu_{01}^n, \nu_{11}^n) + (1 - \lambda_n)(\nu_{02}^n, \nu_{12}^n)$, using Lemma 2.1. Let (μ_0, μ_1) be a strategy corresponding to μ with $\mu_0 \in S_X$ and $\mu_1(x, \cdot) \in S_Y(x)$ for all x. For each $x \in \overline{X}$ with $\lambda_n(x) \neq 1$, $n \in \omega$ and $\varepsilon > 0$, let $A(n, \varepsilon, x) \subset Y$ have $\mu_1(x, A(n, \varepsilon, x)) < \varepsilon$ and $\nu_{12}^n(x, A(n, \varepsilon, x))$. If

$$A(n, \varepsilon) = \bigcup_{x \in X} \{x\} \times A(n, \varepsilon, x)$$

then

$$\mu(A(n, \varepsilon)) < \varepsilon \text{ and } (\nu_{02}^n, \nu_{12}^n)(A(n, \varepsilon)) > 1 - \varepsilon.$$

Letting $\varepsilon > 0$ vary it follows that $\mu \perp (\nu_{02}^n, \nu_{12}^n)$. Thus,

$$\nu = \lim_n (\nu_{01}^n, \nu_{11}^n).$$

A similar argument using the decomposition of ν_{01}^n into a part ν_{03}^n in S_X and a part in S_X' shows that

$$\nu = \lim_n (\nu_{03}^n, \nu_{11}^n) \in \overline{\mathscr{E}}. \blacksquare$$

COROLLARY 2.2.1. *If $\sigma = (\sigma_0, \sigma_1)$, set $\sigma_0 = \gamma\sigma_{01} + (1 - \gamma)\sigma_{02}$ where $\sigma_{01} \in S_X$ and $\sigma_{02} \in S_X'$. For all $x \in X$, set*

$$\sigma_1(x, \cdot) = \lambda(x)\sigma_{11}(x, \cdot) + [1 - \lambda(x)]\sigma_{12}(x, \cdot)$$

with

$$\sigma_{11}(x, \cdot) \in S_Y(x) \quad \textit{and} \quad \sigma_{12}(x, \cdot) \in S_Y'(x).$$

Then $\sigma \in \mathscr{E}'$ iff $\gamma \cdot \int \lambda(x)\sigma_{01}(x)dx = 0$.

Proof. Set

$$\sigma^{11} = (\sigma_{01}, \sigma_{11}), \sigma^{12} = (\sigma_{01}, \sigma_{12}), \sigma^{21} = (\sigma_{02}, \sigma_{11}) \text{ and}$$
$$\sigma^{22} = (\sigma_{02}, \sigma_{12}).$$

If $\lambda = \int \lambda(x)\sigma_{01}(x)dx$ then

$$\sigma^{11} = \gamma\lambda\sigma^{11} + \gamma(1' - \lambda)\sigma^{12} + (1 - \gamma)\lambda\sigma^{21} + (1 - \gamma)(1 - \lambda)\sigma^{22}.$$

Since $\{\sigma^{12}, \sigma^{21}, \sigma^{22}\}$ are all in $\mathscr{C}'$ and $\sigma^{11} \in \mathscr{C}$ it follows that $\sigma \in \mathscr{C}'$ iff $0 = \gamma \int \lambda(x)\sigma_{01}(dx)$. ∎

COROLLARY 2.2.2. *Let $\mathscr{C}'_X$ be the split face which is the norm closure of those strategic measures σ with $\sigma_0 \in S'_X$ and let $\mathscr{C}'_Y$ be the split face which is the norm closure of those strategic measures with $\sigma_1(x, \cdot) \in S'_Y(x)$ for all $x \in X$. Then $\mathscr{C}'$ is the convex hull of $\mathscr{C}'_X \cup \mathscr{C}'_Y$.*

Remark. The split faces of $\overline{\Sigma}$ which arise in Proposition 2.2 are ubiquitous but not all encompassing. If $\sigma = (\sigma_0, \sigma_1)$ is a strategic measure, the smallest split face $\overline{\mathscr{C}}$ of $\overline{\Sigma}$ with $\sigma \in \overline{\mathscr{C}}$ of the form described in Proposition 2.2 has

$$S_X = \text{sface}(\sigma_0) \quad \text{and} \quad S_Y(x) = \text{sface}(\sigma_1(x, \cdot))$$

for all $x \in X$, yet it will be seen that the resulting split face $\overline{\mathscr{C}}$ of $\overline{\Sigma}$ may contain measures in $\{\sigma\}^{\perp}$ so $\overline{\mathscr{C}} \neq \text{sface}(\sigma)$.

If $\mathscr{P}$ is a property of measures in $P(Y)$ we say that a strategy $\sigma = (\sigma_0, \sigma_1)$ is *conditionally* $\mathscr{P}$ iff $\sigma_1(x, \cdot)$ has property $\mathscr{P}$ for all $x \in X$. If $\sigma_1(x, \cdot)$ has property $\mathscr{P}$ except on a σ_0-negligible set then σ is *essentially conditionally* $\mathscr{P}$. Usually $\mathscr{P}$ will be the property that a measure lies in a certain face or split face of $P(Y)$. For instance we will use the terms conditionally diffuse, conditionally discrete, conditionally countably additive, and conditionally non-atomic. We will say σ is *marginally* $\mathscr{P}$ where $\mathscr{P}$ is a property on $P(X)$ to denote the fact that σ_0 has property $\mathscr{P}$. This terminology extends to the strategic measures induced by the strategies. It is important to note that a strategic measure is conditionally $\mathscr{P}$ iff it is essentially conditionally $\mathscr{P}$.

COROLLARY 2.2.3. *Let S_X be a split face of $P(X)$ and S_Y be a split face of $P(Y)$. The norm closure of those strategic measures which are marginally S_X and conditionally S_Y form a split face of $\overline{\Sigma}$. The complementary split face is the convex hull of the split faces generated in a like manner by* (a) *the marginally S'_X strategic measures and* (b) *the conditionally S'_Y strategic measures.*

Remark. Although it is true that if γ is a limit in norm of marginally S_X strategic measures then $\gamma_X \in S_X$ it is not to be expected that if γ is a limit of conditionally S_Y measures then $\gamma_Y \in S_Y$. In fact, even if γ is a conditionally S_Y strategic measure, γ_Y need not be in S_Y. For instance one may readily construct conditionally discrete strategic measures whose Y-margin is diffuse and in fact non-atomic.

COROLLARY 2.2.4. *Let $\gamma \in P(X \times Y)$ have X marginal γ_X.*

(a) *γ is nearly strategic iff for all $\varepsilon > 0$ there is a strategy $(\sigma_0, \sigma_1) = \sigma$ with $\sigma_0 = \gamma_X$ and $\|\sigma - \gamma\| < \varepsilon$.*

(b) *γ is purely non-strategic iff for any strategy $\sigma = (\sigma_0, \sigma_1)$ with $\sigma_0 = \gamma_X$ one has $\gamma \perp \sigma$.*

Proof. (b) If γ is singular with respect to all strategic measures σ with $\sigma_0 = \gamma_X$ then γ is singular with respect to all strategic measures with $\sigma_0 = f(x)\gamma_X$ with f a simple function with $\int f(x)d\gamma_X = 1$. To see this let $b = \max(f)$ and write

$$\gamma_X = \frac{1}{b}f(x)\gamma_X + \frac{b-1}{b}\frac{b-f(x)}{b-1}\gamma_X.$$

For any choice of $\sigma_1(x, \cdot)$, as strategic measures,

$$(\gamma_X, \sigma_1) = \frac{1}{b}(f(x)\gamma_X, \sigma_1) + \frac{b-1}{b}\left(\frac{b-f(x)}{b-1}\gamma_X, \sigma_1\right)$$

so $\gamma \perp (f(x)\gamma_X, \sigma_1)$. If $\sigma_0 \ll \gamma_X$ there exists a sequence $\{f_n\}$ of simple functions so that $\lim_{n\to\infty}\|f_n(x)\gamma_X - \sigma_0\| = 0$. It is easily checked that

$$lim\|(f_n(x)\gamma_X, \sigma_1) - (\sigma_0, \sigma_1)\| = 0.$$

Since $\gamma \perp (f_n(x)\gamma_X, \sigma_1)$ it follows that $\gamma \perp (\sigma_0, \sigma_1)$. Since (σ_0, σ_1) is an arbitrary strategy marginally absolutely continuous with respect to γ_X and, since γ must be singular with respect to any strategy marginally singular with respect to γ_X, γ is purely non-strategic.

(a) There is a sequence of strategic measures $\sigma^n = (\sigma_0^n, \sigma_1^n)$ so that

$$\lim_{n\to\infty}\|\sigma_0^n - \gamma_X\| = 0$$

hence the part μ_n of σ_0^n absolutely continuous with respect to γ_X must satisfy $\lim_{n\to\infty}\|\mu_n - \gamma_X\| = 0$. As a result, if

$$\hat{\sigma}^n = \left(\frac{\mu_n}{\|\mu_n\|}, \sigma_1\right)$$

then $\lim_{n\to\infty}\|\hat{\sigma}^n - \gamma\| = 0$. Thus, it may be assumed that $\sigma_0^n \ll \gamma_X$ for all n. Furthermore, use of Bochner's finitely additive Radon-Nikodym Theorem allows us to suppose that $\sigma_0^n = f_n(x)\gamma_X$ for a simple function f_n. From the fact that

$$\lim_{n\to\infty}\|f_n(x)\gamma_X - \gamma_X\| = 0$$

it follows that we may replace $f_n(x)\gamma_X$ by γ_X which establishes (a). ∎

COROLLARY 2.2.5. *If $\gamma \in \overline{\Sigma}$ there is, for $\varepsilon > 0$, a strategy $\sigma = (\sigma_0, \sigma_1)$ with $\gamma_X = \sigma_0$, $\gamma \ll \sigma$ and $\|\sigma - \gamma\| < \varepsilon$.*

Proof. Find strategies $\sigma^n = (\gamma_X, \sigma_1^n)$ so that $\|\sigma^n - \gamma\| < \varepsilon \cdot 2^n$. Let

$$\sigma = \left(\gamma_X, \sum_{n=1}^{\infty} 2^{-n}\sigma_1^n\right).$$

It is immediate that, as strategic measures, $\sigma = \sum_{n=1}^{\infty} 2^{-n}\sigma^n$ so $\|\sigma - \gamma\| < \varepsilon$. Since $\lim_{n\to\infty}\sigma^n = \gamma$ in variation norm it is well known that $\gamma \ll \sum_{n=1}^{\infty} 2^{-n}\sigma^n$. ∎

Let $\sigma = (\sigma_0, \sigma_1)$ be a strategy. Let $\nu = (\nu_0, \nu_1)$ be another strategy with $\nu_0 \ll \sigma_0$ and $\nu_1(x, \cdot) \ll \sigma_1(x, \cdot)$ for $x \in X$. For $r \in [0, \infty)$ let

$$\nu_1^r(x, \cdot) = [r\sigma_1(x, \cdot)] \wedge \nu_1(x, \cdot) \quad \text{and} \quad \mu_1^r(x, \cdot) = \nu_1(x, \cdot) - \nu_1^r(x, \cdot).$$

Define $\{\nu^r, \mu^r\} \subset P(X \times Y)$ by setting, for bounded f,

$$\int f(x, y)d\nu^r = \int\left[\int f(x, y)\nu_1^r(x, dy)\right]\nu_0(dx)$$

and

$$\int f(x, y)d\mu^r = \int\left[\int f(x, y)\mu_1^r(x, dy)\right]\nu_0(dx).$$

For each r, $\nu^r \leqslant r(\nu_0, \sigma_1)$ where (ν_0, σ_1) is considered as a strategic measure. Since $(\nu_0, \sigma_1) \ll \sigma$, $\nu^r \ll \sigma$. As $r \to \infty$, ν^r increases to $\nu^\infty \leqslant \nu$ and $\nu^\infty \ll \sigma$.

Proposition 2.3. *ν^∞ is the part of ν absolutely continuous with respect to σ and $\nu - \nu^\infty$ is the part of ν singular to σ.*

Proof. It is only necessary to show that $(\nu - \nu^\infty) \perp \sigma$.

