ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 3, Fall 1992

A VERSALITY THEOREM FOR TRANSVERSELY HOLOMORPHIC FOLIATIONS OF FIXED DIFFERENTIABLE TYPE

BY

J. GIRBAU[1]

Introduction

The aim of this paper is to give a versality theorem (analogous to that of Kuranishi for compact complex manifolds [6]) for transversely holomorphic foliations of fixed differentiable type (Theorem 7.1).

Throughout M will denote a compact C^∞-manifold endowed with a transversely holomorphic foliation $\mathcal{F}$ defined by a foliate cocycle $\{U_i, f_i, Z, \gamma_{ij}\}$, where $\{U_i\}$ is an open covering of M, $f_i\colon U_i \to Z$ are C^∞-submersions, Z is a complex manifold and $\{\gamma_{ij}\}$ are local holomorphic transformations of Z such that $f_i = \gamma_{ij} \circ f_j$. A family of deformations $\mathcal{F}^t$ of $\mathcal{F}$ parametrized by a germ of analytic space (T, o) is defined by a family of C^∞-submersions, $f_i^t\colon U_i \to Z$, parametrized by (T, o), depending holomorphically on t for each $x \in U_i$, and a family γ_{ij}^t of local holomorphic transformations of Z parametrized by the same (T, o), such that $f_i^t = \gamma_{ij}^t \circ f_j^t$, with $f_i^o = f_i$ and $\gamma_{ij}^o = \gamma_{ij}$. Two families of deformations, $\mathcal{F}^t$ and $\mathcal{F}'^t$, parametrized by the same (T, o) are said to be *isomorphic* if there exists a C^∞-family h^t of diffeomorphisms of M parametrized by (T, o), with $\mathcal{F}'^t = (h^t)^* \mathcal{F}^t$.

Girbau, Haefliger and Sundararaman [3] proved the existence of a germ of analytic space (S, o) (versal space) parametrizing a family of deformations $\mathcal{F}^s$ of $\mathcal{F}$ (versal family), with the following property: if $\mathcal{F}'^t$ is another family of deformations of $\mathcal{F}$ parametrized by (T, o), then there exists a morphism of germs of analytic spaces, $f\colon (T, o) \to (S, o)$, such that $\mathcal{F}^{f(t)}$ is isomorphic to $\mathcal{F}'^t$. Moreover the tangent map $d_o f$ of f at o is unique.

Most of the computable examples have a smooth versal space; that is, S is the germ at the origin of a complex vector space, concretely the cohomology space $H^1(M, \Theta^{tr})$, where Θ^{tr} is the sheaf of germs of local C^∞-vector fields generating flows preserving $\mathcal{F}$. There is a useful sufficiency criterion for the

Received October 30, 1990
1991 Mathematics Subject Classification. Primary 58H15; Secondary 57R32, 32G05.
[1]Research supported by the DGICYT.

versality of a family of deformations (Corollary 2 on page 126 of [3]): If (S, o) is a germ of analytic space parametrizing a family of deformations $\mathcal{F}^s$ of $\mathcal{F}$, then (S, o) is the versal space and $\mathcal{F}^s$ the versal family if the two following (versality) conditions are fulfilled:

VC1. (S, o) is smooth.

VC2. The Kodaira-Spencer map $\rho\colon T_oS \to H^1(M, \Theta^{tr})$, associated to the family $\mathcal{F}^s$, is an isomorphism.

What can be said about the deformations of $\mathcal{F}$ that do not change $\mathcal{F}$ as a real foliation (those that only change its transverse complex structure)? These deformations will be called *deformations of fixed differentiable type*. Is there a versality theorem for these deformations? These are, of course, natural questions.

The standard machinery of elliptic operators does not work here. In fact, the equivalence classes of infinitesimal deformations of this type are parametrized by the Dolbeault basic cohomology space $H^1(A_b^{0,*}(N^{1,0}), \bar\partial)$ with values in $N^{1,0}$, the normal bundle of type $(1, 0)$ of $\mathcal{F}$. This cohomology space is finite-dimensional although the Dolbeault basic complex

$$0 \longrightarrow A_b^{0,0}(N^{1,0}) \xrightarrow{\bar\partial} A_b^{0,1}(N^{1,0}) \xrightarrow{\bar\partial} \cdots$$

is not elliptic.

El Kacimi and Nicolau [2] gave a versality theorem for these deformations with the assumption that the initial foliation $\mathcal{F}$ is hermitian. It follows from a careful reading of their proof that it is not necessary to require $\mathcal{F}$ to be hermitian, but only to satisfy the following two hypotheses:

A. There exists a transversely projectable connection (this means that the tangent bundle of the model manifold Z admits a connection invariant under the γ_{ij}).

B. There is a (sufficiently large) positive real number r such that $\bar\partial({}^rA_b^{0,1}(N^{1,0}))$ is closed in ${}^{r-1}A_b^{0,2}(N^{1,0})$, where ${}^rA_b^{0,1}(N^{1,0})$ denotes the Sobolev r-completion of $A_b^{o,1}(N^{1,0})$ (this condition is fulfilled, for example, when $H^2(A_b^{0,*}(N^{1,0}), \bar\partial)$ is finite-dimensional).

When the initial foliation $\mathcal{F}$ satisfies conditions A and B (this is the case if $\mathcal{F}$ is hermitian) then El Kacimi and Nicolau [2] proved a weak versality theorem for deformations of fixed differentiable type. Concretely they constructed a family $\mathcal{F}^{s_b}$ of deformations of $\mathcal{F}$ of fixed differentiable type, parametrized by a germ of analytic space (S_b, o), with the following versality property: If $\mathcal{F}'$ is close enough to $\mathcal{F}$, and $\mathcal{F}'$ is conjugate to $\mathcal{F}$ as a C^∞-foliation, then there exists a diffeomorphism h of M close to the identity and $s_b \in S_b$ such that $\mathscr{F}' = h^*(\mathscr{F}^{s_b})$. They were not able to prove a strong versality theorem; that is, given any family of deformations $\mathscr{F}'^t$ of fixed differentiable type parametrized by (T, o'), to prove the existence of a (holomorphic) morphism f: $(T, o') \to (S_b, o)$ such that $\mathscr{F}^{f(t)}$ is isomorphic to $\mathscr{F}'^t$.

The purpose of this paper is to prove such a strong versality theorem for deformations of fixed differentiable type (Theorem 7.1). Hypotheses analogous to A and B above are needed also.

From the intuitive point of view it does not seem that the passage from a weak versality theorem to a strong one means any significant gain. Nevertheless, in the general transversely holomorphic case, the proof of the versality criterion quoted above when conditions VC1 and VC2 are fulfilled needs the existence of a strong versality theorem. In the case of deformations of fixed differentiable type the same thing happens. We give such a criterion in Section 8 (Theorem 8.1). Since this is one of the criteria most used in practice, this justifies our interest in proving a strong versality theorem instead of a weak one.

The proof given here of Theorem 7.1 is based on a Hodge splitting theorem obtained in Section 2 and on a careful analysis of the construction of the versal space for transversely holomorphic foliations given in [3]. For this reason we need to rewrite the construction of [3] and modify some minor points of that presentation in order that all details work. This is done in Sections 3, 4 and 5. Section 6 is a preparation for the proof of the versality theorem given in Section 7.

I thank M. Nicolau for his invaluable help in the preparation of this paper, for his careful reading of the previous versions of the manuscript as well as his criticisms and useful observations.

1. Derivations and the space of infinitesimal deformations

A transversely holomorphic foliation $\mathscr{F}$ on M is determined by the sheaf $\mathscr{O}_{\mathscr{F}}$ of germs of functions f which are constant along the leaves and transversely holomorphic. Denote by $A^{\star}(M)$ the algebra of complex-valued C^∞-forms and by I_F the ideal of $A^{\star}(M)$ generated by the differentials of elements of $\mathscr{O}_{\mathscr{F}}$. In each local chart (x^u, z^a) adapted to the foliation, where the x^u are coordinates along leaves and the z^a are transverse complex coordinates, I_F is generated by $\{dz^a\}$. Let F be the subbundle of ${}^cTM$ whose sheaf of germs of sections is the sheaf of germs of smooth vector fields which annihilate $\mathscr{O}_{\mathscr{F}}$. In each $\mathscr{F}$-adapted local chart (x^u, z^a), F is generated by $\{\partial/\partial x^u, \partial/\partial \overline{z^a}\}$.

