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A COBORDISM OBSTRUCTION TO EMBEDDING MANIFOLDS

BY

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1. Introduction

Let M be a smooth, compact, stably almost complex manifold and υ a complex normal bundle for M, of large dimension. Let RP^n denote real projective space of dimension n and ξ the nontrivial real line bundle over RP^n. Then the external tensor product $\upsilon \otimes \xi$ is a bundle over $M \times RP^n$, and it is the real bundle underlying the complex bundle $\upsilon \otimes_C (\xi \otimes C)$; thus $\upsilon \otimes \xi$ is oriented in complex cobordism theory $MU^*(\ )$. We shall prove:

THEOREM 1.1. *If M embeds in euclidean space with codimension $2l$ and the complex dimension of υ is $l + k$, then the Euler class of $\upsilon \otimes \xi$ in $MU^*(M \times RP^{2k})$ vanishes, provided the natural map*

$$MU^{ev}(M) \otimes_{\pi_* MU} MU^*(RP^{2k}) \to MU^{ev}(M \times RP^{2k}) \tag{1.2}$$

is an isomorphism.

It is well known that if M immerses in codimension $2l$ then the Euler class $e(\upsilon \otimes \xi)$ vanishes over $M \times RP^{2k-1}$. For if υ_0 is the normal bundle of an immersion then one has $\upsilon \cong \upsilon_0 \oplus 2k$ as real bundles, and thus

$$\upsilon \otimes \xi \cong \upsilon_0 \otimes \xi \oplus 1 \otimes 2k\xi;$$

this implies that $\upsilon \otimes \xi$ has a nonvanishing section over $M \times RP^{2k-1}$, because $2k\xi$ has a nonvanishing section over RP^{2k-1}. The content of (1.1) is then that, under appropriate hypotheses, if M embeds in codimension $2l$ then $e(\upsilon \otimes \xi)$ must vanish over the larger space $M \times RP^{2k}$. In the case of Euler classes in singular cohomology with Z_2 coefficients this amounts to the fact that the highest Stiefel Whitney class of the normal bundle of an embedding is zero. In Z_2 cohomology one always has the Künneth theorem in its strong form, and all manifolds are oriented; in this light the hypotheses of (1.1) seem reasonable. The map (1.2) is injective by the Künneth theorem for MU theory [6]; the

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hypothesis on (1.2) is then that the Tor term in the Künneth sequence should vanish in even degrees. This is satisfied, for example, if M has integral homology free of torsion, or if it is a real projective space.

Theorem 1.1 implies the classical result of Atiyah and Hirzebruch [4] that complex projective space of dimension n does not embed with codimension $2n - 2\alpha(n)$ where $\alpha(n)$ is the number of ones in the binary expansion of n.

A more interesting example is that of real projective space. If $M = RP^n$ with n odd then (1.1) applies and yields a strong nonembedding result via Davis's recent calculations [5]. However, Davis's results are best for projective spaces of even dimension. To take better advantage of these calculations we work a little harder and stretch (1.1) a bit to make it apply to even dimensional projective spaces, which are not stably almost complex manifolds. Let $\alpha(m)$ denote the number of ones in the binary expansion of m.

THEOREM 1.3. *Real projective space of dimension* $2(m + \alpha(m) - 1)$ *does not embed in euclidean space of dimension* $4m - 2\alpha(m) + 1$.

This is the nonembedding version of Davis's nonimmersion Theorem [5].

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2. Proofs

We shall work in the category of CW spectra as described in Part III of [1].

Recall that a complex oriented spectrum is a commutative ring spectrum E equipped with a ring map

$$MU \to E.$$

Examples of such spectra are MU itself, the spectrum bu for complex connective K theory, and the smash product $bu \wedge MU$, as well as the spectra for singular integral and mod 2 cohomology. We shall have occasion to use all these in this section.