It is convenient to work in the Stonian setting. 2^Y is considered as the clopen algebra of βY. For any $\mu \in BA(2^Y)$, $\tilde{\mu}$ denotes the corresponding Radon measure on βY. For each $x \in X$ let $h(x, \cdot)$ be a Radon-Nikodym derivative of $\tilde{\nu}_1(x, \cdot)$ with respect to $\tilde{\sigma}_1(x, \cdot)$. For any $r \in [0, \infty)$,

$$\tilde{\nu}_1^r(x, \cdot) = [h(x, \cdot) \wedge r]\tilde{\sigma}_1(x, \cdot) \quad \text{and}$$

$$\tilde{\mu}_1^r(x, \cdot) = [h(x, \cdot) - h(x, \cdot) \wedge r]\tilde{\sigma}_1(x, \cdot).$$

For $r \in [0, \infty)$ let

$$A(x, r) = \{z \in \beta Y : \mathrm{h}(x, z) \leqslant r\}.$$

Since $r\tilde{\sigma}_1(x, A^c(x, r)) \leqslant \tilde{\nu}_1(x, A^c(x, r)) \leqslant 1$,

$$\tilde{\sigma}_1(x, A(x, r)) \geqslant 1 - r^{-1}.$$

Also,

$$\mu_1^r(x, A^c(x, r)) = \|\mu_1^r\| \quad \text{for } x \in X.$$

For $\varepsilon > 0$ find $A(x, r, \varepsilon) \subset Y$ so that, considered as a clopen set in βY,

$$\tilde{\sigma}_1(x, A(x, r, \varepsilon)\Delta A(x, r)) < \varepsilon \quad \text{and} \quad \tilde{\mu}_1^r(x, A^c(x, r, \varepsilon)\Delta A^c(x, r)) < \varepsilon.$$

Set $A(r, \varepsilon) \subset X \times Y$ equal to $\cup\ \{\{x\} \times A(x, r, \varepsilon) : x \in X\}$. We have

$$\begin{aligned}\sigma(A(r,\varepsilon)) &= \int \sigma_1(x, A(x,r,\varepsilon))\sigma_0(dx)\\ &\geqslant \int [\sigma_1(x, A(x,r)) - \varepsilon]\sigma_0(dx)\\ &\geqslant \int [1 - r^{-1} - \varepsilon]\sigma_0(dx) = 1 - r^{-1} - \varepsilon.\end{aligned}$$

We also have

$$\begin{aligned}\mu^r(A^c(r,\varepsilon)) &= \int \mu_1^r(x, A^c(x,r,\varepsilon))\nu_0(dx)\\ &\geqslant \int (\|\mu_1^r\| - \varepsilon)\nu_0(dx) = \|\mu^r\| - \varepsilon.\end{aligned}$$

Select r so that $r^{-1} < \varepsilon$ and so that $\|\mu^r - (\nu - \nu^\infty)\| < \varepsilon$. Then,

$$\sigma(A(r,\varepsilon)) \geqslant 1 - 2\varepsilon \quad \text{and}$$

$$(\nu - \nu^\infty)(A^c(r,\varepsilon)) \geqslant \mu^r(A^c(r,\varepsilon)) - \varepsilon \geqslant \|\mu^r\| - 2\varepsilon \geqslant \|\nu - \nu^\infty\| - 2\varepsilon.$$

Since ε is arbitrary $(\nu - \nu^\infty) \perp \sigma$. ∎

Let ν and σ be as above. For any r let $f_r(x) = \|\nu_1^r(x, \cdot)\|$. The norm of ν^r is $\int f_r(x)\nu_0(dx)$. If $r > 0$ then $f_r(x) > 0$ for all r and x. If $\nu \perp \sigma$ then $\int f_r(x)dx = 0$ for all r.

LEMMA 2.4. *$\mu \in P(X)$ is strongly finitely additive iff there is a strictly positive f on X such that $\int f(x)\mu(dx) = 0$.*

Proof. Let such an f exist. If

$$A_n = \left\{\frac{1}{n-1} > f \geqslant \frac{1}{n}\right\} \quad \text{for} \quad n \in N\left(\frac{1}{0} = \infty\right),$$

then $\{A_n : n \in N\}$ is a partition of X into μ-negligible sets so μ is strongly finitely additive. Conversely, if μ is strongly finitely additive and $\{A_n : n \in N\}$ is a partition of X into μ-negligible sets one may set $f = 1/n$ on A_n to obtain a strictly positive f with $\int f(x)dx = 0$. ∎

COROLLARY 2.3.1. (a) *If, in Proposition 2.3, $\nu \perp \sigma$ then ν_0 is strongly finitely additive*

(b) *If σ_0 is countably additive then $\nu \ll \sigma$.*

Proof. (a) f_r is strictly positive for $r > 0$ and $\int f_r(x)\nu_0(dx) = 0$.

(b) If σ_0 is countably additive so is ν_0. As a result, since $\lim_{r\to\infty} f_r = 1$ the monotone convergence theorem implies that $\|\nu^\infty\| = 1$ so $\nu^\infty = \nu$. ∎

We call a $\mu \in P(X)$ *molecular* iff it is a finite convex combination of $\{0, 1\}$-valued measures. A $\mu \in P(X)$ fails to be molecular iff it has an infinite range iff $\inf\{\mu(A) : \mu(A) > 0\} = 0$ [18].

COROLLARY 2.3.2. *If σ_0 is strongly finitely additive and σ is conditionally non-molecular there is a $\nu = (\sigma_0, \nu_1)$ conditionally absolutely continuous with respect to σ so that $\nu \perp \sigma$.*

Proof. Let f be a function strictly positive on X so that

$$\int f(x)\sigma_0(dx) = 0.$$

For each x let $A(x) \subset Y$ satisfy $0 < \sigma_1(x, A(x)) \leqslant f(x)$. Let

$$A = \cup \{\{x\} \times A(x) : x \in X\}$$

so $0 \leqslant \sigma(A) = \int \sigma_1(x, A(x))\sigma_0(dx) \leqslant \int f(x)\sigma_0(dx) = 0$. Let

$$\nu_1(x, \cdot) = \chi_{A(x)}[\sigma_1(x, A(x))]^{-1}\sigma_1(x, \cdot) << \sigma_1(x, \cdot).$$

Then $\nu(A) = \int \nu_1(x, A(x))\sigma_0(dx) = \int 1\ \sigma_0(dx) = 1$. Thus, $\nu \perp \sigma$. ∎

COROLLARY 2.3.3. *If σ_0 isn't countably additive and σ is conditionally non-molecular there is a $\nu = (\nu_0, \nu_1)$ marginally absolutely continuous with respect to σ_0 and conditionally absolutely continuous with respect to σ_1 so that $\nu \perp \sigma$.*

Proof. There is a $\nu_0 << \sigma_0$ which is strongly finitely additive. Apply Corollary 2.3.2 to (ν_0, σ_1). ∎

If σ isn't conditionally non-molecular there is a set $A \subset X$ so that $\sigma_0(A) > 0$ and $\sigma_1(x, \cdot)$ is molecular for all $x \in A$. Let

$$f(x) = \inf\{\sigma_1(x, E) : \sigma_1(x, E) > 0\} \quad \text{if } x \in A$$

and set $f(x) = 0$ otherwise. Let

$$A_n = \left\{\frac{1}{n-1} > f \geqslant \frac{1}{n}\right\} \cap A$$

so $\{A_n : n \in N\}$ partitions A.

COROLLARY 2.3.4. *Suppose that σ_0 is purely finitely additive.*

(a) *If $\sum_{n=1}^{\infty} \sigma_0(A_n) < \sigma_0(A)$ then there is a $\nu = (\nu_0, \nu_1)$ marginally absolutely continuous with respect to σ_0 and conditionally absolutely continuous with respect to σ_1 with $\nu \perp \sigma$.*

(b) *If $\sum_{n=1}^{\infty} \sigma_0(A_n) = \sigma_0(A) = 1$ then any $\nu = (\nu_0, \nu_1)$ marginally absolutely continuous with respect to σ_0 and conditionally absolutely continuous with respect to σ_1 satisfies $\nu << \sigma$.*

Proof. (a) Let $\mu_1 = \sum_{n=1}^{\infty}(\chi_{A_n}\sigma_0)$ and let $\mu_2 = \chi_A\sigma_0 - \mu_1$. Since $\mu_2(A^c) = 0$ and $\mu_2(A) \neq 0$ we may normalize μ_2 to get $\nu_0 = \mu_2 \cdot [\mu_2(A)]^{-1} \in P(X)$. Since $\mu_2(A_n) = 0$ for all n it follows that $\int f(x)d\nu_0 = 0$. Since

$\sigma_1(x, \cdot)$ is molecular for $x \in A$ there exists an $A(x) \subset Y$ with $\sigma_1(x, A(x)) = f(x) > 0$. Let

$$\mathcal{A} = \cup \{\{x\} \times A(x) : x \in A\}$$

and let $\nu_1(x, \cdot) = \chi_{A(x)}[f(x)]^{-1}\sigma_1(x, \cdot)$. As in the proof of Corollary 2.3.2, $\sigma(\mathcal{A}) = 0$ and $\nu(\mathcal{A}) = 1$ if $\nu = (\nu_0, \nu_1)$.

(b) Let $x \in A_n$ and let $\nu_1(x, \cdot) << \sigma_1(x, \cdot)$. Write $\sigma_1(x, \cdot)$ as $\Sigma_{i=1}^m \lambda_i \chi_{\mathcal{U}_i}$ where each $\mathcal{U}_i$ is an ultrafilter and $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m = f(x)$. Then $\nu_1(x, \cdot)$ is the convex combination $\Sigma_{i=1}^m \gamma_i \chi_{\mathcal{U}_i}$. If E is in $\mathcal{U}_i$ but in no other $\mathcal{U}_i$ then

$$\nu_1(x, E) = \gamma_i \leq 1 \leq n\sigma_1(x, E).$$

As a consequence, $\nu_1(x, F) \leq n\sigma_1(x, F)$ for any $F \subset Y$. As a result, if $B \subset A_n \times Y$ then $\nu(B) \leq n\sigma(B)$. That is, on $A_n \times Y$, $\nu << \sigma$, hence, on $(\cup_{n=1}^m A_n) \times Y$, $\nu << \sigma$. Fix $\varepsilon > 0$. Pick m so that $\nu(X \setminus \cup_{n=1}^m A_n) < \varepsilon$. Pick $\delta < \varepsilon m^{-1}$ so that $\delta_0(E) < \delta$ implies $\nu_0(E) < \varepsilon$. Let $B \subset X \times Y$ with $\sigma(B) < \delta$. We have

$$\sigma\left(B \cap \left(\bigcup_{n=1}^m A_n \times Y\right)\right) \leq \delta$$

so

$$\nu\left(B \cap \left(\bigcup_{n=1}^m A_n \times Y\right)\right) \leq m\delta < \varepsilon.$$

We also have

$$\nu\left(B \setminus \left(\bigcup_{n=1}^m A_n \times Y\right)\right) \leq \upsilon_0\left(X \setminus \bigcup_{n=1}^m A_n\right) \leq \varepsilon.$$

Thus $\nu(B) < \varepsilon + \varepsilon = 2\varepsilon$. That is, $\nu << \sigma$. ∎

Remark. If $\nu = (\nu_0, \nu_1)$ is a strategy marginally absolutely continuous with respect to the strategy $\sigma = (\sigma_0, \sigma_1)$ it is possible that as strategic measures $\nu << \sigma$ with $\nu_1(x, \cdot)$ not absolutely continuous with respect to $\sigma_1(x, \cdot)$ for any x. If $\nu_1^s(x, \cdot)$ is the part of $\nu_1(x, \cdot)$ singular to $\sigma_1(x, \cdot)$ we must have, in this case, $\int \|\nu_1^s(x, \cdot)\|\nu_0(dx) = 0$.

