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IDEAL PROPERTIES OF REGULAR OPERATORS BETWEEN BANACH LATTICES

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1. Introduction

Suppose E and F are Banach lattices such that E^* and F have order-continuous norms. In [4] Dodds and Fremlin (cf. also [1]) showed that if T: $E \to F$ is a positive compact operator and $0 \leq S \leq T$ then S is also compact. Aliprantis and Burkinshaw [1] showed by examples that the hypotheses on E and F are necessary. In [2] they asked whether a similar result is true for Dunford-Pettis operators, under the same hypotheses on E and F.

In this paper we give a positive answer to the question of Aliprantis and Burkinshaw. However, after the initial preparation of the paper we learned of the work of W. Haid [6] who also had answered the question in the form stated a little before our work (see also de Pagter [9]). Haid's theorem is:

THEOREM 1.1. *Let E and F be Banach lattices so that E^* and F have order-continuous norm. Let T: $E \to F$ be a positive Dunford-Pettis operator. If $0 \leq S \leq T$ then S is a Dunford-Pettis operator.*

Our methods are similar in spirit to those of Haid, but yield a more powerful result (Theorem 4.4 below) in that the hypotheses on E^* can be eliminated.

We also strengthen another result of [2]. In [2] it is shown that for any Banach lattice E if T: $E \to E$ is a positive Dunford-Pettis operator and $0 \leq S \leq T$ then S^3 is Dunford-Pettis; we show (Corollary 4.7) that in fact S^2 is Dunford-Pettis. Again examples in [1] and [2] show that S need not be Dunford-Pettis.

The argument for these results hinges on Theorem 3.2, a technical result which has many other applications to similar problems. Some of these are examined in Section 5. For an example we mention Theorem 5.4. Suppose E is any Banach lattice and F is a Banach lattice with order-continuous norm. Suppose further there is no disjoint sequence in F equivalent to the unit vector basis of l_2. Suppose R, S: $E \to F$ are regular operators with $|S| \leq |R|$. Suppose there is a closed subspace H of E, isomorphic to l_2, such that S is an

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isomorphism on H. Then we can conclude that there is a closed subspace H_1 of E, isomorphic to l_2, so that R is an isomorphism on H_1.

2. Notation

Let X and Y be Banach spaces. We denote by $\mathscr{L}(X,Y)$ the space of bounded linear operators from X into Y and abbreviate $\mathscr{L}(X,X)$ to $\mathscr{L}(X)$. We recall that $T \in \mathscr{L}(X,Y)$ is a *Dunford-Pettis operator* if T maps weakly compact sets into norm compact sets, or equivalently if $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly. In [2], T is said to be a *weak-Dunford-Pettis* operator if ST is a Dunford-Pettis operator for every weakly compact operator $S \in \mathscr{L}(Y,Z)$ for some Banach space Z. Alternatively T is a weak-Dunford-Pettis operator if whenever $x_n \to 0$ weakly in X and $y_n^* \to 0$ weakly in Y^* then $\lim_{n\to\infty} y_n^*(Tx_n) = 0$.

Suppose now E is a Banach lattice. The positive cone of E is denoted by E_+. If $u \in E_+$ then E_u denotes the principal ideal generated by E, i.e.,

$$E_u = \{x \in E : |x| \le mu \text{ for some } m \in \mathbf{N}\}.$$

If E is separable then E certainly has a quasi-interior positive element [11, p. 97].

For general $u \in E_+$, E_u considered with the order-interval $[-u,u]$ as its unit ball is a abstract M-space and hence can be identified with a space $C(K_u)$ of continuous functions on some compact Hausdorff space K_u [11, p. 165]. Precisely there is a lattice isomorphism J_u of $C(K_u)$ onto E_u mapping the constant function 1 onto u. We shall refer to this isomorphism J_u as the Kakutani isomorphism associated to u.

A Banach lattice E has *order-continuous norm* if every descending sequence $e_n \in E_+$ is norm convergent. E is then order-complete and forms an ideal in E^{**} [11, p. 89]. We note that for any Banach lattice E, E^* has order-continuous norm if and only if every disjoint bounded sequence e_n in E is weakly convergent to zero. [4, Corollary 2.9]. In particular for any closed sublattice E_0 of E, E_0^* will also have order-continuous norm.

If E and F are both Banach lattices then a linear operator $T \in \mathscr{L}(E,F)$ is called *regular* if $T = P_1 - P_2$ where $P_1, P_2 \in \mathscr{L}(E,F)$ are positive; alternatively T is regular if for some positive P we have $|Te| \le P|e|$ for $e \in E$. The subspace of regular operators is denoted by $\mathscr{L}_r(E,F)$. If F is order-complete then $\mathscr{L}_r(E,F)$ is a lattice [11, p. 230]; in fact $\mathscr{L}_r(E,F)$ is a Banach lattice under the norm $\|T\|_r = \|\,|T|\,\|$.

In general, $\mathscr{L}_r(E,F)$ need not be a lattice, but, since F^{**} is order-complete, $\mathscr{L}_r(E,F^{**})$ is a lattice. Thus if $T \in \mathscr{L}_r(E,F)$ then we can define $|T| \in \mathscr{L}(E,F^{**})$. If F has order-continuous norm then $|T|$ (in $\mathscr{L}(E,F^{**})$) maps E into F and hence coincides with $|T|$ in the lattice $\mathscr{L}(E,F)$.

For any Banach lattice E we shall define a multiplier $M \in \mathscr{L}_r(E)$ to be an operator such that for some $m \in \mathbf{N}$,

$$|Me| \leq m|e|, \quad e \in E.$$

Thus $-mI \leq M \leq mI$. If E is order-complete, M is a multiplier if it belongs to the principal ideal generated by the identity operator.

LEMMA 2.1. *Let K be a compact Hausdorff space. Then $M \in \mathscr{L}(C(K))$ is a multiplier if and only if there exists $f \in C(K)$ so that $Mh(s) = f(s)h(s)$, $s \in K, h \in C(K)$.*

Proof. Suppose M is a multiplier. Then for $s \in K$ the linear functional $h \to Mh(s)$ satisfies $|Mh(s)| \leq m|h(s)|, h \in C(K)$. Thus there exists $f(s)$ with $-m \leq f(s) \leq m$ so that $Mh(s) = f(s)h(s)$. Since $M1 \in C(K)$, $f \in C(K)$ and M has the prescribed form. The converse is trivial.

LEMMA 2.2. *Let E be a Banach lattice with a quasi-interior positive element u and let J_u: $C(K_u) \to E_u$ be the associated Kakutani isomorphism. Then there is an isometric isomorphism of $C(K_u)$ onto the space of multipliers of E given by $f \to \hat{f}$ where $\hat{f}[J_u g] = J_u(fg)$, $g \in C(K_u)$. Further the map $f \to \hat{f}$ is an algebra isomorphism.*

Proof. If $f \in C(K_u)$ then the formula $\hat{f}(J_u g) = J_u(fg)$, $g \in C(K_n)$, defines a linear operator $\hat{f}$: $E_u \to E_u$. Clearly $\|\hat{f}\| \leq \|f\|$ and so $\hat{f}$ extends to a linear operator in $\mathscr{L}(E)$ which is clearly a multiplier. The map $f \to \hat{f}$ is clearly an injective algebra homomorphism.

