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ON LIE GROUPS THAT ADMIT LEFT-INVARIANT LORENTZ METRICS OF CONSTANT SECTIONAL CURVATURE

BY

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1. Introduction

We say that a non-abelian Lie group G is of special type if its Lie algebra, $\mathfrak{g}$, has the property that $[x, y]$ is a linear combination of x and y for all x, y in $\mathfrak{g}$. In [3], J. Milnor proved that every left-invariant Riemannian metric on a Lie group of special type is of constant negative sectional curvature, and, in [4], K. Nomizu proved that every left-invariant Lorentz metric on such a Lie group is also of constant sectional curvature, but, depending on the choice of left-invariant Lorentz metric, the sign of the constant sectional curvature may be positive, negative, or zero.

Lie groups of special type belong to a larger class, studied by E. Heintze in [2], of Lie groups that admit some left-invariant Riemannian metric of constant negative sectional curvature. A natural question is which Lie groups in the larger class admit left-invariant Lorentz metrics of constant sectional curvature and what are the possible signs of those curvatures. In this paper we answer this question completely by proving the following theorem.

THEOREM 1.1. *Let G be a Lie group that admits a left-invariant Riemannian metric of constant negative sectional curvature. Then*:

(i) *G admits a left-invariant Lorentz metric of constant positive sectional curvature.*

(ii) G *admits a left-invariant Lorentz metric of constant negative, or zero, sectional curvature if, and only if, $\mathfrak{g}$ contains a one-dimensional ideal.*

In [2], E. Heintze proved that $\mathfrak{g}$ contains an abelian ideal $\mathfrak{u}$ of codimension 1 and that for any b not in $\mathfrak{u}$, $\operatorname{ad}(b)|\mathfrak{u} = \lambda I + B$, where B is a linear transformation which is skew-adjoint with respect to the inner product the left-invariant Riemannian metric induces on $\mathfrak{g}$, and where λ is non-zero. Lie groups of special type are precisely those for which B is identically zero. Together with Theorem 1.1, this result of Heintze gives us these two corollaries.

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Corollary 1.2. *Let G be an even-dimensional Lie group. If G admits a left-invariant Riemannian metric of constant negative sectional curvature, then G admits left-invariant Lorentz metrics of constant sectional curvature of each possible sign.*

Proof. It suffices to show that $\mathfrak{g}$ contains a one-dimensional ideal. Write $\operatorname{ad};(b)|\mathfrak{u} = \lambda I + B$, where B is skew-adjoint with respect to some Riemannian inner product. To show the existence of the one-dimensional ideal, it suffices to show that $Bc = 0$ for some non-zero element $c \in \mathfrak{u}$. But this is clearly true since $\mathfrak{u}$ is odd-dimensional and thus B has zero as an eigenvalue.

Corollary 1.3. *There are odd-dimensional Lie groups that admit left-invariant Riemannian metrics of constant negative sectional curvature, but which do not admit any left-invariant Lorentz metrics of constant negative, or zero sectional curvature.*

Proof. Let V be an even dimensional real vector space and let b be any nonzero vector in V, and $\langle\ ,\ \rangle$ any Riemannian inner product on V. Let U denote the orthogonal complement of b, and let B be a skew-adjoint linear transformation on U. Since U is even dimensional, we may choose B to be non-singular. Now define a Lie algebra on V by setting $[b, u] = u + Bu$ for all $u \in U$, and $[u, v] = 0$ for all $u, v \in U$. Then $\mathfrak{g} = V$ and $\mathfrak{u} = U$ are as above, and the Lie group G with Lie algebra $\mathfrak{g}$ and left-invariant Riemannian metric induced by $\langle\ ,\ \rangle$ is of constant negative sectional curvature. But since B is non-singular, then $\mathfrak{g}$ contains no one-dimensional ideals, and hence G admits no left-invariant Lorentz metrics of constant negative sectional curvature.

