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NILPOTENT ELEMENTS IN GROTHENDIECK RINGS

BY

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This work was inspired by the following theorem of Bass and Guralnick [3]: Let P and Q be finitely generated projective modules over a commutative ring. Then $\otimes^m P \cong \otimes^m Q$ for some $m > 0$ if and only if $\oplus^n P \cong \oplus^n Q$ for some $n > 0$. Their proof depends on the fact [1, Ch. IX, 4.4] that $([P] - [Q])^{d+1} = 0$ in $K_0(R)$, whenever P and Q are finitely generated projective modules with the same rank and R is Noetherian and d-dimensional.

In Section 2 of this paper we show that the nilpotency theorem, suitably interpreted, is valid without the assumption that P and Q be projective. In Section 3 we investigate to what extent the Bass-Guralnick theorem generalizes to non-projective modules, and we obtain some positive results in dimensions one and two. In the fourth section we prove the nilpotency theorem for finitely presented modules over arbitrary (non-Noetherian) rings. The bound $d =$ Krull dimension always suffices, and in at least one case we can do much better: For the ring of continuous real-valued functions on a compact Hausdorff space X one can take $d =$ covering dimension of X.

I want to thank the referee for reading this paper carefully and for suggesting some substantial improvements, particularly in Section 3.

1. Notation and preliminaries

All rings are commutative, and modules are always finitely generated. If M and N are R-modules, we write MN for the tensor product $M \otimes_R N$, and M^r for the r-fold tensor power, with $M^0 = R$. Similarly, $M + N = M \oplus N$ and $rM = \oplus^r M$. If $f \in \mathbf{Z}^+[X_1, \ldots, X_n]$, the set of polynomials with nonnegative integer coefficients, and $M_1, \ldots, M_n$ are R-modules, the expression $f(M_1, \ldots, M_n)$ makes sense. For an arbitrary polynomial $f \in \mathbf{Z}[X_1, \ldots, X_n]$, let f^+ (respectively $-f^-$) be the sum of all the monomials of f with positive (respectively negative) coefficients. Then $f = f^+ - f^-$, and f^+ and f^- are in

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$\mathbf{Z}^+[X_1, \ldots, X_n]$.

1.1. *Notation.* Let $M_1, \ldots, M_n$ be R-modules, and let $f \in \mathbf{Z}[X_1, \ldots, X_n]$. We write $f(M_1, \ldots, M_n) \equiv 0$ provided $f^+(M_1, \ldots, M_n) \cong f^-(M_1, \ldots, M_n)$. Let

$$\mathscr{R}(M_1, \ldots, M_n) = \{ f \in \mathbf{Z}[X_1, \ldots, X_n] | f(M_1, \ldots, M_n) \equiv 0 \},$$

and let

$$\mathscr{I}(M_1, \ldots, M_n) = \{ g - h | g, h \in \mathbf{Z}^+[X_1, \ldots, X_n] \text{ and } g(M_1, \ldots, M_n) \cong h(M_1, \ldots, M_n) \}.$$

The latter set is an ideal (see (1.2)), and we let

$$\mathbf{Z}[M_1, \ldots, M_n] = \mathbf{Z}[X_1, \ldots, X_n] / \mathscr{I}(M_1, \ldots, M_n).$$

Of course $\mathscr{R}(M_1, \ldots, M_n) \subseteq \mathscr{I}(M_1, \ldots, M_n)$, but the reverse inclusion can fail. Suppose, for example, that $M \oplus R \cong 3R$ but $M \not\cong 2R$. Then $X - 2$ is in $\mathscr{I}(M)$ but not in $\mathscr{R}(M)$. Fortunately, failure of direct-sum cancellation is the only thing that distinguishes $\mathscr{I}$ from $\mathscr{R}$. We leave the easy proof of the next lemma to the reader.

1.2. LEMMA. *$\mathscr{I}(M_1, \ldots, M_n)$ is an ideal of $\mathbf{Z}[X_1, \ldots, X_n]$. Moreover, a polynomial f is in $\mathscr{I}(M_1, \ldots, M_n)$ if and only if*

$$f^+(M_1, \ldots, M_n) \oplus p(M_1, \ldots, M_n) \cong f^-(M_1, \ldots, M_n) \oplus p(M_1, \ldots, M_n)$$

for some $p \in \mathbf{Z}^+[X_1, \ldots, X_n]$.

The reason for working with $\mathscr{R}$ rather than the more manageable $\mathscr{I}$ is that we want to get information about the real world of isomorphism classes of modules, not just about relations in Grothendieck rings. I have been unable to decide whether or not $\mathscr{I}$ is always generated by $\mathscr{R}$ as an ideal (or even as an additive group). If the M_i are projective and $f \in \mathscr{I}(M_1, \ldots, M_n)$ then by the next remark we have $kf \in \mathscr{R}(M_1, \ldots, M_n)$ for large k. Therefore $f = (k+1)f - kf$ is the difference of two elements of $\mathscr{R}$.

1.3. *Remark.* Let $M_1, \ldots, M_n$ be projective R-modules, and let

$$f \in \mathscr{I}(M_1, \ldots, M_n).$$

Then

$$kf \in \mathscr{R}(M_1, \ldots, M_n)$$

for all sufficiently large k.

Proof. Let $A^+ = f^+(M_1, \ldots, M_n)$, $A^- = f^-(M_1, \ldots, M_n)$. By (1.2) there is a projective module B such that $A^+ \oplus B \cong A^- \oplus B$. We want to prove that $kA^+ \cong kA^-$ for large k. We may assume A^+ and A^- have constant positive rank, and, by restricting to a suitable finitely generated subring, we may assume R is Noetherian with finite Krull dimension d. Now take $k > d$ and use the Bass cancellation theorem [1, Ch. IX, 4.1].