We show that $f \to \hat{f}$ is in fact an isometry. Suppose $\|f\| = 1$ but $\|\hat{f}\| = r < 1$. Choose ρ with $r < \rho < 1$ and then $h \in C(K_u)$ with $\|h\| = 1$ and $h(s) = 0$ whenever $|f(s)| < \rho$. Then if $e = J_u h$, $|\hat{f} \cdot e| \geq \rho|e|$ so that $\|\hat{f}\| \geq \rho$ contrary to our assumption.

A Banach lattice E is said to satisfy an upper-p-estimate where $1 \leq p < \infty$ if there is a constant C so that for disjoint set $e_1, \dots, e_n$ in E,

$$\|e_1 + \cdots + e_n\| \leq C\left(\sum_{i=1}^{n} \|e_i\|^p\right)^{1/p}.$$

E is said to satisfy a lower-q-estimate for $1 \leq q < \infty$ if there exists $c > 0$ so that for any disjoint set $e_1, \dots, e_n$ in E,

$$\|e_1 + \cdots + e_n\| \geq c\left(\sum_{i=1}^{n} \|e_i\|^q\right)^{1/q}.$$

See [6, p. 82].

If $e_1, \dots, e_n \in E$ then for $0 < p < \infty$, the element $(|e_1|^p + \cdots + |e_n|^p)^{1/p} \in E$ is unambiguously defined (see pp. 40–42 of [7]).

A subset A of E is called *solid* if whenever $a \in A$ and $|e| \leq |a|$ then $e \in A$. The *solid hull* of the set B is the set $A = \{e \in E\colon |e| \leq |b| \text{ for some } b \in B\}$. We set $B^+ = B \cap E_+$.

Finally we note that it will often be convenient to use $\langle\, , \rangle$ for the natural pairing between E and E^* or between E^{**} and E^*.

3. The basic approximation theorem

We start with a lemma which follows from work of Dodds and Fremlin [4]:

LEMMA 3.1. *Let E and F be Banach lattices and suppose $A \subset E$ and $B \subset F^*$ are bounded solid sets. Suppose $T_n\colon E \to F$ are positive operators so that $T_n \to 0$ in the weak-operator topology, i.e., $\langle T_n e, f^* \rangle \to 0$ for $e \in E, f^* \in F^*$. Suppose further whenever $\{a_n\}$ is a disjoint sequence in A^+ and $\{b_n\}$ is a disjoint sequence in B^+ we have*

(i) $\langle T_n a_n, b \rangle \to 0, \quad b \in B,$

(ii) $\langle T_n a, b_n \rangle \to 0, \quad a \in A,$

(iii) $\langle T_n a_n, b_n \rangle \to 0.$

Then

$$\lim_{n \to \infty} \sup_{a \in A} \sup_{b \in B} |\langle T_n a, b \rangle| = 0.$$

Proof. For $a \in A^+$ note that

$$\lim_{n \to \infty} \langle T_n a, b \rangle = 0, b \in B, \quad \text{and} \quad \lim_{n \to \infty} \langle T_n a, b_n \rangle = 0$$

for (b_n) disjoint in B^+. Thus by Theorem 2.4 of [4].

$$\lim_{n \to \infty} \sup_{b \in B} \langle T_n a, b \rangle = 0.$$

If (a_n) is disjoint in A^+ then using conditions (i) and (iii) above and Theorem 2.4 of [4],

$$\lim_{n \to \infty} \sup_{b \in B} \langle T_n a_n, b \rangle = 0.$$

Now let d_n by *any* sequence in B^+. We have

$$\lim_{n \to \infty} \langle a, T_n^* d_n \rangle = 0 \quad \text{and} \quad \lim_{n \to \infty} \langle a_n, T_n^* d_n \rangle = 0$$

where $a \in A^+$ and $\{a_n\}$ is disjoint in A^+. Thus by Theorem 2.4 of [4] again,

$$\lim_{n\to\infty} \sup_{a\in A} \langle a, T_n^* d_n\rangle = 0$$

and the lemma follows.

THEOREM 3.2. *Let E and F be Banach lattices each with a quasi-interior positive element. Let T be a positive operator $T\colon E \to F$ and let $A \subset E$, $B \subset F^*$ be a solid bounded sets. Suppose that whenever $\{a_n\}$ is disjoint in A^+ and $\{b_n\}$ is disjoint in B^+ then*

(i) $\lim_{n\to\infty} Ta_n = 0$ *weakly*,

(ii) $\lim_{n\to\infty} T^*b_n = 0$ *weak**,

(iii) $\lim_{n\to\infty}\langle Ta_n, b_n\rangle = 0$.

Suppose further that $R, S \in \mathscr{L}(E,F)$ satisfy $|S| \le |R| \le T$ in $\mathscr{L}(E, F^{**})$. Then given $\varepsilon > 0$ there exist multipliers $M_1, \dots, M_k \in \mathscr{L}(E)$, $L_1, \dots, L_k \in \mathscr{L}(F)$ so that if

$$S_0 = \sum_{i=1}^{k} L_i R M_i$$

then

$$|\langle Sa - S_0 a, b\rangle| \le \varepsilon, \quad a \in A, b \in B.$$

Proof. We let $u \in E_+$ and $v \in F_+$ be quasi-interior elements such that $Tu \le v$. Let

$$J_u\colon C(K_u) \to E_u \quad \text{and} \quad J_v\colon C(K_v) \to F_v$$

be the associated Kakutani isomorphisms. As in Section 2 there is an isometric algebra isomorphism of $C(K_u)$ onto the multipliers of E given by $f \to \hat{f}$ where

$$J_u(fh) = \hat{f} J_u(h), \quad h \in C(K_u),$$

and a similar isomorphism $g \to \hat{g}$ of $C(K_v)$ onto the multipliers of F.

We shall break up the proof into several lemmas. Before proving the first we note a fact which will be used several times. Let $\hat{F}$ denote the order-ideal in F^{**} generated by F; i.e., $x \in \hat{F}$ if $|x| \le w$ for some $w \in F$. If $\phi_n \ge 0$ is a monotone increasing sequence in F^* and $\phi = \sup_{n\ge 1}\phi_n$ then

$$\langle x, \phi_n\rangle \to \langle x, \phi\rangle \quad \text{for all } x \in \hat{F}.$$

In fact if $|x| \le w \in F$ then $\langle x, \phi - \phi_n\rangle \le \langle w, \phi - \phi_n\rangle$ since $\phi_n \to \phi$ weak*.

LEMMA 3.3. *There exists* $\phi \in F_+^*$ *so that if* $0 \le x \le Tu$ *in* F^{**} *and* $\langle x, \phi\rangle = 0$ *then* $\langle x, b\rangle = 0$ *for all* $b \in B$.

