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ON THE COHOMOLOGY OF FINITE GROUPS OF LIE TYPE

BY

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Introduction

Let $\mathbf{F}_q$ denote the finite field with q elements and let $G = X_{\mathbf{F}_q}$ be a connected (not necessarily split) reductive group scheme over spec($\mathbf{F}_q$). We will be interested in the cohomology of the finite groups $G(\mathbf{F}_{q^n})$ of $\mathbf{F}_{q^n}$-rational points of G, with coefficients in $\mathbf{Z}/l$ where l is a prime different from $p = \operatorname{char}(\mathbf{F}_q)$. These groups are closely related and present special cases of the groups referred to in the title. A finite group of Lie type is, by definition, a central quotient of a group of the form $G(\mathbf{F}_{q^n})$. For instance, all finite Chevalley groups (as defined in [Gor] or [Car]) are of this kind. For simplicity, we will formulate our results for the groups $G(\mathbf{F}_{q^n})$ rather than for these more general central quotients; it should be clear to the reader how to apply the results, mutatis mutandis, to the general finite groups of Lie type. The basic references for reductive group schemes are [DeGr] or [Dem] (see also [Jan] for an account of the basic results).

It is well known that if a homomorphism $\phi: \tau \to \pi$ of finite groups induces an isomorphism in $H^*(;\mathbf{Z}/l)$, l a prime, then the kernel of ϕ has order prime to l and the image of ϕ has an index prime to l in π (cf. [Jac]); in particular, it will follow that τ and π will have isomorphic l-Sylow subgroups. Obviously, the converse statement is in general false, as one can see by looking at the inclusion map of an l-Sylow subgroup. Our main theorem shows, however, that for the natural inclusions of groups of Lie type a converse statement holds. The precise statement is as follows.

THEOREM. *Let G be a connected reductive $\mathbf{F}_q$-group scheme and let l be a prime different from* $\operatorname{char}(\mathbf{F}_q) = p$. *Then the following are equivalent*:

(i) *The inclusion $G(\mathbf{F}_q) \to G(\mathbf{F}_{q^n})$ induces an $H^*(;\mathbf{Z}/l)$-isomorphism.*

(ii) *The groups $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$ have isomorphic l-Sylow subgroups.*

In Section 1 we will discuss trace formulas and prove the theorem. We will also point out the relationship of the theorem with conjugacy questions

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concerning the l-subgroups of $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$. In Section 2 we show how to adapt the proof of the theorem to cover the Suzuki and Ree groups (as defined in [Gor]).

The notational conventions of the Introduction will be kept throughout this paper.

1. Trace formulas

As in the Introduction, $G = X_{\mathbf{F}_q}$ denotes always a connected reductive $\mathbf{F}_q$-group scheme. Let $\overline{\mathbf{F}}_q$ be the algebraic closure of $\mathbf{F}_q$ and put

$$\overline{G} = G \times_{\operatorname{spec}(\mathbf{F}_q)} \operatorname{spec}\left(\overline{\mathbf{F}}_q\right),$$

the reductive $\overline{\mathbf{F}}_q$-group scheme obtained by base change from G. We will write $\phi\colon \overline{G} \to \overline{G}$ for the Frobenius endomorphism associated with the $\mathbf{F}_q$-form G of $\overline{G}$. Similarly, ϕ^n will denote the Frobenius of the $\mathbf{F}_{q^n}$-form $G \times_{\operatorname{spec}(\mathbf{F}_q)} \operatorname{spec}(\mathbf{F}_{q^n})$ of $\overline{G}$. Thus, if $G = \operatorname{spec}(A)$ and $\overline{G} = \operatorname{spec}(A \otimes \overline{\mathbf{F}}_q)$ then ϕ is given by the $\mathbf{F}_q$-homomorphism which maps $x \in A \otimes \overline{\mathbf{F}}_q$ to x^q. One has therefore a natural diagram of Lang maps

$$\begin{tikzcd} G(\mathbf{F}_q) \arrow[r] \arrow[d] & \overline{G} \arrow[r, "1/\phi"] \arrow[d, equal] & \overline{G}, \arrow[d, "\psi_n"] \\ G(\mathbf{F}_{q^n}) \arrow[r] & \overline{G} \arrow[r, "1/\phi^n"'] & \overline{G} \end{tikzcd} \tag{1}$$