3. Atomic and non-atomic elements of $\bar{\Sigma}$

In Corollary 2.2.5 it was shown that any $\gamma \in \bar{\Sigma}$ is absolutely continuous with respect to some $\sigma = (\sigma_0, \sigma_1)$ with $\sigma_0 = \gamma_X$. When γ_X is molecular we show that γ is strategic in Corollary 3.4.2. In fact if δ is any $\{0, 1\}$-valued element of $P(X \times Y)$ either δ is strategic or it is purely non-strategic (Corollary 3.4.1). These results have been obtained for $X = N$ by Schervish, Seidenfeld and Kadane [22]. Along the way we characterize which $\sigma = (\sigma_0, \sigma_1)$ are non-atomic.

The results of this section dealing with strategic measures σ which are conditionally {0, 1}-valued and marginally {0, 1}-valued are essentially dealing with the construction of ultrafilters on the product $X \times Y$ from an ultrafilter on X and a family of ultrafilters on Y, hence is closely related to certain constructions in Comfort and Negrepontis [12]. These constructions may be of some interest to logical model theorists in the study of ultrapowers or others whose main interests are ultrafilters rather than measures.

PROPOSITION 3.1. *If $\sigma = (\sigma_0, \sigma_1)$ is a marginally non-atomic strategic measure or is a conditionally non-atomic strategic measure, then σ is a non-atomic measure.*

Proof. We only establish the proposition in the harder case where σ is conditionally non-atomic. Fix $\varepsilon > 0$. It is easy to see that if $n > 2/\varepsilon$ and μ is non-atomic there is a partition $\{A_1(x), \ldots, A_n(x)\}$ of Y so that $\sigma_1(x, A_j(x)) < \varepsilon$ for all j and each x. Set $A_j \subset X \times Y$ equal to $\cup \{\{x\} \times A_j(x), x \in X\}$ for all j. It is easily verified that $\sigma(A_j) \leqslant \varepsilon$ for all j. Since $\varepsilon > 0$ is arbitrary, σ is non-atomic. ∎

Via Lemma 2.1, any strategic measure decomposes into a marginally non-atomic part, a marginally atomic and conditionally non-atomic part, and a marginally atomic and conditionally atomic part. Can the marginally atomic and conditionally atomic part be a non-atomic measure? The answer is yes. We let $\sigma = (\sigma_0, \sigma_1)$ be such a measure and let $\sigma_1(x, \cdot) = \Sigma_{i=1}^{\infty} \lambda_i(x)\sigma_1^i(x, \cdot)$ where each $\sigma_1^i(x, \cdot)$ is a {0, 1}-valued measure, $\lambda_1(x) \geqslant \cdots \geqslant \lambda_n(x) \cdots \geqslant 0$, and $\Sigma_{i=1}^{\infty} \lambda_i(x) = 1$.

PROPOSITION 3.2. *Let $\sigma = (\sigma_0, \sigma_1)$ be as above. Then $\int \lambda_1(x)\sigma_0(dx) = 0$ iff σ is non-atomic.*

Proof. Suppose that $\int \lambda_1(x)\sigma_0(dx) = 0$. For any n, $\sigma_0\{x : \lambda_1(x) \geqslant 1/n\} = 0$. Fix n and suppose that $\lambda_1(x) < 1/n$ for all x.

For each j, let $m_j(x)$ be the last m, possibly ∞, so that $\Sigma_{i=1}^{m} \lambda_i(x) < j/n$. As a result,

$$\frac{j}{n} \sum \{\lambda_i(x) : 1 \leqslant i \leqslant m_j(x)\} > \frac{j}{n} - \lambda_1(x)$$

and

$$\frac{1}{n} + \lambda_i(x) \geqslant \sum \{\lambda_i(x) : m_j(x) + 1 \leqslant i \leqslant m_{j+1}(x)\}$$

$$\geqslant \frac{1}{n} - \lambda_1(x) \quad \text{for all } x \in X.$$

Pick a partition $\{A_1(x), \ldots, A_n(x)\}$ so that

$$\sigma_1^i(x, A_j(x)) = 1 \quad \text{if} \quad m_{j-1}(x) < \mathrm{i} \leqslant \mathrm{m}_j(x)$$

and

$$\sigma_1^i(x, A_j(x)) = 0 \quad \text{if} \quad m_{l-1}^{(x)} < i \leq m_l(x) \quad \text{and} \quad j \neq l \leq n$$

We have

$$\begin{aligned}\sigma_1(x, A_j(x)) &= \sum \{\lambda_i(x) : m_{j-1}(x) < i \leq m_j(x)\} \\ &\quad + \sum \{\lambda_i(x)\sigma_1^i(x, A_j(x)) : m_n(x) < j\} \\ &\leq \left[\frac{1}{n} + \lambda_1(x)\right] + \lambda_1(x) \\ &\leq \frac{3}{n}.\end{aligned}$$

Set $A_j = \cup_X \{x\} \times A_j(x)$ for $j = 1, \ldots, n$. We have $\sigma(A_j) \leq 3/n$. Since n is arbitrary, σ is non-atomic.

Conversely, if $\lambda_1 = \int \lambda_1(x)\sigma_0(dx) > 0$ let ν be the strategy (ν_0, ν_1) with $\nu_1(x, \cdot) = \sigma_1^1(x, \cdot)$ for all x and $\nu_0 = \lambda_1^{-1}\lambda_1(x)\sigma_0$. By Lemma 2.1, $\sigma = \lambda_1\nu + (1 - \lambda_1)\gamma$ for some strategic measure γ. ν is conditionally $\{0, 1\}$-valued and ν_0 is atomic since σ_0 is atomic. Since ν_0 is a countable convex combination of $\{0, 1\}$-valued measures, so is ν. Thus, if $\int \lambda_1(x)\sigma_0(dx) \neq 0$ then σ is not non-atomic. ■

If a strategic measure, $\sigma = (\sigma_0, \sigma_1)$ is to be atomic then, by Lemma 2.1 and Proposition 3.1, it may be taken to be conditionally and marginally atomic. In this case, write

$$\sigma_1(x, \cdot) = \sum_{i=1}^{\infty} \lambda_i(x)\sigma_1^i(x, \cdot)$$

where each $\sigma_1^i(x, \cdot)$ is $\{0, 1\}$-valued and $\{\lambda_i(x)\}$ is a decreasing sequence in $[0, 1]$ summing to 1. Write $\lambda_i = \int \lambda_i(x)\sigma_0(dx)$ and $\lambda = \Sigma_{i=1}^{\infty} \lambda_i$ and, if $\lambda_i \neq 0$, let σ_i be the strategy $(\lambda_i^{-1} \lambda_i(x)\sigma_0, \sigma_1^i)$. Each σ_i is atomic and we may write

$$\sigma = \sum^{\infty} \lambda_i\sigma^i + (1 - \lambda)\sigma^0$$

for some $\sigma^0 \in \overline{\Sigma}$. If σ_0 is $\{0, 1\}$-valued then $\lambda_i^{-1}\lambda_i(x)\sigma_0 = \sigma_0$ if $\lambda_i \neq 0$. If $\lambda = 1$ then σ is atomic.

PROPOSITION 3.3. *Let $\sigma = (\sigma_0, \sigma_1)$ be a marginally atomic, marginally countably additive and conditionally atomic strategic measure. Then σ is atomic and equal to $\Sigma_{i=1}^{\infty} \lambda_i\sigma^i$.*

Proof. Since $\Sigma_{i=1}^{\infty} \lambda_i(x) = 1$, the monotone convergence theorem guarantees that $\lambda = 1$. ■

PROPOSITION 3.4. *Let $\sigma = (\sigma_0, \sigma_1)$ be a conditionally atomic and marginally* *atomic strategic measure. Then* $\Sigma_{i=1}^{\infty} \lambda_i \sigma^i$ *is the atomic part of* σ *and* $(1 - \lambda)\sigma^0$ *is the non-atomic part.*

Proof. Assume that σ_0 is $\{0, 1\}$-valued. We need to show that σ^0 is non-atomic. If $\lambda = 1$ the assertion is vacuous. If $\lambda < 1$ choose an integer m and an $\varepsilon > 0$ so that $\varepsilon < 1/2m$ and $\varepsilon < (1 - \lambda)/m$. Choose an integer n so that $\Sigma_{i=1}^{n-1} \lambda_i > \lambda - \varepsilon$ and, as a result, $\lambda_n < \varepsilon$. Since σ_0 is $\{0, 1\}$-valued,

$$\sum_{i=n+1}^{\infty} \lambda_i(x) \geqslant 1 - (\lambda_1 + \ldots + \lambda_n) \geqslant 1 - \lambda$$

and $\lambda_j(x) < \varepsilon$ for σ_0-almost all x and all $j \geqslant n$. We will suppose this holds for all x.

For each x let $m(l, x)$ be the first integer k with

$$\sum_{j=n+1}^{k} \lambda_j(x) \geqslant \frac{l}{m} \sum_{i=n+1}^{\infty} \lambda_i(x).$$

Then

$$\sum_{j=n+1}^{m(l,x)} \lambda_j(x) \leqslant \frac{l}{m} \sum_{i=n+1}^{\infty} \lambda_i(x) + \lambda_n(x)$$

and $\Sigma\{\lambda_j(x) : m(l - 1, x) < j \leqslant m(l, x)\}$ is between

$$\frac{1}{m} \sum_{i=n+1}^{\infty} \lambda_i(x) - \lambda_n(x) \quad \text{and} \quad \frac{1}{m} \sum_{i=n+1}^{\infty} \lambda_i(x) + \lambda_n(x).$$

Since $\lambda_n(x) < \varepsilon < (1 - \lambda)/m$ it follows that

$$\Sigma\{\lambda_j(x) : m(l - 1, x) < j \leqslant m(l, x)\} > 0.$$

Let $\omega^l(x, \cdot) = \Sigma\{\lambda_j(x)\sigma_1^j(x, \cdot) : m(l - 1, x) < j \leqslant m(l, x)\}$ so

$$\|\omega^l(x, \cdot)\| = \Sigma\{\lambda_j(x) : m(l - 1, x) < j \leqslant m(l, x)\}.$$

Set $\nu_1^l(x, \cdot) = \omega^l(x, \cdot)\|\omega^l(x, \cdot)\|^{-1}$ and let ν_l be the strategy (σ_0, ν_1^l). Set

$$\omega_l = \int \|\omega_l(x, \cdot)\|\sigma_0(dx).$$

We have

$$\frac{1 - \lambda}{m} - \varepsilon \leqslant \omega_l \leqslant \frac{1 - \lambda}{m} + \varepsilon \quad \text{for all } l.$$

We write σ as $\Sigma_{i=1}^{n} \lambda_i \sigma^i + \Sigma_{j=1}^{m} \omega_l \nu^l$ and note that $(1 - \lambda)\sigma^0 \leqslant \Sigma_{j=1}^{m} \omega_l \nu^l$. For any x let

$$\{A_1(x), \ldots, A_n(x), B_1(x), \ldots B_m(x)\}$$

be a partition of Y such that

$$\sigma_1^i(x, A_i(x)) = 1 \quad \text{for all } i \quad \text{and} \quad \nu_1^l(x, B_l(x)) > 1 - \varepsilon \quad \text{for all } l.$$

Set $A_i = \cup_X\{x\} \times A_i(x)$ for all i and $B_l = \cup_X\{x\} \times B_l(x)$ for all l. Consequently, $\nu^l(A_i) < \varepsilon$ for all i and $\nu^l(B_j) < \varepsilon$ if $j \neq l$. We have, for any i,

$$(1 - \lambda)\sigma^0(A_i) \leq \sum_{l=1}^{m} \omega_l \nu^l(A_i) \leq \varepsilon \sum_{l=1}^{m} \omega_l \leq \varepsilon < \frac{1}{2m}.$$

We have, for any j,

$$\begin{aligned}(1 - \lambda)\sigma^0(B_j) &\leq \sum_{l=1}^{m} \omega_l \nu^l(B_j) \leq \sum_{l \neq j} \omega_l \varepsilon + \omega_j \nu^j(B_j) \\ &\leq \varepsilon + \omega_j \leq \varepsilon + \left(\frac{1}{m} + \varepsilon\right) \\ &= 2\varepsilon + \frac{1}{m} < \frac{2}{m}.\end{aligned}$$

Since m is arbitrary, σ^0 is non-atomic.

