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KÄHLER CURVATURE IDENTITIES FOR TWISTOR SPACES[1]

JOHANN DAVIDOV, OLEG MUŠKAROV AND GUEO GRANTCHAROV

1. Introduction

In order to generalize some results of Kähler geometry, A. Gray [10] introduced and studied three classes of almost-Hermitian manifolds whose curvature tensor resembles that of Kähler manifolds. They are defined by the following curvature identities:

$$\begin{aligned}
\mathscr{AH}_1 &: R(E,F,G,H) = R(E,F,JG,JH)\\
\mathscr{AH}_2 &: R(E,F,G,H) = R(JE,JF,G,H)\\
&\qquad\qquad + R(JE,F,JG,H) + R(JE,F,G,JH)\\
\mathscr{AH}_3 &: R(E,F,G,H) = R(JE,JF,JG,JH)
\end{aligned}$$

(here J is the almost-complex structure).

These identities are very useful in the study of the action of the unitary group on the space of curvature tensors (cf. [16]) as well as for characterizing the Kähler manifolds in various classes of almost-Hermitian manifolds (for example, see [9], [10], [15], [17], [18]). By a result of S. Goldberg [9] (see also [10]) every compact almost-Kähler manifold of class $\mathscr{AH}_1$ is Kählerian and it is an open question raised by A. Gray [10, Th. 5.3] whether the same is true under the weaker condition $\mathscr{AH}_2$. We answer negatively to this question showing that the twistor space of a compact Einstein and self-dual 4-manifold with negative scalar curvature provides an example of a compact non-Kähler almost-Kähler manifold of class $\mathscr{AH}_2$.

Recall that the twistor space of an oriented Riemannian 4-manifold M is the (2-sphere) bundle $\mathscr{Z}$ on M whose fibre at any point $p \in M$ consists of all complex structures on T_pM compatible with the metric and the opposite

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orientation of M. The 6-manifold $\mathscr{Z}$ admits a 1-parameter family of Riemannian metrics h_t, $t > 0$, such that the natural projection $\pi : \mathscr{Z} \to M$ is a Riemannian submersion with totally geodesic fibres (for example, see [7], [8], [19]). These metrics are compatible with the almost-complex structures J_1 and J_2 on $\mathscr{Z}$ introduced, respectively, by Atiyah, Hitchin and Singer [1] and Eells-Salamon [6].

The purpose of this note is to investigate the twistor spaces as a source of examples of almost-Hermitian manifolds of the classes $\mathscr{AH}_i$. Our main result is the following.

THEOREM. *Let M be a (connected) oriented Riemannian 4-manifold with scalar curvature s. Then:*

(i) $(\mathscr{Z}, h_t, J_n) \in \mathscr{AH}_3$ *if and only if* $(\mathscr{Z}, h_t, J_n) \in \mathscr{AH}_2$ *if and only if M is Einstein and self-dual* ($n = 1$ *or* 2).

(ii) $(\mathscr{Z}, h_t, J_1) \in \mathscr{AH}_1$ *if and only if M is Einstein and self-dual with* $s = 0$ *or* $s = 12/t$.

(iii) $(\mathscr{Z}, h_t, J_2) \in \mathscr{AH}_1$ *if and only if M is Einstein and self-dual with* $s = 0$.

The proof is based on an explicit formula for the sectional curvature of $(\mathscr{Z}, h_t)$ in terms of the curvature of M [3]

REMARKS. Let M be an Einstein self-dual manifold with scalar curvature s.

(1) If $s < 0$ and $t = -12/s$, then $(\mathscr{Z}, h_t, J_2)$ is an almost-Kähler manifold [14] of class $\mathscr{AH}_2$. This manifold is not Kählerian since the almost-complex structure J_2 is never integrable [6]. So $(\mathscr{Z}, h_t, J_2)$ gives a negative answer to the Gray question.

Note that the only known examples of compact Einstein and self-dual manifolds with negative scalar curvature are compact quotients of the unit ball in $\mathbf{C}^2$ with the metric of constant negative curvature or the Bergman metric (for a description of the twistor space of the unit ball in $\mathbf{C}^2$, see [19]).

