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PREDICTION FROM PART OF THE PAST OF A STATIONARY PROCESS

BY

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1. Introduction

Let w be the spectral density of a stationary process $X(t)$ $(-\infty < t < \infty)$. It will be assumed that $(\log w)/(1 + x^2)$ is integrable on R with respect to Lebesgue measure. Thus $w(x) = |h(x)|^2$ where h is an outer function in H^2, the Hardy space for the upper half-plane. Let Z denote the space of measurable functions which are square-integrable against the measure $w(x)dx$, and let $Z(a, b)$ denote the closed subspace of Z generated by $\{e_t : a \leqslant t \leqslant b\}$ $(e_t(x) \equiv e^{itx})$. The obvious meanings will be ascribed if a or b is $\pm\infty$. The problem studied here is that of approximating orthogonal projection on $Z(-a, a)$. In [5], Dym and McKean worked out a recipe for projection on $Z(-a, a)$, but their solution is difficult to apply. A less general approach was adopted by Segier [10] to work out a projection formula in the case $w = |P|^2/|B|^2$ where P is a polynomial and B is an entire function of finite exponential type. The latter approach will be followed here: under mild assumptions, $Z(-a, a) = Z(-a, \infty) \cap Z(-\infty, a)$, so the desired projection may be approximated by "projecting back and forth" on $Z(-a, \infty)$ and $Z(-\infty, a)$; projection onto these last subspaces is straightforward. How good this scheme is depends upon structural properties of the weight w. This is discussed in Sections 3 and 4; a connection between these approximations and strong mixing is given in Section 5.

2. Preliminaries

Let L^2 denote the Hilbert space of functions on R which are square-summable with respect to Lebesgue measure. Then the map $S : f \to hf$ is an isometry of Z into L^2. Moreover, since h is outer, S is surjective (See [5], p. 97) and maps $Z(-\infty, a)$ and $Z(a, \infty)$ respectively onto $(e_a h/\overline{h})\overline{H^2}$ and $e_a H^2$ (where the bar denotes complex conjugation). The following notation will be used:

(1) P_a is projection onto $e_a H^2$ in L^2;
(2) Q_a is projection onto $(e_a h/\overline{h})\overline{H^2}$;

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(3) $M_a = e_{-a}H^2 \cap (e_a h/\bar{h})\bar{H}^2$;
(4) π_a is projection onto M_a;
(5) H^∞ is the space of essentially bounded functions on R whose Poisson extensions to the upper half-plane are analytic.

Then $S^{-1}\pi_a S$ is the projection from Z onto $Z(-\infty, a) \cap Z(-a, \infty)$. This is the projection which will be studied in light of the following theorem due to Dym [3].

THEOREM. *If $1/w$ is locally integrable and $\|e_{2s}h/\bar{h} - F\|_\infty < 1$ for some $s > 0$ and $F \in H^\infty$, then $Z(-a, a) = Z(-\infty, a) \cap Z(-a, \infty)$ for every $a > s$.*

For a fixed outer function h and a real number T let $r(T)$ and $r_*(T)$ be the numbers defined by

$$r(T) = \operatorname{dist}(e_T\bar{h}/h, H^\infty) = \inf\{\|e_T\bar{h}/h - g\|_\infty : g \in H^\infty\}$$

and

$$r_*(T) = \operatorname{dist}(e_T h/\bar{h}, H^\infty).$$

A well-known duality argument of Helson-Szego [7] shows that

$$r(2a) = \cos_Z(Z(-\infty, -a), Z(a, \infty))$$

and

$$r_*(2a) = \cos_{L^2}(e_{-a}\bar{H}^2, (e_a h/\bar{h})H^2)$$

where $\cos_H(A, B)$ denotes the cosine of the angle between the subspaces A and B of H.

3. Approximation of π_a

The operator studied here is $Q_a P_{-a}$. The number $r_*(2a)$ gives an estimate of $\|(Q_a P_{-a})^n - \pi_a\|$.

THEOREM 1. *Let w^{-1} be locally integrable. Then the following are equivalent:*

(i) $r^*(2a) < 1$;
(ii) $(Q_a P_{-a})^n \to \pi_a$ *exponentially fast in operator norm;*
(iii) *w may be written in the form*

$$w(x) = |B(x)|^{-2}\exp[u(x) + \tilde{v}(x)] \quad (-\infty < x < \infty);$$

where B is entire of exponential type at most a, and where u and v are real functions in L^∞ with $\|v\|_\infty < \pi/2$ ($\tilde{v}$ denotes the harmonic conjugate of v).