If σ_0 is not $\{0, 1\}$-valued, let it be the countable convex combination $\Sigma_{j=1}^{\infty}\gamma_j \sigma_0^j$ of the $\{0, 1\}$-valued $\{\sigma_0^j\}$. Let $\sigma^j = (\sigma_0^j, \sigma_1)$ and $\sigma^{ji} = (\sigma_0^j, \sigma_1^i)$ for all i, j. Write

$$\sigma^j = \sum_{i=1}^{\infty} \lambda_{ji}\sigma^{ji} + \left(1 - \sum_{i=1}^{\infty}\lambda_{ji}\right)\sigma^{j0}$$

where σ^{j0} is the non-atomic part of σ^j and

$$\lambda_{ji} = \int \lambda_i(x)\sigma_0^j(dx) \quad \text{for all } i \text{ and } j.$$

We have $\lambda_i = \Sigma_{j=1}^{\infty} \gamma_j \lambda_{ji}$ for all i. The atomic part of σ is

$$\sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{\infty} \lambda_{ji}\sigma^{ji}$$

and the non-atomic part is

$$\sum_{j=1}^{\infty} \gamma_j \left(1 - \sum_{i=1}^{\infty} \lambda_{ji}\right)\sigma^{j0}.$$

The norm of the non-atomic part is

$$\begin{aligned}\sum_{j=1}^{\infty} \gamma_j \left(1 - \sum_{i=1}^{\infty} \lambda_{ji}\right) &= 1 - \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{\infty} \lambda_{ji} \\ &= 1 - \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \gamma_j \lambda_{ji}\right) \\ &= 1 - \sum_{i=1}^{\infty} \lambda_i.\end{aligned}$$

The norm of the atomic part of σ is $\Sigma_{i=1}^{\infty} \lambda_i$. Since $\Sigma_{i=1}^{\infty} \lambda_i \sigma^i$ has norm $\Sigma_{i=1}^{\infty} \lambda_i$ and is atomic, it is the atomic part of σ, and σ^0 is the non-atomic part. ∎

COROLLARY 3.4.1. *If γ is a* {0, 1}*-valued element of $\bar{\Sigma}$ it is strategic.*

Proof. If γ is not in Σ it is absolutely continuous with respect to a strategic measure (σ_0, σ_1) which, by Lemma 2.1 and Proposition 3.1, may be taken to be marginally and conditionally atomic, for γ is singular to all non-atomic measures. Writing $(\sigma_0, \sigma_1) = \Sigma \lambda_i \sigma^i + (1 - \lambda)\sigma^0$, as in Proposition 3.4, it follows that $\gamma \ll \sigma^i$ for some i, hence $\gamma = \sigma^i$ for some i. ∎

COROLLARY 3.4.2. *Let γ be an atomic element of $\bar{\Sigma}$ such that γ_X is molecular. Then γ is strategic.*

Proof. Suppose that γ_X is {0, 1}-valued. Write γ as $\Sigma_{i=1}^{\infty} \lambda_i \gamma_i$ where each γ_i is {0, 1}-valued. We have $(\gamma_i)_X = \gamma_X$ for all i. Each $\gamma_i \in \bar{\Sigma}$ hence is strategic, corresponding to a strategy (γ_X, γ_1^i). γ is the strategic measure for the strategy $(\gamma_X, \Sigma_{i=1}^{\infty} \lambda_i \gamma_1^i)$.

If $\gamma_X = \Sigma_{i=1}^{n} \lambda_i \gamma_X^i$, where each γ_X^i is a distinct {0, 1}-valued measure, find a partition $\{A_1, \ldots, A_n\}$ of X so that $\gamma_X^i(A_i) = 1$. On $A_i \times Y$, $\lambda_i^{-1}\gamma$ is a nearly strategic measure with $(\lambda_i^{-1}\gamma)_X = \gamma_X^i$. It is, in fact, strategic and corresponds to a strategy (γ_X^i, γ_1^i) on $A_i \times Y$. Let $\gamma_1(x, \cdot) = \gamma_1^i(x, \cdot)$ if $x \in A^i$. γ is the strategic measure on $X \times Y$ for the strategy (γ_X, γ_1). ∎

COROLLARY 3.4.3. *If there is a partition of X into atoms for γ_X (for instance if γ_X is countably additive and atomic) then γ is strategic.*

Proof. The proof of Corollary 3.4.2 only required the existence of a partition $\{A_n\}$ so that if $\gamma_X = \Sigma_{n=1}^{\infty} \lambda_n \gamma_X^n$ where each γ_X^n was {0, 1}-valued then $\gamma_X^n(A_n) = 1$ for all n. ∎

COROLLARY 3.4.4. *If (σ_0, σ_1) is a conditionally discrete and marginally atomic strategic measure then there exist a sequence $\{f_n\}$ of functions from X to Y and a decreasing sequence of functions $\{\lambda_n(\cdot)\}$ of functions from X to $[0, \infty)$ so that if $\lambda_n = \int \lambda_n(x)\sigma_0(dx)$ then the atomic part of σ is given by $\Sigma_{n=1}^{\infty} \lambda_n \sigma^n$ where*

$$\int g(x, g)d\sigma^n = \int g(x, f_n(x))\frac{\lambda_n(x)}{\lambda_n}\sigma_0(dx) \quad \textit{for all } g.$$

Proof. λ_n is defined as before and $f_n(x)$ is the y in Y with $\sigma_1^i(x, \cdot) = \delta_y(\cdot)$. ∎

COROLLARY 3.4.5. *If $\sigma = (\sigma_0, \sigma_1)$ is a conditionally discrete and marginally countably additive strategic measure then $\sigma = \Sigma_{i=1}^{\infty} \lambda_i \sigma^i$ where $\sigma^i =$*

$(\lambda_i(x)\lambda_i^{-1}\sigma_0, \sigma_1^i)$, *where each* $\lambda_i = \int \lambda_i(x)\sigma_0(dx)$. *Each* σ^i *gives measure* 1 *to the graph of some function* $f_i : X \to Y$.

4. κ-additivity of strategic measures

We start this section with a characterization of which strategic measures are countably additive and which are purely finitely additive.

PROPOSITION 4.1. *Let* $\sigma = (\sigma_0, \sigma_1)$ *belong to* Σ.

(a) σ *is countably additive iff it is marginally countably additive and conditionally countably additive.*

(b) σ *is purely (strongly) finitely additive if it is either marginally purely (strongly) finitely additive or conditionally purely (strongly) finitely additive.*

Proof. (a) is nearly immediate.

(b) Suppose that σ is conditionally strongly finitely additive. For each $x \in X$ let $\{A_n(x)\}$ be a countable partition of Y' into $\sigma_1(x, \cdot)$-negligible sets. If $A_n = \cup_X\{x\} \times A_n(x)$ for $n \in N$ then $\{A_n\}$ is a partition of $X \times Y$ into σ-negligible sets, so σ is strongly finitely additive.

If each $\sigma_1(x, \cdot)$ is purely finitely additive let $\varepsilon > 0$ be given and let

$$\sigma_1(x, \cdot) = \lambda(x)\sigma_{11}(x, \cdot) + [1 - \lambda(x)]\sigma_{12}(x, \cdot)$$

where $\sigma_{11}(x, \cdot)$ is strongly finitely additive and $\lambda(x) > 1 - \varepsilon$. As in Lemma 2.1, write $\sigma = \lambda\sigma^1 + (1 - \lambda)\sigma^2$ where $\sigma^1 = (\lambda(x)\sigma_0, \sigma_{11})$, $\sigma^2 = ((1 - \lambda(x))\sigma_0, \sigma_{12})$ and $\lambda = \int \lambda(x)d\sigma_0$. σ^1 is strongly additive and $\|\sigma - \sigma^1\| < \varepsilon$. Since ε is arbitrary, σ is purely finitely additive. This establishes the hardest parts of (b). The rest of the assertions are easily verified. ∎

If κ is an infinite cardinal number and μ is a finite positive measure then μ is said to be *κ-additive* iff $\Sigma\{\mu(A_\alpha) : \alpha \in \Gamma\} = \mu(\cup\{A_\alpha : \alpha \in \Gamma\})$ for any disjoint family $\{A_\alpha : \alpha \in \Gamma\}$ with $|\Gamma| \leq \kappa$. The κ-additive elements of $P(X)$ form a split face of $P(X)$ [6]. The elements of the complementary split face are called *purely non-κ-additive* probabilities [6]. If there is a partition of X into κ or fewer μ-negligible sets then the positive measure μ is called *strongly non-κ-additive* and is purely non-κ-additive.

A cardinal κ is said to be *Ulam real valued measurable* (URVM) iff there is a countably additive diffuse probability μ on a set of cardinality κ. If μ is $\{0, 1\}$-valued then κ is said to be *Ulam measurable* (UM). If μ is a diffuse probability on κ which is κ-complete then κ is said to be *real valued measurable* (RVM), and if μ is $\{0, 1\}$-valued then κ is said to be *measurable*. The first URVM is RVM and the first UM is measurable. Any cardinal larger than a URVM (UM) is again a URVM (UM). It is consistent with ZFC that no URVM exists. It is also consistent that $2_0^{\aleph_0}$ be a RVM. However, no RVM can be a successor cardinal so this violates the continuum hypothesis. Any measurable cardinal κ is inaccessible in that if $\lambda < \kappa$ then $2^\lambda < \kappa$. In fact κ must be preceeded by an inaccessible number of inaccessible cardinals. A *measurable ideal cardinal* (MIC) is a κ so that there is a set X with $|X| = \kappa$ admitting a diffuse probability μ with $\mathcal{N}_\mu$ a κ-complete ideal hence, if μ is countably additive, with μ κ-complete. Any RVM is an MIC and any MIC larger than $2^{\aleph_0}$ is a measurable ideal cardinal. Solovay [28] shows that if existence of MIC's is consistent then so is the existence of measurable cardinals. Baumgartner, in unpublished notes, shows that if it is consistent that an MIC exist then it is also consistent that they exist yet no RVM exists. Solovay [28] shows that it is consistent that an RVM exist yet no measurable cardinal exists.

In [6] it is shown that $\lambda_p(\mu)$ is a countable sum of RVM's or is $\aleph_0$[6]. A positive finite measure μ is purely non-κ-additive iff it is a sum of countably many strongly non-κ-additive positive measures iff it is a countable convex combination of strongly non-κ-additive measures iff for all $\varepsilon > 0$ there is a strongly non-κ-additive ν, a $\lambda > 1 - \varepsilon$, a ν_0 with $\mu = \lambda\nu + (1 - \lambda)\nu_0$ [6]. Weaker than κ-additivity is κ-completeness. If μ is a finite positive measure then μ is *κ-complete* iff it is λ-additive for all cardinals $\lambda < \kappa$. Any finitely additive measure is $\aleph_0$-complete. Countably additive measures are the $\aleph_1$-complete measures. In general, if κ^+ denotes the successor to κ then μ is κ^+complete iff it is κ-additive. The κ-complete μ in $P(X)$ form a split face of $P(X)$ which is the (decreasing) intersection of the split faces of λ-additive probabilities for $\lambda < \kappa$. The complementary split face of *purely non-κ-complete* probabilities is the σ-convex hull of the face which is the union of the purely non-λ-additive probabilities as λ ranges over the cardinals less than κ. Here, purely non-λ-additive probabilities may be replaced by strongly non-λ-additive probabilities since a σ-convex hull is involved.