The proof of the nilpotency theorem depends on several well-known elementary results, which we collect here for ease of reference.

1.4. LEMMA [2, Ch. II, Prop. 19]. *Let S be a multiplicative subset of R, and let M, N be R-modules, with M finitely presented. Then the natural homomorphism $S^{-1}\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ is an isomorphism.*

1.5. LEMMA. *Let M and N be finitely presented R-modules such that $M_P \cong N_P$ for all P in some finite set Ω of prime ideals. Then there exists a homomorphism ϕ: $M \to N$ and a neighborhood U of Ω in* spec(R) *such that ϕ_Q: $M_Q \to N_Q$ is an isomorphism for all $Q \in U$.*

Proof. Using (1.4) one can find homomorphisms ϕ: $M \to N$ and ψ: $N \to M$ and an element $s \in R - \cup\Omega$, such that ϕ/s and ψ/s are reciprocal isomorphisms between $S^{-1}M$ and $S^{-1}N$. Take

$$U = D(s) - \left(\mathrm{Supp}\left(\phi\psi - s^2 1_N\right) \cup \mathrm{Supp}\left(\psi\phi - s^2 1_M\right)\right).$$

1.6. LEMMA. *Let $M.$ be a complex of R-modules, with connecting maps*

$$\partial_n\colon M_n \to M_{n-1}, \quad n \in \mathbf{Z}.$$

The following are equivalent:

(1) *$M.$ is exact, and* $\ker \partial_n$ *is a direct summand of M_n for every $n \in \mathbf{Z}$.*

(2) *The identity map on $M.$ is chain homotopic to* 0.

Proof. (1) $\Rightarrow$ (2) Let $u_{n-1}p_n$ be the canonical factorization of ∂_n through $\ker \partial_{n-1}$. Each of the short exact sequences

$$0 \to \ker \partial_n \xrightarrow{u_n} M_n \xrightarrow{p_n} \ker \partial_{n-1} \to 0 \tag{1.6.1}$$

splits, so there are maps f_n: $M_n \to \ker \partial_n$ and g_n: $\ker \partial_{n-1} \to M_n$ satisfying

$$f_n u_n = 1, \; p_n g_n = 1, \; u_n f_n + g_n p_n = 1.$$

Set $d_n = g_{n+1} f_n$: $M_n \to M_{n+1}$. Then

$$\partial_{n+1} d_n + d_{n-1}\partial_n = u_n p_{n+1} g_{n+1} f_n + g_n f_{n-1} u_{n-1} p_n = 1,$$

as desired.

(2) $\Rightarrow$ (1) Given maps $d_n\colon M_n \to M_{n+1}$ such that $\partial_{n+1}d_n + d_{n-1}\partial_n = 1$, one checks that $M.$ is exact. Also, the surjection p_n in (1.6.1) is split by $d_{n-1}u_{n-1}$, so $\ker \partial_n$ is a summand of M_n.

DEFINITION. If $M.$ is a complex satisfying (1) and (2) of (1.6), we say $M.$ *splits*.

1.7. LEMMA. *Let $M.$ be a complex of finitely presented R-modules. If, for every maximal ideal $\mathscr{M}$, the localized complex $(M.)_{\mathscr{M}}$ splits, then $M.$ splits.*

Proof. Certainly $M.$ is exact. We have to show that the short exact sequences (1.6.1) split. But (1.6.1) splits if and only if the map

$$\phi\colon \operatorname{Hom}_R(\ker \partial_{n-1}, M_n) \to \operatorname{Hom}_R(\ker \partial_{n-1}, \ker \partial_{n-1}),$$

induced by p_n, is surjective. Now $\ker \partial_n$ is finitely generated, being a homomorphic image of M_{n+1}, and it follows that $\ker \partial_{n-1}$ is finitely presented. Since (1.6.1) splits locally, (1.4) implies that $\phi_{\mathscr{M}}$ is surjective for all $\mathscr{M}$. Hence ϕ is surjective.

We conclude this introductory section with a proof of the nilpotency theorem for projective modules, even though a much more general result will be proved in Section 2. The proof given here is easier than existing proofs in the literature [1], [14] and is conceptually cleaner than the proof given in Section 2. For simplicity we work with the whole spectrum, rather than just the maximal ideal space, though the modifications required to get the sharper theorem of [1] and [14] are easy and standard.

1.8. THEOREM [1, Ch. IX, 4.4]. *Let M and N be projective modules over a d-dimensional ring with Noetherian spectrum. If M and N have the same rank (i.e., are locally isomorphic), then $(M - N)^{d+1} \equiv 0$.*

Proof. By (1.5) there exist a homomorphism $f\colon M \to N$ and an open set $U = D(I)$ containing all minimal primes, such that f_Q is an isomorphism for all $Q \in U$. Now R/I has dimension less than d, so

$$((M/IM) - (N/IN))^d \equiv 0$$

by induction. This means, according to (1.1), that $U/IU \cong V/IV$, where U (respectively V) is the direct sum of the modules $\binom{d}{i}M^{d-i}N^i$ over all even (respectively odd) subscripts i. Since U is projective there is a map $g\colon U \to V$ inducing an isomorphism modulo I. Then g_Q is surjective, hence an isomorphism, for every prime $Q \supseteq I$. Form the complex

$$0 \to MU \xrightarrow{\alpha} MV \oplus NU \xrightarrow{\beta} NV \to 0 \qquad (1.8.1)$$

where $\alpha = [1 \otimes g,\ f \otimes 1]^t$ and $\beta = [f \otimes 1,\ -1 \otimes g]$. We want to show that (1.8.1) splits, and by (1.7) it is enough to check this at each maximal ideal $\mathscr{M}$. Fortunately, either $f_{\mathscr{M}}$ or $g_{\mathscr{M}}$ is an isomorphism. If, say, $f_{\mathscr{M}}$ is an isomorphism, the maps $[0, f_{\mathscr{M}}^{-1} \otimes 1]$ and $[f_{\mathscr{M}}^{-1} \otimes 1, 0]^t$ split (1.8.1) at $\mathscr{M}$.