Denote by $\mathscr{D}^p$ the space of degree p derivations of $A^{\star}(M)$. Following Kodaira and Spencer [5], a degree p derivation δ is given in each local chart by a pair (φ, ξ) of smooth vector forms of degrees p and $p+1$ respectively, with

$$\varphi = \varphi^u \frac{\partial}{\partial x^u} + \varphi^a \frac{\partial}{\partial z^a} + \varphi^{\bar{a}} \frac{\partial}{\partial \overline{z^a}}$$

$$\xi = \xi^u \frac{\partial}{\partial x^u} + \xi^a \frac{\partial}{\partial z^a} + \xi^{\bar{a}} \frac{\partial}{\partial \overline{z^a}},$$

where $\varphi^u = \delta x^u$, $\varphi^a = \delta z^a$, $\varphi^{\bar{a}} = \delta \overline{z^a}$, $\xi^u = (-1)^p \delta\, dx^u$, $\xi^a = (-1)^p \delta\, dz^a$, $\xi^{\bar{a}} = (-1)^p \delta\, d\overline{z^a}$. All the φ corresponding to various local charts give rise to a global vector p-form. But the ξ do not correspond to a global vector $(p+1)$-form. The differences $d\varphi - \xi$ are global, where $d\varphi$ means the local vector $(p+1)$-form whose components are $(d\varphi^u, d\varphi^a, d\varphi^{\bar{a}})$.

Denote by $\mathscr{D}^p_{\mathscr{F}}$ the subspace of $\mathscr{D}^p$ consisting of those δ such that $\delta(I_F) \subset I_F$. Then $\delta \in \mathscr{D}^p_{\mathscr{F}}$ if and only if the local pairs (φ, ξ) of δ are such that $\xi^a \in I_F$ for all a.

$\mathscr{D}^\star_{\mathscr{F}}$ is endowed with the bracket operation on the derivations, and with the differential D: $\mathscr{D}^p_{\mathscr{F}} \to \mathscr{D}^{p+1}_{\mathscr{F}}$ defined by $D\delta = [d, \delta]$, where d is the exterior differential. D acts on the pairs (φ, ξ) which determine δ in the following way:

$$\delta = (\varphi, \xi) \xrightarrow{D} (d\varphi - \xi, -d\xi).$$

Denote by $A^{p,q}_b(M)$ the space of basic (p,q)-forms; that is, those forms φ which have the following expression in an adapted local chart (x^u, z^a):

$$\varphi = \frac{1}{p!}\frac{1}{q!}\varphi_{a_1 \cdots a_p \overline{b_1} \ldots \overline{b_q}}(z, \bar{z})\, dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\overline{z^{b_1}} \wedge \cdots \wedge d\overline{z^{b_q}}.$$

Let $A^i_b = \sum_{p+q=i} A^{p,q}_b(M)$. Let $\mathscr{D}^p_b$ be the subspace of $\mathscr{D}^p_{\mathscr{F}}$ consisting of those $\delta \in \mathscr{D}^p_{\mathscr{F}}$ which satisfy $\delta(A^\star_b) \subset A^\star_b$. The elements of $\mathscr{D}^p_b$ will be called *basic derivations of degree p*. Given $\delta = (\varphi, \xi) \in \mathscr{D}^p_{\mathscr{F}}$ then $\delta \in \mathscr{D}^p_b$ if and only if in each $\mathscr{F}$-adapted local chart (x^u, z^a), the components $\varphi^a, \xi^a, \varphi^{\bar{a}}, \xi^{\bar{a}}$ are basic forms and $\xi^a \in I_F$.

The bracket of two elements of $\mathscr{D}^\star_b$ belongs to $\mathscr{D}^\star_b$ and the differential D maps $\mathscr{D}^p_b$ to $\mathscr{D}^{p+1}_b$, giving rise to an operator D_b: $\mathscr{D}^p_b \to \mathscr{D}^{p+1}_b$.

El Kacimi and Nicolau use the complex

$$0 \longrightarrow A^{0,0}_b(N^{1,0}) \xrightarrow{\bar{\partial}} A^{0,1}_b(N^{1,0}) \xrightarrow{\bar{\partial}} \cdots$$

where $N^{1,0} = {}^cTM/F$ is the normal bundle of type $(1,0)$ and $A^{0,q}_b(N^{1,0})$ denotes the subspace of $\Gamma^\infty((\wedge^q \overline{N^{1,0}})^* \otimes N^{1,0})$ consisting of those elements which are basic. The Dolbeault operator $\bar{\partial}$ is well-defined on $A^{0,q}_b(N^{1,0})$ and the cohomology space $H^1(A^{0,\star}_b(N^{1,0}), \bar{\partial})$ is isomorphic to the space of infinitesimal deformations of $\mathscr{F}$ of fixed differential type ([2], [4]).

Choose a splitting ${}^cTM = F \oplus N^{1,0}$ of ${}^cTM$ to be fixed throughout. If (U, x^u, z^a) is an adapted local chart, let

$$Z_a = \frac{\partial}{\partial z^a} + \lambda^u_a \frac{\partial}{\partial x^u}$$

be the vector field in U defined by the condition that Z_a is a section of $N^{1,0} \subset {}^cTM$. The dual basis of $\{\partial/\partial x^u, Z_a, \partial/\partial \overline{z^a}\}$ is $\{\theta^u, dz^a, d\overline{z^a}\}$, with $\theta^u = dx^u - \lambda^u_a\, dz^a$.

Given $\omega \in A^{0,q}_b(N^{1,0})$, its local expression will be

$$\omega = \frac{1}{q!}\omega^a_{\bar{b}_1 \cdots \bar{b}_q}(z, \bar{z})\, d\overline{z^{b_1}} \wedge \cdots \wedge d\overline{z^{b_q}} \otimes \left[\frac{\partial}{\partial z^a}\right]$$

where $[\partial/\partial z^a]$ is the class of $\partial/\partial z^a$ in ${}^cTM/F$. Denote by $\iota\omega$ the vector form defined in each local chart by

$$\iota\omega = \frac{1}{q!}\omega^a_{\bar{b}_1 \cdots \bar{b}_q}(z, \bar{z})\, d\overline{z^{b_1}} \wedge \cdots \wedge d\overline{z^{b_q}} \otimes Z_a.$$

Then $\iota\omega$ is a global vector form. Moreover the pair $j\omega = (\iota\omega, d\iota\omega - \iota\bar{\partial}\omega)$ defines an element of $\mathscr{D}^q_{\mathscr{F}}$ since $\iota\bar{\partial}\omega$ is global. It is easy to see that $j(A^\star_b(N^{1,0})) \subset \mathscr{D}^\star_b$ and that $D \circ j = j \circ \bar{\partial}$. Therefore j induces a map

$$j_*\colon H^q\big(A^{0,\star}_b(N^{1,0}), \bar{\partial}\big) \to H^q(\mathscr{D}^\star_b, D_b).$$

PROPOSITION 1.1. *j_* is an isomorphism.*

Proof. For simplicity we shall suppose $q = 1$, the only case needed. The injectivity being trivial, we shall prove that the map is onto. Let δ be an element in $\mathscr{D}^1_b$ with $D\delta = 0$. In each local chart, δ will be a pair $(\varphi, d\varphi)$ with the φ^a and $\varphi^{\bar{a}}$ basic forms. We have $\varphi^a = \varphi^a_c\, dz^c + \varphi^a_{\bar{c}}\, d\overline{z^c}$, with $\bar{\partial}(\varphi^a_{\bar{c}}\, d\overline{z^c}) = 0$. Let $\omega \in A^{0,1}_b(N^{1,0})$ be defined in each local chart by $\omega = \varphi^a_{\bar{c}}\, d\overline{z^c} \otimes [\partial/\partial z^a]$. ω is global. Let $\varphi' = \varphi - \iota\omega$. We have

$$\begin{aligned}\delta = (\varphi, d\varphi) &= (\iota\omega + \varphi', d\iota\omega + d\varphi') \\ &= \Big(\iota\omega, d\iota\omega - \underbrace{\iota\bar{\partial}\omega}_{0}\Big) + (\varphi', d\varphi') = j\omega + (\varphi', d\varphi')\end{aligned}$$

But the pair $(0, -\varphi')$ defines a derivation $\delta' \in \mathscr{D}^0_{\mathscr{F}}$ since $d0 - (-\varphi') = \varphi'$ is global. Moreover $\varphi'^a = \varphi^a_c\, dz^c$ belongs to $I_F \cap A^\star_b$. We have $D\delta' = D(0, -\varphi') = (\varphi', d\varphi')$. Therefore $\delta = j\omega + D\delta'$. Thus δ and $j\omega$ determine the same cohomology class. $\square$

It follows from the proposition that $H^1(\mathscr{D}_b^\star, D_b)$ is isomorphic to the space of infinitesimal deformations of $\mathscr{F}$ of fixed differentiable type. Moreover, since $H^1(A_b^{0,\star}(N^{1,0}), \bar\partial)$ can be imbedded in $H^1(M, \Theta^{tr})$ in a natural way, the space $H^1(\mathscr{D}_b^\star, D_b)$ is finite-dimensional. But the $H^i(\mathscr{D}_b^\star, D_b)$ are not finite-dimensional in general for $i > 1$.