For diffuse measures μ there is a least cardinal $\lambda_s(\mu)$ so that μ is strongly-non-$\lambda_s(\mu)$-additive. This is the least cardinal number of a partition of X into μ-negligible sets [6]. There is a least cardinal $\lambda_p(\mu)$ so that μ is purely non-$\lambda_p(\mu)$-additive [6]. $\lambda_p(\mu) \leq \lambda_s(\mu) \leq |X|$. Neither $\lambda_p(\mu)$ nor $\lambda_s(\mu)$ are limit cardinals [6]. There is a unique cardinal $\lambda_c(\mu)$ so that μ is κ-complete but not κ^+-complete. We have $\lambda_c(\mu) \leq \lambda_p(\mu)$ with $\lambda_c(\mu) = \lambda_p(\mu) = \kappa$ if and only if μ is purely non-κ-additive but is κ-complete. The set of $\mu \in P(X)$ with $\lambda_p(\mu) = \lambda_c(\mu) = \kappa$ form a split face of $P(X)$. All such split faces are disjoint and the σ-convex hull of these faces is the split face of all diffuse measures. Notice that the discrete measures are those which are κ-additive for all cardinals κ. $\lambda_c(\mu) \geq \aleph_0$ for any diffuse μ. Of course $\kappa = \lambda_p(\mu) = \lambda_c(\mu)$ if μ is a diffuse κ-complete probability on a set of cardinality κ. Conversely, if $\kappa = \lambda_p(\mu) = \lambda_s(\mu)$ there is a surjection Φ of X onto the pointset of κ so that the image measure is κ-complete. The cardinal $\lambda_s(\mu)$ is shown in [6] to be an at most countable sum of MICs or to be $\aleph_0$.

It is possible, if an RVM exists, for a probability μ to be non-countably additive yet to have $\mathcal{N}_\mu$ be countably additive.

Example 4.1. Let κ be an RVM and let β be a diffuse κ-complete probability measure on Y where $|Y| = \kappa$. Let α_1 be a discrete probability on $X = N$ with $\alpha_1(n) > 0$ for all $n \in N$ and let α_2 be a diffuse probability on X. Then

$$\mu = \frac{1}{2}\sigma(\alpha_1, \beta) + \frac{1}{2}\sigma(\alpha_2, \beta)$$

is not countably additive yet $\mathcal{N}_\mu$ is countably additive. To see this note that from Proposition 4.1, $\sigma(\alpha_2, \beta)$ is purely finitely additive so μ isn't countably additive. Now suppose that $\{A(n) : n \in N\}$ is a sequence in $\mathcal{N}_\mu$. We have $\beta((A(n))_m) = 0$ for all $m \in N$ since $\alpha_1(\{m\}) > 0$ for all $m \in N$. Thus,

$$0 = \beta\left(\bigcup_{n=1}^{\infty} (A(n))_m\right) = \beta\left(\left[\bigcup_{m=1}^{\infty} A(n)\right]_m\right).$$

From this it follows that

$$\sigma(\alpha_1, \beta)\left(\bigcup_{m=1}^{\infty} A(n)\right) = \sigma(\alpha_2, \beta)\left(\bigcup_{n=1}^{\infty} A(n)\right) = 0.$$

Thus, $\bigcup_{n=1}^{\infty} A(n) \in \mathcal{N}_\mu$ which establishes countable additivity of $\mathcal{N}_\mu$.

PROPOSITION 4.2. *Any diffuse $\mu \in P(X)$ admits a unique decomposition as a countable convex combination $\Sigma\{\lambda_\kappa \mu_\kappa : \kappa$ a cardinal$\}$ where $\lambda_p(\mu_\kappa) = \lambda_c(\mu_\kappa) = \kappa$.*

Proof. If $\lambda_p(\mu) = \lambda_c(\mu) = \kappa$ set $\lambda_\kappa = 1$ and $\mu_\kappa = \mu$. Otherwise $\kappa = \lambda_c(\mu) < \lambda_p(\mu)$. For $\lambda < \kappa$, μ is λ-additive yet μ isn't κ-additive. Write

$$\mu = \lambda_\kappa \mu_\kappa + (1 - \lambda_\kappa)\mu'$$

where μ' is κ-additive and μ_κ is purely non κ-additive. We have $\lambda_p(\mu_\kappa) = \lambda_c(\mu_\kappa) = \kappa$ since μ_κ is λ-additive for $\lambda < \kappa$. We have

$$\lambda_p(\mu) \geqslant \lambda_p(\mu') \geqslant \lambda_c(\mu') > \kappa = \lambda_c(\mu).$$

Replace μ by μ' and proceed by induction to obtain $\lambda_{\kappa'}$, and $\mu_{\kappa'}$ for κ' between κ and $\lambda_p(\mu)$. ∎

COROLLARY 4.2.1. *If μ is $\{0, 1\}$-valued then $\lambda_p(\mu) = \lambda_c(\mu)$.*

COROLLARY 4.2.2. *If λ is a cardinal number then $\Sigma\{\lambda_\kappa \mu_\kappa : \kappa \leqslant \lambda\}$ is the purely non-λ-additive part of μ,* $\Sigma\{\lambda_\kappa \mu_\kappa : \kappa > \lambda\}$ is the λ-additive part of μ, *$\Sigma\{\lambda_\kappa \mu_\kappa : \kappa \geqslant \lambda\}$ is the λ-complete part of μ and $\Sigma\{\lambda_\kappa \mu_\kappa : \kappa < \lambda\}$ is the purely non-λ-complete part of μ.*

Degrees of additivity and of completeness are defined for ideals and for filters analogously to the corresponding definitions for measures. For instance an ideal (filter) is κ-additive iff the union (intersection) of any subfamily of cardinality at most equal to κ is an element of the ideal (filter). An ideal (filter) is κ-complete iff it is λ-additive for $\lambda < \kappa$.

The definition of κ-completeness of an ideal or filter here is that of κ-additivity in [6] and elsewhere but is now consistent with that for measures. For countably additive probabilities μ, the degree of additivity of μ is that either of its ideal $\mathcal{N}_\mu$ of negligible sets or, dually, that of its filter F_μ of sets of measure 1. This is a well known fact which we record as a lemma and shall prove.

LEMMA 4.3. *Let μ be a countably additive measure and κ an infinite cardinal number. Then μ is κ-additive (κ-complete) iff $\mathcal{N}_\mu$ is κ-additive (κ-complete).*

Proof. Let $\{A_\alpha : \alpha \in \Gamma\}$ be a disjoint collection in 2^X. There are at most countably many members of this collection with $\mu(A_\alpha) > 0$, say

$$\{A_{\alpha_1}, \dots, A_{\alpha_n}, \dots\}.$$

Let $\Gamma' = \{\alpha \in \Gamma : \alpha \neq \alpha_i \text{ all } i\}$. We have

$$\mu\left(\bigcup_\Gamma A_\alpha\right) = \sum_{i=1}^\infty \mu(A_{\alpha_i}) + \mu\{\cup A_\alpha : \alpha \in \Gamma'\}.$$

If $\mathcal{N}_\mu$ is $|\Gamma'|$ additive then

$$\mu\left(\bigcup_\Gamma A_\alpha\right) = \sum_{i=1}^\infty \mu(A_{\alpha_i}) = \sum \{\mu(A_\alpha) : \alpha \in \Gamma\}.$$

This suffices to establish the lemma.

PROPOSITION 4.4. *Let $\sigma = (\sigma_0, \sigma_1)$ be countably additive.*

(a) *σ is marginally and conditionally κ-additive iff σ is κ-additive.*

(b) *If σ is marginally or conditionally purely (strongly) non-κ-additive it is purely (strongly) non-κ-additive.*

(c) *If σ is strongly (purely) non-κ-additive and ν is the normalized κ-additive part of σ_0 then (ν, σ_1) is conditionally strongly (purely) non-κ-additive.*

Proof. To establish (a), it must first be shown that if σ is marginally and conditionally κ-additive then $\mathcal{N}_\sigma$ is κ-additive. Let $\{A_\lambda : \lambda < \kappa\}$ be a family in $\mathcal{N}_\sigma$ indexed by κ and let $A_\kappa = \cup \{A_\lambda : \lambda < \kappa\}$. We must show that $\sigma(A_\kappa) = 0$. For any $\lambda < \kappa$, $\int \sigma_1(x, (A_\lambda)_x)\sigma_0(dx) = 0$ so $\sigma_1(x,(A_\lambda)_x) = 0$ for σ_0-almost all x. Since σ_0 is κ-additive there is an $N \in \mathcal{N}_{\sigma_0}$ so that if $x \in X \setminus N$ then $\sigma_1(x, (A_\lambda)_x) = 0$ for all $\lambda < \kappa$. Since $\sigma_1(x, \cdot)$ is κ-additive $\sigma_1(x, (A_\kappa)_x) = 0$ for $x \in X \setminus N$. As a result,

$$\sigma(A_\kappa) = \int_{X \setminus N} \sigma_1(x, (A_\kappa)_x)\sigma_0(dx) = 0.$$

Thus, σ is κ-additive if it is marginally and conditionally κ-additive.

Conversely, assuming (b), if σ is κ-additive, σ_0 can't have a non-trivial purely non-κ-additive part, nor, using Lemma 2.1 can it be true that $\sigma_1(x, \cdot)$ has a non-trivial purely non-κ-additive part for a set of x with σ_0 positive measure.

(b) Suppose that σ is conditionally strongly non-κ-additive. For each $x \in X$ let $\{A_\lambda(x) : \lambda < \kappa\}$ be a partition of Y into κ sets, some of which may be $\emptyset$, each of which is $\sigma_1(x, \cdot)$-negligible. Set $A_\lambda = \cup \{\{x\} \times A_\lambda(x)\}$ for all $\lambda < \kappa$ to obtain a partition of $X \times Y$ into κ sets in $\mathcal{N}_\sigma$. Thus, σ is strongly non-κ-additive if it is conditionally strongly non-κ-additive. An easier demonstration shows that σ is strongly non-κ-additive if it is marginally strongly non-κ-additive.

Suppose that σ is conditionally purely non-κ-additive. For an $\varepsilon > 0$ and each $x \in X$, let $\sigma_1(x, \cdot) = \lambda(x)\sigma_{11}(x, \cdot) + (1 - \lambda(x))\sigma_{12}(x, \cdot)$ where $\lambda(x) > 1 - \varepsilon$ and $\sigma_{11}(x, \cdot)$ is strongly non-κ-additive. By Lemma 2.1, the strongly non-κ-additive strategic measure (σ_0, σ_{11}) is within ε of σ in variation norm. Since ε is arbitrary, σ is purely non-κ-additive. Similarly, if σ is marginally purely non-κ-additive then it is purely non-κ-additive.

(c) Suppose that σ is purely non-κ-additive. Then (ν, σ_1) is also purely non-κ-additive. Thus we may assume that $\nu = \sigma_0$. If σ is purely non-κ-additive then it is impossible that, for a set of x with positive σ_0 measure, $\sigma_1(x, \cdot)$ has a non-trivial κ-additive part. Thus, σ is conditionally purely non-κ-additive.

Now assume that σ is strongly non-κ-additive. Let $\{A_\lambda : \lambda < \kappa\}$ be a partition of $X \times Y$ into κ sets in $\mathcal{N}_\sigma$. For each $\lambda < \kappa$,

$$\int \sigma_1(x, (A_\lambda)_x)\sigma_0(dx) = 0$$

so $\sigma_1(x, (A_\lambda)_x) = 0$ for σ_0-almost all x. Since σ_0 is κ-additive we have, for σ_0-almost all x and for all λ, $\sigma_1(x, (A_\lambda)_x) = 0$. Thus, for σ_0-almost all x, $\sigma_1(x, \cdot)$ is strongly non-κ-additive. Thus, if σ is strongly non-κ-additive it is conditionally strongly non-κ-additive. ∎

COROLLARY 4.4.1. (a) *σ is κ-complete iff it is marginally and conditionally κ-complete.*

(b) *If σ is marginally or conditionally purely non-κ-complete, it is purely non-κ-complete.*

(c) *If σ is purely non-κ-complete and ν is the normalized κ-complete part of σ_0 then (ν, σ_1) is conditionally purely non-κ-complete.*

Let μ be countably additive and diffuse. For each cardinal $\kappa \geq \aleph_0$, let $\lambda_\kappa \mu_\kappa$ be the κ-complete purely non-κ-additive part of μ so $\mu_\kappa \in P(X)$ and $\lambda_\kappa \neq 0$ for at most countably many κ. There is a partition $\{A_\kappa : \kappa$ an infinite cardinal$\}$ with $\lambda_\kappa \mu_\kappa$ equal to μ on A_κ for all κ. This is a consequence of the Hahn decomposition theorem. We shall set $A_\kappa = \emptyset$ if $\lambda_\kappa = 0$ and call A_κ the *κ-complete purely non-κ-additive set* for μ noting that it is unique modulo $\mathcal{N}_\mu$.