2. The nilpotency theorem

If M and N are not projective, the simple inductive argument of (1.8) breaks down. The proof we will give is similar to the proof for projectives given in [14, Part II, Ch. 10].

2.1. LEMMA. *Let $X.$ be any complex, and let $Y.$ be a complex with $Y_i = 0$ for $i \neq 0, 1$ and with connecting map $f\colon Y_1 \to Y_0$ an isomorphism. Then $X.\otimes Y.$ splits.*

Proof. Let $Z. = X.\otimes Y.$; so $Z_n = X_n Y_0 + X_{n-1} Y_1$, and the connecting map $Z_n \to Z_{n-1}$ is

$$\Delta_n = \begin{bmatrix} \partial_n \otimes 1 & \pm 1 \otimes f \\ 0 & \partial_{n-1} \otimes 1 \end{bmatrix}.$$

The chain homotopy is given by the maps

$$D_n = \begin{bmatrix} 0 & 0 \\ \pm 1 \otimes f^{-1} & 0 \end{bmatrix} : Z_n \to Z_{n+1}.$$

2.2. THEOREM. *Let R be a commutative ring with d-dimensional Noetherian j-spectrum, let M_i, N_i be finitely presented R-modules $0 \le i \le d$, and assume $(M_i)_P \cong (N_i)_P$ for all j-primes P of j-height $\le i$. Then*

$$(M_0 - N_0) \ldots (M_d - N_d) \equiv 0.$$

Proof. By (1.5) there is a map $f_0\colon M_0 \to N_0$ such that $(f_0)_p$ is an isomorphism on some open set U_0 containing all minimal j-primes. Then U_0 contains all but a finite set Ω of j-primes of j-height 1, so there is a map $f_1\colon M_1 \to N_1$ which is an isomorphism at all j-primes in some neighborhood U_1 of Ω. Then $U_0 \cup U_1$ contains all but finitely many j-primes of j-height ≤ 2. Continuing, we get $f_i\colon M_i \to N_i$, $0 \le i \le d$, such that at each j-prime at least one of the f_i is an isomorphism. Form $d + 1$ complexes with M_i in degree 1, N_i in degree 0 and f_i as connecting map, and tensor them all together. The resulting complex $U.$ splits, by (2.1) and (1.7). This means

$$\oplus \{C_m\colon m \text{ even}\} \cong \oplus \{C_n\colon n \text{ odd}\},$$

and the desired conclusion follows.

The theorem is of course true if all occurrences of "j-" are deleted.

2.3. COROLLARY. *Let R be a regular Noetherian domain of dimension d, let $0 \to P_d \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M, and let $S_i = \operatorname{Im}(P_i \to P_{i-1})$ be the ith syzygy (with $S_0 = M$). Let r_i be the torsionfree rank of S_i. Then $(S_0 - r_0R)\dots(S_d - r_dR) \equiv 0$.*

2.4. COROLLARY. *Let M and N be locally isomorphic finitely presented modules over any commutative ring R. Then $(M - N)^n = 0$ for some integer $n \geq 1$.*

Proof. We may replace R by a sufficiently large finitely generated subring, and then apply (2.2). The only difficulty is in preserving the hypothesis that M and N be locally isomorphic. This is done as follows: Using (1.4), we find ϕ_i: $M \to N$ and $s_i \in R$ such that the maps $M[s_i^{-1}] \to N[s_i^{-1}]$ induced by ϕ_i are isomorphisms, and such that $Rs_1 + \cdots + Rs_m = R$. We include, in our finitely generated subring, all the s_i, all scalars necessary to define the ϕ_i and their "inverses", and elements r_i such that $r_1s_1 + \cdots + r_ms_m = 1$.

We conclude this section with the stable version of the theorem of Bass and Guralnick. The delicate question of direct-sum cancellation will be postponed until the next section. Write $\langle S \rangle$ for the ideal of $\mathbf{Z}[X, Y]$ generated by S.

2.5. LEMMA. *For integers $n \geq 1$ and $e \geq 0$,*

$$X^{n^e} - Y^{n^e} \in \langle n(X - Y), (X - Y)^{2^e} \rangle.$$

Proof [3, Part (b)]. We may assume $n \geq 2$ and $e \geq 1$. By induction on e, write $X^{n^{e-1}} = Y^{n^{e-1}} + F$, where

$$F \in \langle n(X - Y), (X - Y)^{2^{e-1}} \rangle.$$

Taking the nth power of both sides, we get

$$X^{n^e} = Y^{n^e} + nY^{n^{e-1}}F + G, \quad \text{where } G \in \langle F^2 \rangle.$$

Now $F \in \langle X - Y \rangle$, so $nF \in \langle n(X - Y) \rangle$; and clearly

$$F^2 \in \langle n(X - Y), (X - Y)^{2^e} \rangle.$$

Therefore

$$nY^{n^{e-1}}F + G \in \langle n(X - Y), (X - Y)^{2^e} \rangle,$$

as desired.

2.6. LEMMA. *For integers $e \geq 0$, $c \geq 1$ and $m \geq 1$,*

$$(mc^{m-1})^e(X - Y) \in \langle X^m - Y^m \rangle + \langle X - c, Y - c\rangle^e \langle X - Y \rangle.$$

Proof. Let $h(X, Y) = (X^m - Y^m)/(X - Y)$. Then $h(c, c) = mc^{m-1}$, so

$$h(X, Y) - mc^{m-1} \in \langle X - c, Y - c \rangle.$$

Therefore $(mc^{m-1})^e \in \langle h \rangle + \langle X - c, Y - c\rangle^e$. Now multiply by $X - Y$.