2. A Hodge splitting of $\mathscr{D}_{\mathscr{F}}^1$ adapted to basic derivations

For large enough $r \in \mathbf{R}^+$ denote by ${}^r\mathscr{D}_{\mathscr{F}}^p$ and ${}^r\mathscr{D}_b^p$ the Sobolev r-completions of $\mathscr{D}_{\mathscr{F}}^1$ and $\mathscr{D}_b^1$ respectively. Endow the spaces ${}^r\mathscr{D}_{\mathscr{F}}^p$ with hermitian metrics. The operator $D\colon {}^r\mathscr{D}_{\mathscr{F}}^p \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^{p+1}$ has an adjoint D^* with respect to these metrics. It is well-known that for r large enough there is a splitting

$$(1)\qquad {}^r\mathscr{D}_{\mathscr{F}}^p = H^p \oplus D\left({}^{r+1}\mathscr{D}_{\mathscr{F}}^{p-1}\right) \oplus D^*\left({}^{r+1}\mathscr{D}_{\mathscr{F}}^{p+1}\right),$$

where the sum is orthogonal.

LEMMA 2.1. $D({}^{r+1}\mathscr{D}_{\mathscr{F}}^0) \cap {}^r\mathscr{D}_b^1 = D({}^{r+1}\mathscr{D}_b^0)$.

Proof. Let $\delta = (\varphi, \xi)$ be in $\mathscr{D}_{\mathscr{F}}^0$ with $D\delta = (d\varphi - \xi, -d\xi) \in \mathscr{D}_b^1$. This means that $d\varphi^a - \xi^a$, $d\varphi^{\bar a} - \xi^{\bar a}$, $-d\xi^a$ and $-d\xi^{\bar a}$ are basic. Since $\xi^a \in I_F$ and $d\varphi^a - \xi^a$ is basic then φ^a is basic and so is ξ^a. Set $\psi = \varphi^{\bar a} \otimes \overline{Z_a}$ (in the same notation as in the preceding section). The components of ψ with respect to the basis $\{\partial/\partial z^a, \partial/\partial \overline{z^a}, \partial/\partial x^u\}$ will be $\psi^{\bar a} = \varphi^{\bar a}$, $\psi^a = 0$, $\psi^u = \varphi^{\bar a}\overline{\lambda_a^u}$. Let δ' be the derivation given by the pair $(-\psi, -d\psi)$. We have $D\delta = D(\delta + \delta')$, but $\delta + \delta' = (\varphi - \psi, \xi - d\psi)$ is basic since $\varphi^a - \psi^a = \varphi^a$ (basic), $\varphi^{\bar a} - \psi^{\bar a} = 0$, $\xi^a - d\psi^a = \xi^a$ (basic) and $\xi^{\bar a} - d\psi^{\bar a} = \xi^{\bar a} - d\varphi^{\bar a}$ (basic). $\square$

From the splitting (1) it follows that $D({}^{r+1}\mathscr{D}_{\mathscr{F}}^0)$ is a closed subspace of ${}^r\mathscr{D}_{\mathscr{F}}^1$. From the lemma it follows that $D({}^{r+1}\mathscr{D}_b^0)$ is closed in ${}^r\mathscr{D}_b^1$. Since ${}^r\mathscr{D}_{\mathscr{F}}^1$ and ${}^r\mathscr{D}_b^1$ are Hilbert spaces, all closed subspaces admit a closed complement. Let B_b be such a complement of $D({}^{r+1}\mathscr{D}_b^0)$ in ${}^r\mathscr{D}_b^1$.

The following two lemmas concern well-known facts we shall need on topological vector spaces.

LEMMA 2.2. *Let $h\colon E_1 \to E_2$ be a continuous linear map between two Fréchet spaces. If $h(E_1)$ is finite-codimensional than $h(E_1)$ is closed in E_2.*

LEMMA 2.3. *Let V and W be closed subspaces of a topological vector space E. If $W \subset W'$ with W' a closed subspace and $E = V \oplus W'$ then $V \oplus W$ is closed in E.*

PROPOSITION 2.4. *If $D({}^r\mathscr{D}_b^1)$ is closed in ${}^{r-1}\mathscr{D}_b^2$ then $B_b \oplus D({}^{r+1}\mathscr{D}_{\mathscr{F}}^0)$ is closed in ${}^r\mathscr{D}_{\mathscr{F}}^1$.*

Proof. ${}^r\mathscr{D}_b^1 + \operatorname{Ker}\{D: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^2\} = D^{-1}D_b({}^r\mathscr{D}_b^1)$ is closed (recall that D_b denotes the restriction of D to $\mathscr{D}_b^1$). On the other hand

$$\left({}^r\mathscr{D}_b^1 + \operatorname{Ker} D\right)/\left({}^r\mathscr{D}_b^1 + \operatorname{Im}\left\{D: {}^{r+1}\mathscr{D}_{\mathscr{F}}^0 \to {}^r\mathscr{D}_{\mathscr{F}}^1\right\}\right)$$

is a quotient of $H^1(M, \Theta^{tr})$ which is finite-dimensional (see [3]). By virtue of lemma 2.2 applied to the inclusion ${}^r\mathscr{D}_b^1 + \operatorname{Im} D \subset {}^r\mathscr{D}_b^1 + \operatorname{Ker} D$, the space

$$B_b \oplus \operatorname{Im} D = {}^r\mathscr{D}_b^1 + \operatorname{Im} D$$

is closed. $\square$

Henceforth we shall suppose that $D({}^r\mathscr{D}_b^1)$ is closed in ${}^{r-1}\mathscr{D}_b^2$. Since $B_b \oplus D({}^{r+1}\mathscr{D}_{\mathscr{F}}^0)$ is closed in ${}^r\mathscr{D}_{\mathscr{F}}^1$, it has a topological complement A. From Lemma 2.3 (with $V = A$, $W = B_b$ and $W' = B_b \oplus D({}^{r+1}\mathscr{D}_{\mathscr{F}}^0)$) it follows that $B_b \oplus A$ is closed. Denote by $B = B_b \oplus A$. Define $H = \operatorname{Ker}\{D: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^2\} \cap B$ and $H_b = H \cap {}^r\mathscr{D}_b^1$. We have $H_b \subset B_b$. On the other hand, $B_b = B \cap {}^r\mathscr{D}_b^1$ is closed. Let C_b be a complement of H_b in B_b; that is, $B_b = H_b \oplus C_b$. We shall prove that $H \cong H^1(\mathscr{D}_{\mathscr{F}}^\star, D)$ and $H_b \cong H^1(\mathscr{D}_b^\star, D_b)$. We have,

$$\begin{aligned}
\operatorname{Ker}\left\{D: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^2\right\} &= \operatorname{Ker} D \cap \left(B \oplus D\left({}^{r+1}\mathscr{D}_{\mathscr{F}}^0\right)\right) = H \oplus D\left({}^{r+1}\mathscr{D}_{\mathscr{F}}^0\right) \\
\operatorname{Ker}\left\{D_b: {}^r\mathscr{D}_b^1 \to {}^{r-1}\mathscr{D}_b^2\right\} &= \operatorname{Ker}\left\{D: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^2\right\} \cap {}^r\mathscr{D}_b^1 \\
&= \operatorname{Ker}\left\{D: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^{r-1}\mathscr{D}_{\mathscr{F}}^2\right\} \cap \left(B_b \oplus D\left({}^{r+1}\mathscr{D}_b^0\right)\right) \\
&= H_b \oplus D\left({}^{r+1}\mathscr{D}_b^1\right).
\end{aligned}$$

$C_b \oplus H$ is closed because $C_b \oplus H = \pi^{-1}\pi(H)$, where $\pi: {}^r\mathscr{D}_{\mathscr{F}}^1 \to {}^r\mathscr{D}_{\mathscr{F}}^1/C_b$ is the canonical projection, and H is finite-dimensional, thus closed. Let D be a complement of $C_b \oplus H$ in B. By Lemma 2.3 (with $V = D$, $W = C_b$ and $W' = C_b \oplus H$) $C_b \oplus D$ is closed. Let C denote $C_b \oplus D$. We have proved the following result.