Proof. If $\{b_n\}$ is disjoint in B^+ then $\langle Tu, b_n\rangle \to 0$. Hence there is a maximal countable disjoint set $\{b_n\}$ in B^+ with $\langle Tu, b_n\rangle > 0$ for each $n \in \mathbf{N}$. Set $\phi = \Sigma 2^{-n} b_n$. Thus if $b \in B$ and $b \wedge \phi = 0$, $\langle Tu, b\rangle = 0$. Now if $0 \le x \le Tu$ and $\langle x, \phi\rangle = 0$ then if $b \in B^+$, since $x \in \hat{F}$,

$$\left\langle x, \sup_m b \wedge m\phi\right\rangle = \sup_m \langle x, b \wedge m\phi\rangle = 0.$$

However $\langle Tu, b - \sup_m b \wedge m\phi\rangle = 0$ so that $\langle x, b\rangle = 0$.

Now let P be the band projection onto the band generated by ϕ in F^*. Thus if $f^* \ge 0$, $Pf^* = \sup_m f^* \wedge m\phi$. Again if $x \in \hat{F} \subset F^{**}$,

$$\langle x, Pf^*\rangle = \lim_{m \to \infty} \langle x, f^* \wedge m\phi\rangle.$$

LEMMA 3.4. *Suppose* $V \in \mathscr{L}(E, F^{**})$ *and* $-T \le V \le T$. *Suppose*

$$\langle V\hat{f}u, \hat{g}^*\phi\rangle \ge 0, \quad f \in C(K_u)_+, g \in C(K_v)_+.$$

Then $P^*V \ge 0$ *in* $\mathscr{L}(E, F^{**})$.

Proof. We need only show $P^*Ve \ge 0$ if $0 \le e \le u$. Pick $f \in C(K_u)_+ \ge 0$ so that $\hat{f}u = e$.

Now suppose $0 \le \psi \le \phi$. Then $0 \le J_v^*\psi \le J_v^*\phi$ in $C(K_v)^*$. Now by the Radon-Nikodym theorem given $\varepsilon > 0$ there exists $g \in C(K_v)$ so that $0 \le g \le 1$ and

$$|J_v^*\psi(h) - J_v^*\phi(gh)| \le \varepsilon\|h\|, \quad h \in C(K_v).$$

Hence if $w \in [-v, v]$ in F, $|\psi(w) - \hat{g}^*\phi(w)| \le \varepsilon$. By a weak*-density argument if $-v \le x \le v$ in F^{**} $|\langle x, \psi - \hat{g}^*\phi\rangle| \le \varepsilon$. Thus

$$\langle Ve, \psi\rangle \ge \langle Ve, \hat{g}^*\phi\rangle - \varepsilon = \langle V\hat{f}u, \hat{g}^*\phi\rangle - \varepsilon \ge -\varepsilon.$$

As $\varepsilon > 0$ is arbitrary $\langle Ve, \psi\rangle \ge 0$, for $0 \le \psi \le \phi$. Now if $\psi \in F^*$, with $\psi \ge 0$,

$$\langle P^*Ve, \psi\rangle = \langle Ve, P\psi\rangle = \lim_{m \to \infty} \langle Ve, \psi \wedge m\phi\rangle \ge 0.$$

For $f \in C(K_u)$ and $g \in C(K_v)$ we define $f \otimes g \in C(K_u \times K_v)$ by $f \otimes g(s, t) = f(s)g(t)$.

LEMMA 3.5. *Suppose $V \in \mathscr{L}(E, F)$ with $-\alpha T \leq V \leq \alpha T$ for some $\alpha \geq 0$. Then there is a unique bounded linear operator Γ_V: $C(K_u \times K_v) \to \mathscr{L}(E, F)$ such that*

$$\Gamma_V(f \otimes g) = \hat{g} V \hat{f}.$$

If $V \geq 0$ then Γ_V is a positive operator.

Proof. Define W_i: $C(K_u) \to C(K_v)$ for $i = 1, 2$ by $W_1 = J_v^{-1} T J_u$, $W_2 = J_v^{-1} V J_u$. Then $W_1 \geq 0$ and $W_1 1 \leq 1$. Hence for each $t \in K_v$ there is a positive Borel measure $\mu_t \in M(K_u)$ with $\mu_t(K_u) \leq 1$ so that

$$W_1 h(t) = \int h(s)\, d\mu_t(s).$$

Now if $h \geq 0$, $-\alpha W_1 h \leq W_2 h \leq \alpha W_1 h$ so that for any $h \in C(K_u)$,

$$|W_2 h(t)| \leq \alpha \int |h(s)|\, d\mu_t(s).$$

Hence for each t there exists a Borel function ϕ_t on K_u with $-\alpha \leq \phi_t \leq \alpha$ everywhere so that

$$W_2 h(t) = \int \phi_t(s) h(s)\, d\mu_t(s).$$

For $\sum_{i=1}^n f_i \otimes g_i \in C(K_u) \otimes C(K_v)$ define $\Gamma_V(\sum_{i=1}^n f_i \otimes g_i) = \sum_{i=1}^n \hat{g}_i V \hat{f}_i$. Then

$$J_v^{-1} \Gamma_V\left(\sum_{i=1}^n f_i \otimes g_i \right) J_u h(t) = \int \phi_t(s) h(s) \sum_{i=1}^n f_i(s) g_i(t)\, d\mu(s).$$

Hence if

$$\left\| \sum_{i=1}^n f_i \otimes g_i \right\| = \max_{(s,t) \in K_u \times K_v} \left| \sum_{i=1}^n f_i(s) g_i(t) \right|,$$

$$\left| J_v^{-1} \Gamma_V\left(\sum_{i=1}^n f_i \otimes g_i \right) J_u h(t) \right| \leq \alpha \left\| \sum_{i=1}^n f_i \otimes g_i \right\| \int |h(s)|\, d\mu_t(s).$$

It follows that

$$\left| \Gamma_V\left(\sum_{i=1}^n f_i \otimes g_i \right) e \right| \leq \alpha \left\| \sum_{i=1}^n f_i \otimes g_i \right\| T|e|$$

for any $e \in E_u$ and hence for any $e \in E$. In particular

$$\left\| \Gamma_V\left(\sum_{i=1}^{n} f_i \otimes g_i \right) \right\| \le \alpha \|T\| \left\| \sum_{i=1}^{n} f_i \otimes g_i \right\|$$

so that Γ_V extends uniquely to a bounded linear operator

$$\Gamma_V\colon C(K_u \times K_v) \to \mathscr{L}(E, F)$$

with $\|\Gamma_v\| \le \alpha \|T\|$. If $V \ge 0$ then we may take $0 \le \phi_t \le \alpha$ everywhere and it is not difficult to check that

$$J_v^{-1}\Gamma_V(k) J_u h(t) = \int k(s,t)\phi_t(s)\, d\mu_t(s)$$

for $h \in C(K_u)$, $k \in C(K_u \times K_v)$. Hence $\Gamma_V(k) \ge 0$ if $k \ge 0$.