Combining these lemmas with (2.2), we obtain:

2.7. PROPOSITION. *Let R be a ring with d-dimensional Noetherian j-spectrum, and let M and N be locally isomorphic finitely presented R-modules.*

(a) **In the ring $\mathbf{Z}[M, N]$**, *if $nM = nN$ for some $n \geq 1$ and $2^e > d$, then $M^{n^e} = N^{n^e}$.*

(b) *Assume $M_p \cong cR_p$ for each j-prime of j-height $< d$.* **In the ring $\mathbf{Z}[M, N]$**, *if $M^m = N^m$ for some $m \geq 1$, then*

$$(mc^{m-1})^d M = (mc^{m-1})^d N.$$

3. Rings of dimension one and two

In order to get concrete information from (2.7), we need results on direct-sum cancellation for non-projective modules. For one-dimensional rings, Guralnick has proved the following result:

3.1. THEOREM (Guralnick). *Let S be a one-dimensional, reduced, Noetherian ring with finitely generated normalization. Let M, N, U be S-modules, and assume that there is an integer m such that for each maximal ideal P, U_P is isomorphic to a direct summand of mM_P. Then $M \oplus U \cong N \oplus U$ implies $M \cong N$.*

Proof. By the proof of [6, Lemma 3.1] or by the methods of §1 one checks that U is (globally) a direct summand of $2mM$. Now apply [6, Theorem 6.1]. (The assumption that S is a module-finite algebra over a Dedekind domain is not necessary in this set-up.)

3.2. THEOREM. *Let R be a one-dimensional, reduced, Noetherian ring with finitely generated normalization, and let M and N be R-modules such that $nM \cong nN$ for some $n \geq 1$. Then $M^n \cong N^n$.*

Proof. Let X and Y be indeterminates, and write

$$X^n = Y^n + n(X - Y)Y^{n-1} + \binom{n}{2}(X - Y)^2 Y^{n-2} + \cdots.$$

Moving all the negative terms to the left, we get

$$X^n + (nY)Y^{n-1} + \binom{n}{2}(2XY)Y^{n-2} + \cdots$$
$$= Y^n + (nX)Y^{n-1} + \binom{n}{2}(X^2 + Y^2)Y^{n-2} + \cdots \qquad (3.2.1)$$

Now M and N are locally isomorphic, [11, §5], so $(M - N)^r \equiv 0$ for $r \geq 2$. This means $2MN \cong M \oplus N$, etc., so when we plug M and N into (3.1.1) and use the fact that $nN \cong nM$, we get $M^n \oplus U \cong N^n \oplus U$, where U is a direct sum of modules of the form M^iN^j with $i + j = n$. But M^iN^j is locally isomorphic to M^n, and we can apply (3.1), obtaining $M^n \cong N^n$.

The reverse implication in the Bass-Guralnick theorem can fail, even for ideals in a one-dimensional ring. The following example was suggested by L.S. Levy.

3.3. *Example.* Let $R = k[T^3, T^4, T^5]$, where k is a field of characteristic 0. Then R has fractional ideals I, J such that $I \otimes_R I \cong J \otimes_R J$, I and J are locally isomorphic, and $nI \not\cong nJ$ for all $n \geq 1$.

Proof. The conductor $\mathfrak{c}$ of R in its normalization $\tilde{R} = k[T]$ is $T^3k[T]$; and $\tilde{R}/\mathfrak{c} = k[t]$, with $t^3 = 0$. There is an action $(M, g) \to M^g$ of $(\tilde{R}/\mathfrak{c})^*$, the group of units, on the set of isomorphism classes of torsionfree R-modules. (See [17] and [18].) Also, to each torsionfree R-module is associated a certain subgroup Δ_U of $(\tilde{R}/\mathfrak{c})^*$ in such a way that (i) $\Delta_{U \oplus V} = \Delta_U\Delta_V$, and (ii) $U^3 \cong U \Leftrightarrow g \in \Delta_U H$, where H is the image of $\tilde{R}^*$ in $(\tilde{R}/\mathfrak{c})^*$. (See [18, 1.7, 1.6].) For a fractional ideal I, the subgroup Δ_I is easily described: Let S be the endomorphism ring of I; this is a ring between R and $\tilde{R}$, and $\Delta_I = (S/\mathfrak{c})^*$.

Let $I = R + RT$, $g = 1 + t \in (\tilde{R}/\mathfrak{c})^*$ and $J = I^g$. The endomorphism ring of I is just R, so $\Delta_I = (R/\mathfrak{c})^* = k^*$, and of course $H = k^*$ as well. Also, $\Delta_{(mI)} = k^*$ for all $m \geq 1$, by (i). Now $mJ = m(I^g) \cong (mI)^{g^m}$ by [18, 1.4.1]; and since g has infinite order modulo k^*, it follows from (ii) that $mI \not\cong mJ$ for all $m \geq 1$. Since the group action does not disturb local isomorphism classes, [17, 2.2], I and J are locally isomorphic. It remains to be shown that $I \otimes I \cong J \otimes J$.

Temporarily abandoning our established notation, we let I^2 and J^2 denote the ordinary ideal products. Clearly $I^2 = k[T]$, and since J^2 is locally isomorphic to I^2 and $k[T]$ is a principal ideal domain, I^2 and J^2 are isomorphic. Also, the torsion submodules of $I \otimes I$ and $J \otimes J$, being of finite length, are isomorphic. Now the torsion submodule W of $I \otimes I$ is clearly the kernel of the multiplication map $I \otimes I \twoheadrightarrow I^2$, so it will suffice to show that the exact sequence

$$0 \to W \to I \otimes I \to I^2 \to 0 \qquad (*)$$

splits. For then the analogous sequence for J splits locally, hence globally by (1.7).