THEOREM 2.5. *If $D({}^r\mathscr{D}_b^1)$ is closed in ${}^{r-1}\mathscr{D}_b^2$ then there are topological splittings*

$$\begin{aligned}
{}^r\mathscr{D}_{\mathscr{F}}^1 &= C \oplus H \oplus \operatorname{Im} D \\
{}^r\mathscr{D}_b^1 &= C_b \oplus H_b \oplus \operatorname{Im} D_b,
\end{aligned}$$

with

$$\operatorname{Ker}\{D: {}^{r}\mathscr{D}^{1}_{\mathscr{F}} \to {}^{r-1}\mathscr{D}^{2}_{\mathscr{F}}\} = H \oplus \operatorname{Im} D$$

$$\operatorname{Ker}\{D_b: {}^{r}\mathscr{D}^{1}_{b} \to {}^{r-1}\mathscr{D}^{2}_{b}\} = H_b \oplus \operatorname{Im} D_b$$

$$C_b = C \cap {}^{r}\mathscr{D}^{1}_{b}, \qquad H_b = H \cap {}^{r}\mathscr{D}^{1}_{b}.$$

Remark. The above splitting of ${}^{r}\mathscr{D}^{1}_{\mathscr{F}}$ is similar to the usual Hodge splitting. Here C plays the role of $\operatorname{Im} D^*$ and $B = C \oplus H$ the role of $\operatorname{Ker} D^*$. The advantage of the splitting in theorem 2.5 is that it restricts to an analogous splitting for ${}^{r}\mathscr{D}^{1}_{b}$.

3. Vector 1-forms and transversely holomorphic foliations

To each transversely holomorphic foliation $\mathscr{F}$ we associated in Section 1 the subbundle F of ${}^{c}TM$ generated by $\{\partial/\partial x^u, \partial/\partial \overline{z^a}\}$ in each $\mathscr{F}$-adapted local chart. This subbundle satisfies the following two conditions:

THF1. $F + \overline{F} = {}^{c}TM$.
THF2. $[F, F] \subset F$.

A subbundle F of ${}^{c}TM$ which satisfies THF1 will be called a *transversely holomorphic distribution*.

Let $\mathscr{F}$ be a transversely holomorphic foliation that we take as an initial foliation. Let F be its associated bundle. Let $A^1({}^{c}TM)$ be the space of vector 1-forms. If $\varphi \in A^1({}^{c}TM)$ is close enough to the identity in such a way that $\varphi(F) + \overline{\varphi(F)} = {}^{c}TM$ then $\varphi(F)$ is a transversely holomorphic distribution. Denote by β the map $\varphi \to \varphi(F)$ which carries vector 1-forms close to the identity to transversely holomorphic distributions close to F.

By using the splitting ${}^{c}TM = F \oplus N^{1,0}$ that we chose in section 1, each transversely holomorphic distribution F' close to F is the graph of a morphism of vector bundles ω: $F \to N^{1,0}$; that is, $F' = \{X + \omega(X) | X \in F\}$. ω can be extended to a morphism ${}^{c}TM \to {}^{c}TM$ with the condition $\omega | N^{1,0} = 0$. Denote by α the map $F \to \mathrm{id} + \omega$ which carries transversely holomorphic distributions close to F to vector 1-forms close to the identity. Obviously $\beta \circ \alpha = \mathrm{id}$, but $\alpha \circ \beta \neq \mathrm{id}$, in general.

The diffeomorphisms f of M close to the identity act on transversely holomorphic distributions close to F in the following way: $(f, F') \to f^*(F')$, where $f^*(F')$ is the subbundle of ${}^{c}TM$ with $f^*(F')_x = f^{-1}(F'_{f(x)})$ at each point $x \in M$. The diffeomorphisms f of M close to the identity act on vector 1-forms close to the identity by $(f, \varphi) \to f^*(\varphi)$, where $f^*(\varphi) = \alpha f^*(\beta(\varphi))$.

4. Derivations and transversely holomorphic foliations

As in the preceding section let F be the subbundle of ${}^cTM$ corresponding to the initial foliation $\mathcal{F}$. Let φ be a vector 1-form close to the identity. In each $\mathcal{F}$-adapted local chart (U, x^u, z^a) set $\varphi_u = \varphi(\partial/\partial x^u)$, $\varphi_a = \varphi(\partial/\partial z^a)$, $\varphi_{\bar{a}} = \varphi(\partial/\partial \overline{z^a})$. We have $[\varphi_\lambda, \varphi_\mu] = C^\nu_{\lambda\mu}\varphi_\nu$, where the indices λ, μ, ν denote all the indices a, $\bar{a}$, and u. Let C be the vector 2-form

$$C = \frac{1}{2} C^\nu_{\lambda\mu}\, dz^\lambda \wedge dz^\mu \otimes \frac{\partial}{\partial z^\nu},$$

where z^λ is to be interpreted as x^u when $\lambda = u$. The pair $\delta = (\varphi, C)$ gives a 1-derivation $\delta \in \mathscr{D}^1$ such that $[\delta, \delta] = 0$ (see [5]). Then $\delta \in \mathscr{D}^1_{\mathcal{F}}$ if $\varphi(F)$ is integrable. Let $A^1_{\text{int}}({}^cTM)$ be the subbundle of $A^1({}^cTM)$ consisting of those φ close to the identity such that $\beta(\varphi)$ is a transversely holomorphic foliation. The map $\varphi \to \delta = (\varphi, C)$ is a bijection between $A^1_{\text{int}}({}^cTM)$ and the set of elements $\delta \in \mathscr{D}^1_{\mathcal{F}}$ close to the exterior differential d such that $[\delta, \delta] = 0$.

To work in a neighbourhood of the origin rather than in a neighbourhood of d, we take the composite of the two bijections

$$\varphi \to \delta = (\varphi, C) \to d - \delta$$

which gives a bijection between $A^1_{\text{int}}({}^cTM)$ and the analytic subspace J of $\mathscr{D}^1_{\mathcal{F}}$ consisting of those $\delta \in \mathscr{D}^1_{\mathcal{F}}$ close to the identity and satisfying

$$D\delta - \tfrac{1}{2}[\delta, \delta] = 0.$$

To each $\delta \in J$ corresponds (by the composite of the inverse bijection and β) the transversely holomorphic foliation given by $(\mathrm{id} - \varphi)(F)$. We shall denote this map also by β.