2. The curvature tensor R

From now on let G be a Lie group that admits a left-invariant Riemannian metric of constant negative sectional curvature, let $\mathfrak{g}$ denote its Lie algebra, and let $\mathfrak{u}$ denote the abelian ideal of codimension one in $\mathfrak{g}$ guaranteed by Heintze's result. In this section we compute the curvature tensor R for a left-invariant Lorentz metric $\langle\ ,\ \rangle$ on G which is non-degenerate on $\mathfrak{u}$.

Choose b in $\mathfrak{g}$ with b orthogonal to $\mathfrak{u}$ and $\langle b, b\rangle = \pm 1$. Let $L = \operatorname{ad}(b)|\mathfrak{u}$. The transpose L^* of L is defined by

$$\langle L^*(x), y\rangle = \langle x, L(y)\rangle \quad \text{for all } x, y \text{ in } \mathfrak{u}.$$

Then $S = (L + L^*)/2$ and $B = (L - L^*)/2$ are the self-adjoint and skew-adjoint parts, respectively, of L with respect to $\langle\ ,\ \rangle$.

For all $x, y, z \in \mathfrak{g}$ the metric connection ∇ of $\langle\ ,\ \rangle$ satisfies the fundamental formula for left-invariant metrics.

$$2\langle \nabla_x y, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle. \tag{2.1}$$

With this formula we compute ∇. We have the following slight generalization of Lemma 5.5 of [3].

LEMMA 2.2. *For all $u, v \in \mathfrak{u}$ we have*

$$\nabla_b b = 0, \quad \nabla_b u = Bu, \quad \nabla_u b = -Su,$$

and

$$\nabla_u v = r\langle Su, v\rangle b \quad \text{where } r = \langle b, b\rangle.$$

Proof. A sample computation goes as follows. The component of $\nabla_u v$ in the direction of b is equal to $r\langle \nabla_u v, b\rangle$. By (2.1) we have

$$\begin{aligned} 2\langle \nabla_u v, b\rangle &= \langle [u, v], b\rangle - \langle [v, b], u\rangle + \langle [b, u], v\rangle \\ &= 0 + \langle [b, v], u\rangle + \langle [b, u], v\rangle \\ &= \langle Sv + Bv, u\rangle + \langle Su + Bu, v\rangle \\ &= 2\langle Su, v\rangle, \end{aligned}$$

the last step following from the self-adjointness of S and the skew-adjointness of B.

The component of $\nabla_u v$ in any direction orthogonal to b is zero since in this case all computations take place inside the abelian ideal $\mathfrak{u}$. Therefore, $\nabla_u v = r\langle Su, v\rangle b$. The other computations are handled in a similar manner.

The curvature tensor R is defined by

$$R(x, y)\, z = \nabla_{[x, y]} z - \nabla_x \nabla_y z + \nabla_y \nabla_x z,$$

for all smooth vector fields x, y, z on G. Let u, v, w be arbitrary elements of $\mathfrak{u}$. Using Lemma 2.2 we can easily show that R satisfies:

LEMMA 2.3. (a) $R(u, v)w = r\{\langle Sv, w\rangle Su - \langle Su, w\rangle Sv\}$.
(b) $R(u, v)b = 0$.
(c) $R(b, u)b = -S^2 u - SBu + BSu$.
(d) $R(b, u)v = r\{\langle S^2 u + SBu - BSu, v\rangle\} b$.

3. Construction of left-invariant Lorentz metrics of constant sectional curvature

In this section we construct on G a left-invariant Lorentz metric of constant positive sectional curvature, and under the assumption that $\mathfrak{g}$ contains a one-dimensional ideal, we construct a left-invariant Lorentz metric of constant negative sectional curvature, and a flat left-invariant Lorentz metric.