Let σ be the automorphism of $I \otimes I$ taking $a \otimes b$ to $b \otimes a$, and let C be the image of $1 - \sigma$. Since $(1-\sigma)^2 = 2(1-\sigma)$, the inclusion $C \hookrightarrow I \otimes I$ is split by $\frac{1}{2}(1-\sigma)$. The proof will be complete if we can show that $C = W$. Since C is generated by $1 \otimes T - T \otimes 1$ we have $C \subseteq W$. For the reverse inclusion, let $x \in W$, say

$$x = a(1 \otimes 1) + b(1 \otimes T) + c(T \otimes 1) + d(T \otimes T),$$

with $a, b, c, d \in R$, and $a + (b+c)T + dT^2 = 0$. Write $a = \sum_n a_n T^n$, $a_n \in k$, etc., and let

$$\begin{aligned} x_n = a_n T^n(1 \otimes 1) &+ b_{n-1}T^{n-1}(1 \otimes T) \\ &+ c_{n-1}T^{n-1}(T \otimes 1) + d_{n-2}T^{n-2}(T \otimes T) \end{aligned}$$

Then $x = \sum_n x_n$, so it will suffice to show that $x_N \in C$ for each $n \geq 1$. We know that $a_n + b_{n-1} + c_{n-1} + d_{n-2} = 0$ for all n, and it follows easily that $x_n = 0$ for $n \geq 4$. Also $x_3 = a_3 T^3(1 \otimes 1) = 0$, since a_3 is forced to be zero by the requirement that $b, c, d \in R$. Similar arguments yield $x_0 = x_2 = 0$, and $x_1 = b_0(1 \otimes T) + c_0(T \otimes 1)$, which is in C because $b_0 + c_0 = 0$.

It is interesting to note that the analogous calculation does not work over the ring $S = k[T^3, T^4]$. If we take $I = S + ST$, $(*)$ does *not* split, and W needs *two* generators:

$$1 \otimes T - T \otimes 1 \quad \text{and} \quad T^6(1 \otimes 1) - T^4(T \otimes T).$$

For torsionfree modules over coordinate rings of smooth surfaces, we actually get both implications of the Bass-Guralnick theorem, provided we kill the torsion in the tensor powers. For an R-module U, let $\mathscr{F}(U)$ denote the module $U/\text{torsion}$.

3.4. THEOREM. *Let R be a two-dimensional regular domain, finitely generated as an algebra over an algebraically closed field K, and let M and N be torsionfree R-modules of rank c, not a multiple of the characteristic of K. Then*:

(a) *If $nM \cong nN$ for some $n \geq 1$, then $\mathscr{F}(M^{n^2}) \cong \mathscr{F}(N^{n^2})$.*

(b) *If $\mathscr{F}(M^m) \cong \mathscr{F}(N^m)$ for some m prime to p, and M and N are locally isomorphic, then $(mc^{m-1})^2 M \cong (mc^{m-1})^2 N$.*

Proof. By (2.7) and (1.2) we have, in part (a), $M^{n^2} \oplus U \cong N^{n^2} \oplus U$ for a suitable module U. (We need [11, §5] again to conclude that M and N are locally isomorphic.) Similarly, in part (b), we get

$$(mc^{m-1})^2 M \oplus V \cong (mc^{m-1})^2 M \oplus V \quad \text{for some } V.$$

(For each prime P of height 1, M_P and N_P are free because R_P is a discrete valuation ring.) Now apply $\mathscr{F}$ to both of these isomorphisms. By [17, 1.2] we can cancel $\mathscr{F}(U)$ and $\mathscr{F}(V)$, since the torsionfree ranks of M^{n^2} and $(mc^{m-1})^2M$, namely c^{n^2} and $(mc^{m-1})^2c$, are both prime to p.

4. Non-Noetherian rings

Our goal is a proof of the nilpotency theorem, $(M - N)^{d+1} = 0$, for locally isomorphic finitely presented modules over a ring whose spectrum is d-dimensional but not necessarily Noetherian. In some situations d can be chosen much smaller than Krull dimension. For example, if R is the ring of continuous functions on a compact Hausdorff space X, one can take d to be the covering dimension of X.

Along the way we obtain a few results of independent interest, for example, a local-global bound on the number of generators of a module. The method we employ was pioneered by Heitmann in his 1976 paper on Prüfer domains, [8], and was refined and generalized in [15]. Here the method receives a further cleansing, and the main theorem of this section, (4.5), is an abstract formulation of Heitman's idea. The abstract form is rather easy to prove and very easy to apply in a wide variety of situations.

We now review the required background on the patch topology. See [10] for the details. A *spectral space* is a topological space that is homeomorphic to spec(R) for some commutative ring R. A *constructible set* in a spectral space X is a subset of the form $D_1 \cap D_2^c$, where D_1 and D_2 are open and quasi-compact. (The "c" denotes "complement".) A *patch* is an intersection of constructible sets, and the *patch topology* on X is obtained by taking the closed sets to be the patches. A patch, when endowed with the Zariski topology inherited from X, is again a spectral space. If $x, y \in X$ we write "$x \geq y$" provided $x \in \mathrm{cl}\{y\}$, where cl denotes closure in the Zariski topology. (Unless the patch topology is mentioned explicitly, all topological notions refer to the Zariski topology.) The *height* of an element x is defined by declaring that $\mathrm{ht}(x) \geq n$ if and only if there is a chain $x_0 < \cdots < x_n = x$; and dim X is the supremum of the heights of elements of X.