Proof. First, note (as in [4]) that

$$M_a = [e_{-a}\overline{H}^2 + (e_a h/\overline{h})H^2]^{\perp}.$$

If (i) is true, then the bracketed summands above are at a positive angle so their sum is closed. Thus,

$$L^2 = M_a \oplus [e_{-a}\overline{H}^2 + (e_a h/\overline{h})H^2].$$

Next note that the bracketed summands may also be identified with $(e_{-a}H^2/M_a)^{\perp}$ and $[(e_a h/\overline{h})\overline{H}^2/M_a]^{\perp}$, respectively, in L^2/M_a. Hence,

$$\begin{aligned} 1 > r_*(2a) &= \cos((e_{-a}H^2/M_a)^{\perp}, [(e_a h/\overline{h})\overline{H}^2/M_a]^{\perp}) \\ &= \cos(e_{-a}H^2/M_a, (e_a h/\overline{h})\overline{H}^2/M_a) \end{aligned}$$

(see [6]). The last quantity is just the norm of the operator

$$(Q_a - \pi_a)(P_{-a} - \pi_a).$$

Thus,

$$\begin{aligned} \|(Q_aP_{-a})^n - \pi_a\| &= \|(Q_aP_{-a} - \pi_a)^n\| \\ &= \|[(Q_a - \pi_a)(P_{-a} - \pi_a)]^n\| \\ &\leqslant r_*(2a)^n, \end{aligned}$$

so (ii) follows. If, on the other hand, $\|(Q_aP_{-a})^n - \pi_a\| < 1$ for some n, so is the norm of the positive operator $(P_{-a}Q_aP_{-a} - \pi_a)^n$, so

$$\|Q_aP_{-a} - \pi_a\| = \|P_{-a}Q_aP_{-a} - \pi_a\|^{1/2} < 1$$

and (i) is true.

The equivalence of (i) and (iii) relies on a standard analytic continuation argument. If (i) holds, it is possible to write $h/\overline{h} = e_{-2a}\, b \exp[i(\tilde{u} - v)]$ where b is an inner function and where u and v are real functions in L^∞ with $\|v\|_\infty < \pi/2$ (see [7]). It then follows that

$$F = e_{-2a}\, b \exp[(u + \tilde{v}) + i(\tilde{u} - v)]/h^2 \geqslant 0 \quad \text{a.e.}$$

on R and extends analytically into the upper half-plane. Since $1/h^2$ is locally integrable on R, and the other factor is essentially in H^1, it is possible to continue F analytically into the lower half-plane (see [8]). Furthermore, F is of bounded type $\leqslant 2a$ in both half-planes, so by a theorem of Krein, F has exponential type $\leqslant 2a$. (See [1, p. 38] for a discussion of this). Also, F may be factored: $F(x) = |B(x)|^2$ $(-\infty < x < \infty)$ where B is entire and of exponential type $\leqslant a$. Thus, $w = |h|^2 = |B|^{-2}\exp(u + \tilde{v})$ as desired. Conversely, if (iii) holds, then

$$h/\overline{h} = e_{-2\tau} b \exp[i(\tilde{u} - v)]$$

where $\tau \leq a$ and b is a Blaschke product whose zeroes arise from the zeroes of B. Then

$$e_{2a}h/\bar{h} = e_{2(a-\tau)}\, b \exp[i(\tilde{u} - v)]$$

whose distance to H^∞ is less than unity so $\rho_*(2a) < 1$, and the theorem is proved.

4. The Compactness of $Q_aP_{-a} - \pi_a$

The result of this section relies on properties of Toeplitz operators and functions on the unit circle T. If ϕ is an essentially bounded function on R, let $W(\phi)$ denote the Wiener-Hopf operator on H^2 defined by $W(\phi)f = P(\phi f)$ where P is the orthogonal projection from L^2 onto H^2. For each function f on R let Vf denote the function on T given by

$$Vf(e^{i\theta}) = f[i(1 + e^{i\theta})/(1 - e^{i\theta})].$$

Then V induces an isometry from $L^\infty(R) \to L^\infty(T)$ which maps $H^\infty(R) + C_0(R)$ onto $H^\infty(T) + C(T)$, where $C_0(R)$ denotes the continuous functions on R which vanish at ∞, and $C(T)$ denotes the continuous functions on T. Devinatz [2, p. 83] showed that $W(\phi)$ is unitarily equivalent to the Toeplitz operator on $H^2(T)$ with symbol $V(\phi)$. Thus, properties of Toeplitz operators can be carried over to Wiener-Hopf operators. The following facts will be needed: Let ϕ be a unimodular function on R.