LEMMA 3.1. *Let $\langle\ ,\ \rangle$ be a left-invariant Lorentz metric on G such that the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is non-degenerate. Suppose that $b \in \mathfrak{g}$ is orthogonal to $\mathfrak{u}$ and that $r = \langle b, b\rangle = \pm 1$. If* $\operatorname{ad}(b)|\mathfrak{u} = \lambda I + B$, *where B is a skew-adjoint linear transformation, then $\langle\ ,\ \rangle$ is of constant sectional curvature $-r\lambda^2$.*

Proof. For all $x, y \in \mathfrak{g}$ define the linear transformation $x \wedge y$ by

$$(x \wedge y)z = \langle x, z\rangle y - \langle y, z\rangle x \quad \text{for all } z \in \mathfrak{g}.$$

The left-invariant Lorentz metric $\langle\ ,\ \rangle$ is of constant sectional curvature $-r\lambda^2$ if, and only if, $R(x, y) = -r\lambda^2(x \wedge y)$ for all x, y in $\in \mathfrak{g}$.

Let u, v, w be arbitrary elements of $\mathfrak{u}$. Using Lemma 2.3 we can easily compute

$$\begin{aligned} R(u, v)w &= r\lambda^2\{\langle v, w\rangle u - \langle u, w\rangle v\},\\ R(u, v)b &= 0,\\ R(b, u)b &= -\lambda^2 u,\\ R(b, u)v &= r\lambda^2\langle u, v\rangle b. \end{aligned}$$

Thus,

$$\begin{aligned} R(u, v)w &= -r\lambda^2(u \wedge v)w,\\ R(u, v)b &= -r\lambda^2(u \wedge v)b,\\ R(b, u)b &= -r\lambda^2(b \wedge u)b,\\ R(b, u)v &= -r\lambda^2(b \wedge u)v. \end{aligned}$$

Using the linearity of R in each of its components we get the desired result.

We now go ahead to the constructions of the left-invariant metrics. Let $\langle\langle\ ,\ \rangle\rangle$ be a left-invariant Riemannian metric on G of constant negative sectional curvature. By Heintze [2] we can write $\mathfrak{g} = b\mathbf{R} + \mathfrak{u}$, where $\operatorname{ad}(b)|\mathfrak{u} = \lambda I + B$, and where B is a skew-adjoint linear transformation with respect to $\langle\langle\ ,\ \rangle\rangle$, and λ is non-zero. Define a Lorentz inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by setting

$$\begin{aligned}
\langle b, b\rangle &= -1, \\
\langle b, u\rangle &= 0 \quad \text{for all } u \in \mathfrak{u}, \\
\langle u, v\rangle &= \langle\langle u, v\rangle\rangle \quad \text{for all } u, v \in \mathfrak{u}.
\end{aligned}$$

The linear transformation B is clearly also skew-adjoint with respect to $\langle\ ,\ \rangle$, and by the previous lemma, the left-invariant Lorentz metric that $\langle\ ,\ \rangle$ induces on G is of constant positive sectional curvature λ^2.

Suppose that $\mathfrak{u}$ contains a one-dimensional ideal spanned by $c \in \mathfrak{u}$. Let V be the orthogonal complement in $\mathfrak{u}$, with respect to $\langle\langle\ ,\ \rangle\rangle$, of c. Suppose that u and v are arbitrary elements of V. Define a Lorentz inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by setting

$$\begin{aligned}
\langle c, c\rangle &= -1, \\
\langle b, c\rangle &= 0, \\
\langle b, v\rangle &= \langle c, v\rangle = 0 \quad \text{for all } v \in V, \\
\langle u, v\rangle &= \langle\langle u, v\rangle\rangle \quad \text{for } u, v \in V, \\
\langle b, b\rangle &= 1.
\end{aligned}$$

Since c spans an ideal of $\mathfrak{g}$ then Bc is a multiple of c. But B is skew-adjoint with respect to $\langle\langle\ ,\ \rangle\rangle$, so then $Bc = 0$. It now easily follows that B is also skew-adjoint with respect to $\langle\ ,\ \rangle$, so by Lemma 3.1 this left-invariant Lorentz metric has constant negative sectional curvature $-\lambda^2$.