Since the diffeomorphisms of M close to the identity act on $A^1_{\text{int}}({}^cTM)$ by the action defined in the preceding section, they also act on J through the bijection

$$A^1_{\text{int}}({}^cTM) \to J.$$

5. Construction of the versal space for the transversely holomorphic foliations

Let $\mathcal{F}$ be a transversely holomorphic foliation on M that we shall take as an initial foliation. Without loss of generality we can assume that M and $\mathcal{F}$ are of class C^ω, because any transversely analytic foliation is isotopic to a real analytic foliation. Fix a large enough positive r and endow the

Sobolev completions ${}^r\mathscr{D}^p_{\mathscr{F}}$ with hermitian metrics. Denote by D^* the adjoint of D: ${}^r\mathscr{D}^p_{\mathscr{F}} \to {}^{r-1}\mathscr{D}^{p+1}_{\mathscr{F}}$ with respect to these metrics. Set $\Sigma = \{\delta \in {}^r\mathscr{D}^1_{\mathscr{F}} | D^*(D\delta - \frac{1}{2}[\delta, \delta]) = 0\}$. Σ is a Banach manifold in a neighbourhood of the origin, with $\operatorname{Ker} D = T_o\Sigma$. We shall work here with germs of manifolds and germs of analytic spaces. Thus we shall substitute, without explicit mention, a manifold or an analytic space with a neighbourhood of a distinguished point of this manifold or analytic space. Set $\tilde{H} = \Sigma \cap \operatorname{Ker} D^*$. $\tilde{H}$ can be defined alternatively (see [3] or [1]) as the set of those $\delta \in {}^r\mathscr{D}^1_{\mathscr{F}}$ satisfying the elliptic equation

$$D^*(D\delta - \tfrac{1}{2}[\delta, \delta]) + DD^*\delta = 0.$$

Thus all $\delta \in \tilde{H}$ are C^ω. One can easily see that $\tilde{H}$ is an analytic Banach submanifold of Σ (in a neighbourhood of $0 \in \tilde{H}$) having $H = \operatorname{Ker} D \cap \operatorname{Ker} D^*$ as a tangent space at 0. Set $S = \tilde{H} \cap J$. S is an analytic space with $T_o(S) = H$. We want to see that the family of foliations $\{\mathscr{F}_s\}_{s \in S}$, with $\mathscr{F}_s = \beta(s)$, is versal (where S can be replaced with a suitable neighbourhood of 0 in S).

As in Section 1.9 of [3] one can construct a C^∞-morphism

$$g: S \times N^{1,0} \to M$$

(defined only in a neighbourhood of the origin of $\{0\} \times \gamma(M)$, where γ is the zero section of $N^{1,0}$) depending holomorphically on the variable S and such that:

(i) $g(0, \gamma(x)) = x \ \forall x \in M$.

(ii) For each $s \in S$ and $x \in M$ let $g_{s,x}$ be the restriction of g to $\{s\} \times N^{1,0}_x$. For each local chart (U, x^u_s, z^a_s) of M adapted to the foliation $\mathscr{F}_s$, with $x \in U$, let $\pi_s(u) = (z^a_s(u))$. With these notations $\pi_s \circ g_{s,x}$ is holomorphic.

(iii) The tangent map of $g_{o,x}$ at $(o, \gamma(x))$ is the canonical inclusion $N^{1,0}_x \subset {}^cT_x(M)$.

As in Section 1.9 of [3] let ${}^r\mathrm{Diff}^S(M)$ be the family $\{f_s\}$ of diffeomorphisms of M of r-Sobolev class which are of the form $x \to g(s, \xi(x))$, where ξ is a section of $N^{1,0}$ close to the zero section. Let ${}^rA$ be (a neighbourhood of 0 in) the vector space of r-Sobolev class 1-forms mapping F to F. Define

$$\rho: {}^{r+1}\mathrm{Diff}^S(M) \times {}^{r+1}A \to {}^rA^1({}^cTM)$$

by $\rho(f_s, a) = f_s^*(\varphi_s) \circ b_s^{-1} \circ \varphi_s \circ (\mathrm{id} + a)$, where φ_s is the first component of the derivation s (that is, a vector 1-form), $b_s = \alpha\beta(\varphi_s)$ and the symbol $\circ$

means composition of endomorphisms. Notice that $\rho(\mathrm{id}_s, 0) = \varphi_s$ because (by definition) $\mathrm{id}_s^*(\varphi_s) = \alpha(\mathrm{id}^*(\beta(\varphi_s)) = b_s$.

Let ϕ be the map $S \times {}^{r+1}\Gamma(N^{1,0}) \to {}^{r+1}\mathrm{Diff}^S(M)$ given by

$$(s, \xi) \to (x \to g(s, \xi(x))$$

in a suitable neighbourhood of $(0,0)$ in $S \times {}^{r+1}\Gamma(N^{1,0})$. Endow ${}^{r+1}\mathrm{Diff}^S(M)$ with the complex structure given by the local chart ϕ. With this structure one can prove that ρ is holomorphic.

Since φ_s corresponds to an integrable distribution so does $\rho(f_s, a)$; that is, $\rho(f_s, a)$ belongs to $A^1_{\mathrm{int}}({}^cTM)$ in the notation of Section 4. Through the bijection $A^1_{\mathrm{int}}({}^cTM) \cong J$, ρ gives a holomorphic morphism

$$\rho\colon {}^{r+1}\mathrm{Diff}^S(M) \times {}^{r+1}A \to {}^rJ.$$

Here $\rho(\mathrm{id}_s, 0) = s$. If we identity ${}^{r+1}\mathrm{Diff}^S(M)$ with $S \times {}^{r+1}\Gamma(N^{1,0})$ in a neighbourhood of $(0, \mathrm{id}_o)$ through the local chart ϕ we shall have

$$\rho\colon S \times {}^{r+1}\Gamma(N^{1,0}) \times {}^{r+1}A \to {}^rJ.$$

In each adapted local chart (U, x^u, z^a) take the vector fields Z_a and the 1-forms θ^u defined in Section 1. If $\xi \in \Gamma(N^{1,0})$ is expressed on U by $\xi = \xi^a Z_a$, define

$$d_F\xi = \left(\overline{Z_a}(\xi^b)d\overline{z^a} + \frac{\partial \xi^b}{\partial x^u}\theta^u\right) \otimes Z_b.$$

Then $d_F\xi$ is a global vector 1-form. Let ${}^{r+1}\mathscr{D}^0_N$ be the subspace of ${}^{r+1}\mathscr{D}^0_{\mathscr{F}}$ consisting of those elements $\delta \in {}^{r+1}\mathscr{D}^0_{\mathscr{F}}$, $\delta = (\varphi, \psi)$, with $\varphi \in {}^{r+1}\Gamma(N^{1,0}) \subset {}^{r+1}\Gamma({}^cTM)$. The morphism

$$k\colon {}^{r+1}\Gamma(N^{1,0}) \times {}^{r+1}A \to {}^{r+1}\mathscr{D}^0_N$$

given by

$$k(\xi, a) = \text{derivation}\,(\xi, d\xi - d_F\xi + a),$$

is an isomorphism. Identifying ${}^{r+1}\Gamma(N^{1,0}) \times {}^{r+1}A$ with ${}^{r+1}\mathscr{D}^0_N \subset {}^{r+1}\mathscr{D}^0_{\mathscr{F}}$ through k, one can prove that the tangent map of ρ at $(\mathrm{id}_o, 0) \in {}^{r+1}\mathrm{Diff}^S \times {}^{r+1}A$ is given by

$$(d\rho)_{(\mathrm{id}_o, 0)}(h, \delta) = h + D\delta,$$

with $h \in H = T_oS$ and D the differential ${}^{r+1}\mathscr{D}^0_{\mathscr{F}} \to {}^r\mathscr{D}^1_{\mathscr{F}}$. Let

$$\tilde{\rho}\colon \tilde{H} \times {}^{r+1}\Gamma(N^{1,o}) \times {}^{r+1}A \to {}^r\mathscr{D}^1_{\mathscr{F}}$$

be a holomorphic extension of

$$\rho\colon S \times {}^{r+1}\Gamma(N^{1,o}) \times {}^{r+1}A \to {}^rJ \subset {}^r\mathscr{D}^1_{\mathscr{F}}$$

to the ambient space. Let $\bar{\rho}$ be the composition

$$\tilde{H} \times {}^{r+1}\Gamma(N^{1,o}) \times {}^{r+1}A \xrightarrow{\tilde{\rho}} {}^r\mathscr{D}^1_{\mathscr{F}} \xrightarrow{\pi_{\ker}} \operatorname{Ker} D \xrightarrow{\gamma} \Sigma,$$

where $\pi_{\ker}$ is the projection on $\operatorname{Ker} D$ given by the usual Hodge splitting of ${}^r\mathscr{D}^1_{\mathscr{F}}$ and γ is a parametrization of Σ by its tangent space $\operatorname{Ker} D$.