Now Γ_V has an extension $\tilde{\Gamma}_V$: $C(K_u \times K_v)^{**} \to \mathscr{L}(E, F^{**})$ which is continuous for the weak*-topology on $C(K_u \times K_v)^{**}$ and the weak*-operator topology on $\mathscr{L}(E, F^{**})$. We identify the space $B(K_u \times K_v)$ of bounded Borel functions on $K_u \times K_v$ as a linear subspace of $C(K_u \times K_v)^{**}$ in the natural way. Note that if $V \ge 0$ then $\tilde{\Gamma}_V \ge 0$.

LEMMA 3.7. *Suppose $R, S \in \mathscr{L}(E, F)$ with $|S| \le |R| \le T$ in $\mathscr{L}(E, F^{**})$. Then there exists $h \in B(K_u \times K_v)$ so that $P^*\tilde{\Gamma}_R(h) = P^*S$ in $\mathscr{L}(E, F^{**})$.*

Proof. For $V \in \mathscr{L}(E, F)$ with $-mT \le V \le mT$ for some $m \in \mathbf{N}$ we define a measure

$$\mu(V) \in M(K_u \times K_v)$$

by

$$\int k\, d\mu(V) = \langle \Gamma_V(k)u, \phi \rangle, \quad k \in C(K_u \times K_v).$$

If $f \in C(K_u)$ and $g \in C(K_v)$,

$$\int f \otimes g\, d\mu(v) = \langle \hat{g} V \hat{f} u, \phi \rangle.$$

It follows that $\mu(\hat{g} V \hat{f}) = f \otimes g \cdot \mu(V)$ for $f \in C(K_u)$, $g \in C(K_v)$. Thus

$$\mu(\Gamma_V(f \otimes g)) = f \otimes g \cdot \mu(V).$$

Now suppose $k \in B(K_u \times K_v)$ and that k_α is a bounded net in $C(K_u) \otimes$

$C(K_v)$ such that $k_\alpha \to k$ weak*. Then $\Gamma_V(k_\alpha) \to \tilde{\Gamma}_V(k)$ in the weak*-operator topology. For $f \otimes g \in C(K_u) \otimes C(K_v)$ we have

$$\langle \hat{g}\Gamma_V(k_\alpha)\hat{f}u, \phi\rangle = \langle \Gamma_V(k_\alpha)\hat{f}u, \hat{g}^*\phi\rangle \to \langle \tilde{\Gamma}_V(k)\hat{f}u, \hat{g}^*\phi\rangle.$$

However $k_\alpha \cdot \mu(V) \to k \cdot \mu(V)$. Hence

$$\int (f \otimes g) \cdot k\, d\mu\,(V) = \langle \tilde{\Gamma}_V(k)\hat{f}u, \hat{g}^*\phi\rangle.$$

Now take $V = R$ and choose k so that $|k| = 1$ and $k \cdot \mu(R) = |\mu(R)|$. Then

$$\langle (\tilde{\Gamma}_R(k) \pm R)\hat{f}u, \hat{g}^*\phi\rangle \geq 0$$

for all $f \geq 0$, $g \geq 0$. Hence $P^*\tilde{\Gamma}_R(k) \geq |P^*R|$ in $\mathscr{L}(E, F^{**})$. Thus

$$P^*\tilde{\Gamma}_R(k) + (I - P^*)|R| \geq \pm S \quad \text{and} \quad P^*\tilde{\Gamma}_k(k) \geq \pm P^*S.$$

Again if $f, g \geq 0$, $\langle \tilde{\Gamma}_R(k)\hat{f}u, \hat{g}^*\phi\rangle \geq |\langle S\hat{f}, \hat{g}^*\phi\rangle|$ so that $|\mu(S)| \leq |\mu(R)|$.

Now select h so that $|h| \leq 1$, $h \in B(K_u \times K_v)$ and $h \cdot \mu(R) = \mu(S)$. Then

$$\langle \tilde{\Gamma}_R(h) - S\hat{f}u, \hat{g}^*\phi\rangle = 0$$

for all f, g. Hence $P^*\tilde{\Gamma}_R(h) = P^*S$.

We are finally in position to complete the proof. We define the map

$$\Delta\colon C(K_u \times K_v) \to l_\infty(A \times B)$$

by

$$\Delta h(a, b) = \langle \Gamma_R(h)a, b\rangle, \quad a \in A \quad b \in B.$$

Δ is clearly bounded; we shall show that Δ is weakly compact. It suffices to take a sequence $\{h_n\}$ with disjoint supports and $0 \leq h_n \leq 1$ and show that $\|\Delta h_n\| \to 0$. In fact

$$|\langle \Gamma_R(h_n)a, b\rangle| \leq \langle \Gamma_T(h_n)|a|, |b|\rangle, \quad a \in A, \quad b \in B,$$

since $\Gamma_{T-R}(h_n) \geq 0$ and $\Gamma_{T+R}(h_n) \geq 0$.

Let $\Gamma_T(h_n) = T_n$. Since $h_n \to 0$ weakly in $C(K_u \times K_v)$, $T_n \to 0$ weakly in $\mathscr{L}(E, F)$ and hence $T_n \to 0$ in the weak-operator topology. If $\{a_n\}$ is disjoint in A^+ and $\{b_n\}$ is disjoint in B^+

$$\langle T_n a_n, b\rangle \leq \langle Ta_n, b\rangle \to 0,\ b \in B^+, \quad \langle T_n a, b_n\rangle \leq \langle Ta, b_n\rangle \to 0,\ a \in A,$$

and

$$\langle T_n a_n, b_n \rangle \leq \langle T a_n, b_n \rangle \to 0.$$

Thus by Lemma 3.1,

$$\lim_{n\to\infty} \sup_{a\in A} \sup_{b\in B} |\langle T_n a, b\rangle| = 0$$

and hence $\|\Delta h_n\| \to 0$ as required.

Since Δ is weakly compact it has a weak*-weak continuous extension

$$\tilde{\Delta}: C(K_u \times K_v)^{**} \to l_\infty(A \times B).$$

By continuity, if $h \in B(K_u \times K_v)$,

$$\tilde{\Delta}h(a,b) = \langle \tilde{\Gamma}_R(h)a, b\rangle, \quad a \in A,\ b \in B.$$

Fix h so that $-1 \leq h \leq 1$ and $P^*\tilde{\Gamma}_R(h) = P^*S$ in $\mathscr{L}(E, F^{**})$ (by Lemma 3.7). For $e \in E$,

$$\langle |\tilde{\Gamma}_R(h)e - Se|, \phi\rangle = \sup_{|\psi|\leq\phi} \langle \tilde{\Gamma}_R(h)e - Se, \psi\rangle = 0.$$

However $\tilde{\Gamma}_T - \tilde{\Gamma}_R \geq 0$ and $\tilde{\Gamma}_T + \tilde{\Gamma}_R \geq 0$. Hence

$$|\tilde{\Gamma}_R(h_+)| \leq \tilde{\Gamma}_T(h_+) \quad \text{and} \quad |\tilde{\Gamma}_R(h_-)| \leq \tilde{\Gamma}_T(h_-).$$

Thus $|\tilde{\Gamma}_R(h)| \leq \tilde{\Gamma}_T(|h|) \leq T$. Also $|S| \leq T$. Thus $|\tilde{\Gamma}_R(h)e - Se| \leq 2T|e|$.