We now define a flat left-invariant Lorentz metric $\langle\ ,\ \rangle$ as follows. Set

$$\begin{aligned}
\langle u, v\rangle &= \langle\langle u, v\rangle\rangle, \\
\langle u, c\rangle &= 0, \\
\langle b, c\rangle &= 1, \\
\langle b, b\rangle &= \langle c, c\rangle = 0.
\end{aligned}$$

This defines a Lorentz inner product since in the two dimensional orthogonal complement of the Riemannian subspace V, the orthonormal basis

$$\left\{e_1 = \frac{b+c}{2},\quad e_2 = \frac{b-c}{2}\right\}$$

satisfies

$$\langle e_1, e_1\rangle = 1 \quad \text{and} \quad \langle e_2, e_2\rangle = -1,$$

thus showing that the index of $\langle\ ,\ \rangle$ is 1.

The inner product $\langle\ ,\ \rangle$ is degenerate on $\mathfrak{u}$ so we cannot use Lemma 2.3 to compute the curvature tensor R. We now compute ∇ for this particular case. Since $Bc = 0$ and B is skew-adjoint with respect to $\langle\langle\ ,\ \rangle\rangle$, then B maps V into itself. Keeping this in mind and setting $[b, c] = \lambda c$, one easily computes, using (2.1) that

$$\begin{aligned}
&\nabla_c \equiv 0\\
&\nabla_u c = 0 \quad \text{for } u \in V,\\
&\nabla_u v = \lambda\langle u, v\rangle c \quad \text{for } u, v \in V,\\
&\nabla_b u = Bu \quad \text{for } u \in V,\\
&\nabla_b b = -\lambda b.
\end{aligned}$$

Since ∇_c is identically zero and c spans a one-dimensional ideal of $\mathfrak{g}$, then $R(x, c)y = 0$ for all $x, y \in \mathfrak{g}$. Since $\nabla_u c = 0$ for all $u \in V$, then $R(u, v)c = 0$ for all $u, v \in V$. Also,

$$\begin{aligned}
R(u, v) &= -\nabla_u\nabla_v b + \nabla_v\nabla_u b\\
&= -\nabla_u(-\lambda v) + \nabla_v(-\lambda u)\\
&= \lambda\{\nabla_u v - \nabla_v u\}\\
&= \lambda[u, v]\\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
R(b, u)b &= \nabla_{[b,u]}b - \nabla_b\nabla_u b + \nabla_u\nabla_b b\\
&= \nabla_{(\lambda u + Bu)}b - \nabla_b(-\lambda u) + \nabla_u(-\lambda b)\\
&= \nabla_{\lambda u}b + \nabla_{Bu}b + \lambda\{\nabla_b u - \nabla_u b\}\\
&= -\lambda^2 u - \lambda Bu + \lambda[b, u]\\
&= -\lambda^2 u - \lambda Bu + \lambda\{Su + Bu\}\\
&= 0.
\end{aligned}$$

Thus $R(x, y)z = 0$ for all $x, y, z \in \mathfrak{g}$, and $\langle\ ,\ \rangle$ is the desired flat left-invariant Lorentz metric.

4. The existence of a one-dimensional ideal of $\mathfrak{g}$ for flat left-invariant Lorentz metrics

Assume that G admits a flat left-invariant Lorentz metric $\langle\ ,\ \rangle$. In this section we show that $\langle\ ,\ \rangle$ is necessarily degenerate on $\mathfrak{u}$. This means that $\mathfrak{u} = \mathbf{R}c + V$ (direct sum), where c is a light-vector, $\langle c, u\rangle = 0$ for all $u\mathfrak{u}$, and the restriction of $\langle\ ,\ \rangle$ to V is Riemannian. We then prove that c spans a one-dimensional ideal of $\mathfrak{g}$.