4.1. Proposition [10, §2]. *Let Y be a patch in a spectral space X, and let $x \in X$. Then $x \in \mathrm{cl}\, Y$ if and only if $x \geq y$ for some $y \in Y$.*

4.2. *Notation* [8], [15]. Let $\mathscr{A}$ be a finite covering of a topological space X by quasi-compact open sets, and define closed sets $E(D_0, \ldots, D_n)$, for $D_i \in \mathscr{A}$, by the following rules: $E(D_0) = \mathrm{cl}\, D_0$, and

$$E(D_0, \ldots, D_n) = \mathrm{cl}[E(D_0, \ldots, D_{n-1}) \cap D_{n-1}^c \cap D_n].$$

Let $m_{\mathscr{A}}(x)$, or simply $m(x)$ if $\mathscr{A}$ is understood, be the largest integer n for which there exist $D_i \in \mathscr{A}$ such that $x \in E(D_0, \ldots, D_n)$. Clearly $E(D_0, \ldots, D_n) = \varnothing$ if $D_i = D_j$ for some $i \neq j$, so $m_{\mathscr{A}}(x) < |\mathscr{A}|$. Let

$$m_{\mathscr{A}} = \sup\{m_{\mathscr{A}}(x) \colon x \in X\}.$$

We assign to each spectral space X an invariant

$$\mu(X) \in \{0, 1, 2, \ldots, \infty\}$$

as follows: $\mu(X) \leq n$ if and only if each open covering of X can be refined by a finite quasi-compact open covering $\mathscr{A}$ such that $m_{\mathscr{A}} \leq n$.

4.3. PROPOSITION. *With X, $\mathscr{A}$ as in* (4.2), *we have $m(x) \leq \operatorname{ht} x$ for each $x \in X$. Therefore $\mu(X) \leq \dim X$.*

Proof. Given $D_0, \ldots, D_n \in \mathscr{A}$, let $B_0 = D_0$, and for $1 \leq k \leq n$ let

$$B_k = E(D_0, \ldots, D_{k-1}) \cap D_{k-1}^c \cap D_k.$$

If $k \geq 1$ and $x \in B_k$, then (4.1) implies that x is in the closure of some $y \in B_{k-1}$ (since $x \in E(D_0, \ldots, D_{k-1}) = \operatorname{cl} B_{k-1}$). Further, $x \neq y$ since $x \in D_{k-1}^c$ and $y \in D_{k-1}$. By induction, $\operatorname{ht} x \geq k$ for each $y \in B_k$, and by (4.1) $\operatorname{ht} z \geq n$ for each $z \in E(D_0, \ldots, D_n)$.

4.4. LEMMA (Heitmann, [8]). *Let X and $\mathscr{A}$ be as in* (4.2) *and let F be a closed set on which $m_{\mathscr{A}}$ is constant. Then $D \cap F$ is closed for each $D \in \mathscr{A}$.*

Proof. Given $p \in \operatorname{cl}(D \cap F)$ we want to show that $p \in D$. By (4.1) there is a point of $q \in D \cap F$ such that $p \geq q$. Choose $D_i \in \mathscr{A}$ such that $q \in E(D_0, \ldots, D_n)$, where $m(p) = m(q) = n$. Then $q \in E(D_0, \ldots, D_{n-1}, D)$ as well. If $p \notin D$, select any $D' \in \mathscr{A}$ with $p \in D'$, and check that

$$p \in E(D_0, \ldots, D_{n-1}, D, D'),$$

contradicting $m(p) = n$.

4.5. THEOREM. *Let X be a spectral space, and let $\mathscr{U}$ be a quasi-compact open covering satisfying the following two properties*:

(a) *Every quasi-compact open subset of a member of $\mathscr{U}$ is in $\mathscr{U}$.*

(b) *If A and B are disjoint closed sets, each contained in a member of $\mathscr{U}$, then $A \cup B$ is contained in a member of $\mathscr{U}$.*

Then X is covered by $\mu(X) + 1$ members of $\mathscr{U}$.

Proof. Let $m = \mu(X)$, and choose a finite, quasi-compact open refinement (= subcover) $\mathscr{A} = \{A_1, \ldots, A_t\}$, with $m_{\mathscr{A}} = m$. For $k = 0, \ldots, m$, let

$$F_k = \{p \in X | m(p) \geq k\}.$$

Each F_k is closed, being the union of the closed sets $E(D_0, \ldots, D_n)$, the D_i ranging over the finite set $\mathscr{A}$. By (4.4), the pairwise disjoint sets

$$A_1 \cap F_m, A_1^c \cap A_2 \cap F_m, \ldots, A_1^c \cap \cdots \cap A_{t-1}^c \cap A_t \cap F_m$$

are closed, and their union is F_m. Each of these sets is contained in some A_j, so $F_m \subseteq E_0$ for some $E_0 \in \mathscr{U}$, by assumption (b). Now $m_{\mathscr{A}}$ has constant value $m - 1$ on $F_{m-1} \cap E_0^c$, so the same argument provides a set $E_1 \in \mathscr{U}$ such that $F_{m-1} \cap E_0^c \subseteq E_1$. Similarly,

$$F_{m-2} \cap E_0^c \cap E_1^c \subseteq E_2 \in \mathscr{U},$$

etc., and eventually we get $X = F_0 \subseteq E_0 \cup \cdots \cup E_m$.

For module theoretic purposes we need to work with a patch X containing all maximal ideals of a ring R. We call the smallest patch containing all maximal ideals the *h-spectrum* of R and denote it by h-spec(R). (Heitmann calls this set the j-spectrum in [9], but I prefer to use j-spec(R) to denote, as usual, the set of primes that are intersections of maximal ideals.) One always has j-spec(R) $\subseteq$ h-spec(R), and the two sets agree if j-spec(R) is Noetherian, [16, 1.5]. In general they may be unequal. (See (4.12).)