If $0 \leq |e|u$ we use Lemma 3.3 to conclude that $\langle |\tilde{\Gamma}_R(h)e - Se|, b\rangle = 0$, $b \in B$, and hence, since B is solid,

$$\langle \tilde{\Gamma}_R(h)e, b\rangle = \langle Se, b\rangle, \quad b \in B.$$

Now this equation holds, by density, for all $e \in E$ and in particular for all $a \in A$; i.e., $\tilde{\Delta}h(a,b) = \langle Sa, b\rangle$, $a \in A$, $b \in B$.

Since Δ is weakly compact $\tilde{\Delta}h$ is in the closed linear span of

$$\{\Delta(f \otimes g) : f \in C(K_u), g \in C(K_v)\};$$

i.e., for $\varepsilon > 0$ there exists $f_1, \ldots, f_k \in C(K_u)$ and $g_1, \ldots, g_k \in C(K_v)$ so that

$$\left\| \Delta\left(\sum_{i=1}^k f_i \otimes g_i \right) - \tilde{\Delta}h \right\| \leq \varepsilon.$$

However if $L_i = \hat{g}_i \in \mathscr{L}(F)$, $M_i = \hat{f}_i \in \mathscr{L}(E)$ and $S_0 = \Sigma_{i=1}^k L_i R M_i$ then this implies that $|\langle Sa - S_0 a, b\rangle| \leq \varepsilon$, $a \in A$, $b \in B$.

4. Applications to Dunford-Pettis operators

In order to apply Theorem 3.2, we shall need a lemma which helps to establish conditions under which the hypotheses of 3.2 can be verified.

LEMMA 4.1. *Let F be a Banach lattice and let X be a Banach space. Let Q: $F \to X$ be an operator which maps order-intervals into relatively weakly compact sets. Let $B \subset F^*$ be the solid hull of $Q^*(U_X^*)$ where U_X^* is the closed unit ball of X^*. Then if $\{b_n\}$ is a disjoint sequence in B^+, $b_n \to 0$ weak*.*

Remark. Q maps order intervals into relatively weakly compact sets if for every majorized disjoint sequence f_n in F, $\|Qf_n\| \to 0$ (cf. [2] Theorem 1.2).

Proof. Consider the map V: $F \to l_\infty(B)$ given by $Vf(b) = \langle f, b\rangle$. If $\{f_n\}$ is a disjoint majorized sequence, $0 \leq |f_n| \leq f$ say, then

$$\begin{aligned}\|Vf_n\| &= \sup_{b\in B}|\langle f_n, b\rangle| \\ &= \sup_{\|x^*\|\leq 1}\langle |f_n|, |Q^*x^*|\rangle \\ &= \sup_{|g_n|\leq|f_n|}\ \sup_{\|x^*\|\leq 1}|\langle g_n, Q^*x^*\rangle| \\ &= \sup_{|g_n|\leq|f_n|}\|Qg_n\| \\ &\to 0.\end{aligned}$$

Thus V also maps order-intervals to relatively weakly compact sets. Assume now for some $f \in F^+$ and $\{b_n\}$ disjoint in B^+ we have $\langle f, b_n\rangle = 1$ for all n. We can find a weak*-cluster point β of the point-evaluations $\varepsilon_n(\phi) = \phi(b_n)$ on $l_\infty(B)$, and since $V[-f, f]$ is relatively weakly compact there exist convex combinations

$$\delta_n = \sum_{P_{n-1}+1}^{p_n} \alpha_i \varepsilon_i$$

where $p_0 = 0 < p_1 < p_2 < \ldots$ so that $\delta_n \to \beta$ uniformly on $V[-f, f]$. Thus $\delta_n - \delta_{n+1} \to 0$ uniformly on $V[-f, f]$ and so for suitable n,

$$\sup_{-f\leq g\leq f}|\delta_n(Vg) - \delta_{n+1}(Vg)| < \tfrac{1}{2};$$

i.e.,

$$\sup_{-f \le g \le f} |\langle g, c_n - c_{n+1} \rangle| < \tfrac{1}{2}$$

where $c_n = \sum_{p_{n-1}}^{p_n} \alpha_i b_i$. Thus since $|c_n - c_{n+1}| = c_n + c_{n+1}$,

$$\langle f, c_n + c_{n+1} \rangle < \tfrac{1}{2}$$

and hence $\langle f, c_n \rangle < \frac{1}{2}$. However $\langle f, c_n \rangle = 1$ for all n. This contradiction shows that $\langle f, b_n \rangle \to 0$ for all $f \in F^+$.

LEMMA 4.2. *Let E be a Banach lattice and let $A \subset E$ be the solid hull of some relatively weakly compact set W. If $\{a_n\}$ is a disjoint sequence in A^+ then $a_n \to 0$ weakly.*

Proof. Let R: $Y \to E$ be a weakly compact operator such that $R(U_Y) \supset W$. Then R^*: $E^* \to Y^*$ is weakly compact and hence if a_n^{**} is disjoint in C^+ where C is the solid hull of $R^{**}(U_Y^{**})$ then $a_n^{**} \to 0$ weak* by Lemma 4.1. Since $A \subset E \cap R^{**}(U_Y^{**})$ the lemma follows.

Before giving our main results we derive the Dodds-Fremlin theorem [1], [4] from our techniques.

THEOREM 4.3. *Let E and F be Banach lattices so that E^* and F have order-continuous norm. Suppose T: $E \to F$ is a positive compact operator. If $0 \le S \le T$, then S is compact.*

Proof. First we note that E^* has order-continuous norm if and only if every disjoint bounded sequence in E_+ is weakly null [4]; equivalently F has order-continuous norm if and only if every disjoint bounded sequence in F_+^* is weak* null.

It clearly suffices to show S is compact on any subspace E_0 of E of the form $E_0 = \overline{E}_u$ where $u \ge 0$. Then replace F by $F_0 = \overline{F}_v$ where $v = Tu$. E_0^* and F_0 also have order-continuous norms while E_0 and F_0 have quasi-interior positive elements. Thus we can reduce the theorem to the case when E and F have quasi-interior positive elements.

Now let $A = U_E$ and $B = U_F^*$. We apply Theorem 3.2. Clearly (i) holds since $a_n \to 0$ weakly; similarly (ii) holds. For (iii) note that $a_n \to 0$ weakly implies $\|Ta_n\| \to 0$. Take $R = T$ in the theorem. Then there exists multipliers $L_1, \ldots, L_k, M_1, \ldots, M_k$ so that $\|S - \sum_{i=1}^k L_i T M_i\| < \varepsilon$ and so S is compact.

THEOREM 4.4. *Let E and F be Banach lattices so that F has order-continuous norm. Suppose T: $E \to F$ is a positive Dunford-Pettis operator and $0 \le S \le T$. Then S is a Dunford-Pettis operator.*

Proof. As in the previous theorem it suffices to take E and F with quasi-interior positive elements. Suppose $e_n \in E$ and $e_n \to 0$ weakly. Let A be the solid hull of $\{e_n: n \in \mathbf{N}\}$ and let $B = U_F^*$. We again apply Theorem 3.2. If $\{a_n\}$ is disjoint in A^+ then $a_n \to 0$ weakly (Lemma 4.2) and so $Ta_n \to 0$ weakly. If $\{b_n\}$ is disjoint in B^+ then $b_n \to 0$ weak* and so $T^*b_n \to 0$ weak*. Finally since $a_n \to 0$ weakly, $\|Ta_n\| \to 0$ as T is a Dunford-Pettis operator.