(2) Let $s = 0$. Then $(\mathscr{Z}, h_t, J_2)$, $t > 0$, is a quasi-Kähler manifold [14] of class $\mathscr{AH}_1$ which is not Kählerian. So the Goldberg result cannot be extended to quasi-Kähler manifolds. In the case when $M = \mathbf{R}^4$, the twistor space is $\mathscr{Z} = \mathbf{R}^4 \times S^2$ and we recover an example found by A. Gray [10].

By a result of Vaisman [18] any compact Hermitian surface of class $\mathscr{AH}_1$ is Kählerian. This is not true in higher dimensions since $(\mathscr{Z}, h_t, J_1)$ is a Hermitian manifold [1] of class $\mathscr{AH}_1$ which is not Kählerian [8].

Note that by a result of Hitchin [12] the only compact Einstein self-dual manifolds with $s = 0$ are the flat 4-tori, the $K3$-surfaces with the Calabi-Yau metric and the quotients of $K3$-surfaces by $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

(3) If $s > 0$ and $t = 12/s$, then $(\mathscr{Z}, h_t, J_1)$ not only belongs to the class $\mathscr{AH}_1$ but it is actually a Kähler manifold ([8]). In fact in this case $M = S^4$ or $M = \mathbf{CP}^2$ ([8], [13]) and $\mathscr{Z} = \mathbf{CP}^3$ or $\mathscr{Z} = SU(3)/S(U(1) \times U(1) \times U(1))$ with their standard Kähler structures.

2. Preliminaries

Let M be a (connected) oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by the formula

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = 1/2 \det\big(g(X_i, X_j)\big).$$

The Riemannian connection of M determines a connection on the vector bundle $\Lambda^2 TM$ (both denoted by ∇) and the respective curvatures are related by

$$R(X \wedge Y)(Z \wedge T) = R(X,Y)Z \wedge T + X \wedge R(Y,Z)T$$

for $X, Y, Z, T \in \chi(M)$; $\chi(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R of M we adopt the following definition $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$). The curvature operator $\mathscr{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathscr{R}(X \wedge Y), Z \wedge T) = g(R(X,Y)Z, T)$$

for all $X, Y, Z, T \in \chi(M)$. The Hodge star operator defines an endomorphism $*$ of $\Lambda^2 TM$ with $*^2 = \mathrm{Id}$. Hence

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

where $\Lambda^2_\pm TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1)-eigenvectors of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$\begin{aligned} s_1 &= E_1 \wedge E_2 - E_3 \wedge E_4, & \bar{s}_1 &= E_1 \wedge E_2 + E_3 \wedge E_4, \\ s_2 &= E_1 \wedge E_3 - E_4 \wedge E_2, & \bar{s}_2 &= E_1 \wedge E_3 + E_4 \wedge E_2, \\ s_3 &= E_1 \wedge E_4 - E_2 \wedge E_3, & \bar{s}_3 &= E_1 \wedge E_4 + E_2 \wedge E_3. \end{aligned} \tag{2.1}$$

Then (s_1, s_2, s_3) (resp. $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a local oriented orthonormal frame of $\Lambda^2_- TM$ (resp. $\Lambda^2_+ TM$). The block-decomposition of $\mathscr{R}$ with respect to the above splitting of $\Lambda^2 TM$ is

$$\mathscr{R} = \begin{bmatrix} s/6\,\mathrm{Id} + \mathscr{W}_+ & \mathscr{B} \\ {}^t\mathscr{B} & s/6\,\mathrm{Id} + \mathscr{W}_- \end{bmatrix}$$

where s is the scalar curvature of M; $s/6.\mathrm{Id} + \mathscr{B}$ and $\mathscr{W} = \mathscr{W}_+ + \mathscr{W}_-$ represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is said to be self-dual (anti-self-dual) if $\mathscr{W}_- = 0$ ($\mathscr{W}_+ = 0$). It is Einstein exactly when $\mathscr{B} = 0$.

The twistor space of M is the 2-sphere bundle $\pi : \mathscr{Z} \to M$ consisting of all unit vectors of $\Lambda^2_- TM$. The Riemannian connection ∇ of M gives rise to a splitting $T\mathscr{Z} = \mathscr{H} \oplus \mathscr{V}$ of the tangent bundle of $\mathscr{Z}$ into horizontal and vertical components. Further we consider the vertical space $\mathscr{V}_\sigma$ at $\sigma \in \mathscr{Z}$ as the orthogonal complement of σ in $\Lambda^2_- T_pM$, $p = \pi(\sigma)$.