Let CP^n denote complex projective space of (complex) dimension n. From (2.5) of Part II of [1] we have

$$E^*(CP^n) = \pi_*(E)[y]/(y^{n+1})$$

where $y \in \tilde{E}^2(CP^n)$ is the Euler class of the canonical complex line bundle η over CP^n. Let x denote the Euler class

$$e(\xi \otimes C) \in \tilde{E}^2(RP^{2n}).$$

and q: $RP^{2n} \to CP^n$ the natural map. Since $q^*\eta = \xi \otimes C$ we have $q^*y = x$.

LEMMA 2.1. *If 2 is not a divisor of zero in $\pi_*(E)$ and $\pi_{odd}(E) = 0$ then*

$$E^*(RP^{2n}) = \pi_*(E)[x]/(x^{n+1}, \mu^E(x))$$

where $\mu^E(x) = q^*e(\eta \otimes_C \eta)$. In particular,

$$q^*\colon E^*(CP^n) \to E^*(RP^{2n})$$

is onto.

This is well known and its proof is as in (3.5) of [3].

Proof of (1.1). We assume M embeds in euclidean space with codimension $2l$; let v_0 be the normal bundle of an embedding. Then

$$v \cong v_0 \oplus 2k \tag{2.2}$$

as real bundles. Thus v_0 is a stably complex bundle, so it is oriented in MU cohomology. Note that $v_0 \otimes \xi$ is also stably complex. Indeed, adding a trivial bundle of dimension $2^N - 2k$ to equation (2.2) and then taking tensor product with ξ we obtain

$$\big(v \oplus (2^N - 2k)\big) \otimes \xi \cong v_0 \otimes \xi \oplus 1 \otimes 2^N\xi \tag{2.3}$$

over $M \times RP^{2k}$. Since the left hand side of (2.3) is a complex bundle and $2^N\xi$ is trivial when N is large, $v_0 \otimes \xi$ is stably complex. Thus from (2.2) we have, after taking tensor product with ξ and taking Euler classes in MU cohomology

$$e(v \otimes \xi) = e(v_0 \otimes \xi)e(1 \otimes 2k\xi)$$

Now, $2k\xi$ is the real bundle underlying $k\xi \otimes C$, so its Euler class is x^k, in the notation of (2.1); thus

$$e(v \otimes \xi) = e(v_0 \otimes \xi)x^k \tag{2.4}$$

in $MU^{ev}(M \times RP^{2k})$. By Lemma 2.1 and the hypotheses of (1.1) we have

$$MU^{ev}(M \times RP^{2k}) = MU^{ev}(M)[x]/(x^{k+1}, \mu(x)).$$

Thus we may write

$$e(v_0 \otimes \xi) = \sum_{i=0}^{k} a_i x^i \tag{2.5}$$

with $a_i \in MU^{ev}(M)$. Clearly (2.4) becomes

$$e(v \otimes \xi) = a_0 x^k \tag{2.6}$$

because $x^{k+1} = 0$. Now consider the inclusion

$$j\colon M \to M \times RP^{2k}$$

defined using a base point in RP^{2k}. We have $j^*(v_0 \otimes \xi) = v_0$, so

$$j^*e(v_0 \otimes \xi) = e(v_0).$$

But from (2.5) it is clear that $j^*e(v_0 \otimes \xi) = a_0$, so that (2.6) turns into

$$e(v \otimes \xi) = e(v_0)x^k. \tag{2.7}$$

Since v_0 is the normal bundle of an embedding in euclidean space, $e(v_0)$ is zero. The proof of this is exactly as in (11.3) and (11.4) of [7], replacing H by MU throughout. This concludes the proof of (1.1).

Next we must prove (1.3). We shall deduce it from the results of Davis [5] and from Proposition 2.8 below.

Let N be a large positive integer and let $v = (2^N - 2n - 1)\xi$. This is a normal bundle for RP^{2n}. It is not orientable, even in singular cohomology, but $v \oplus \xi$ has a complex structure, so it is oriented in MU cohomology.