Now for $\varepsilon > 0$ there exist multipliers $L_1, \dots, L_k$ of F and $M_1, \dots, M_k$ of E so that if $S_0 = \Sigma L_i T M_i$ then

$$|\langle Sa - S_0 a, b\rangle| \leq \varepsilon, \quad a \in A, b \in B.$$

Thus $\|Se_n - S_0 e_n\| \leq \varepsilon$, $n \in \mathbf{N}$. However $\lim_{n \to \infty} \|S_0 e_n\| \to 0$ since T is Dunford-Pettis and so $\lim_{n \to \infty} \sup \|Se_n\| \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary S is Dunford-Pettis.

THEOREM 4.5. *Let E and F be Banach lattices and T: $E \to F$ be a positive weak Dunford-Pettis operator. If $0 \leq S \leq T$ then S is a weak Dunford-Pettis operator.*

Proof. Suppose first both E and F have quasi-interior positive elements. Suppose $e_n \to 0$ weakly in E and $f_n^* \to 0$ weakly in F^*. Let A be the solid hull of $\{e_n: n \in \mathbf{N}\}$ and B be the solid hull of $\{f_n^*: n \in \mathbf{N}\}$. If a_n is disjoint in A^+ and b_n is disjoint in B^+ then $a_n \to 0$ weakly and $b_n \to 0$ weakly so that $\langle Ta_n, b_n\rangle \to 0$.

Now applying Theorem 3.2, if $\varepsilon > 0$ there exist multipliers $L_1, \dots, L_k$ of F and $M_1, \dots, M_k$ of E so that if $S_0 = \Sigma L_i T M_i$,

$$|\langle Se_n - S_0 e_n, f_n^*\rangle| \leq \varepsilon.$$

Now S_0 is weak-Dunford-Pettis so

$$\limsup |\langle Se_n, f_n^*\rangle| \leq \varepsilon.$$

We conclude that $\langle Se_n, f_n^*\rangle \to 0$; i.e., S is weak-Dunford-Pettis.

For the general case it suffices to show that S: $E_0 \to F_0$ is weak Dunford-Pettis whenever $E_0 = \overline{E}_u$ and $F_0 = \overline{F}_v$ where $u \geq 0$, $v = Tu$. This will follow from the preceding argument if we show that T: $E_0 \to F_0$ is weak Dunford-Pettis. Suppose $e_n \to 0$ weakly in E_0 and $f_n^* \to 0$ weakly in F_0^*. Then there is bounded operator Q: $F_0^* \to F^*$ so that for $f \in F_0$, $\langle f, Qf^*\rangle = \langle f, f^*\rangle$. In fact if $f^* \geq 0$ and $f \geq 0$ we define

$$\langle f, Qf^*\rangle = \sup_n \langle f \wedge nu, f^*\rangle$$

and extend Q by linearity. Now $Qf_n^* \to 0$ weakly in F^* and $\langle Te_n, Qf_n^*\rangle = \langle Te_n, f_n^*\rangle \to 0$ as required.

We shall need the following extension of Theorem 4.4.

THEOREM 4.6. *Let E and F be Banach lattices and let X be any Banach space. Suppose T: $E \to F$ is positive Dunford-Pettis operator and $0 \leq S \leq T$. Let Q: $F \to X$ be any operator which maps order-intervals to relatively weakly compact sets. Then QS is a Dunford-Pettis operator.*

Proof. Again it suffices to consider the case when E and F have quasi-interior positive elements. Let $e_n \to 0$ weakly in E and let A be the solid hull of $\{e_n\}$. Let B be the solid hull of $Q^*(U_X^*)$. Applying Lemma 4.1 we see that the conditions of Theorem 3.2 hold. Hence for $\varepsilon > 0$ there exist multipliers $L_1, \ldots, L_k \in \mathscr{L}(F)$, $M_1, \ldots, M_k \in \mathscr{L}(E)$ so that if $S_0 = \Sigma L_i T M_i$ then

$$|\langle Se_n - S_0 e_n, Q^* x^* \rangle| \leq \varepsilon, \quad n \in \mathbf{N}, \|x^*\| \leq 1.$$

However S_0 is Dunford-Pettis so that $\|S_0 e_n\| \to 0$. Hence $\limsup \|QSe_n\| \leq \varepsilon$. Again we conclude that $\|QSe_n\| \to 0$.

COROLLARY 4.7. *If E is a Banach lattice, T: $E \to E$ is a Dunford-Pettis operator and $0 \leq S \leq T$ then S^2 is a Dunford-Pettis operator.*

Proof. For any disjoint majorized positive sequence e_n, $e_n \to 0$ weakly and so $\|Te_n\| \to 0$. Thus $\|Se_n\| \to 0$ and so S maps order-intervals into weakly compact sets. Hence S^2 is Dunford-Pettis.

As in [2] we can restate Corollary 4.7 for the case of products $S_1 S_2$ where $0 \leq S_1 \leq T_1$ and $0 \leq S_2 \leq T_2$ and T_1 and T_2 are Dunford-Pettis.

If E is an AL-space and F is weakly sequentially complete then $\mathscr{L}(E, F) = \mathscr{L}_r(E, F)$ and is thus a Banach lattice (see [11, p. 232 and p. 95]). It has been shown by Dodds and Fremlin [4] (cf. also Bourgain [3]) that if F is also an AL-space then the Dunford-Pettis operators in $\mathscr{L}(E, F)$ form a band. See [2, Corollary 3.6] for an extension of this result.

THEOREM 4.8. *Let E be an AL-space and suppose F is a weakly sequentially complete Banach lattice. Then the Dunford-Pettis operators form an order-ideal in $\mathscr{L}(E, F)$.*

Proof. We suppose $R \in \mathscr{L}(E, F)$ is a Dunford-Pettis operator and $S \in \mathscr{L}(e, F)$ with $|S| \leq |R|$. We let $T = |R|$. As usual if $e_n \to 0$ weakly in E we can find closed order-ideals E_0 in E and F_0 in F each with quasi-interior positive elements so that $T(E_0) \subset F_0$ and $e_n \in E_0$ for all $n \in \mathbf{N}$. In $\mathscr{L}(E_0, F_0)$ we also have $|S| \leq |R| = T$.

Now let A be the solid hull of $\{e_n: n \in \mathbf{N}\}$ in E_0 and let B be the unit ball of F_0^*. If $\{a_n: n \in \mathbf{N}\}$ is disjoint in A^+ then $a_n \to 0$ weakly and hence as E_0 is an AL-space, $\|a_n\| \to 0$. Conditions (i) and (iii) of Theorem 3.2 now follow immediately. Furthermore F and hence F_0 have order-continuous norm and hence $T^*b_n \to 0$ for any disjoint $\{b_n\}$ in B^+.