where ψ_n is given on $\overline{\mathbf{F}}_q$-rational points by $x \mapsto x \cdot \phi(x) \cdot \phi^2(x) \cdot \dots \cdot \phi^{n-1}(x)$. We will write $H^*_{\mathrm{et}}(\overline{G}; \mathbf{Z}/l^j)$ for the etale cohomology of $\overline{G}$ with coefficients in the constant sheaf with stalks $\mathbf{Z}/l^j$, l a prime different from $p = \operatorname{char}(\mathbf{F}_q)$. As usual, the l-adic cohomology $H^*_{\mathrm{et}}(\overline{G}; \mathbf{Q}_l)$ is defined as

$$\left(\varprojlim_j H^*_{\mathrm{et}}\left(\overline{G}; \mathbf{Z}/l^j\right)\right) \otimes \mathbf{Q}.$$

For the convenience of the reader, we recall some basic facts on $H^*_{\mathrm{et}}(\overline{G}; \mathbf{Q}_l)$. By the classification of connected reductive group schemes over an algebraically closed field, there exists a unique reductive algebraic group L over $\mathbf{C}$ with the same *Root Data* as $\overline{G}$. The associated Lie group of complex points $L(\mathbf{C})^{\mathrm{top}}$, with the strong topology, satisfies (cf. [FrPa])

$$H^*_{\mathrm{sing}}\left(L(\mathbf{C})^{\mathrm{top}}; \mathbf{Q}_l\right) \cong H^*_{\mathrm{et}}\left(\overline{G}; \mathbf{Q}_l\right).$$

Moreover, the Lie group $L(\mathbf{C})^{\text{top}}$ has complex dimension $N = \dim(\overline{G})$, and $L(\mathbf{C})^{\text{top}}$ is homeomorphic to $K \times \mathbf{R}^N$, where K denotes a maximal compact subgroup of $L(\mathbf{C})^{\text{top}}$ and, of course,

$$H^*_{\text{sing}}\big(L(\mathbf{C})^{\text{top}}; \mathbf{Q}_l\big) \cong H^*_{\text{sing}}(K; \mathbf{Q}_l).$$

It is well known that $H^*_{\text{sing}}(K; \mathbf{Q}_l)$ is an exterior algebra over $\mathbf{Q}_l$ on odd dimensional generators. Thus

$$H^*_{\text{et}}\big(\overline{G}; \mathbf{Q}_l\big) \cong \wedge W$$

with W a graded vector space over $\mathbf{Q}_l$. In particular,

$$H^N_{\text{et}}\big(\overline{G}; \mathbf{Q}_l\big) \cong \mathbf{Q}_l,$$

where $N = \dim(\overline{G})$, and therefore a morphism $f\colon \overline{G} \to \overline{G}$ of schemes over $\operatorname{spec}(\overline{\mathbf{F}}_q)$ has a well-defined degree

$$d(f) \in \mathbf{Z}_l,$$

$\mathbf{Z}_l \subset \mathbf{Q}_l$ the l-adic integers, satisfying

$$f^*(x) = d(f) \cdot x$$

for all $x \in H^N_{\text{et}}(\overline{G}; \mathbf{Q}_l)$. Because the algebra $H^*_{\text{et}}(\overline{G}; \mathbf{Z}/l)$ satisfies Poincaré duality it is clear that

$$H^*_{\text{et}}(f; \mathbf{Z}/l)\colon H^*_{\text{et}}\big(\overline{G}; \mathbf{Z}/l\big) \to H^*_{\text{et}}\big(\overline{G}; \mathbf{Z}/l\big)$$

is an isomorphism if and only if $d(f) \in \mathbf{Z}_l$ is a unit. It is also useful to observe that $H^*_{\text{et}}(\overline{G}; \mathbf{Q}_l)$ has the structure of a Hopf-algebra, with coalgebra structure induced by the multiplication in $\overline{G}$. If we put