LEMMA 4.1. *Let $\langle\ ,\ \rangle$ be a flat left-invariant Lorentz metric on G. Then, the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is degenerate.*

Proof. Assume that $\langle\ ,\ \rangle$ is in fact non-degenerate on $\mathfrak{u}$. We then derive a contradiction, namely, that for $b \notin \mathfrak{u}$, $\mathrm{tr}(\mathrm{ad}(b)|\mathfrak{u}) = 0$.

Write $\mathrm{ad}(b)|\mathfrak{u} = S + B$, where S is the self-adjoint part of $\mathrm{ad}(b)|\mathfrak{u}$ and B is the skew-adjoint part. Since B is skew-adjoint, then $\mathrm{tr}(B) = 0$, so to derive the desired contradiction it suffices to show that $\mathrm{tr}(S) = 0$. If S is not identically zero, choose $w \in \mathfrak{u}$ such that $Sw \neq 0$. By the assumption of the non-degenerary of $\langle\ ,\ \rangle$ on $\mathfrak{u}$, there exists $v \in \mathfrak{u}$ with $\langle v, Sw\rangle \neq 0$. Without loss of generality we may assume that b is orthogonal to $\mathfrak{u}$ and that $r = \langle b, b\rangle = \pm 1$. Let U denote the orthogonal complement of Sw in $\mathfrak{g}$. By Lemma 2.3 (a) it follows that for $u \in U$,

$$\begin{aligned} R(u, v)w &= r\{\langle Sv, w\rangle Su - \langle Su, w\rangle Sv\} \\ &= r\{\langle V, Sw\rangle Su - \langle u, Sw\rangle Sv\} \\ &= r\langle v, Sw\rangle Su. \end{aligned}$$

Since we are assuming that R is identically zero, we have that $Su = 0$. Therefore, S is identically zero on U.

From Lemma 2.3(c) and the fact that R is identically zero, we have $\mathrm{tr}(S^2) = 0$. U is of codimension 1 in $\mathfrak{g}$ whether Sw is a light-vector or not. Thus we can choose a basis of $\mathfrak{u}$ such that its first element is not in U but the remaining elements are in U. If (s_{ij}) denotes the matrix of S with respect to this basis, then $\mathrm{tr}(S^2) = s_{11}^2 = 0$. Therefore, $\mathrm{tr}(S) = s_{11} = 0$.

LEMMA 4.2. *Suppose $\langle\ ,\ \rangle$ is a left-invariant Lorentz metric on G such that the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is degenerate. Suppose b, c are light-vectors in $\mathfrak{g}$, with $c \in \mathfrak{u}$, $b \notin \mathfrak{u}$, and $\langle b, c\rangle = 1$. Then,*

$$\langle R(b, c)b, c\rangle = -\frac{3}{4}\langle [b, c], [b, c]\rangle.$$

Proof. We can write $\mathfrak{u} = \mathbf{R}c + V$ (direct sum), where c is a light-vector such that $\langle c, u\rangle = 0$ for all $u \in V$ and V is a Riemannian subspace. Any light-vector b' with $\langle b', c\rangle = 0$ is contained in the orthogonal complement of V, so b is contained in that complement.

Using (2.1) we obtain

$$\begin{aligned} &\nabla_c c = 0, \\ &\langle \nabla_c b, b\rangle = \langle \nabla_c b, c\rangle = 0, \\ &\langle \nabla_c b, u\rangle = -\frac{1}{2}\langle [b, c], u\rangle \quad \text{for } u \in V, \end{aligned}$$

and it follows that

$$\nabla_c b = -\frac{1}{2}[b, c] + \frac{1}{2}\langle [b, c], b\rangle c.$$

Since $\nabla_b c - \nabla_c b = [b, c]$ then

$$\nabla_b c = \frac{1}{2}[b, c] + \frac{1}{2}\langle [b, c], b\rangle c.$$

For any left-invariant metric the linear transformation ∇_x is skew-adjoint for all $x \in \mathfrak{g}$. Therefore,

$$\langle \nabla_b \nabla_c b, c\rangle = -\langle \nabla_b c, \nabla_c b\rangle = \frac{1}{4}\langle [b, c], [b, c]\rangle,$$

and

$$\langle \nabla_c \nabla_b b, c\rangle = -\langle \nabla_c c, \nabla_b b\rangle = 0.$$