(4.1) (Nehari's Theorem) $\|I - W(\bar{\phi})W(\phi)\|^{1/2} = \text{dist}\,(\phi, H^\infty)$.
(4.2) (Hartman) $I - W(\bar{\phi})W(\phi)$ is compact if and only if $\phi \in H^\infty + C_0$.
(4.3) $W(\phi)$ is left invertible if and only if $\text{dist}(\phi, H^\infty) < 1$.
(4.4) $W(\phi)$ is left Fredholm if and only if $\text{dist}(\phi, H^\infty + C_0) < 1$.
(4.5) (Wolff) $\phi \in H^\infty + C_0$ if and only if ϕ can be written as

$$\phi = [(x + i)/(x - i)]^n \cdot b \cdot \exp[i(v - \tilde{u})]$$

where b is an inner function, and u, v are real functions in C_0 (n a positive integer).
(4.6) (Coburn) $W(\phi)$ and $W(\bar{\phi})$ cannot both have nontrivial kernels.
(4.7) A function of the form $\exp(u + \tilde{v})$ with u and v in C_0 is locally in L^p for every finite p.

Wolff's factorization can be found in [11]; a nice discussion including the rest of the results can be found in [9].

THEOREM 2. *A necessary and sufficient condition for $Q_aP_{-a} - \pi_a$ to be compact is that w can be written in the form*

$$w(x) = |B(x)|^{-2} \exp\,(u + \tilde{v})\ (-\infty < x < \infty) \tag{4.8}$$

where B is entire of exponential type $\leq a$ and where u and v are real functions in $C_0(R)$.

Proof. Let $\phi = e_{2a}h/\bar{h}$ and suppose that $Q_aP_{-a} - \pi_a$ is compact. If $\|Q_aP_{-a} - \pi_a\| = 1$, then there is a function f in $M_a^{\perp}$ with unit norm such that $\|Q_aP_{-a}f\| = 1$. Since Q_a and P_{-a} are projections, $f \in e_{-a}H^2 \cap (e_ah/\bar{h})\bar{H}^2 = M_a$. This is absurd, so it follows that $\|Q_aP_{-a} - \pi_a\|$ and hence $\rho_*(2a)$ are less than unity so by 4.3, $W(\phi)$ is left invertible. Because $(e_ah/\bar{h})H^2$ is contained in the kernel of π_a, $P_{-a}Q_aP_{-a} \mid (e_ah/\bar{h})H^2$ is compact. For a function f in L^∞, let the symbol f also denote the multiplication operator $g \to fg$ on l^2. Then we have

$$P_{-a}Q_aP_{-a} \mid (e_ah/\bar{h})H^2 = e_{-a}Pe_a(e_ah/\bar{h})(I - P)(e_{-a}\bar{h}/h)e_{-a}Pe_a \mid (e_ah/\bar{h})H^2$$

so that $P\phi(I - P)\bar{\phi}\, P\, \phi \mid H^2$ is compact. This last operator equals

$$W(\phi)[I - W(\bar{\phi})W(\phi)];$$

since $W(\phi)$ is left invertible, it follows that $I - W(\bar{\phi})W(\phi)$ is compact. By (4.2), $\phi \in H^\infty + C_0$, so

$$e_{2a}h/\bar{h} = [(x + i)/(x - i)]^n \cdot b \cdot \exp[i(v - \tilde{u})]$$

where b is inner, and where u and v are real functions in C_0. An application of (4.7) allows the analytic continuation argument of Theorem 1 to be carried out and we get

$$w(x) = (1 + x^2)^n\, e^{u+\tilde{v}}/|B|^2.$$

The factor $(1 + x^2)^n$ may be absorbed into the exponent with no harm at the expense of the required number of zeroes from the denominator, $|B|^2$. (B must have at least $n + 1$ zeroes, or

$$(1 + x^2)^n\, e^{u+\tilde{v}}/|B|^2$$

would not be integrable.)

Suppose, conversely, that w is of the form (4.8). Then

$$\phi = e_{2a}h/\bar{h} = e_s\, b \exp[i(v - \tilde{u})] \quad \text{where } s \geq 0;$$

b is a Blaschke product whose zeroes arise from the zeroes of B. Thus, $\phi \in H^\infty + C_0$ so $W(\phi)$ is left Fredholm by (4.4) so has closed range. Note also that h is in the kernel of $W(\bar{\phi})$ so, by (4.6), $W(\phi)$ is one to one and hence left invertible. Therefore, $\operatorname{dist}(\phi, H^\infty) < 1$. This last condition implies that

$$L^2 = M_a \oplus [(e_ah/\bar{h})H^2 + e_{-a}\bar{H}^2]$$

where the bracketed summands are at a positive angle. Now, $M_a + e_{-a}\bar{H}^2$ is contained in the kernel of $Q_aP_{-a} - \pi_a$ and

$$[W(\phi) - W(\phi)W(\bar{\phi})W(\phi)]$$

is compact, so $P_{-a}Q_aP_{-a} \mid (e_ah/\bar{h})H^2$ is also compact. Thus $P_{-a}Q_aP_{-a} - \pi_a$ is compact on L^2. This last operator is just $(Q_aP_{-a} - \pi_a)^*(Q_aP_{-a} - \pi_a)$, so $Q_aP_{-a} - \pi_a$ is compact as well. This completes the proof of Theorem 2.