$$V = PH^*_{\text{et}}\big(\overline{G}; \mathbf{Q}_l\big),$$

the subspace of primitive elements, then we have a natural isomorphism

$$\wedge V = H^*_{\text{et}}\big(\overline{G}; \mathbf{Q}_l\big).$$

It follows that if $f\colon \overline{G} \to \overline{G}$ is a morphism of schemes over $\overline{\mathbf{F}}_q$, such that the induced map $f^*\colon H^*_{\text{et}}(\overline{G}; \mathbf{Q}_l) \to H^*_{\text{et}}(\overline{G}; \mathbf{Q}_l)$ maps $V = PH^*_{\text{et}}(\overline{G}; \mathbf{Q}_l)$ to itself (e.g., if f is a morphism of group schemes), then

$$d(f) = \det(f^* | V),$$

because $H^N_{\mathrm{et}}(\overline{G};\mathbf{Q}_l) \cong \wedge^{\max} V$, the largest non-vanishing exterior power of V. Moreover, if we put $f^* = 1 - g$, then the linear map g maps V into itself and satisfies

$$\det(f^*|V) = \det(1 - g|V) = \mathrm{grTr}(g),$$

where by $\mathrm{grTr}(g)$ we mean the *graded trace*

$$\sum(-1)^i \mathrm{Tr}\big(g\colon H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l) \to H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l)\big).$$

This is a consequence of the fact that V has a bases consisting of elements of odd degrees, so that $\mathrm{grTr}(g)$, as defined above, is also equal to

$$\sum(-1)^j \mathrm{Tr}(\wedge^j Pg\colon \wedge^j V \to \wedge^j V),$$

where Pg denotes the restriction of g to V. Our next goal is to analyse the degree of the morphism ψ_n in the diagram (1). An easy spectral sequence argument, applied to the diagram (1), shows the following (see [FrMi] for the case of an $\mathbf{F}_q$-split G).

PROPOSITION 1.1. *Let $G = X_{\mathbf{F}_q}$ and $\psi_n\colon \overline{G} \to \overline{G}$ be as above. Then for every prime l different from $p = \mathrm{char}(\mathbf{F}_q)$ the following are equivalent:*
(i) *$G(\mathbf{F}_q) \to G(\mathbf{F}_{q^n})$ induces an $H_*(;\mathbf{Z}/l)$-isomorphism.*
(ii) *The degree $d(\psi_n) \in \mathbf{Z}_l$ is an l-adic unit.*

To prove the theorem of the Introduction, we need to express the degree $d(\psi_n)$ in terms of the orders of the groups $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$. This will be done by interpreting $d(\psi_n)$ in terms of graded traces and by relating these to the orders of the groups $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$ using the Lefschetz trace formula applied to the Frobenius endomorphism of $\overline{G}$.

LEMMA 1.2. *Let $\overline{G}$ be a connected reductive $\overline{\mathbf{F}}_q$-group scheme and $\phi\colon \overline{G} \to \overline{G}$ the Frobenius endomorphism associated with an $\mathbf{F}_q$-form of $\overline{G}$. Then the degree $d(1/\phi)$ of the Lang map $1/\phi\colon \overline{G} \to \overline{G}$ is given by*

$$d(1/\phi) = \sum(-1)^i \mathrm{Tr}\big(\phi^*\colon H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l) \to H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l)\big).$$

Proof. As observed above, the Hopf-algebra $H^*_{\mathrm{et}}(\overline{G};\mathbf{Q}_l)$ is an exterior algebra $\wedge V$ on $V = PH^*_{\mathrm{et}}(\overline{G};\mathbf{Q}_l)$, the subspace of primitive elements. If we denote by $\mathrm{grTr}(\phi^*)$ the graded trace

$$\sum(-1)^i \mathrm{Tr}\big(\phi^*\colon H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l) \to H^i_{\mathrm{et}}(\overline{G};\mathbf{Q}_l)\big)$$

then, because all homogeneous elements of V have odd cohomological degree, we have

$$\operatorname{grTr}(\phi^*) = \sum(-1)^j \operatorname{Tr}(\wedge^j P\phi^* : \wedge^j V \to \wedge^j V), \tag{2}$$

where $P\phi^*$: $PH^*_{\mathrm{et}}(\overline{G}; \mathbf{Q}_l) \to PH^*_{\mathrm{et}}(\overline{G}; \mathbf{Q}_l)$ is the map induced from the morphism of group schemes ϕ: $\overline{G} \to \overline{G}$; as observed earlier, the right hand side of (2) is therefore just $\det(1 - P\phi^*)$. On the other hand, $1/\phi$: $\overline{G} \to \overline{G}$ maps $x \in PH^*_{\mathrm{et}}(\overline{G}; \mathbf{Q}_l)$ to $x - \phi^* x$, which is equal to $(1 - P\phi^*)x$. Thus $\det(1 - P\phi^*) = d(1/\phi)$ which shows that $d(1/\phi) = \operatorname{grTr}(\phi^*)$ as claimed.