By (2.1) we obtain

$$\langle \nabla_{[b, c]} b, c\rangle = -\frac{1}{2}\langle [b, c], [b, c]\rangle.$$

Thus,

$$\begin{aligned}\langle R(b, c)b, c\rangle &= \langle \nabla_{[b, c]} b, c\rangle - \langle \nabla_b \nabla_c b, c\rangle \\ &= -\frac{3}{4}\langle [b, c], [b, c]\rangle\end{aligned}$$

LEMMA 4.3. *If G admits a flat left-invariant Lorentz metric then, $\mathfrak{g}$ contains a one-dimensional ideal.*

Proof. By Lemma 4.1 and Lemma 4.2 there exist light-vectors b and c, with $c \in \mathfrak{u}$ and $b \notin \mathfrak{u}$ such that

$$\langle R(b, c)b, c\rangle = -\frac{3}{4}\langle [b, c], [b, c]\rangle.$$

Since R is identically zero, this means that $[b, c]$ is a light-vector in $\mathfrak{u}$, or else is the zero vector. Thus $[b, c]$ is a multiple of c. This suffices to show that c spans an ideal of $\mathfrak{g}$ since $\mathfrak{u}$ is an abelian ideal of codimension 1 in $\mathfrak{g}$.

5. The existence of a one-dimensional ideal of $\mathfrak{g}$ for left-invariant Lorentz metrics of constant negative sectional curvature

In this section we finish the proof of Theorem 1.1 by showing that if G admits a left-invariant Lorentz metric of constant negative sectional curvature then, $\mathfrak{g}$ contains a one-dimensional ideal. We begin with:

LEMMA 5.1. *If $\langle\ ,\ \rangle$ is a left-invariant Lorentz metric on G of constant negative sectional curvature, then the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is non-degenerate and Lorentz.*

Proof. Suppose that the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is degenerate. Let c be a light-vector in $\mathfrak{u}$ and let b be a light-vector with $\langle b, c\rangle = 1$. The sectional curvature of the plane spanned by b and c is

$$\frac{\langle R(b,c)b,c\rangle}{(\langle b,b\rangle\langle c,c\rangle - \langle b,c\rangle^2)},$$

which is non-negative from Lemma 4.2. Thus, $\langle\ ,\ \rangle$ cannot be degenerate on $\mathfrak{u}$.

If the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is Riemannian, there exists b orthogonal to $\mathfrak{u}$ with $\langle b, b\rangle = -1$. Let S and B denote the self-adjoint and skew-adjoint parts, respectively, of $\mathrm{ad}(b)|\mathfrak{u}$ and let $u \in \mathfrak{u}$ be a unit eigenvector of S with corresponding eigenvalue λ. From Lemma 2.3(b) it follows that the sectional curvature of the plane spanned by b and u is λ^2. Thus, the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ cannot be Riemannian, either. It must be non-degenerate and Lorentz.

LEMMA 5.2. *Let $\langle\ ,\ \rangle$ be a left-invariant Lorentz metric on G of constant negative sectional sectional curvature. Then, for any b not in $\mathfrak{u}$, the self-adjoint part of $\mathrm{ad}(b)|\mathfrak{u}$ is a non-zero multiple of the identity.*

