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AN INEQUALITY IN INTEGRAL REPRESENTATION THEORY

BY

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Dedicated to the Memory of Irving Reiner
Colleague and Friend

Let p be a prime number and G an abelian group of order p^n, $n \geq 1$. Let M be a ZG module on which G acts faithfully and which is a free Z-module. Let F denote the field of p elements and Q the field of rational numbers. For any G module X, X^G denotes the submodule of X consisting of elements left fixed by the action of G. In connection with work on transformation groups acting on topological spaces, A. Adem raised the question of the validity of the bound

$$\text{rank}_Z(M) - \dim_F(M/pM)^G \geq n \tag{1}$$

in the case that G is elementary abelian, p is odd, and M is a Z-free ZG module on which G acts faithfully. The need for this result arose in a generalization of a theorem of P. Smith which asserts that an abelian group acting freely on a sphere S^m, must be cyclic. Adem and Browder have shown that an elementary abelian group of order p^n can act freely on the product of k spheres, $(S^m)^k$, only when $n \leq k$. In this paper we shall prove a result which implies that the bound (1) does hold. We obtain a result like (1) which holds for all finite abelian groups G, in which the n is replaced by a function that is easily computed from the elementary divisors of G. The inequality is sharp in the sense that for every G, there are modules for which equality holds.

Before stating the precise result, some notation is needed. For a finite abelian p-group A, let $d_i(A)$ be the number of elementary divisors of A which equal p^i. Thus in a decomposition of A into a direct sum of cyclic groups, exactly $d_i(A)$ summands have order p^i. Finally, ϕ is Euler's function: we have

$$\phi(p^i) = p^{i-1}(p-1) \quad \text{for } i \geq 1.$$

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THEOREM 1. *Let G be a finite abelian p-group, p any prime, and let M be a Z-free ZG module on which G acts faithfully. Then*

$$\text{rank}_Z(M) - \dim_F(M/pM)^G \geq \sum_{i=1}^{e} d_i(G)(\phi(p^i) - 1), \tag{2}$$

where p^e is the exponent of G.

We begin with some notation for the simple QG modules. Let ζ_i denote a primitive p^i-th root of unity and let S_i denote the field $Q(\zeta_i)$ viewed as a vector space over Q on which G acts as the group $\langle\zeta_i\rangle$ by way of some homomorphism ψ_i mapping G onto $\langle\zeta_i\rangle$. Every simple QG module is obtained as a pair $\{S_i, \psi_i\}$ for some ψ_i and some i with $0 \leq i \leq e$. The Q dimension of S_i is $\phi(p^i)$.

We first prove a result about the dimension of a faithful module.

PROPOSITION. *Let G be as in Theorem 1 and let V be a QG module on which G acts faithfully. Then*

$$\dim_Q(V) \geq \sum_{i=1}^{e} d_i(G)\phi(p^i).$$

Proof. The QG module V may be decomposed as a direct sum

$$V = A_1 \oplus \cdots \oplus A_t \tag{3}$$

with each A_i a simple QG module. Let A_i afford the matrix representation α_i so that $\alpha_i(G)$ is a cyclic group by Schur's lemma. Let P denote the direct product

$$P = \alpha_1(G) \times \cdots \times \alpha_t(G). \tag{4}$$

The condition that G acts faithfully on V translates into the statement that the mapping

$$g \to (\alpha_1(g), \ldots, \alpha_t(g))$$

is a one-to-one homomorphism of G into P. The condition that the abelian group P contains an isomorphic copy of G is expressible in terms of the elementary divisors of the two groups [1, p. 107]. We obtain

$$(5)\qquad \sum_{j=k}^{e} d_j(P) \geq \sum_{j=k}^{e} d_j(G)$$

and this must hold for each $k = 1, \ldots, e$.

For later use, we derive a consequence of (5) slightly more general than is needed immediately.

LEMMA. *Assume* (5) *holds for* $1 \leq k \leq e$. *If* $0 \leq f_1 < \cdots < f_e$ *is an increasing sequence of real numbers, then*

$$(6)\qquad \sum_{i=1}^{e} d_i(P) f_i \geq \sum_{i=1}^{e} d_i(G) f_i.$$

Proof. Let $a_1 = f_1$ and $a_i = f_i - f_{i-1}$ for $1 < i \leq e$. Then each a_i is nonnegative. Multiply the inequality (5) by a_k and add the resulting inequalities for $k = 1, \ldots, e$. After interchanging the order of summation, (6) is obtained.

Now the proposition follows by taking $f_i = \phi(p^i)$; then $d_i(P)$ counts the number of A_j which have dimension f_i and $\dim_Q(V)$ is the sum of the terms $d_i(P) f_i$.

For a ZG module M, let $\delta(M) = \dim_F (M/pM)^G$ and let $\lambda(M)$ denote the composition length of the QG module $Q \otimes_Z M = QM$.