From the diagram (1) above we see that $\psi_n \circ (1/\phi) = 1/\phi^n$. The following corollary is then immediate.

Corollary 1.3. *The degree of the map ψ_n in the diagram* (1) *is given by*

$$d(\psi_n) = \frac{\operatorname{grTr}(\phi^n)^*}{\operatorname{grTr}(\phi^*)} \in \mathbf{Z}_l.$$

The Lefschetz trace formula in H^*_c, etale cohomology with compact supports, permits to relate the number of $\mathbf{F}_{q^n}$-rational points of certain schemes to a graded trace. The following proposition is well known (see [Mil], and also [DeLu] for a general discussion of the Lefschetz trace formula; the formula we use here is 1.9.4, page 174, of [Del]).

Proposition 1.4. *Let $X_{\mathbf{F}_q}$ be a quasi-projective $\mathbf{F}_q$-scheme,*

$$\overline{X} = X_{\mathbf{F}_q} \times_{\operatorname{spec}(\mathbf{F}_q)} \operatorname{spec}\left(\overline{\mathbf{F}}_q\right),$$

and ϕ: $\overline{X} \to \overline{X}$ the associated Frobenius endomorphism. Then

$$|X\mathbf{F}_q(\mathbf{F}_q)| = \sum(-1)^i \operatorname{Tr}\left(\phi^*_c : H^i_c\left(\overline{X}; \mathbf{Q}_l\right) \to H^i_c\left(\overline{X}; \mathbf{Q}_l\right)\right). \tag{3}$$

In accordance with the notation used above we will write $\operatorname{grTr}(\phi^*_c)$ for the right hand side of the formula (3). It remains to relate $\operatorname{grTr}(\phi^*)$ to $\operatorname{grTr}(\phi^*_c)$ in case $X_{\mathbf{F}_q} = G$. Since $\overline{G}$ is a smooth quasi-projective variety of dimension N over $\overline{\mathbf{F}}_q$, there is a natural Poincaré-Duality pairing (cf. [Mil])

$$\langle\ ,\ \rangle: H^j_{\mathrm{et}}\left(\overline{G}; \mathbf{Q}_l\right) \times H^{2N-j}_c\left(\overline{G}; \mathbf{Q}_l\right) \to \mathbf{Q}_l(-N)$$

satisfying $\langle \phi^* x, \phi^*_c y\rangle = q^N\langle x, y\rangle$. Therefore $\langle x, \phi^*_c y\rangle = \langle q^N(\phi^*)^{-1}(x), y\rangle$ which, since the pairing $\langle\ ,\ \rangle$ is non-degenerate, implies that

$$\operatorname{Tr}\left(\phi^*_c | H^{2N-j}_c\left(\overline{G}; \mathbf{Q}_l\right)\right) = q^N \operatorname{Tr}\left((\theta^*)^{-1} | H^j_{\mathrm{et}}\left(\overline{G}; \mathbf{Q}_l\right)\right).$$

Thus, taking graded traces, we infer

$$\operatorname{grTr}(\phi_c^*) = q^N \cdot \operatorname{grTr}\big((\phi^*)^{-1}\big). \tag{4}$$

From the computations in the proof of the Lemma 1.2 we see that

$$\begin{aligned}\operatorname{grTr}\big((\phi^*)^{-1}\big) &= \det\big(1-(P\phi^*)^{-1}\big) = (-1)^r \det(P\phi^*)^{-1}\det(1-P\phi^*)\\ &= (-1)^r \det(P\phi^*)^{-1}\cdot \operatorname{grTr}(\phi^*)\end{aligned}$$

where $r = \dim PH_{\mathrm{et}}^*(\overline{G};\mathbf{Q}_l)$. Note that $\det P\phi^* = u \in \mathbf{Z}_l$ is a unit, because ϕ induces an isomorphism in $H_{\mathrm{et}}^*(\overline{G};\mathbf{Z}/l)$ for any l prime to p. Applying the same reasoning to ϕ^n, we obtain

$$\operatorname{grTr}\big(((\phi^n)^*)^{-1}\big) = (-1)^r u^{-n}\cdot \operatorname{grTr}(\phi^n)^*.$$