Remark. The above proof also shows that $Q_a P_{-a} - \pi_a$ is trace-class if and only if

$$\int_T^\infty t|(\bar{h}/h)^\vee(t)|^2\,dt < \infty$$

for some finite T (see [5, p. 135]).

Example. If $w = 1/(x^2+1)^{3/2}$, then $h/\bar{h} = (x-i)^{3/2}/(x+i)^{3/2}$. It is not hard to see that $r_*(2a) = e^{-2a}$ but that $e_{2a}h/\bar{h}$ is not in $H^2 + C_0$ for any a. Thus, $(Q_a P_{-a})^n$ provides a good approximation of π_a for all positive a, but the remainder is never compact.

5. The relation between r_* and strong mixing

The quantity $r(a)$ measures the dependence of the "future" of the process from time a upon the "past" of the process. If $r(a) \to 0$ as $a \to \infty$, then the process is said to be strongly mixing or completely regular (see [7]). It was shown in [6] that if either $r(a)$ or $r_*(a)$ tends to zero and the other is eventually less than unity, then both quantities tend to zero. It turns out that a quantitative relation exists between the rates of decay of r and r_*. The following lemma generalizes a result proved by Dym [5, p. 132].

LEMMA. *If a, b, and c are positive real numbers, then*

$$r(a+b+c) \leq r(a)r(c) + r_*(b).$$

Proof. Let f and g belong to the unit spheres of $Z(a+b+c, \infty)$ and $Z(-\infty, 0)$ respectively. Let $\pi_{a,b}$ denote the orthogonal projection on $e_a H^2 \cap (e_{a+b}h/\bar{h})\overline{H^2}$ in L^2. Then if $\langle\ ,\ \rangle_w$ and $\langle\ ,\ \rangle$ respectively denote the inner products in Z and L^2, we have

$$\begin{aligned}
|\langle f, g\rangle_w| &= |\langle fh, gh\rangle| \\
&= |\langle P_a fh, Q_{a+b} gh\rangle| \\
&= |\langle fh, P_a Q_{a+b} gh\rangle| \\
&\leq |\langle fh, (P_a Q_{a+b} - \pi_{a,b})gh\rangle| + |\langle fh, \pi_{a,b} gh\rangle| \\
&\leq r_*(b) + r(a)\,r(c).
\end{aligned}$$

Since $r(a+b+c)$ is the supremum of all such quantities, the lemma is proved.

As a consequence, we have the following theorem.

THEOREM 3. *If $\lim_{t\to\infty} r_*(t) = 0$ and $r(a) < 1$, then there exist constants K and c such that*

$$r(n^2a) \leqslant K(e^{-cn} + r_*(cn))$$

for every positive integer n.

Proof. Let $\alpha = r(a)$. Then from the preceding lemma,

$$r(3a) \leqslant \alpha^2 + r_*(a),$$

$$r((3 + 2 + 1)a) \leqslant r(3a)r(a) + r_*(2a) \leqslant \alpha^3 + \alpha r_*(a) + r_*(2a),$$

and, inductively,

$$r\left(\sum_{k=1}^{n} ka\right) \leqslant \alpha^n + \alpha^{n-2}r_*(a) + \alpha^{n-3}r_*(2a) + \cdots + \alpha r_*(n-2)a) + r_*((n-1)a).$$

Since $r_*(t)$ is a non-increasing function of t,

$$\begin{aligned} r\left(\frac{n^2+n}{2}\cdot a\right) &\leqslant \{\alpha^n + \alpha^{n-2} + \cdots + \alpha^{[n/2]}\} + r_*([n/2]a)\{\alpha^{[n/2]} \\ &\quad + \alpha^{[n/2]-1} + \cdots + 1\} \\ &\leqslant \alpha^{[n/2]}\frac{1}{1-\alpha} + r_*([n/2]a)\frac{1}{1-\alpha}. \end{aligned}$$

This proves the theorem with $K = (1-\alpha)^{-1}$ and $c = 3^{-1}\cdot\min(1, -\ln\alpha)$.

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