PROPOSITION 2.8. *Suppose RP^{2n} embeds in euclidean space with codimension $2l - 1$. Then*

$$e((v \oplus \xi) \otimes \xi) = 0$$

in $MU^*(RP^{2n} \times RP^{2k})$, where $k = 2^{N-1} - n - l$.

Proof. Let v_0 be the normal bundle of an embedding. The argument used in the proof of (1.1) to justify (2.7) is valid, replacing v by $v \oplus \xi$ and v_0 by $v_0 \oplus \xi$. Thus

$$e((v \oplus \xi) \otimes \xi) = e(v_0 \oplus \xi)x^k$$

in $MU^*(RP^{2n} \times RP^{2k})$, where x is the Euler class of $\xi \otimes C$ over RP^{2k}. The class of x^k has order 2; we will show that

$$e(v_0 \oplus \xi) \in MU^{2l}(RP^{2n})$$

is even. We will prove this first for the Euler class

$$e_b(v_0 \oplus \xi) \in bu^{2l}(RP^{2n}) \tag{2.9}$$

of $v_0 \oplus \xi$ in connective K theory.

Under the standard map into integral cohomology

$$bu^*(RP^{2n}) \to H^*(RP^{2n}) \tag{2.10}$$

the class $e_b(v_0 \oplus \xi)$ goes into the cohomology Euler class $e_H(v_0 \oplus \xi)$. It is easy to see that this is zero. Indeed, modulo 2 one has

$$e_H(v_0 \oplus \xi) \equiv W_{2l}(v_0 \oplus \xi) = W_{2l-1}(v_0)W_1(\xi) = 0;$$

$W_{2l-1}(v_0)$ vanishes because v_0 is the normal bundle of an embedding. This means that $e_H(v_0 \oplus \xi)$ is zero, because $H^{2l}(RP^{2n})$ has order 2. Now, by Bott periodicity $\pi_*(bu)$ is a polynomial ring over the integers on one generator t of degree minus two, and by (2.1),

$$bu^*(RP^{2n}) = \pi_*(bu)[x]/(x^{n+1}, 2x + tx^2).$$

The fact that $\mu^{bu}(x) = 2x + tx^2$ follows from (2.9) of Part II of [1]. Thus $bu^{2l}(RP^{2n})$ is cyclic of order 2^{n+1-l} and it is generated by x^l. The relation $2x^l + tx^{l+1} = 0$ shows that all multiples of t lying in this group are divisible by 2. But the kernel of the map (2.10) consists precisely of the multiples of t. Thus the Euler class (2.9) is even, as claimed.

To show that the MU Euler class is even consider now the maps of ring spectra α_M and α_b defined to be the compositions

$$MU \xrightarrow{\cong} S^0 \wedge MU \xrightarrow{i \wedge 1} bu \wedge MU$$

and

$$bu \xrightarrow{\cong} bu \wedge S^0 \xrightarrow{1 \wedge i} bu \wedge MU$$

where the maps i are the unit maps of bu and MU. It is the map α_M that makes $bu \wedge MU$ a complex oriented spectrum. These maps give rise to a diagram

$$\begin{tikzcd} MU^*(RP^{2n}) \arrow[r, "\alpha_M"] & (bu \wedge MU)^*(RP^{2n}) \\ & bu^*(RP^{2n}). \arrow[u, "\alpha_b"'] \end{tikzcd}$$

A map of ring spectra sends Euler classes in one theory into Euler classes in the other theory. Thus $\alpha_M e(v_0 \oplus \xi)$ and $\alpha_b e_b(v_0 \oplus \xi)$ are both Euler classes for $v_0 \oplus \xi$ in the theory $bu \wedge MU$. These correspond to two (very likely different) orientations. By virtue of the existence of Thom isomorphisms, any two orientations are the same up to multiplication by a unit, and therefore the same is true of any two Euler classes. Since we have shown that $\alpha_b e_b(v_0 \oplus \xi)$

is even, we conclude that

$$\alpha_M e(v_0 \oplus \xi) \in (bu \wedge MU)^*(RP^{2n})$$

is also even.