Therefore, combining this with Corollary 1.3, Proposition 1.4 and formula (4), we obtain

$$\begin{aligned}d(\psi_n) &= \frac{u^n\cdot \operatorname{grTr}((\phi^n)^*)^{-1}}{u\cdot \operatorname{grTr}(\phi^*)^{-1}} = \frac{q^{-nN}\cdot u^n\cdot \operatorname{grTr}(\phi_c^*)^n}{q^{-N}\cdot u\cdot \operatorname{grTr}(\phi_c^*)}\\ &= \left(\frac{u}{q^N}\right)^{n-1}\cdot\frac{|G(\mathbf{F}_{q^n})|}{|G(\mathbf{F}_q)|},\quad u\in \mathbf{Z}_l^*.\end{aligned}$$

It is now plain that $d(\psi_n)$ is an l-adic unit if and only if $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$ have isomorphic l-Sylow subgroups and, in view of Proposition 1.1 the proof of the theorem is therefore completed.

The following example illustrates the general result. Take $G = Sl_2$. It is easy to check the theorem in this case directly because the cohomology rings $H^*(Sl_2(\mathbf{F}_q);\mathbf{Z}/l)$, l a prime different from the characteristic of $\mathbf{F}_q$, are completely known (cf. [FiPr]). The cohomology is periodic of period 4, and for l an odd (prime prime to q) one has

$$H^*\big(Sl_2(\mathbf{F}_q);\mathbf{Z}/l\big) \cong \begin{cases} E(u)\otimes P(v),\, u\in H^3 \text{ and } v\in H^4 \ (\text{if } l \text{ divides } q^2-1)\\ \mathbf{Z}/l\ (\text{if } l \text{ does not divide } q^2-1).\end{cases}$$

Here, $E(u)$ denotes an exterior algebra on u, and $P(v)$ a polynomial algebra on v (over $\mathbf{Z}/l$). Since the order of $Sl_2(\mathbf{F}_q)$ is $q(q^2-1)$ we see that the l-Sylow subgroup of $Sl_2(\mathbf{F}_q)$ is non-trivial if and only if l divides q^2-1 and thus, by the formula above, one has abstract isomorphisms of rings

$$H^*\big(Sl_2(\mathbf{F}_q);\mathbf{Z}/l\big) \cong H^*\big(SL_2(\mathbf{F}_{q^n});\mathbf{Z}/l\big) \tag{5}$$

if $l|q^2-1$. But $Sl_2(\mathbf{F}_q)$ and $Sl_2(\mathbf{F}_{q^n})$ have isomorphic l-Sylow subgroups if and only if q^2-1 and $q^{2n}-1$ contain the same power of l, that is (assuming that l divides q^2-1), if and only if n is relatively prime to l. Thus, in most cases, the isomorphism (5) is not induced by the restriction map

$$\text{res: } H^*\big(Sl_2(\mathbf{F}_{q^n});\mathbf{Z}/l\big) \to H^*\big(Sl_2(\mathbf{F}_q);\mathbf{Z}/l\big).$$

Our general setting will take the following form in case $G = Sl_2$. One has

$$H^*_{\text{et}}\big(\overline{Sl}_2;\mathbf{Q}_l\big) = E(w),$$

an exterior algebra over $\mathbf{Q}_l$ in $w \in H^3_{\text{et}}$. (Recall that $H^*_{\text{et}}(\overline{SL}_2;\mathbf{Q}_l) \cong H^*_{\text{top}}(Sl_2(\mathbf{C});\mathbf{Q}_l)$, and $Sl_2(\mathbf{C})$ is homotopy equivalent to a three-dimensional sphere).