Notice that $\operatorname{Ker} D \cap {}^{r+1}\mathscr{D}^0_N$ is finite dimensional. Let E be a complement of $\operatorname{Ker} D \cap {}^{r+1}\mathscr{D}^0_N$ in ${}^{r+1}\mathscr{D}^0_N$. Let $\mathscr{G}$ be the subspace of ${}^{r+1}\mathscr{D}^0_{\mathscr{F}}$ consisting of those elements $\delta \in {}^{r+1}\mathscr{D}^0_{\mathscr{F}}$ of the form $\delta = (\varphi, d\varphi)$ with $\varphi \in {}^{r+1}\Gamma(F) \subset {}^{r+1}\Gamma({}^cTM)$. Notice that ${}^{r+1}\mathscr{D}^0_{\mathscr{F}} = \mathscr{G} \oplus {}^{r+1}\mathscr{D}^0_N$; for if $\delta \in {}^{r+1}\mathscr{D}^0_{\mathscr{F}}$, $\delta = (\varphi, \psi)$, $\varphi = \varphi^a Z_a + \varphi^{\bar{a}} \overline{Z_a} + \varphi^u \partial/\partial x^u$, and if we set $\varphi_N = \varphi^a Z_a$ and $\varphi_F = \varphi - \varphi_N$, then

$$(\varphi, \psi) = (\varphi_F, d\varphi_F) + (\varphi_N, \psi - d\varphi_F).$$

Observe that the pairs $(\varphi_F, d\varphi_F)$ and $(\varphi_N\ \psi - \varphi_F)$ define global derivations and that $(\varphi_F, d\varphi_F) \in \mathscr{G}$ and $(\varphi_N, \psi - d\varphi_F) \in \mathscr{D}^0_N$.

D maps E onto $\operatorname{Im} D$ isomorphically because $\mathscr{G}$ is contained in $\operatorname{Ker} D$. The restriction of $\bar{\rho}$ to $\tilde{H} \times E$ gives a holomorphic isomorphism $\bar{\rho}\colon \tilde{H} \times E \cong \Sigma$ in a neighbourhood of the origin (by the inverse function theorem). One has

$$S \times E \overset{\rho}{\subset} J \subset \Sigma \cong (\text{by } \bar{\rho}^{-1}) \cong \tilde{H} \times E$$

and the lemma of Douady [1] asserts that $S \times E \overset{\rho}{\cong} J$ in a neighbourhood of the origin. The holomorphic projection π

$$J \xrightarrow{\rho^{-1}} S \times E \to S$$

gives the versality of the family $\{\mathscr{F}_s\}_{s \in S}$ in a neighbourhood of 0 because if $\delta \in J$ corresponds to a transversely holomorphic foliation $\mathscr{F}_\delta$ then $\pi(\delta)$ corresponds (by definition of ρ) to a foliation obtained from $\mathscr{F}_\delta$ by a diffeomorphism.

6. Construction of another versal space adapted to basic derivations

Let S be the versal space of the preceding section. $T_o(S \cap {}^r\mathscr{D}_b^1)$ is not isomorphic to $H^1(\mathscr{D}_b^\star, D_b)$ in general. We would like to find another versal space S' fulfilling $T_o(S' \cap {}^r\mathscr{D}_b^1) \cong H^1(\mathscr{D}_b^\star, D_b) =$ the space of infinitesimal deformations of fixed differentiable type of $\mathscr{F}$.

Take a Hodge splitting compatible with basic derivations, as in theorem 2.5:

$${}^r\mathscr{D}_{\mathscr{F}}^1 = C' \oplus H' \oplus \operatorname{Im} D$$

(for this we must assume, although, that $D({}^r\mathscr{D}_b^1)$ is closed in ${}^{r-1}\mathscr{D}_b^2$). Here we write C' and H' instead of C and H, since H has a precise meaning in the preceding section. Define $\tilde{H}' = \Sigma \cap (C' \oplus H')$, $S' = J \cap (C' \oplus H')$. We have $T_o\tilde{H}' = H' \cong H = T_o\tilde{H}$. In the preceding section we had

$$\begin{array}{ccc} S \times E & \xrightarrow{\rho} & J \\ \downarrow & & \downarrow \\ \tilde{H} \times E & \xrightarrow{\tilde{\rho}} & \Sigma \end{array}$$

where horizontal arrows are isomorphisms and vertical arrows are injections. Since $\tilde{H}'$ is a submanifold of Σ and $\tilde{\rho}$ is a bioholomorphism then $\tilde{K} = \tilde{\rho}^{-1}(\tilde{H}')$ is a submanifold of $\tilde{H} \times E$ that can be described as the graph of a holomorphic map $\tilde{f}\colon \tilde{H} \to E$. Set

$$\tilde{\sigma}\colon \tilde{H} \to \tilde{K} \subset \tilde{H} \times E,$$
$$h \to \big(h, \tilde{f}(h)\big).$$

Set $K = \tilde{K} \cap (S \times E) =$ graph of $f =$ image of σ, where $f = \tilde{f}|_S$ and $\sigma = \tilde{\sigma}|_S$. Then

$$\sigma\colon S \to K \subset S \times E$$

is a biholomorphism. We have $\rho(K) \subset S' = J \cap (C' \oplus H')$. Therefore ρ induces a morphism $K \to S'$. Let λ be the composite

$$S \xrightarrow{\sigma(\cong)} K \xrightarrow{\rho} S'.$$

Let μ be the composite

$$S' \hookrightarrow J \xrightarrow{\rho^{-1}} S \times E \xrightarrow{\mathrm{pr}_1} S,$$

where pr_1 is the projection onto the first factor. One has $\mu \circ \lambda = \mathrm{id}_S$ and

$d_o\mu$ is injective. This implies that λ and μ are both isomorphisms. Let P be the composite

$$P\colon J \xrightarrow{\rho^{-1}} S \times E \xrightarrow{\mathrm{pr}_1} S \xrightarrow{\lambda} S'.$$

Then P is a projection. If $\delta \in J$ corresponds to a foliation $\mathscr{F}_\delta$ then $P(\delta)$ corresponds to a diffeomorphic foliation by virtue of the definitions of ρ and λ. The family $\{\beta(s')\}_{s' \in S'}$ is thus versal.

Notice that $S' \subset J \subset {}^r\mathscr{D}^1_{\mathscr{F}}$. Let n be the dimension of M. Choose a real number $k > n/2$. Sobolev's lemma says that the elements of S' are $C^{[r-k]}$-derivations when $r > k$, $[r-k]$ being the integer part of $r-k$. Then Newlander-Nirenberg's theorem for class $C^{[r-k]}$ tells us that each $\beta(s')$ with $s' \in S'$ is a $C^{[r-k]}$-transversely holomorphic foliation. Recall that in the preceding section the elements $s \in S$ were of class C^ω. Here we cannot say the same for the elements $s' \in S'$. In the preceding section techniques of elliptic operators were used and we are not able to use such techniques here.

7. Versality theorem for fixed differentiable type

The construction of the preceding section leads to the following definitions: Given a germ of analytic space (T, o') and a large enough positive r, a C^r-family of deformations of the initial foliation $\mathscr{F}$ parametrized by $(T, 0)$ is a family $\{\delta_t\}_{t \in T}$ of elements of $J \subset {}^r\mathscr{D}^1_{\mathscr{F}}$, depending holomorphically on t, with $\delta_o = 0$. Each $\beta(\delta_t)$ is then a $C^{[r-k]}$-transversely holomorphic foliation by virtue of Sobolev's lemma and Newlander-Nirenberg's theorem. As there is no Newlander-Nirenberg's theorem with parameters for class C^r we are not able to find a family of deformations of $\mathscr{F}$ in the usual sense of [3] corresponding to $\{\delta_t\}_{t \in T}$. A C^r-family $\{\delta_t\}_{t \in T}$ of deformations of $\mathscr{F}$ is said to be of *fixed differentiable type if* for each t there is a transversely holomorphic foliation $\mathscr{F}_t$ close to $\mathscr{F}$ with $\delta_t = \alpha(\mathscr{F}_t)$ and $\mathscr{F}_t = \mathscr{F}$ as real foliations. This is equivalent to saying that $\delta_t \in J \cap {}^r\mathscr{D}^1_b$, $\forall t$. So a C^r-family $\{\delta_t\}_{t \in T}$ of deformations of $\mathscr{F}$ of *fixed differentiable type* is a family of elements δ_t of $J \cap {}^r\mathscr{D}^1_b$ depending holomorphically on t, with $\delta_o = 0$. Two families $\{\delta_t\}_{t \in T}$ and $\{\delta'_t\}_{t \in T}$ of fixed differential type are said to be *equivalent* if there exists a family of diffeomorphisms $\{f_t\}$ preserving $\mathscr{F}$ as a real foliation such that $f_t^*(\beta(\delta_t)) = \beta(\delta'_t)$. With this notation we are able now to state the following.