4.6. THEOREM (See [5, 3.2]). *Let M and N be R-modules (M need not be finitely generated), and suppose that for each maximal ideal p there is a map f: $M \to N$ such that f_p: $M_p \to N_p$ is surjective. Then there are maps f_i: $M \to N$, $i = 1, \ldots, \mu = \mu(h\text{-spec } R)$, such that at each maximal ideal p at least one of the f_i is surjective.*

Proof. Let $X = h$-spec(R), and let $\mathscr{U}$ be the set of all quasi-compact open sets D of X for which there exists f: $M \to N$ such that f_p: $M_p \to N_p$ is surjective for all $p \in D$. Since N is finitely generated, $\mathscr{U}$ covers X. Clearly (a) of (4.5) is satisfied, and all we need to check is that (b) holds.

Let A and B be disjoint closed subsets of X, with $A \subseteq U$, $B \subseteq V$, and $U, V \in \mathscr{U}$. Choose f, g: $M \to N$, surjective on U, V, respectively. There are ideals I, J such that $A = V(I) \cap X$ and $B = V(J) \cap X$ (where $V(I)$ denotes the set of prime ideals of R containing I). Then $I + J = R$, say $b + a = 1$ with $b \in I$ and $a \in J$. By Nakayama's lemma, $h = af + bg$ is surjective at every prime in $A \cup B$. There is an open (in spec(R)) neighborhood W of $A \cup B$ on which g is surjective, and, since $A \cup B$ is quasi-compact, W can be taken to be quasi-compact too. Then $A \cup B \subseteq W \cap X \in \mathscr{U}$, and (b) is proved.

4.7. COROLLARY (See [15, 2.1]). *Let N be a finitely generated R-module, and let $\mu = \mu(h\text{-spec}\, R)$. If N is locally n-generated, then N is generated by $n(\mu + 1)$ elements.*

Proof. Apply (4.6) with M a free R-module of rank n.

4.8 COROLLARY. *Let M and N be locally isomorphic finitely presented R-modules, and let $\mu = \mu(h\text{-spec}\, R)$. Then $(M - N)^{\mu+1} \equiv 0$.*

Proof. By (1.4) or (1.5) the hypotheses of (4.6) are satisfied. Let $f_0, \dots, f_\mu$ be the maps promised by 4.6. At each maximal ideal p, at least one of the f_i must be a surjection, hence an isomorphism, since $M_p \cong N_p$ and surjective endomorphisms of finitely generated modules are isomorphisms. Now proceed exactly as in the last part of the proof of (2.2).

Combining (4.3) and (4.8) we have:

4.9. COROLLARY. *Let M and N be locally isomorphic finitely presented modules over a ring R with $\dim(h\text{-spec}(R)) = d$. Then $(M - N)^{d+1} \equiv 0$.*

4.10. THEOREM. *Let R be the ring of continuous (real-valued or complex-valued) functions on a compact Hausdorff space of covering dimension d. If M and N are locally isomorphic finitely presented R-modules, then $(M - N)^{d+1} \equiv 0$.*

Proof. It is enough to prove that $\mu(h\text{-spec}(R)) \leq d$. In fact, a stronger assertion is true:

4.11. PROPOSITION. *Let R be the ring of continuous functions on a compact Hausdorff space of covering dimension d, and let X be any patch in* $\text{spec}(R)$ *containing all the maximal ideals of R. Then $\mu(X) \leq d$.*

Proof. Let $\mathcal{U}$ be any open covering of X. We seek a finite quasi-compact open refinement $\mathcal{A}$ such that $m_{\mathcal{A}} \leq d$. We may assume $\mathcal{U}$ is finite, say $\mathcal{U} = \{W_1, \dots, W_k\}$. Let Y be the maximal ideal space of R, and set $U_i = W_i \cap Y$. Since Y is a normal space of covering dimension d, there is an open covering $\{V_1, \dots, V_k\}$ of Y such that $\text{cl}_Y V_i \subseteq U_i$ for each i and such that any $d + 2$ of the sets $\text{cl}_Y V_i$ have empty intersection. (See [12, Chapter 3.1.6].) Write $V_i = V_i' \cap Y$, where the V_i are open in X. Since Y is dense in X (because the Jacobson radical of R is 0), $\text{cl}_Y V_i = (\text{cl}\, V_i') \cap Y$ for each i. Now each nonempty closed subset of X meets Y, so any $d + 2$ of the sets $\text{cl}\, V_i'$ have empty intersection. Also, the V_i' cover X for the same reason. I claim that $V_i' \subseteq W_i$. For, let $x \in V_i'$, and choose $y \in \text{cl}\{x\} \cap Y$. Then

$$y \in (\text{cl}\, V_i') \cap Y \subseteq U_i \subseteq W_i.$$

It follows that $x \in W_i$, and the claim is proved. Since X is homeomorphic to the spectrum of some ring, the open cover $\{V_1', \dots, V_k'\}$ can be shrunk to a quasi-compact open cover

$$\mathscr{A} = \{A_1, \dots, A_k\}, \quad \text{with } A_i \subseteq V_i'.$$

To see that $m_{\mathscr{A}} \leq d$, let $D_0, \dots, D_{d+1} \in \mathscr{A}$. If $D_i = D_j$ for some $i \neq j$ we know that $E(D_0, \dots, D_{d+1}) = \varnothing$. If, on the other hand, the D_i are distinct, then $(\mathrm{cl}\, D_0) \cap \cdots \cap \mathrm{cl}(D_{d+1}) = \varnothing$. But it is clear from (4.2) that this intersection contains $E(D_0, \dots, D_{d+1})$, which must therefore be empty.