The proof is now exactly as the proof of Theorem 4.4. We deduce from Theorem 3.2 that since R: $E_0 \to F_0$ is Dunford-Pettis we must have $\|Se_n\| \to 0$.

5. Other applications

Let us call a linear subspace $\mathscr{I}$ of $\mathscr{L}(X, Y)$ (for X and Y Banach spaces) an *ideal* if $STV \in \mathscr{I}$ whenever $S \in \mathscr{L}(Y)$, $T \in \mathscr{I}$ and $V \in \mathscr{L}(X)$.

If E and F are Banach lattices an operator T: $E \to F$ is *M-weakly compact* [8] if $\|Ta_n\| \to 0$ whenever a_n is a disjoint bounded sequence in E. If F has order-continuous norm then the Banach lattice $\mathscr{L}_r(E, F)$ has order-continuous norm if and only if every $T \in \mathscr{L}_r(E, F)$ is M-weakly compact [4, Theorem 5.1].

THEOREM 5.1. *If E and F are Banach lattices such that F has order-continuous norm then each of the following conditions suffices to ensure that $\mathscr{L}_r(E, F)$ has order-continuous norm.*

(a) *E satisfies an upper p-estimate and F satisfies a lower q-estimate where $1 \leq q < p$.*

(b) *E^* has order-continuous norm and F is an AL-space.*

Proof. (a) See Theorem 7.7 of [4].

(b) Suppose T: $E \to F$ is a positive linear operator and $\{a_n\}$ is a disjoint bounded sequence in E_+. Then $a_n \to 0$ weakly and hence $Ta_n \to 0$ weakly so that $\|Ta_n\| \to 0$; i.e., T is M-weakly compact.

THEOREM 5.2. *Suppose E and F are separable Banach lattices such that $\mathscr{L}_r(E, F)$ has order-continuous norm. Let $\mathscr{I}$ be closed ideal of $\mathscr{L}(E, F)$. Then $\mathscr{I} \cap \mathscr{L}_r(E, F)$ is a band.*

Proof. Both E and F have quasi-interior positive elements. Since $\mathscr{L}_r(E, F)$ has order-continuous norm, we need only show that $\mathscr{I} \cap \mathscr{L}_r(E, F)$ is an order-ideal. Suppose $R \in \mathscr{I}$ and $S \in \mathscr{L}_r(E, F)$ with $|S| \leq |R| = T$. Let $A = U_E$ and $B = U_F^*$ in Theorem 3.2. Since T is M-weakly compact and F has order-continuous norm, the hypotheses of Theorem 3.2 are satisfied. Hence, for $\varepsilon > 0$, there exist $L_1, \ldots, L_k \in \mathscr{L}(F)$ and $M_1, \ldots M_k \in \mathscr{L}(E)$ so that if $S_0 = \sum_{i=1}^k L_i R M_i$ then $\|S - S_0\| < \varepsilon$. Thus $S \in \mathscr{I}$.

Remark. Theorem 5.2 applies to the case $E = L_p$ and $F = L_q$ where $1 \leq q < p$.

We shall say that a linear operator T: $X \to Y$ is l_p-singular (where $1 \leq p < \infty$) if there is no infinite dimensional subspace X_0 of X isomorphic to l_p such that $T|X_0$ is an isomorphism. The case $p = 2$ is of special interest here in view of Rosenthal's characterization of the Dunford-Pettis operators in $\mathscr{L}(L_1)$ as the l_2-singular operators [10].

PROPOSITION 5.3. *Suppose $1 \leq p < \infty$ and that F is a Banach lattice with order-continuous norm. Suppose there is no sequence of disjoint vectors in F_+ equivalent to the unit vector basis of l_p. Then if F_0 is a closed subspace of F isomorphic to l_p, there exists $\phi \in F_+^*$ so that for some $c > 0$, $c\|f\| \leq \phi(|f|)$, $f \in F_0$.*

Proof. We need only consider the case when F has a weak-order unit and then take ϕ to be any strictly positive linear functional on F. Then the result follows simply from a result of Figiel, Johnson and Tzafriri [5], or [7, Proposition 1.c.8, p. 38].

THEOREM 5.4. *Suppose E and F are Banach lattices with F having order-continuous norm. Suppose $1 < p < \infty$ and F contains no closed sublattice, lattice-isomorphic to l_p. Then the l_p-singular operators in $\mathscr{L}_r(E, F)$ form an order-ideal.*

Remark. By Proposition 5.3 (or Proposition 1.c.8 of [7]). This theorem is trivial if $2 < p < \infty$ since every operator is l_p-singular.

Proof. Again, it will suffice to consider the case when E and F have quasi-interior positive elements. Suppose $S, R \in \mathscr{L}_r(E, F)$ where $|S| \leq |R|$ and R is l_p-singular. Let E_0 be a closed subspace of E isomorphic to l_p such that $S|E_0$ is an isomorphism. According to Proposition 5.3 there exists $\phi \in F_+^*$ so that for some $c > 0$,

$$\phi(|Se|) \geq c\|Se\|, \quad e \in E_0.$$

Let A be the solid hull of U_{E_0} and let $B = [-\phi, \phi]$. Let $T = |R|$ and apply Theorem 3.2. We note first that by Lemma 4.2, $Ta_n \to 0$ weakly for every disjoint sequence $\{a_n : n \in \mathbf{N}\}$ in A^+. If $\{b_n\}$ is disjoint in B^+ then $b_n \to 0$ weak* and so $T^*b_n \to 0$ weak*. Finally $\langle Ta_n, b_n \rangle \leq \langle Ta_n, \phi \rangle \to 0$.

Now there exist multipliers $L_1, \ldots, L_k \in \mathscr{L}(F)$ and $M_1, \ldots, M_k \in \mathscr{L}(E)$ so that if $S_0 = \sum_{i=1}^k L_i R M_i$ then $|\langle Sa - S_0 a, b \rangle| \leq \frac{1}{2}c$, $a \in A$, $b \in B$. Hence for $e \in E_0$,

$$\phi(|Se - S_0 e|) \leq \tfrac{1}{2}c\|e\|.$$

Thus $\phi(|S_0 e|) \geq \frac{1}{2}c\|e\|$ and hence S_0 is also an isomorphism on E_0. Thus there is a closed subspace E_1 of E_0 with $E_1 \cong l_p$ and $1 \leq j \leq k$ so that $L_j R M_j | E_1$ is an isomorphism. Hence $M_j(E_1) \cong l_p$ and $R|M_j(E_1)$ is an isomorphism contrary to our assumption. Hence S is also l_p-singular.

We now consider a slight modification of Theorem 5.4. Let us say that an operator $T\colon X \to Y$ is *complementably l_p-singular* if there is no infinite-dimensional subspace X_0 of X isomorphic to l_p so that $T|X_0$ is an isomorphism and $T(X_0)$ is complemented in Y. An l_p-singular operator is complementably l_p-singular; the converse is true if the range space Y has the property that every subspace isomorphic to l_p contains a complemented infinite-dimensional subspace. In the case $p = 2$, it can be shown that this latter property holds for the spaces $Y = L_r$ where $1 < r < \infty$.