THEOREM 7.1. *Let $\mathscr{F}$ be a C^∞-transversely holomorphic foliation on a compact C^∞-manifold M. Suppose that the following conditions hold*:

A *There exists a transversely projectable connection. This means that the tangent bundle of the model manifold Z admits a connection invariant under the γ_{ij} (notation of Introduction).*

B *There is a (sufficiently large) positive real number r such that* $D({}^r\mathscr{D}_b^1)$ *is closed in* ${}^{r-1}\mathscr{D}_b^2$ *(this condition holds, for example, when* $H^2(\mathscr{D}_b^\star, D_b)$ *is finite-dimensional).*

Then there is a germ of analytic space (S_b', o) *parametrizing a* C^r*-family of deformations of* $\mathscr{F}$ *of fixed differentiable type,* $\{\delta'_{s'}\}_{s' \in S_b'}$, *such that for any other* C^r*-family of deformations of* $\mathscr{F}$ *of fixed differentiable type,* $\{\delta_t'\}_{t\in T}$, *parametrized by a germ of analytic space* (T, o), *there is a holomorphic morphism* $f\colon (T, o) \to (S_b', o')$ *such that* $\{\delta_t'\}_{t\in T}$ *is equivalent to* $\{\delta_{f(t)}\}_{t\in T}$; *moreover the tangent map* $d_o f$ *is unique.*

Proof. We use the notation of the preceding section. Let $\pi_{\ker}$ denote the projection ${}^r\mathscr{D}^1_{\mathscr{F}} \to \operatorname{Ker} D$ given by the Hodge splitting

$$ {}^r\mathscr{D}^1_{\mathscr{F}} = C' \oplus \underbrace{H' \oplus \operatorname{Im} D}_{\operatorname{Ker} D} $$

Let γ be a parametrization of $\Sigma, \gamma\colon \operatorname{Ker} D \to \Sigma$, such that $\pi_{\ker} \circ \gamma = \mathrm{id}$ and $\gamma \circ \pi_{\ker}|_\Sigma = \mathrm{id}_\Sigma$. Define $\Sigma_b = \gamma(\operatorname{Ker} D_b) \subset \Sigma$. $\tilde{H}_b' = \gamma(H_b')$ with $H_b' = H' \cap {}^r\mathscr{D}_b^1$. Set $S_b' = S' \cap {}^r\mathscr{D}_b^1$ and $J_b = J \cap {}^r\mathscr{D}_b^1$. To continue the proof we need the following.

PROPOSITION 7.2. (a) $\Sigma \cap {}^r\mathscr{D}_b^1 \subset \Sigma_b$.
(b) $\tilde{H}' \cap {}^r\mathscr{D}_b^1 = \tilde{H}_b' \cap {}^r\mathscr{D}_b^1$.
(c) $\tilde{H}_b' \cap J_b = S_b'$.

Proof. (a) If $\delta \in \Sigma \cap {}^r\mathscr{D}_b^1$ then $\delta = c' + k$ with $k \in \operatorname{Ker} D$, $k = \pi_{\ker}(\delta)$. Since $\delta \in {}^r\mathscr{D}_b^1$ then $k \in \operatorname{Ker} D_b$, $c' \in C_b'$, $\gamma \circ \pi_{\ker}(\delta) = \delta$, $\delta = \gamma(k)$, so $\delta \in \Sigma_b$.

(b) is easy.

(c) One has $\tilde{H}_b' \cap J_b = \tilde{H}_b' \cap J \cap {}^r\mathscr{D}_b^1 =$ (by (b)) $= \tilde{H}' \cap {}^r\mathscr{D}_b^1 \cap J = S' \cap {}^r\mathscr{D}_b^1 = S_b'$. □

Continuation of proof of the theorem. Let g be the exponential $S \times N^{1,0} \to M$ used in Section 5 (defined only in a neighbourhood of $(0, 0)$). Let $\tilde{g}$ be a C^∞-extension of g to the ambient space:

$$ \tilde{g}\colon \tilde{H} \times N^{1,0} \to M. $$

As in Section 5 denote by ${}^{r+1}\mathrm{Diff}^{\tilde{H}}(M)$ the family of C^{r+1}-diffeomorphisms of the form $x \to \tilde{g}(h, \xi(x))$, where ξ is a C^{r+1}-section of $N^{1,0}$ close enough to the zero section, and $h \in \tilde{H}$. To simplify the notation we shall use $\tilde{\mathscr{M}}$ instead of ${}^{r+1}\mathrm{Diff}^{\tilde{H}}(M)$. $\tilde{\mathscr{M}}$ is a differentiable submanifold of

$\tilde{H} \times {}^{r+1}\mathrm{Diff}(M)$ with the following local chart

$$\begin{aligned} \tilde{H} \times {}^{r+1}\Gamma(N^{1,0}) &\xrightarrow{\phi} \tilde{\mathscr{M}}, \\ (h, \xi) &\longrightarrow (x \to \tilde{g}(h, \xi(x))). \end{aligned}$$

Denote by ${}^{r+1}\mathrm{Diff}(M, \mathscr{F})$ the set of C^{r+1}-diffeomorphisms preserving $\mathscr{F}$ as a real foliation. Hypothesis A implies that ${}^{r+1}\mathrm{Diff}(M, \mathscr{F})$ is a differentiable submanifold of ${}^{r+1}\mathrm{Diff}(M)$ (by using the exponential given by the transversely projectable connection). Then the intersection $\tilde{\mathscr{M}} \cap (\tilde{H} \times {}^{r+1}\mathrm{Diff}(M, \mathscr{F}))$ is a submanifold of $\tilde{H} \times {}^{r+1}\mathrm{Diff}(M)$ in a neighbourhood of $(0, \mathrm{id})$ because the following transversality condition holds:

$$T_{(0,\mathrm{id})}(\tilde{\mathscr{M}}) + T_{(0,\mathrm{id})}(\tilde{H} \times \mathrm{Diff}(M, \mathscr{F})) = T_{(0,\mathrm{id})}(\tilde{H} \times \mathrm{Diff}(M)).$$

Set $\mathscr{N} = \phi^{-1}(\tilde{\mathscr{M}} \cap (\tilde{H} \times {}^{r+1}\mathrm{Diff}(M, \mathscr{F})))$. Then $\mathscr{N}$ is a submanifold of $\tilde{H} \times {}^{r+1}\Gamma(N^{1,0})$ in a neighbourhood of $(0, 0)$.

Until now E (Sections 5 and 6) were a topological complement of $\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0$ in ${}^{r+1}\mathscr{D}_N^0$. Take this complement in the following way. Let

$$\left({}^{r+1}\mathscr{D}_N^0\right)_b = {}^{r+1}\mathscr{D}_N^0 \cap {}^{r+1}\mathscr{D}_b^0.$$

Then

$$\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0 \subset \left({}^{r+1}\mathscr{D}_N^0\right)_b.$$

Take a topological complement E_b of $\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0$ in $({}^{r+1}\mathscr{D}_N^0)_b$. Since $\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0$ is finite-dimensional, $E_b \oplus (\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0)$ is closed in ${}^{r+1}\mathscr{D}_N^0$. Let E' be a topological complement of $E_b \oplus (\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0)$ in ${}^{r+1}\mathscr{D}_N^0$. Take $E = E' \oplus E_b$. We have by construction:

$$\begin{aligned} {}^{r+1}\mathscr{D}_N^0 &= \left(\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0\right) \oplus E_b \oplus E', \\ \left({}^{r+1}\mathscr{D}_N^0\right)_b &= \left(\mathrm{Ker}\, D \cap {}^{r+1}\mathscr{D}_N^0\right) \oplus E_b. \end{aligned} \tag{2}$$

Recall that ${}^{r+1}A$ denotes the vector space of C^{r+1}-vector 1-forms mapping F to F. Denote now by ${}^{r+1}A_b$ the subspace of ${}^{r+1}A$ consisting of those vector 1-forms mapping F to F and L to L, where L is the subbundle of F of vectors tangent to the leaves of $\mathscr{F}$.