4.12. *Remarks.* Let Y be any infinite compact Hausdorff space and let R be its ring of continuous real-valued functions. By [7, 4K.1] there is a point $p \in Y$ such that the maximal ideal $\mathscr{M}^p$ is not equal to $\mathscr{O}^p$, the set of functions vanishing in a neighborhood of p. From the discussion in [7, 14.19] it follows that there is a chain $\mathscr{C}$ in spec$R$ with $|\mathscr{C}| = 2^{\aleph_1}$. In particular, $\dim R = \infty$.

The h-spectrum of R is easily seen to be exactly the set of z-ideals. (I am indebted to Chuck Weibel for pointing this out.) Its dimension appears to be harder to compute. Certainly $\dim(h\text{-spec}\, Y) \geq 1$ always, since minimal primes are always z-ideals [7, 14.7]; and for $Y = \mathbf{N} \cup \{\infty\}$, the dimension is exactly one [7, 14G]. I will show that when Y is the unit interval, $\dim(h\text{-spec}(R)) \geq 2$, so that by (4.11) we have $\mu(h\text{-spec}(R)) < \dim(h\text{-spec}(R))$. To see this, identify Y with the maximal ideal space and let X be the set of minimal primes. It will suffice to prove that $X \cup Y$ is not a patch, and I will do this by producing a function $f \in R$ such that $D(f) \cap (X \cup Y)$ is not quasi-compact. In fact, let f be any function vanishing only at 0 and 1. Choose g_1 with zero-set

$$\{0\} \cup \{1/n \colon n \geq 1\},$$

and for $n \geq 2$ let g_n have

$$[0, 1/(n+1)] \cup [1/(n-1), 1]$$

as its zero-set. Clearly $D(f) \cap Y \subseteq \bigcup_{n \geq 1} D(g_n)$, and $X \subseteq D(f_1)$ because g_1 is a non-zerodivisor. Thus $D(f) \cap (X \cup Y) \subseteq \bigcup_{n \geq 1} D(g_n)$, but no finite union of $D(g_n)$'s will contain $D(f) \cap Y$.

Finally, we remark that any infinite compact Hausdorff space Y gives an example in which h-spec(R) contains j-spec(R) properly, since the latter set is just the maximal ideal space.

5. Relations among projective modules

For one-dimensional domains we will prove that the only relations among projective modules are the quadratic ones given by the nilpotency theorem and the linear ones coming from relations among their determinants. Here is the precise result:

5.1. PROPOSITION. *Let $M_1, \ldots, M_n$ be projective modules over a one-dimensional domain R. Let c_j be the rank of M_j and let*

$$\alpha_j = \det M_j \in \operatorname{Pic} R.$$

Then $\mathscr{R}(M_1, \ldots, M_n) = \mathscr{I}(M_1, \ldots, M_n)$, and this is the ideal of

$$\mathbf{Z}[X_1, \ldots, X_n]$$

generated by all elements $(X_i - c_i)(X_j - c_j)$ and all elements

$$a_1(X_1 - c_1) + \cdots + a_n(X_n - c_n),$$

where $\alpha_1^{a_n} \ldots \alpha_n^{a_n} = 1$ in Pic R.

Proof. If M is a projective R-module of rank c, then

$$M \cong (c-1)R \oplus \Lambda^c M \tag{5.1.1}$$

(This is Serre's theorem, [13], if R is Noetherian, and in the general case it follows from [9, 2.6].) Therefore M is determined, up to isomorphism, by its rank and its determinant. It follows that projective modules satisfy direct-sum cancellation, so

$$\mathscr{I}(M_1, \ldots, M_n) = \mathscr{R}(M_1, \ldots, M_n)$$

by (1.2). Therefore $a_1M_1 + \cdots + a_nM_n$ is free, since its determinant is 1, and it follows that the purported linear generators

$$a_1(X_1 - c_1) + \cdots + a_n(X_n - c_n)$$

are all in $\mathscr{I}(M_1, \ldots, M_n)$. For the quadratic ones, we obtain easily from (5.1.1) the formula

$$\det(M_iM_j) = \alpha_i^{c_j}\alpha_j^{c_i} \tag{5.1.2}$$

and it follows that $M_iM_j + c_ic_jR \cong c_iM_j + c_jM_i$, since both sides have the same rank and the same determinant. This says

$$(X_i - c_i)(X_j - c_j) \in \mathscr{I}(M_1, \ldots, M_n) \quad \text{for all } i, j.$$

Now let $h = f - g \in \mathscr{I}(M_1, \ldots, M_n)$, where $f, g \in \mathbf{Z}^+[X_1, \ldots, X_n]$ and

$$f(M_1, \ldots, M_n) \cong g(M_1, \ldots, M_n).$$

The rank of $f(M_1, \ldots, M_n)$ is obviously $f(c_1, \ldots, c_n)$, and from (5.1.2) it follows that

$$\det(f(M_1, \ldots, M_n)) = \alpha_1^{e_1} \ldots \alpha_n^{e_n} \quad \text{where } e_j = \frac{\partial f}{\partial X_j}(c_1, \ldots, c_n).$$

Applying the same reasoning to g, and equating ranks and determinants, we get

$$h(c_1, \ldots, c_n) = 0,$$

and

$$\alpha_1^{a_1} \ldots \alpha_n^{a_n} = 1 \text{ in } \operatorname{Pic} R \quad \text{where } a_j = \frac{\partial h}{\partial X_j}(c_1, \ldots, c_n).$$

Expanding h about the point $(c_1, \ldots, c_n)$, we have

$$h = a_1(X_1 - c_1) + \cdots + a_n(X_n - c_n) + q,$$

where q is in the ideal generated by the elements $(X_i - c_i)(X_j - c_j)$.

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