PROPOSITION 4.5. *Let* $(\sigma_0, \sigma_1) = \sigma$ *be a countably additive strategic measure. For an infinite cardinal* λ *let* $A^{\circ}_{\lambda} \subset X$ $(A^1_{\lambda}(x) \subset Y)$ *be the* λ*-complete purely non-*λ*-additive set for* σ_0 $(\sigma_1(x, \cdot))$. *For an infinite cardinal* κ,

$$A_\kappa = \{(x, y) : x \in A^0_\kappa, y \in A^1_\lambda(x), \lambda \geqslant \kappa\}$$
$$\cup \{(x, y) : x \in A^0_\lambda, y \in A^1_\kappa(x), \lambda > \kappa\}$$

is the κ*-complete purely non-*κ*-additive set for* σ.

Proof. It follows from Proposition 4.4 and Corollary 4.4.1 and their proofs that the restriction of σ to A_κ is κ-complete and purely non-κ-additive. The complement of this set is the union of

$$\{(x, y) : x \in A^0_\lambda, \lambda < \kappa\} = E_1, \quad \{(x, y) : y \in A^1_\lambda(x), \lambda < \kappa\} = E_2$$

and

$$E_3 = \{(x, y) : x \in A^0_{\lambda_1}, \lambda_1 > \kappa, y \in A^1_{\lambda_2}(x), \lambda_2 > \kappa\}.$$

On $E_1 \cup E_2$, σ is purely non-κ-complete, and, on E_3, σ is κ-additive. Thus, A_κ is the κ-complete purely non-κ-additive set for σ. ∎

5. Uniform strategic measures

A diffuse measure μ on X is said to be *κ-uniform* if it annihilates all subsets of X of cardinality smaller than κ. Denote by $X^{<\kappa}$ the ideal of subsets of X of cardinality less than κ. μ is κ-uniform iff $X^{<\kappa} \subset \mathcal{N}_\mu$. If $\kappa = |X|$ then μ is said to be a *uniform measure* on X if it is κ-uniform. The *uniform ultrafilters* on X are those ultrafilters whose dual maximal ideals contain $X^{<|X|}$ [12]. The *κ-uniform ultrafilters* $\mathcal{U}_\kappa$ are those whose dual maximal ideals contain $X^{<\kappa}$ [12]. If we regard βX as the Stone space of 2^X then $\mathcal{U}_\kappa$ is a closed subset of βX corresponding to the filter dual to the ideal $X^{<\kappa}$. A measure μ is κ-uniform iff the measure $\tilde{\mu}$ corresponding to it under the Stone correspondence has $\text{supp}(\tilde{\mu}) \subset \mathcal{U}_\kappa$.

If μ is any finite diffuse measure then it is $\aleph_0$-uniform. There is a least cardinal number κ of a set A with $\mu(A) > 0$. This is the largest cardinal so that μ is κ-uniform. For this cardinal, there is a maximal disjoint collection of sets A with $|A| = \kappa$ and $\mu(A) > 0$. This collection is at most enumerable. The union A_κ of this family has the property that if $|A| = \kappa$ then $\mu(A \setminus A_\kappa) = 0$. Furthermore, μ, when restricted to A_κ, is uniform and, when restricted to $X \setminus A_\kappa$, is κ^+ uniform. If we let $\kappa_1 = \kappa$ and μ_{κ_1} be the restruction of μ to A_{κ_1}, we may find a smallest cardinal $\kappa_2 > \kappa_1$ of a subset of $A^c_{\kappa_1}$ with positive μ measure and a maximal set $A_{\kappa_2} \subset A^c_{\kappa_1}$ of cardinality κ_2 on which μ is uniform. Proceeding by induction we obtain an increasing sequence $\{\kappa_n : n \in N\}$ of cardinals, a disjoint sequence $\{A_{\kappa_n} : n \in N\}$ of sets with $|A_{\kappa_n}| = \kappa_n$ each of which is maximal in that μ is uniform on it. On

$X \setminus \bigcup_{n=1}^{m} A_{\kappa_n}$ (where $m = \omega$ is allowed and $\kappa_\omega = \sup_n \kappa_n$), μ is κ_m^+-uniform or κ_ω-uniform if $m = \omega$. Set $\mu' = \mu - \Sigma_{n=1}^{\infty} \mu_{\kappa_n}$ where μ_{κ_n} is the restriction of μ to A_{κ_n} so μ' is κ_ω-uniform. Replacing μ by μ', one may repeat the preceding procedure getting a new sequence of cardinals $\{\kappa_\omega, \kappa_{\omega+1}, \ldots\}$ and a new disjoint sequence $\{A_{\kappa_\omega}, A_{\kappa_{\omega+1}}, \ldots\}$ so that $|A_{\kappa_{\omega+1}}| = \kappa_{\omega+j}$, μ' is uniform on $A_{\kappa_{\omega+j}}$, and $A_{\kappa_{\omega+j}}$ is maximal in this regard. Furthermore $A_{\kappa_{\omega+j}}$ is disjoint from $\{A_{k_1}, \ldots, A_{k_n}, .\}$ if $j \geq 1$. Proceeding by transfinite induction, we have this proposition.

PROPOSITION 5.1. *Let μ be a diffuse measure on X. There is a countable ordinal α_0, an increasing sequence $\{\kappa_\alpha : \alpha < \alpha_0\}$ of cardinals, and a corresponding sequence $\{\mu_{\kappa_\alpha} : \alpha < \alpha_0\}$ of positive measures on X so that $\mu = \Sigma \{\mu_{\kappa_\alpha} : \alpha < \alpha_0\}$ and so that each μ_{κ_α} is κ_α-uniform. There is a corresponding sequence $\{A_{\kappa_\alpha} : \alpha < \alpha_0\}$ of subsets of X so that*

$$|A_{\kappa_\alpha}| = \kappa_\alpha,\ \mu_{\kappa_\alpha}(A_{\kappa_\alpha}) = \|\mu_{\kappa_\alpha}\|$$

for all $\alpha < \alpha_0$ and so that $A_{\kappa_\alpha} \cap A_{\kappa_\beta} = \emptyset$ if $\alpha > \beta$ and α is a successor ordinal (or for any α, if μ is countably additive). Furthermore, μ_{κ_α} is the restriction to A_{κ_α} of $\mu - \Sigma \{\mu_{\kappa_\beta} : \beta < \alpha\}$ for $\alpha < \alpha_0$.

Remark. We may call this proposition the "uniform decomposition proposition" since each μ_{κ_α} is the κ_α-uniform purely non-κ_α^+-uniform part of μ. The λ-uniform part of μ is

$$\Sigma \{\mu_{\kappa_\alpha} : \kappa_\alpha \geq \lambda\}$$

and the purely non-λ-uniform part of μ is

$$\Sigma \{\mu_{\kappa_\alpha} : \kappa_\alpha < \lambda\}.$$

If α is a limit ordinal less than α_0 and $\kappa_\alpha \neq \sup\{\kappa_\beta : \beta < \alpha\}$, then A_{κ_α} is disjoint from A_{κ_β} if $\beta < \alpha$. In particular, if the cofinality of κ_α isn't $\aleph_0$, or if κ_α isn't a limit cardinal, this is the case. If $\{A_{\kappa_\alpha} : \alpha < \alpha_0\}$ is disjoint it will be called the *uniform decomposition partition* for μ. Notice that if μ_1 is the purely non-κ_α-uniform part of μ and μ_2 is the purely non-κ_α^+-uniform part of μ then $\mu_1 - \mu_2 = \mu_{\kappa_\alpha}$. Hence, to determine the uniform decomposition of μ it is only necessary to know the purely non-λ-uniform parts of μ for all infinite cardinals λ. It is useful to know that a measure μ is purely non-λ-uniform for some λ iff

$$\mu(A) = \sup\{\mu(A^1): A^1 \subset A, |A^1| < \lambda\}$$

iff $\int f\, d\mu = \sup\{\int_A f\, d\mu : |A| < \lambda\}$ for any bounded function f. These suprema are maxima if the cofinality of λ is not $\aleph_0$.

COROLLARY 5.2.1. *In order that μ be purely non-λ-uniform for some cardinal λ it is necessary and sufficient that for any $A \subset X$ with $\mu(A) > 0$ there exist an $A' \in A^{<\lambda}$ with $\mu(A') > 0$.*

Proof. Suppose that there is an A with $\mu(A) > 0$ yet with $\mu(A') = 0$ when $A' \subset A$ with $|A'| < \lambda$. Then μ is λ-uniform on A hence isn't purely non-λ-uniform. Conversely, suppose that for all A with $\mu(A) > 0$ there is an $A' \subset A$ with $\mu(A') > 0$ with $|A'| < \lambda$. This implies that $\mu = \Sigma\{\mu_{\kappa_\alpha} : \alpha < \lambda\}$. Else there is $\kappa_\alpha \geq \lambda$ occurring in the uniform decomposition of μ. For A_{κ_α} we have $\mu(A_{\kappa_\alpha}) > 0$ yet $\mu(A') = 0$ for all $A' \subset A_{\kappa_\alpha}$ with $|A'| < \lambda$. This contradicts our supposition so $\mu = \Sigma\{\mu_{\kappa_\alpha} : \kappa_\alpha < \lambda\}$ is purely non-λ-uniform. ∎

For $\sigma = (\sigma_0, \sigma_1)$ and κ an infinite cardinal let $\lambda^\kappa \sigma_0^\kappa$ denote the diffuse purely non-κ-uniform part of σ_0 and, for $x \in X$, let $\lambda_\kappa(x)\sigma_1^\kappa(x, \cdot)$ denote the diffuse purely non-κ-uniform part of $\sigma_1(x, \cdot)$ where σ_0^κ and $\sigma_1^\kappa(x, \cdot)$ are diffuse purely non-κ-uniform probabilities.

PROPOSITION 5.2. *Let κ be an infinite cardinal number and let $\sigma = (\sigma_0, \sigma_1)$ be a diffuse strategic measure. The purely non-κ-uniform part σ^κ of σ is the measure described in one of* (a), (b), (c) *or* (d).

(a) *If σ is marginally discrete and conditionally diffuse then σ^κ is*

$$\lambda_\kappa(\lambda_\kappa(x)\lambda_\kappa^{-1}\sigma_0, \sigma_1^\kappa)$$

where $\lambda_k = \int \lambda_k(x)\sigma_0(dx)$.

(b) *If σ is marginally diffuse and conditionally discrete then $\sigma^\kappa = \lambda^\kappa(\sigma_0^\kappa, \sigma_1)$.*

(c) *If σ is marginally diffuse and conditionally diffuse then σ^κ is the supremum of the measures $\sigma^{\alpha,\kappa}$ where*

$$\sigma^{\alpha,\kappa} = \lambda^\kappa(\lambda_\alpha(x)\lambda_{\alpha,\kappa}^{-1}\sigma_0^\kappa, \sigma_1^\alpha)$$

with $\lambda_{\alpha,\kappa} = \int \lambda_\alpha(x)\sigma_0^\kappa(dx)$ for $\alpha < \kappa$ if κ is a limit cardinal and σ^κ is $\sigma^{\kappa,\kappa}$ otherwise.

(d) *If σ is the convex combination $\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3$ where γ_1 is a marginally discrete conditionally diffuse strategic measure, γ_2 is a marginally diffuse conditionally discrete strategic measure and γ_3 is a marginally and conditionally diffuse strategic measure then $\sigma^\kappa = \alpha_1\gamma_1^\kappa + \alpha_2\gamma_2^\kappa + \alpha_3\gamma_2^\kappa$.*

Proof. (d) is a consequence of the fact that purely non-κ-uniform measures form a split face of $P(X \times Y)$.

Of (a), (b) and (c), only the hardest, (c), will be established. Here the difficult case is where κ is a limit cardinal and this will be the case established. Let $\nu = \sup\{\sigma^{\alpha,k} : \alpha < k\}$.