Proof. By Lemma 5.1, the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{u}$ is non-degenerate and Lorentz. Without loss of generality we may assume that b is orthogonal to $\mathfrak{u}$ and $\langle b, b\rangle = 1$. Write $\mathrm{ad}(b)|\mathfrak{u} = S + B$, where S is the self-adjoint part of $\mathrm{ad}(b)|\mathfrak{u}$ and B is the skew-adjoint part. If $\dim \mathfrak{u} = 1$ then G is a Lie group of special type and $\mathrm{ad}(b)|\mathfrak{u}$ is itself a non-zero multiple of the identity. If $\dim \mathfrak{u} \geq 3$ and $\langle Sc, c\rangle \neq 0$ for every light-vector c in $\mathfrak{u}$ then, by Section 12.25 of [1] there exists an orthonormal basis of $\mathfrak{u}$ consisting of eigenvectors of S. Thus, at least one of the following three cases must hold:

(i) There exists an orthonormal basis of $\mathfrak{u}$ consisting of eigenvectors of S,
(ii) $\dim \mathfrak{u} \geq 3$ and $\langle Sc, c\rangle = 0$ for some light-vector $c \in \mathfrak{u}$,
(iii) $\dim \mathfrak{u} = 2$.

Case (i) Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of $\mathfrak{u}$ consisting of eigenvectors of S with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. By the assumption of constant negative sectional curvature we have, for some non-zero λ and for all $x, y, z \in \mathfrak{g}$, the curvature tensor R satisfies

$$R(x, y)z = -\lambda^2\{\langle x, x\rangle - \langle y, z\rangle x\}. \tag{5.3}$$

Computing $R(e_i, e_j)e_i$ two ways, by Lemma 2.3(a) and by (5.3), and equating the results, we find that $\lambda^2 = \lambda_i\lambda_j$ if $i \neq j$. Proceeding in similar fashion with $R(b, e_i)b$ we find that $\lambda^2 = \lambda_i^2$ for all i. Thus, the the eigenvalues of S are all equal and $S = \lambda I$.

Case (ii) Let c' be a light-vector in $\mathfrak{u}$ with $\langle c', c\rangle = 1$. The plane V spanned by c and c' is non-degenerate and Lorentz, so $\mathfrak{u} = V + W$ (direct sum), where W is the orthogonal complement of V in $\mathfrak{u}$. Computing $R(c, c')c$ two ways, as above, we find that

$$\langle Sc, c'\rangle Sc = \lambda^2 c. \tag{5.4}$$

Thus, $Sc = \pm\lambda c$. Changing b to $-b$, if necessary, we may assume that $Sc = \lambda c$. Computing $R(c, c')c'$ two ways again gives us

$$\lambda\langle Sc', c'\rangle c - \lambda Sc' = -\lambda^2 c', \tag{5.5}$$

and thus Sc' is a linear combination of c and c'. Therefore, S maps V into itself, and by the self-adjointness of S we have that S also maps W into itself.

The subspace W is Riemannian, so the eigenvectors of $S|W$ are all real. Let u be a unit eigenvector of $S|w$ corresponding to the eigenvalue λ'. Computing $R(c', u)u$ two ways as before and equating results we obtain $\lambda' Sc' = \lambda^2 c'$. Replacing Sc' with $(\lambda^2/\lambda')c'$ in (5.5) and equating coefficients we find that $\lambda' = \lambda$. Thus, $Sc' = \lambda c$ and $S|W = \lambda I_w$. Therefore, $S = \lambda I$.

Case (iii) Let c_1, c_2 be light-vectors in $\mathfrak{u}$ with $\langle c_1, c_2\rangle = 1$. Let (s_{ij}) and (b_{ij}) denote the matrices of S and B, respectively, relative to the basis $\{c_1, c_2\}$ of $\mathfrak{u}$. Using the self-adjointness of S and the skew-adjointness of B we obtain

$$s_{11} = s_{22}, \quad b_{22} = -b_{11}, \quad b_{12} = b_{21} = 0.$$

Using Lemma 2.3(c) one computes

$$R(c_1, c_2)c_1 = \left(s_{11}^2 - s_{12}s_{21}\right)c_1.$$

But (5.4) shows that $R(c_1, c_2)c_1 = \lambda^2$, and thus

$$s_{11}^2 - s_{12}s_{21} = \lambda^2. \tag{5.6}$$

From Lemma 2.3(c) and (5.4) we obtain $\operatorname{tr}(S^2) = \operatorname{tr}(\lambda^2 I)$. Therefore,

$$s_{11}^2 + s_{12}s_{21} = \lambda^2. \tag{5.7}$$

Equations (5.6) and (5.7) imply that $s_{11} = \pm\lambda$ and $s_{12}s_{21} = 0$. Changing b to $-b$, if necessary, we may assume that $s_{11} = \lambda$.