Clearly, $PH^*_{\text{et}} = H^3_{\text{et}} \cong \mathbf{Q}_l$ and one checks easily that $\phi^*w = q^2w$ so that

$$d(1/\phi) = \det(1-P\phi^*) = 1-q^2$$

and thus

$$d(\psi_n) = \frac{q^{2n}-1}{q^2-1} = \frac{1}{q^{n-1}} \cdot \frac{|SL_2(\mathbf{F}_{q^n})|}{|Sl_2(\mathbf{F}_q)|}.$$

According to our general formula, we must have

$$\frac{1}{q^{n-1}} = \left(\frac{u}{q^N}\right)^{n-1}$$

where $u = \det P\phi^*$ and $N = \dim Sl_2$; this is indeed so, as $\det P\phi^* = q^2$ and $\dim Sl_2 = 3$.

The theorem of the Introduction can also be viewed as a result concerning the conjugacy relationship between the various l- subgroups of $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$. For this purpose, denote by $\text{Frob}_l(F)$ the "Frobenius category" of finite l-subgroups of the group F; its objects are the finite l-subgroups of F, and morphisms are induced by inner automorphisms of F. It is a classical result that for a finite group F, the restriction maps fit together to give rise to an isomorphism

$$H^*(F;\mathbf{Z}/l) \to \varprojlim_{\pi\in \mathrm{Frob}_l(F)^{\mathrm{op}}} H^*(\pi;\mathbf{Z}/l).$$

Thus, any homomorphism of finite groups $\varphi\colon F_1 \to F_2$ inducing an equivalence of categories $\mathrm{Frob}_l(F_1) \to \mathrm{Frob}_l(F_2)$ will induce an isomorphism $H^*(F_2;\mathbf{Z}/l) \to H^*(F_1;\mathbf{Z}/l)$. The converse of this statement is also true according to [Mis]. Thus, our theorem implies the following corollary, which in the case of $G = Gl_n$ is a well-known fact on the representation theory of finite l-groups in characteristic p different from l.

COROLLARY 1.5. *Let G be a connected reductive $\mathbf{F}_q$-group scheme and let l be a prime different from $p = \mathrm{char}(\mathbf{F}_q)$. Suppose that $G(\mathbf{F}_q)$ and $G(\mathbf{F}_{q^n})$ have isomorphic l-Sylow subgroups. Then the inclusion $G(\mathbf{F}_q) \to G(\mathbf{F}_{q^n})$ induces an equivalence of Frobenius categories of finite l-subgroups*

$$\mathrm{Frob}_l\big(G(\mathbf{F}_q)\big) \to \mathrm{Frob}_l\big(G(\mathbf{F}_{q^n})\big).$$

2. Suzuki and Ree groups

We will be considering the three families of groups, which in [Gor] are denoted by

$${}^2B_2(2^n),\, {}^2G_2(3^n),\, {}^2F_4(2^n)$$

with n odd, say $n = 2m + 1$; ${}^2B_2(2^n)$ is isomorphic to the Suzuki group $Sz(2^n)$, which is simple for $n > 1$, and the groups ${}^2G_2(3^n), {}^2F_4(2^n)$ are the simple groups of Ree type. The orders of these groups are (cf. [Gor]):

$$|{}^2B_2(q)| = q^2(q^2+1)(q-1) \quad \text{where } q = 2^{2m+1},$$

$$|{}^2G_2(q)| = q^3(q^3+1)(q-1) \quad \text{where } q = 3^{2m+1},$$

$$|{}^2F_4(q)| = q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1) \quad \text{where } q = 2^{2m+1}.$$

Let $G = G_{\mathbf{F}_q}$ denote the split reductive group scheme over $\mathrm{spec}(\mathbf{F}_q)$ of type B_2, G_2 respectively F_4. Then $\overline{G} = G \times_{\mathrm{spec}\mathbf{F}_q} \mathrm{spec}\overline{\mathbf{F}}_q$ admits an exceptional isogeny

$$\psi\colon \overline{G} \to \overline{G}$$

such that $\psi^2 = \phi$, the Frobenius of the $\mathbf{F}_q$-form G of $\overline{G}$. Moreover

$$^2G(q) = \{x \in G(\mathbf{F}_q) | \psi x = x\}$$

agrees with $^2B_2(q), {}^2G_2(q)$ respectively $^2F_4(q)$ for G of type B_2, G_2 respectively F_4 (we always assume $q = 2^{2m+1}, 3^{2m+1}$ respectively 2^{2m+1}, according to the case one considers). As in [FrMi], one gets then a commutative diagram of finite etale maps arising from Lang's construction