Let $|A| = \alpha < \kappa$. For any x, $|A_x| \leq \alpha$ and $|\{x : A_x \neq \emptyset\}| \leq \alpha$. We have

$$\begin{aligned}\sigma(A) &= \int \sigma_1(x, A_x)\sigma_0(dx) = \lambda^\kappa \int \sigma_1(x, A_x)\sigma_0^\kappa(dx)\\ &= \lambda^\kappa \int \lambda_\beta(x)\sigma_1^\beta(x, A_x)\sigma_0^\kappa(dx)\\ &= \sigma^{\beta,\kappa}(A) \quad \text{if } \alpha \leq \beta.\end{aligned}$$

Thus, $(\sigma - \nu)(A) = 0$ and $\sigma - \nu$ is κ-uniform. Now let $A \subset X \times Y$ have $\nu(A) > 0$ hence have $\sigma^{\alpha,\kappa}(A) > 0$ for some $\alpha < \kappa$. There is an $E \subset X$ so that $|E| = \beta < \kappa$ and $\sigma(A') > 0$ if $A' = A \cap [E \times Y]$. This is because σ_0^κ is purely non-κ-uniform as is ν_X. For each $x \in E$ let $E(x) \subset Y$ have $|E(x)| < \alpha$ and $E(x) \subset A'_x = A_x$ with $\sigma_1^\alpha(x, E(x)) \geq \frac{1}{2}\sigma_1^\alpha(x, E(x))$. Set

$$A'' = \cup \{\{x\} \times E(x) : x \in E\} \subset A.$$

We have $\sigma^{\alpha,\kappa}(A'') \geq \frac{1}{2}\sigma^{\alpha,\kappa}(A') > 0$. Thus, $\nu(A'') > 0$. Since $|A''| \leq \alpha \cdot \beta < \kappa$, ν is purely non-κ-uniform by Corollary 5.2.1. Consequently, $\sigma - \nu$ is the κ-uniform part of σ and ν is the purely non-κ-uniform part. ■

COROLLARY 5.2.1. *If σ is marginally diffuse, marginally countably additive, and conditionally diffuse then $\sigma^\kappa = \sigma^{\kappa,\kappa}$ even if κ is a limit cardinal.*

Proof. It is only necessary to show that $\sigma^{\kappa,\kappa}$ is purely non-κ-uniform for $\sigma - \sigma^{\kappa,\kappa}$ is κ-uniform by the same argument as used in the preceding proof. If $\sigma^{\kappa,\kappa}(A) > 0$ for an $A \subset X \times Y$ we must find $A'' \subset A$ with $|A''| < \kappa$ with $\sigma^{\kappa,\kappa}(A'') > 0$. There is an $E \subset X$ with $|E| < \kappa$ so that $\sigma^{\kappa,\kappa}(A') > 0$ where $A' = A \cap (E \times Y)$. For each $\alpha < \kappa$ let E_α be those $x \in E$ so that there is an $A(x, \alpha) \subset A'_x = A_x$ with $\sigma_1^\kappa(x, A(x, \alpha)) \geq \frac{1}{2}\sigma_1^\kappa(x, A_x)$ and $|A(x, \alpha)| \leq \alpha$. Since each $\sigma_1^\kappa(x, \cdot)$ is purely non-κ-uniform, E_α must increase to E as α increases. If κ is of countable cofinality, $\sigma_0^\kappa(E_\alpha) > 0$ for some $\alpha < \kappa$. In this case, set

$$A'' = \cup \{\{x\} \times A(x, \alpha) : X \in E_{\alpha 0}\}$$

and note that, as in the preceding proof, $\sigma^{\kappa,\kappa}(A'') > 0$ so $\sigma^{\kappa,\kappa}$ is purely non-κ-uniform. If κ isn't of countable cofinality then $\sigma^\kappa = \sigma^\alpha$ for some $\alpha < \kappa$ which is either a successor cardinal or is of countable cofinality. In either case $\sigma^\alpha = \sigma^{\alpha,\alpha} = \sigma^{\kappa,\kappa}$ which establishes the corollary. ▮

COROLLARY 5.2.2. *Let σ be marginally and conditionally diffuse and countably additive and let κ be an infinite cardinal. Let $A_0 \subset X$ with $|A_0| \leq \kappa$ be such that σ_0 is purely non-κ-uniform on A_0 with $\sigma_0^\kappa(A_0) = \|\sigma_0^\kappa\|$. For each $x \in A_0$, let $A_1(x) \subset Y$ be such that $|A_1(x)| \leq \kappa$ and $\sigma_1(x, \cdot)$ is purely non-κ-uniform on $A_1(x)$ with*

$$\sigma_1^\kappa(x, A_1(x)) = \|\sigma_1^\kappa(x, \cdot)\|.$$

Set $A = \cup \{\{x\} \times A_1(x) : x \in A_0\}$. Then σ is purely non-κ-uniform on A and $\sigma(A) = \|\sigma^\kappa\|$.

Proof. Immediate.

Remark. If κ is a cardinal whose cofinality is not $\aleph_0$ then the assumption of conditional and marginal countable additivity may be dropped.

6. Singularity and Absolute Continuity of Reverse Strategic Product Measures if One Margin is Diffuse and Countably Additive

We start this section with a result which indicates that reverse strategic product measures may be nearly strategic even if both margins are diffuse. In fact our result is much stronger. If $\alpha \in P(X)$ and $\beta \in P(Y)$ then $\alpha \otimes \beta$ denotes the usual product measure on the product algebra $2^X \otimes 2^Y$. Both $\sigma(\alpha, \beta)$ and $\tau(\alpha, \beta)$ extend $\alpha \otimes \beta$. The $\alpha \otimes \beta$-completion of $2^X \times 2^Y$ consists of all $E \subset X \times Y$ for which, for all $\varepsilon > 0$, there exist $\{E_1, E_2\} \subset 2^X \otimes 2^Y$ with $E_1 \subset E \subset E_2$ with $\alpha \otimes \beta(E_2 \setminus E_1\} < \varepsilon$. The $\alpha \otimes \beta$-completion of $2^X \otimes 2^Y$ is the largest subalgebra of $2^{X \times Y}$ to which $\alpha \otimes \beta$ has a unique extension. When we say below that $\tau(\alpha, \beta) = \alpha \otimes \beta$ we mean that $2^{X \times Y}$ is the $\alpha \otimes \beta$-completion of $2^X \otimes 2^Y$. In this case $\tau(\alpha, \beta)$ hence $\sigma(\alpha, \beta)$ is the unique extension of $\alpha \otimes \beta$ to $2^{X \times Y}$.

Note that if $\mathscr{B}$ is a subalgebra of $2^{X \times Y}$ and $\{\mu_n : n \in N\}$ are finitely additive probabilities on $\mathscr{B}$ then any E in the μ_n-completion of $\mathscr{B}$ for all n is in the μ-completion of $\mathscr{B}$ for any μ which is a countable convex combination of $\{\mu_n : n \in N\}$. To see this, write μ as $\Sigma_{n \in N} \lambda_n \mu_n$ and pick m so that $\Sigma_{n=1}^{m} \lambda_n \geq 1 - \varepsilon$ for a given ε. Pick $\{E_1, E_2\} \subset \mathscr{B}$ with $E_1 \subset E \subset E_2$ and $\mu_n(E_2 \setminus E_1) < \varepsilon$ for all $n = 1, \ldots, m$. Then, $\mu(E_2 \setminus E_1) < \varepsilon$.

Proposition 6.1. *Let κ be an infinite cardinal.*

(a) *Let $\alpha \in P(X)$ be 2^κ-additive and $\beta \in P(Y)$ purely non-κ^+-uniform. If α is atomic or if β is countably additive then $\tau(\alpha, \beta) = \alpha \otimes \beta$.*

(b) *If $\alpha \in P(X)$ is κ-additive and atomic and $\beta \in P(Y)$ is purely non-κ^+-uniform and atomic then $\tau(\alpha, \beta) = \alpha \otimes \beta$.*

Proof. Since β is purely non-κ^+-uniform iff it gives measure 1 to a set of cardinality κ we may assume that $|Y| \leq \kappa$.

(b) If α and β are $\{0,1\}$-valued this is the content of Corollary 7.24 (b) in [12]. If $\alpha = \Sigma_{n=1}^{\infty} \lambda_n \alpha_n$ and $\beta = \Sigma_{m=1}^{\infty} \gamma_m \beta_m$ where α_n and β_m are $\{0, 1\}$-valued then each α_n is κ-additive and $\alpha \otimes \beta = \Sigma_{n=1}^{\infty} \Sigma_{m=1}^{\infty} \lambda_n \gamma_m \alpha_n \otimes \beta_m$. Since $2^{X \times Y}$ is the $\alpha_n \otimes \beta_m$-completion of $2^X \otimes 2^Y$, it is also the $\alpha \otimes \beta$ completion of $2^X \otimes 2^Y$.

(a) Let $E \subset X \times Y$. We may regard E as the graph of a correspondence $E : X \to 2^Y$ given by $E(x) = E_x$. For $A \subset Y$ let $E^{-1}(A) = \{x : E_x = A\}$. Since

$$X = \cup \{E^{-1}(A) : A \in 2^Y\}$$

and α is 2^κ-additive and $|Y| \leq \kappa$, we have

$$\alpha(X) = \Sigma \{\alpha(E^{-1}(A)) : A \in 2^Y\}.$$

Let $\{A_n : n \in N\}$ enumerate the $A \in 2^Y$ with $\alpha(E^{-1}(A)) > 0$, so

$$\alpha(X) = \Sigma \alpha(E^{-1}(A_n)).$$

Set $N = X \setminus \cup_{n=1}^{\infty} \{E^{-1}(A_n)\}$ so $\alpha(N) = 0$. Set $E' = E \cap [(X \setminus N) \times Y] \subset E$ so

$$\tau(\alpha, \beta)(E') = \tau(\alpha, \beta)(E) \quad \text{and} \quad \alpha \otimes \beta(E') = \alpha \otimes \beta(E).$$

We have $E' = \cup_{n=1}^{\infty} E^{-1}(A_n) \times A_n$. When β is countably additive, so is $\tau(\alpha, \beta)$, by Proposition 4.1. As a result, for any $\varepsilon > 0$, there is an m so that

$$\alpha \otimes \beta\left(\bigcup_{n=1}^{m} E^{-1}(A_n) \times A_n\right) \geqslant \tau(\alpha, \beta)(E') - \varepsilon = \tau(\alpha, \beta)(E) - \varepsilon.$$

That is, there is an $E_1 \subset E$ with

$$E_1 \in 2^X \times 2^Y \quad \text{and} \quad \varepsilon + \alpha \otimes \beta(E_1) \geqslant \tau(\alpha, \beta)(E).$$

Similarly, there is an $E_2 \in 2^X \times 2^Y$ with $E \subset E_2$ and $\tau(\alpha, \beta)(E) \geqslant \alpha \otimes \beta(E_2) - \varepsilon$. Thus, $\alpha \otimes \beta(E_2 \setminus E_1) < 2\varepsilon$ which shows, since ε is arbitrary, that E is in the $\alpha \otimes \beta$-completion of $2^X \otimes 2^Y$. This establishes (a) in the case β that is countably additive.

In (a), when α is atomic the proof immediately reduces to the case where α is $\{0,1\}$-valued. Obtain $E' = \cup_{n=1}^{\infty} E^{-1}(A_n) \times A_n$ as before. In this case,

$$\alpha(E^{-1}(A_n)) > 0 \text{ for only one } n,$$

say $n = 1$, and $E' = E^{-1}(A_n) \times A_n$ so $\alpha \otimes \beta(E') = \alpha \otimes \beta(E) = \tau(\alpha, \beta)(E)$ which suffices to establish this case. ∎

Remarks (1) α and β may have discrete parts. If α and β are discrete, the equality $\sigma(\alpha, \beta) = \tau(\alpha, \beta)$ is Fubini's Theorem.

(2) If λ is a real-valued measurable cardinal we may let X be a set with $X = \lambda$ and α be a diffuse λ-complete probability on X. If Y is any set such that $\kappa = |Y|$ satisfies $2^\kappa < \lambda$ then for any $\beta \in P(Y)$, $\sigma(\alpha, \beta) = \tau(A, \beta)$. Of course, if Y is infinite then $\lambda > 2^{\aleph_0}$, hence is a measurable cardinal. In this case, as soon as $|Y| < |X|$ (i.e., $\kappa < \lambda$) we have $2^\kappa < \lambda$.