Recall that G admits a left-invariant Riemannian metric of constant negative sectional curvature, and consequently, we may write

$$\operatorname{ad}(b)|\mathfrak{u} = \lambda' I + B'$$

where B' is a linear transformation which is skew-adjoint with respect to some Riemannian inner product on $\mathfrak{u}$. Since $B' = S - \lambda' I + B$, the matrix of B', relative to the basis $\{c_1, c_2\}$, is

$$\begin{pmatrix} \lambda - \lambda' + b_{11} & s_{12} \\ s_{21} & \lambda - \lambda' - b_{11} \end{pmatrix}.$$

Since B' is a skew-adjoint linear transformation with respect to a Riemannian inner product, then $\operatorname{tr}(B') = 0$. Therefore, $\lambda = \lambda'$ and the matrix for B' is

$$\begin{pmatrix} b_{11} & s_{12} \\ s_{21} & -b_{11} \end{pmatrix}.$$

Recalling that $s_{12}s_{21} = 0$, it follows that b_{11} and $-b_{11}$ are the eigenvalues of B'. But if skew-adjoint linear transformation on a Riemannian vector space has only real eigenvalues, that transformation is identically zero. Therefore, $s_{12} = s_{21} = b_{11} = 0$. Then, B is identically zero and $S = \lambda I$.

We now show that $\mathfrak{g}$ admits a one-dimensional ideal. From Lemma 5.2 and the fact that G admits a left-invariant Riemannian metric of constant negative sectional curvature, we see that if b is not in $\mathfrak{u}$, then

$$\operatorname{ad}(b)|\mathfrak{u} = \lambda I + B,$$

where B is a linear transformation that is skew-adjoint with respect to a Lorentz inner product $\langle\ ,\ \rangle$ and also with respect to a Riemannian inner product $\langle\langle\ ,\ \rangle\rangle$. To establish the existence of the one-dimensional ideal we show that $Bc = 0$ for some non-zero element $c \in \mathfrak{u}$.

Since B is skew-adjoint with respect to $\langle\langle\ ,\ \rangle\rangle$, there exists a plane V contained in $\mathfrak{u}$ so that B maps V into itself. If the restriction of $\langle\ ,\ \rangle$ to V is degenerate, then the intersection of V and its orthogonal complement (with respect to $\langle\ ,\ \rangle$) is one-dimensional. If c spans that one-dimensional subspace, then

$$\langle Bc, v\rangle = -\langle Bv, c\rangle = 0 \quad \text{for all } v \in V.$$

Therefore, Bc is itself orthogonal to V and must be a multiple of c. But $\langle\langle Bc, c\rangle\rangle = 0$, thus we have that $Bc = 0$. If the restriction of $\langle\ ,\ \rangle$ to V is non-degenerate and Lorentz, let $c \in V$ be a light-vector. The orthogonal complement of c (with respect to $\langle\ ,\ \rangle$) in V is a one-dimensional subspace spanned by c. Since $\langle Bc, c\rangle = 0$, we have again that Bc is a multiple of c, and by the previous argument $Bc = 0$. If the restriction of $\langle\ ,\ \rangle$ to V is Riemannian. Let W be the orthogonal complement of V with respect to $\langle\ ,\ \rangle$. Now repeat on W the procedure described above. Continuing in this manner eventually we must find a plane on which $\langle\ ,\ \rangle$ is degenerate or Lorentz, and this plane contains the desired non-zero element c with $Bc = 0$.

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