Now we claim that

$$\alpha_M\colon MU^*(RP^{2n}) \to (bu \wedge MU)^*(RP^{2n}) \tag{2.11}$$

is an injection onto a direct summand. Clearly if this is true we will have finished the proof of (2.8). To prove this consider

$$\begin{array}{ccc} MU^*(CP^n) & \xrightarrow{\alpha_M} & (bu \wedge MU)^*(CP^n) \\ \Big\downarrow q^* & & \Big\downarrow q^* \\ MU^*(RP^{2n}) & \xrightarrow{\alpha_M} & (bu \wedge MU)^*RP^{2n}) \end{array} \quad .$$

Both maps q^* are onto by (2.1). Let K_M and K_{bM} denote the kernels of these maps q^*. It follows from (2.1) that K_M is free over $\pi_*(MU)$ and that

$$\mu(y), y\mu(y), \ldots, y^{n-1}\mu(y)$$

forms a basis. Analogously, K_{bM} is free over $\pi_*(bu \wedge MU)$ with basis

$$\alpha_M(\mu(y)), \ldots, \alpha_M(y^{n-1}\mu(y)).$$

The theorem of Hattori and Stong [2] asserts that

$$\alpha_M\colon \pi_*(MU) \to \pi_*(bu \wedge MU)$$

is an injection onto a direct summand. Let β be a left inverse; this is a homomorphism of abelian groups. Extend β to

$$\beta\colon (bu \wedge MU)^*(CP^n) \to MU^*(CP^n) \tag{2.12}$$

by $\beta(\Sigma a_i \alpha_M(y^i)) = \Sigma \beta(a_i) y^i$. Because we have explicit bases for K_M and K_{bM} it is easy to verify that K_{bM} goes into K_M under (2.12). Then (2.12) induces a left inverse for (2.11), so that (2.11) is an injection onto a direct summand, as advertised. This concludes the proof of (2.8).

To deduce (1.3) from (2.8) we need the following result of Davis [5].

THEOREM 2.13. *If the binomial coefficient* $\binom{r+s}{n-s}$ *is divisible by* 2^s *but not by* 2^{s+1} *then the Euler class of* $2r\xi \otimes \xi$ *is nonzero in* $MU^*(RP^{2n} \times RP^{2k})$, *where* $k = r - n + 3s$.

Proof of (1.3). Apply (2.13) with $n = m + \alpha(m) - 1$ and $r = 2^{N-1} - n$ and $s = \alpha(m) - 1$, and combine with (2.8).

3. Appendix

This appendix contains a conjecture concerning the desuspension of Thom complexes. Let α be a (real) vector bundle over a CW complex X. As indicated following the statement of Theorem 1.1 we have:

LEMMA 3.1. *If α admits k everywhere linearly independent sections over X then $\alpha \otimes \xi$ admits a nonvanishing section over $X \times RP^{k-1}$.*

The Thom complex analogue of this lemma is:

CONJECTURE 3.2. *If the Thom complex of α is a k fold suspension then the Thom complex of the bundle $\alpha \otimes \xi$ over $X \times RP^{k-1}$ is a suspension.*

Observe that since the Euler class is essentially the square of the Thom class, it must vanish for a bundle whose Thom complex is a suspension. Then Conjecture 3.2 and Davis's theorem (Theorem 2.13 above) together imply:

CONJECTURE 3.3. *If $\binom{r+s}{n-s}$ is divisible by 2^s but not by 2^{s+1} then the truncated real projective space RP^{2r+2n}/RP^{2r-1} is not a $2r - 2n + 6s + 1$ fold suspension.*

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