$$\begin{array}{ccccc} ^2G(q) & \longrightarrow & \overline{G} & \xrightarrow{1/\psi} & \overline{G} \\ \downarrow & & \| & & \downarrow{\scriptstyle\theta} \\ ^2G(q^d) & \longrightarrow & \overline{G} & \xrightarrow[1/\psi^d]{} & \overline{G} \end{array}$$

where d is an odd natural number; θ is given on a point $x \in \overline{G}(\mathbf{F}_q)$ by

$$\theta(x) = x \cdot \psi(x) \cdot \psi^2(x) \cdots \psi^{d-1}(x)$$

As before, it follows that the inclusion $^2G(q) \to {}^2G(q^d)$ is an $H_*(;\mathbf{Z}/l)$ isomorphism (l as always prime to q) if and only if θ^*: $H^*_{\text{et}}(\overline{G}; \mathbf{Q}_l) \to H^*_{\text{et}}(\overline{G}; \mathbf{Q}_l)$ has a degree prime to l (the degree of θ^* is an l-adic integer). There are now three cases to consider. For the computation of the relevant degrees of $(B\psi)^*$ we refer the reader to [AdMa].

(i) *G of type B_2*

In this case, $H^*_{\text{et}}(B\overline{G}; \mathbf{Q}_l) \cong \mathbf{Q}_l[x_4, x_8]$ and $(B\psi)^* x_4 = qx_4, (B\psi)^* x_8 = -q^2 x_8$. Therefore

$$\begin{aligned} \deg \theta &= (1 + q + q^2 + \cdots + q^{d-1})(1 - q^2 + q^4 - \cdots + q^{2d-2}) \\ &= q^{2-2d} |^2B_2(q^d)| / |^2B_2(q)|. \end{aligned}$$

We conclude that $\deg \theta$ is prime to l if and only if $^2B_2(q)$ and $^2B_2(q^d)$ have isomorphic l-Sylow subgroups. Thus the following holds.

PROPOSITION 2.1. *Let $q = 2^{2m+1}$ and $d \geq 1$ be an odd integer. Let l be an odd prime. Then $^2B_2(q) \to {}^2B_2(q^d)$ induces an isomorphism in $\mathbf{Z}/l$-homology if and only if $^2B_2(q)$ and $^2B_2(q^d)$ have isomorphic l-Sylow subgroups.*

(ii) *G of type G_2*

We have $H^*_{\text{et}}(B\overline{G};\mathbf{Q}_l) \cong \mathbf{Q}_l[x_4, x_{12}]$ and $(B\psi)^*x_4 = qx_4, (B\psi)^*x_{12} = -q^3x_{12}$ which yields

$$\begin{aligned}\deg\theta &= (1 + q + q^2 + \cdots + q^{d-1})(1 - q^3 + q^6 - \cdots + q^{3d-3})\\ &= q^{3-3d}|{}^2G_2(q^d)|/|{}^2G_2(q)|\end{aligned}$$

and the next result follows.

PROPOSITION 2.2. *Let $q = 3^{2m+1}$ and d be odd. Let l be a prime different from 3. Then ${}^2G_2(q) \to {}^2G_2(q^d)$ induces a $\mathbf{Z}/l$-homology isomorphism if and only if ${}^2G_2(q)$ and ${}^2G(q^d)$ have isomorphic l-Sylow subgroups.*

(iii) *G of type F_4*

We proceed as in the other two cases and obtain

$$H^*(B\overline{G};\mathbf{Q}_l) \cong \mathbf{Q}_l[x_4, x_{12}, x_{16}, x_{24}],$$
$$(B\psi)^*x_4 = qx_4, (B\psi)^*x_{12} = -q^3x_{12}, (B\psi)^*x_{16} = q^4x_{16},$$
$$(B\psi)^*x_{24} = -q^6x_{24}.$$

This implies that

$$\deg\theta = q^{12-12d}|{}^2F_4(q^d)|/|{}^2F_4(q)|$$

and we get our final result.

PROPOSITION 2.3. *Let $q = 2^{2m+1}$, and $d \geq 1$ be odd. Let l be an odd prime. Then ${}^2F_4(q) \to {}^2F_4(q^d)$ induces a $\mathbf{Z}/l$-homology isomorphism if and only if ${}^2F_4(q)$ and ${}^2F_4(q^d)$ have isomorphic l-Sylow subgroups.*

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