(3) β may be countably additive and diffuse or it may be purely finitely additive. However for this result α must be countably additive if β is purely finitely additive.

(4) $\tau(\alpha, \beta) = \alpha \otimes \beta$ if the assumptions on α and β are interchanged.

LEMMA 6.2. *Let $\alpha \in P(X)$ be diffuse and countably additive and let $\beta \in P(Y)$ be diffuse. Then $\tau(\alpha, \beta)$ is singular to all conditionally discrete strategic measures.*

Proof. $\tau(\alpha, \beta)$ is singular to any conditionally discrete $\sigma = (\sigma_0, \sigma_1)$ if $\sigma_0 \perp \alpha$. We may, therefore, only consider σ with $\sigma_0 \ll \alpha$ hence with σ_0 countably additive. By Corollary 3.4.5, σ is a countable convex combination of strategic measures of the form $(\overline{\sigma}_0, \overline{\sigma}_1)$ where $\overline{\sigma}_0 \ll \sigma_0$ and $\sigma_1(x, dy) =$

$\delta_{f(x)}(dy)$ for some function $f : X \to Y$. Therefore we may assume that $\sigma_0 = \bar{\sigma}_0$ and $\bar{\sigma}_1 = \sigma_1$. Let F be the graph of f so $\sigma(F) = 1$. For each $\delta > 0$ there are only finitely many $y \in Y$ so that $\alpha(F^y) = \alpha(f^{-1}(y)) > \delta$. Since β is diffuse it follows that, for β-almost all y, $\alpha(F^y) \leqslant \delta$. Thus, $\tau(\alpha, \beta)(F) = 0$. This shows that $\tau(\alpha, \beta) \perp \sigma$ which establishes the lemma. ∎

LEMMA 6.3. *Suppose that* $|X| = |Y| = \kappa$, $\alpha \in P(X)$ *is* κ*-uniform and* $\beta \in P(Y)$. *If* $\sigma = (\sigma_0, \sigma_1)$ *is conditionally* κ*-uniform then* $\tau(\alpha, \beta) \perp \sigma$.

Proof. Regard κ as the set of ordinals of cardinal smaller than κ. Regard X and Y as equal to κ. Set

$$D = \{(x, y) \in X \times Y : y < x\}.$$

Since $D_x = \{y : y < x\}$ has $|D_x| < \kappa$ we have $\sigma_1(x, D_x) = 0$ for all x, hence $\sigma(D) = 0$. Since $X \setminus D^y = \{x : x \leqslant y\}$ has $|X \setminus D^y| < \kappa$, it follows that $\alpha(D^y) = 1$ for $y \in Y$. Thus, $\tau(\alpha, \beta)(D) = 1$. Consequently, $\tau(\alpha, \beta) \perp \sigma$. ∎

COROLLARY 6.3.1. *If* σ *is a conditionally* κ*-uniform strategic measure and* τ *is a conditionally* κ*-uniform reverse strategic measure then* $\sigma \perp \tau$.

Remark. These results extend Lemma 7.22 (a) of [12], which says that $\sigma(\alpha, \beta) \perp \tau(\alpha, \beta)$ if α and β are uniform $\{0, 1\}$-valued measures on κ.

PROPOSITION 6.4. *Let* $\alpha \in P(X)$ *and* $\beta \in P(Y)$ *be* κ*-complete diffuse measures where* $\kappa = |X| = |Y|$. *Then* $\tau(\alpha, \beta)$ *is purely non-strategic.*

Proof. We must show that $\tau(\alpha, \beta) \perp \sigma = (\sigma_0, \sigma_1)$ for all σ. We may assume that $\sigma_0 \ll \alpha$ hence that σ_0 is κ-complete, diffuse and, as a result κ-uniform. If σ were conditionally purely non-κ-complete then by Lemma 4.3 it would be purely non-κ-complete hence singular to $\tau(\alpha, \beta)$. Thus, we may assume that σ is conditionally κ-complete. If σ is conditionally discrete, Lemma 6.2 shows that $\sigma \perp \tau(\alpha, \beta)$. Thus, we may assume that σ is conditionally diffuse. For all x, $\sigma_1(x, \cdot)$ is a κ-complete diffuse probability hence is κ-uniform. Lemma 6.3 shows that $\tau(\alpha, \beta) \perp \sigma$ which establishes the proposition. ∎

PROPOSITION 6.5. *Let* X *and* Y *be arbitrary. Let* $\alpha \in P(X)$ *and* $\beta \in P(Y)$ *be* κ*-complete purely non-*κ*-additive probabilities. Then* $\tau(\alpha, \beta)$ *is purely non-strategic.*

Proof. The case $\kappa = \aleph_0$ is Theorem 1. Thus we may assume that $\kappa > \aleph_0$, hence that α and β are countably additive. We first examine the case where $Y = \kappa$.

Suppose that $\tau(\alpha, \beta)$ is not singular with respect to $\sigma = (\sigma_0, \sigma_1)$. By

Corollary 2.2.4 it may be assumed that $\sigma_0 = \alpha$. Decompose the strategic measure σ into a conditionally diffuse part σ^{diff} and a conditionally discrete part σ^{disc} using Lemma 2.1. By Lemma 6.2, $\tau(\alpha, \beta) \perp \sigma^{\text{disc}}$. Thus, $\tau(\alpha, \beta)$ and σ^{diff} aren't singular. Thus, we may assume that σ is conditionally diffuse. Decompose σ into conditionally κ-complete and conditionally purely non-κ-complete parts σ^1 and σ^2 respectively. Since $\tau(\alpha, \beta)$ is κ-complete and σ^2 (with X-margin α) is purely non-κ-complete, by Corollary 4.4.1, $\tau(\alpha, \beta)$ must not be singular to σ^1. That is, we may assume that $\sigma = (\alpha, \sigma_1)$ is conditionally κ-complete and conditionally diffuse, hence conditionally uniform. Since α is purely non-κ-additive, it is approximable in variation norm by a sequence $\{\alpha_n : n \in N\}$ of strongly non-κ-additive measures. The sequence $\{\tau(\alpha_n, \beta) : n \in N\}$ approaches $\tau(\alpha, \beta)$ in variation norm and the sequence $\{\sigma_n : n \in N\}$ of strategic measures given by $\sigma_n = (\alpha_n, \sigma_1)$ approaches σ in variation norm. For some n, $\tau(\alpha_n, \beta)$ is not singular with respect to σ_n. Thus, we may assume that α is strongly non-κ-additive. Since α is strongly non-κ-additive there is a decreasing sequence $\{X_\lambda : \lambda \in \kappa\}$ of subsets of X with empty intersection and with $\alpha(X_\lambda) = 1$ for all $\lambda \in \kappa$. Set

$$S = \cup (X_\lambda \times \{\lambda\}) \subset X \times \kappa.$$

Note that, for each $x \in X$, $|S_x| < \kappa$ so $\sigma_1(x, s_x) = 0$ and, as a result, $\sigma(S) = 0$. On the other hand, $\tau(\alpha, \beta)(S) = \int_\kappa \alpha(X_\lambda)\beta(d\lambda) = 1$. Thus, $\tau(\alpha, \beta) \perp \sigma$. As a result, $\tau(\alpha, \beta) \in \Sigma^\perp$. This establishes the proposition if $Y = \kappa$.

Now let Y be arbitrary. Since β is approximable in variation norm by strongly non-κ-additive measures a familiar argument shows that to show that $\tau(\alpha, \beta) \in \Sigma^\perp$ we need only establish the special case where β is strongly non-κ-additive. In this case there is a $\Phi : Y \to \kappa$ so that the image β' of β under Φ is a κ-complete diffuse measure on κ. A repetition of an argument in the proof of Theorem 1.1 shows that since $\tau(\alpha, \beta')$ is purely non-strategic on $X \times \kappa$, so is $\tau(\alpha, \beta)$ on $X \times Y$. ■

We conclude with an example where $\tau(\alpha, \beta)$ is purely non-strategic with α purely finitely additive and β countably additive. For this example α and β are both chosen $\{0,1\}$-valued so $\tau(\alpha, \beta)$ is $\{0,1\}$-valued. By Corollary 3.4.1, it suffices to show that $\tau(\alpha, \beta) \neq (\sigma_0, \sigma_1) = \sigma$ where σ is marginally and conditionally $\{0,1\}$-valued. Here $|Y|$ may be chosen to be a measurable cardinal λ with β the corresponding λ-complete $\{0,1\}$-valued measure on Y. X is chosen with $|X| = \lambda$ and α is chosen so that its ultrafilter of sets of measure 1 is regular. Recall from [12] that an ultrafilter $\mathcal{U}$ is *regular* on a set X iff there is a family $\{X_\alpha\}$ of cardinality $|X|$ in $\mathcal{U}$ so that the intersection of any infinite subfamily is empty. We may imitate the definition of regularity of ultrafilters and say that a measure μ on X is *regular* iff there exists a family $\{X_\alpha\}$ of subsets of X which has cardinality $|X|$ so that $\mu(X_\alpha) = 1$ for all α, yet so that $\cap_{\alpha \in D} X_\alpha = \emptyset$ for any infinite set D of indices. A more general notion of κ-regularity is definable for both ultrafilters and measures where κ is a cardinal, and one requires the family $\{X_\alpha\}$ to be of cardinal κ. Any κ-regular measure is, of necessity, strongly finitely additive. On any infinite set X there exist regular ultrafilters [12, Lemma 7.11.].

PROPOSITION 6.6. *Let* $|X| = |Y| = \lambda$. *Let* β *be a diffuse* $\{0, 1\}$*-valued element of* $P(Y)$ *and let* α *be a regular* $\{0, 1\}$*-valued element of* $P(X)$. *Then* $\tau(\alpha, \beta)$ *is purely non-strategic.*

Proof. Let $\{T(y) : y \in Y\}$ be such that $\alpha(T(y)) = 1$ for all y and such that if $D \subset Y$ is infinite then $\cap \{T(y) : y \in D\} = \emptyset$. Set

$$T = \cup T(y) \times \{y\}.$$

It is immediate that $\tau(\alpha, \beta)(T) = 1$. Let $\sigma = (\sigma_0, \sigma_1)$ be marginally and conditionally $\{0, 1\}$-valued with $\sigma = \tau(\alpha, \beta)$. Note that if $x \in X$ then

$$T_x = \{y : x \in T(y)\}$$

is finite. Since $\sigma(T) = 1$ $\sigma_1(x, T_x) = 1$ for σ_0-almost all x. That is, for σ_0-almost all x, $\sigma_1(x, \cdot) = \delta_{t(x)}$ for a unique $t(x) \in T_x$. Thus, there is a $t : X \rightarrow Y$ so that $\int f(x, y) d\sigma = \int f(x, t(x))\sigma_0(dx)$, for all f. If

$$\text{graph}(t) = \{(x, t(x)) : x \in X\}$$

then

$$1 = \sigma(\text{graph}(t)) = \tau(\alpha, \beta)(\text{graph}(t)) = \int \alpha(t^{-1}(y))\beta(dy).$$

Thus, $\alpha(t^{-1}(y)) = 1$ for β-almost all y. Since $t^{-1}(y_1) \cap t^{-1}(y_2) = \emptyset$ if $y_1 = y_2$, there is only one y with $\alpha(t^{-1}(y)) = 1$. This implies that β is δ_y for some $y \in Y$ which contradicts the fact that β is diffuse. ■

COROLLARY 6.6.1. *Let* λ *be an infinite cardinal number. Let* α *be a* λ*-regular* $\{0, 1\}$*-valued element of* $P(X)$ *and let* β *be a purely non-*λ*-additive* $\{0, 1\}$*-valued element of* $P(Y)$. *Then* $\tau(\alpha, \beta)$ *is purely non-strategic.*

Proof. Since β is $\{0, 1\}$-valued it is strongly non-λ-additive. There is a surjection $\pi_X : X \rightarrow \lambda$ so that the image of β under π_X is diffuse. Thus it may be assumed, to start, that $|X| = \lambda$ and that β is diffuse on X. To finish the proof note that the cardinality of Y wasn't important in the proof of Proposition 6.6; we only needed the fact that α was λ-regular. ■

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