With these notations, $\mathscr{N} \times {}^{r+1}A_b$ and $\tilde{H} \times E$ are submanifolds of

$$\tilde{H} \times {}^{r+1}\Gamma(N^{1,0}) \times {}^{r+1}A.$$

The intersection of these submanifolds is a submanifold in a neighbourhood of the origin because

$$T_{(o,o,o)}\left(\mathscr{N}\times^{r+1}A_b\right) = H\times^{r+1}\Gamma_b(N^{1,0})\times^{r+1}A_b$$
$$\cong (\text{through id}\times k)\cong H\times\left({}^{r+1}\mathscr{D}^0_N\right)_b$$

and $T_{(o,o,o)}(\tilde{H}\times E) = H\times E$. So the sum of these two tangent spaces is (by virtue of (2)) the whole space $H\times^{r+1}\Gamma(N^{1,0})\times^{r+1}A\cong H\times^{r+1}\mathscr{D}^0_N$. Denote by $\tilde{\mathscr{E}}_b$ the intersection $(\mathscr{N}\times^{r+1}A_b)\cap(\tilde{H}\times E)$.

In the notation of the preceding section, set $\tilde{K}_b = \bar{\rho}^{-1}(\tilde{H}'_b)$, $K_b = \rho^{-1}(S'_b)$. Then $\tilde{K}_b$ is a submanifold of $\tilde{H}\times E$ contained in $\tilde{K}$. Let π be the projection

$$\tilde{H}\times^{r+1}\Gamma(N^{1,0})\times^{r+1}A\longrightarrow\tilde{H}\xrightarrow{\sigma}\tilde{K},$$

where the first arrow is projection on the first factor. Then π restricted to $\tilde{\mathscr{E}}_b$ is a projection, $\pi\colon\tilde{\mathscr{E}}_b\to\tilde{K}$. Set $\tilde{\mathscr{E}}_b|\tilde{K}_b = \pi^{-1}(\tilde{K}_b)$, which is a submanifold of $\tilde{\mathscr{E}}_b$:

$$\pi\colon\tilde{\mathscr{E}}_b|\tilde{K}_b\to\tilde{K}_b.$$

A simple computation shows that

$$T_o\left(\tilde{\mathscr{E}}_b|\tilde{K}_b\right)\cong T_o\left(\tilde{K}_b\right)+\left(\{0\}\times E_b\right).$$

By the inverse function theorem, $\bar{\rho}$ restricted to $\tilde{\mathscr{E}}_b|\tilde{K}_b$ gives a local isomorphism

$$\bar{\rho}\colon\tilde{\mathscr{E}}_b|\tilde{K}_b\cong\Sigma_b.$$

Let $\mathscr{P}$ be the composite

$$\Sigma_b\xrightarrow{\bar{\rho}^{-1}}\tilde{\mathscr{E}}_b|\tilde{K}_b\xrightarrow{\pi}\tilde{K}_b\xrightarrow{\bar{\rho}}\tilde{H}'_b.$$

$\mathscr{P}$ maps integrable derivations to integrable derivations. On the other hand, if $\delta\in J_b$ then $\mathscr{P}(\delta)$ is obtained from δ by a diffeomorphism of $\mathrm{Diff}(M,\mathscr{F})$ acting on δ followed by an automorphism of L. So $\mathscr{P}(\delta)\in J_b$. Therefore $\mathscr{P}$ gives a differentiable morphism

$$J_b\to\tilde{H}'_b\cap J_b = S'_b.$$

On the other hand, the composition of the above morphism and the inclusion $S'_b\subset S'$ coincides with the restriction to J_b of the morphism

$$J\xrightarrow{\rho^{-1}}S\times E\longrightarrow S\xrightarrow{\lambda}S',$$

which is holomorphic!! So the projection $\mathscr{P}\colon J_b \to S'_b$ is holomorphic. This is all we need in order to prove the versality of the family $\{s'\}_{s' \in S'_b}$; for if (T, o) is a germ of analytic space and $\{\delta_t\}_{t \in T}$ a family of deformations of fixed differentiable type, since $\delta_t \in J_b$, if f denotes the holomorphic morphism

$$f\colon T \to J_b \xrightarrow{\mathscr{P}} S'_b,$$
$$t \to \delta_t \longrightarrow \mathscr{P}(\delta_t)$$

then the family $\{f(t)\}_{t \in T}$ is equivalent to $\{\delta_t\}_{t \in T}$. □

Remark. An (apparently) more general definition of deformation of $\mathscr{F}$ of fixed differentiable type is the following: A C^r-family $\{\delta_t\}_{t \in T}$ of deformations of $\mathscr{F}$ parametrized by a germ of analytic space (T, o) is said to be of fixed differentiable type if there is a family of diffeomorphisms $\{h_t\}_{t \in T}$ of M parametrized by the same (T, o) such that for each t there is a transversely holomorphic foliation $\mathscr{F}_t$ with $\delta_t = \alpha(\mathscr{F}_t)$ and $h_t^*(\mathscr{F}_t) = \mathscr{F}$ as real foliations (instead of asking $\mathscr{F}_t = \mathscr{F}$). If one takes this definition, one is lead to the following definition of equivalence: Two families $\{\delta_t\}$ and $\{\delta'_t\}$ of deformations of $\mathscr{F}$ of fixed differentiable type are said to be equivalent if there is a family $\{f_t\}$ of diffeomorphisms of M with $f_t^*(\beta(\delta_t)) = \beta(\delta'_t)$. A Kuranishi theorem analogous to Theorem 7.1 with these definitions is immediate. It suffices to see that if $\{\delta_t\}$ is a family of deformations of $\mathscr{F}$ of fixed differentiable type according to the latter definition then $\{\delta'_t\}$ with $\delta'_t = h_t^*(\delta_t)$ is a family of deformations of fixed differentiable type according to the former definition. One then applies Theorem 7.1 to the family $\{\delta'_t\}$.

8. The Kodaira-Spencer map and the versality criterion for smooth Kuranishi spaces

Let $\{\delta_t\}_{t \in T}$ be a family of deformations of $\mathscr{F}$ of fixed differentiable type following the definition of Section 7. Given a derivation $\partial/\partial t \in T_o(T)$, let $\dot{\delta}_t = (\partial/\partial t)_o \delta_t$. As $\delta_t \in J \cap {}^r\mathscr{D}_b^1$, one has

$$[d, \delta_t] - \tfrac{1}{2}[\delta_t, \delta_t] = 0.$$

By derivation of this identity at $t = 0$ one has $[d, \dot{\delta}_t] = 0$; that is, $D\dot{\delta}_t = 0$. So $\dot{\delta}_t \in \operatorname{Ker} D_b$. The map

$$T_o(T) \to H^1(\mathscr{D}_b^\star, D_b),$$
$$\partial/\partial t \to \text{cohomology class of } \dot{\delta}_t$$

is called the Kodaira-Spencer map of the family of deformations $\{\delta_t\}$. An elementary result on analytic spaces (as in [3]) gives the following.

THEOREM 8.1 (Versality criterion). *Let* $\{\delta_t\}_{t\in T}$ *be a family of deformations of* $\mathscr{F}$ *of fixed differentiable type parametrized by a germ* (T, o) *of analytic space. Suppose the following conditions hold:*

VC1. (T, o) *is smooth.*

VC2. *The Kodaira-Spencer map associated to the family* $\{\delta_t\}$ *is an isomorphism.*

Then the family $\{\delta_t\}$ *is versal and* (T, o) *is (isomorphic to) the versal space of* $\mathscr{F}$.

REFERENCES

1. A. DOUADY, *Le problème des modules pour les variétés analytiques complexes*, Sém. Bourbaki 17e année, vol. 277, 1964/65.
2. A. EL KACIMI ALAOUI and M. NICOLAU, *Deformations des feuilletages transversalment holomorphes à type différentiable fixé*, Publicacions Matemàtiques, vol. 33 (1989), pp. 415–430.
3. J. GIRBAU, A. HAEFLIGER and D. SUNDARARAMAN, *On deformations of transversely holomorphic foliations*, J. Reine Angew. Math, vol. 345 (1983), pp. 122–147.
4. X. GÓMEZ-MONT, *Transversal holomorphic structures*, J. Differential Geom., vol. 15 (1980), pp. 161–185.
5. K. KODAIRA and D.C. SPENCER, *Multifoliate structures*, Ann. of Math, vol. 74 (1961), pp. 52–100.
6. K. KURANISHI, *On deformations of compact complex manifolds*, Les Presses de l'Université de Montréal, Montréal, 1971.

UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA, SPAIN