THEOREM 2. *Let M be a Z-free finitely generated ZG module. Then*

$$\delta(M) \leq \lambda(M).$$

Proof. We use induction on $\lambda(M)$.

Suppose $\lambda(M) = 1$. Then $QM = S$ is a simple module containing M and so M can be realized as ZG submodule of $Q(\zeta_i)$ for some i. Since G acts as $\langle \zeta_i \rangle$, it follows that M is isomorphic to an ideal of $Z[\zeta_i]$. Every ideal has the form $\pi^t Z[\zeta_i] B$ for some integer t, $\pi = 1 - \zeta_i$, and B an ideal relatively prime to p (see [3, §I.9]). Then

$$M/pM \cong \pi^t Z[\zeta_i] B / p \pi^t Z[\zeta_i] B \cong Z[\zeta_i]/pZ[\zeta_i].$$

The G fixed point space in this module is the annihilator of π which is easily seen to be one-dimensional over F; thus $\delta(M) = 1 = \lambda(M)$.

Now assume $\lambda(M) > 1$. Then $QM = S \oplus U$ for S a simple QG module and some nonzero QG module U. Let π_1 and π_2 be the projection maps from QM onto the summands S and U respectively. Set $\pi_i(M) = M_i$ for $i = 1, 2$. Then $M_1 \subset QM_1 = S$, and $M_2 \subset QM_2 = U$; each M_i is a Z-torsion free, ZG

module and $M \subset M_1 \oplus M_2$. Let $N = M \cap M_1$ so that N is the kernel of the projection $M \to M_2$ induced by π_2. We have an exact sequence

$$0 \to N \to M \to M_2 \to 0.$$

This gives rise to an exact sequence

$$0 \to (N + pM)/pM \to M/pM \to M_2/pM_2 \to 0$$

of ZG modules (actually FG modules). Now QN is simple and is a module for a cyclic homomorphic image C, of G. The properties of FC modules are well known: every nonzero submodule of an indecomposable FC module is also indecomposable and has a one-dimensional C-fixed point subspace. It follows then that N/pN is indecomposable and so is every homomorphic image of it. In particular, the G-fixed point submodule of any homomorphic image of N/pN has dimension less than or equal to one. Let W be the submodule of M such that $W/pM = (M/pM)^G$. In the module M/pM, either the submodule $(N + pM)/pM$ is (0), or it has a one-dimensional intersection with W/pM, since this is the fixed point space of a homomorphic image of N/pN. This means that

$$\dim_F(W/pM)$$

is either equal to, or is one greater than

$$\dim_F\{(W + N)/(pM + N)\}.$$

In either case, since G acts trivially on both W/pM and $(W + N)/(pM + N)$, we now have

$$(W + N)/(pM + N) \subset (M/(pM + N))^G = (M_2/pM_2)^G$$

and

$$-1 + \dim_F(W/pM) \leq \dim_F(M_2/pM_2)^G = \delta(M_2). \tag{7}$$

Since M_2 is a Z-free ZG module with $\lambda(M_2) = \lambda(M) - 1$, we may apply the induction hypothesis to obtain $\delta(M_2) \leq \lambda(M_2) = \lambda(M) - 1$. Combine this with (7) and the fact that $\delta(M) = \dim_F(W/pM)$ to obtain $-1 + \delta(M) \leq \lambda(M_2)$. Hence $\delta(M) \leq \lambda(M)$ as required to complete the induction.

Now we can prove the main result. Let M satisfy the hypothesis of Theorem 1 and let $V = QM$ have the decomposition (3) into the direct sum of the simple modules A_j. Let P be the group defined as in (4). Then:

$$\begin{aligned}
\lambda(M) = t &= \sum_{i=1}^{e} d_i(P);\\
\operatorname{rank}_Z(M) &= \sum_{i=1}^{e} d_i(P)\phi(p^i);\\
\operatorname{rank}_Z(M) - \delta(M) &\geq \operatorname{rank}_Z(M) - \lambda(M)\\
&= \sum_{i=1}^{e} \{d_i(P)\phi(p^i) - d_i(P)\}\\
&= \sum_{i=1}^{e} d_i(G)\{\phi(p^i) - 1\}.
\end{aligned}$$

This completes the proof of Theorem 1.

Notice that when G is an elementary abelian group of order p^n, then $e = 1$ and $d_1(G) = n$; the right-hand side of (2) becomes $n(p-2)$. For $p = 2$, this allows the possibility that G acts trivially on $M/2M$ even though G acts faithfully on M. This is certainly possible for suitable M. When p is odd, then $n(p-2) \geq n$ and so the bound (1) holds. For general G, we may take M to be the direct sum of certain M_i for which QM_i is simple. The bound in the theorem can be attained by arranging the QM_i so that $V = QM$ is a minimal faithful QG module with equality in Theorem 2. This can easily be accomplished by taking a module which gives $G = P$ in the proof of the proposition.

References

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