We shall require the following lemma.

LEMMA 5.5. *Let E be a Banach lattice and let $V \in \mathcal{L}(l_p, E)$ where $1 \le p < \infty$. Let $(d_n\colon n \ge 1)$ be the unit vector basis of l_p and suppose $e_n \in E$ are disjoint with $|e_n| \le |Vd_n|$. Then there exists $W \in \mathcal{L}(l_p, E)$ with $Wd_n = e_n$.*

Proof. Suppose $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ with $\sum_{i=1}^n |\alpha_i|^p \le 1$. Then

$$\begin{aligned}\left\| \sum_{i=1}^n \alpha_i e_i \right\| &= \left\| \left(\sum_{i=1}^n \alpha_i^2 |e_i^2| \right)^{1/2} \right\| \\ &\le \left\| \left(\sum_{i=1}^n \alpha_i^2 |Vd_i|^2 \right)^{1/2} \right\| \\ &\le K_G \|V\| \left\| \left(\sum \alpha_i^2 |d_i^2| \right)^{1/2} \right\|_{l_p} \\ &\le K_G \|V\|\end{aligned}$$

by [7, Theorem 1.f.14, p. 93] (K_G is the Grothendieck constant).

THEOREM 5.6. *Suppose E and F are Banach lattices with F having order-continuous norm. Suppose $1 < p < \infty$ and that either*

(a) *E contains no complemented sublattice, lattice-isomorphic to l_p, or*

(b) *F contains no complemented sublattice, lattice-isomorphic to l_p.*

Then the complementably l_p-singular operators in $\mathcal{L}_r(E, F)$ form an order-ideal.

Proof. We prove the theorem for the case when E and F have quasi-interior positive elements. As usual the general case can be reduced to this case, noting in particular that the closure of every principal ideal in F is complemented.

Let us suppose $R \in \mathcal{L}_r(E, F)$ is complementably l_p-singular and that $|S| \le |R|$. Let $T = |R|$. We shall show that if $V\colon l_p \to E$ and $W\colon F \to l_p$ are bounded linear operators then $\langle WSVd_n, d_n^* \rangle \to 0$ where d_n is the unit vector basis of l_p and d_n^* is the unit vector basis of l_q where $q^{-1} + p^{-1} = 1$. Since WSV cannot therefore be the identity on l_p this will establish the result.

Let A be the solid hull of the sequence $\{Vd_n: n \in \mathbf{N}\}$ and let B be the solid hull of the sequence $\{W^*d_n^*: n \in \mathbf{N}\}$. Let $\{a_n\}$ be disjoint in A^+ and let $\{b_n\}$ be disjoint in B^+. Then $a_n \to 0$ weakly (Lemma 4.2) and $b_n \to 0$ weakly so that $Ta_n \to 0$ weakly and $T^*b_n \to 0$ weak*. We shall show that $\langle Ta_n, b_n\rangle \to 0$.

Suppose $\langle Ta_n, b_n\rangle \geq \delta > 0$ for all $n \in \mathbf{N}$. By passing to a subsequence we may suppose $a_n \leq |Vd_{r(n)}|$ where $r(n)$ is a strictly increasing sequence and that $b_n \leq |W^*d^*_{s(n)}|$ where $s(n)$ is a strictly increasing sequence. To see this observe that if $a_n \leq |Vd_k|$ for some fixed k, then

$$\langle Ta_n, b_n\rangle \leq \langle |Vd_k|, T^*b_n\rangle \to 0,$$

and a similar argument shows that if $b_n \leq |W^*d_k|$ then $\langle Ta_n, b_n\rangle \to 0$.

Now suppose we have case (a) of the hypotheses. By Lemma 5.5 there exist

$$V_1: l_p \to E \quad \text{and} \quad Q_1: l_q \to F^*$$

so that $V_1 d_n = a_n$ and $Qd_n^* = b_n$. Then $\langle Q^*TV_1 d_n, d_n^*\rangle \geq \delta$ for all $n \in \mathbf{N}$ and hence by a standard gliding hump argument there is a subsequence $d_{k(n)}$ of d_n so that Q^*TV_1 is an isomorphism on the closed linear span $[d_{k(n)}]$. This implies that $[a_{k(n)}]$ is a closed sub-lattice of E lattice-isomorphic to l_p which is complemented (by PQ^*T where P: $[Q^*TV_1 d_{k(n)}] \to [a_{k(n)}]$ is the inverse of $Q^*T|[a_{k(n)}]$). This contradicts hypothesis (a).

In case (b), we may find c_n with $0 \leq c_n \leq Ta_n$ so that c_n are disjoint and

$$\langle c_n, b_n\rangle = \langle Ta_n, b_n\rangle.$$

Indeed let $B_n = \{f \in F: b_n(|f|) = 0\}$ and let P_n: $F \to B_n$ be the band projection. Set $c_n = Ta_n - P_n Ta_n$. If $m \neq n$, since $b_m \wedge b_n = 0$, given $\varepsilon > 0$ we can write $c_m \wedge c_n = u + v$ where $\langle u, b_m\rangle = 0$ and $\langle v, b_n\rangle = 0$ and $u, v \geq 0$ (Note here that the order-interval $[0, c_m \wedge c_n]$ is weakly compact). Thus $P_n v \leq P_n c_n = 0$ and as $v \in B_n$, $v = 0$; similarly $u = 0$ and so $c_m \wedge c_n = 0$. Thus there exist operators V_2: $l_p \to F$ and Q_1: $l_q \to F^*$ so that $V_2 d_n = c_n$ and $Q_1 d_n^* = b_n$. The conclusion of the argument is similar to case (a) and we omit it. We conclude in either case that $\langle Ta_n, b_n\rangle \to 0$.

Now by Theorem 3.2, if $\varepsilon > 0$, there exist $L_1, \ldots, L_k \in \mathscr{L}(F)$, $M_1, \ldots, M_k \in \mathscr{L}(E)$ so that if $S_0 = \sum_{i=1}^k L_i R M_i$ then $|\langle SVd_n - S_0 Vd_n, W^*d_n^*\rangle| \leq \varepsilon$. We claim

$$\langle S_0 Vd_n, W^*d_n^*\rangle \to 0.$$

Indeed, if not we can find a subsequence $d_{r(n)}$ and $1 \leq j \leq k$ so that

$$|\langle L_j R M_j V d_{r(n)}, W^*d^*_{r(n)}\rangle| \geq \delta.$$

Again this means the existence of a further subsequence $d_{s(n)}$ so that

$$WL_jRM_jV\left[d_{s(n)}\right]$$

is an isomorphism. Now if $G = M_jV[d_{s(n)}]$ then $G \cong l_p$, R is an isomorphism on G and $R(G)$ is complemented in F by the map P_1WL_j where P_1 is the inverse of WL_j: $R(G) \to WL_jR(G)$. This contradicts the fact that R is complementably l_p-singular. Hence $\langle S_0Vd_n, W^*d_n^*\rangle \to 0$ and so

$$\limsup_{n\to\infty} |\langle SVd_n, W^*d_n^*\rangle| \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary the proof is complete.

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