Each point $\sigma \in \mathscr{Z}$ defines a complex structure K_σ on T_pM, $p = \pi(\sigma)$, by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y),\ X, Y \in T_pM. \tag{2.2}$$

This structure K_σ is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of K_σ.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_- T_pM$, $p \in M$. Then it is checked easily that

$$g(R(a)b, c) = -g(\mathscr{R}(b \times c), a) \tag{2.3}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_- T_pM$.

Following [1] and [6], define two almost-complex structures J_1 and J_2 on $\mathscr{Z}$ by

$$J_n V = (-1)^n \sigma \times V \text{ for } V \in \mathscr{V}_\sigma$$
$$J_n X^h_\sigma = (K_\sigma X)^h_\sigma \text{ for } X \in T_pM, p = \pi(\sigma).$$

It is well known [1] that J_1 is integrable (i.e., comes from a complex structure) iff M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable [6].

As in [8], define a Riemannian metric h_t on $\mathscr{Z}$ by

$$h_t = \pi^* g + t g^v$$

where $t > 0$, g is the metric of M and g^v is the restriction of the metric of $\Lambda^2 TM$ on the vertical distribution $\mathscr{V}$. The metric h_t is compatible with the almost-complex structures J_1 and J_2.

3. Proof of the theorem

It is easy to see that $\mathscr{AH}_1 \subset \mathscr{AH}_2 \subset \mathscr{AH}_3$. First we prove that if $(\mathscr{Z}, h_t, J_n) \in \mathscr{AH}_3$, then M is Einstein and self-dual. The natural projection $\pi : (\mathscr{Z}, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres. Applying the O'Neill formulas (for example, see [2]) one can obtain coordinate-free formulas for different curvatures of the twistor space $(\mathscr{Z}, h_t)$ in terms of the curvature of the base manifold M. Denote by R and R_t the Riemannian curvature tensors of (M, g) and its twistor space $(\mathscr{Z}, h_t)$, respectively. If

$E, F \in T_\sigma \mathscr{Z}$ and $X = \pi_* E, Y = \pi_* F$, $A = \mathscr{V}E$, $B = \mathscr{V}F$ where $\mathscr{V}$ means "vertical component", then (see [3])

$$
\begin{aligned}
R_t(E,F,E,F) &= R(X,Y,X,Y) - tg((\nabla_X \mathscr{R})(X \wedge Y), \sigma \times B) \\
&\quad + tg((\nabla_Y \mathscr{R})(X \wedge Y), \sigma \times A) \\
&\quad - 3tg(\mathscr{R}(\sigma), X \wedge Y) g(\sigma \times A, B) \\
&\quad - t^2 g(R(\sigma \times A)X, R(\sigma \times B)Y) \\
&\quad + t^2/4 \| R(\sigma \times B)X + R(\sigma \times A)Y \|^2 \\
&\quad - 3t/4 \| R(X \wedge Y)\sigma \|^2 + t\left(\|A\|^2 \|B\|^2 - g(A,B)^2\right).
\end{aligned}
\tag{3.1}
$$

Let $\sigma \in \mathscr{Z}$, $p = \pi(\sigma)$ and $X, Y \in T_pM$. Since $\mathscr{Z} \in \mathscr{AH}_3$, it follows from (3.1) that

$$
\begin{aligned}
&R(X,Y,X,Y) - R(K_\sigma X, K_\sigma Y, K_\sigma X, K_\sigma Y) \\
&\quad = 3t/4\left(\| R(X \wedge Y)\sigma \|^2 - \| R(K_\sigma X \wedge K_\sigma Y)\sigma \|^2\right)
\end{aligned}
\tag{3.2}
$$

where K_σ is the complex structure on T_pM determined by σ via (2.2). Fix $\tau \in \mathscr{Z}_p$, $\tau \perp \sigma$ and $E \in T_pM$, $\|E\| = 1$. Since $K_\sigma \circ K_\tau = -K_{\sigma \times \tau}$, $(E_1, E_2, E_3, E_4) = (E, K_\sigma E, K_\tau E, K_{\sigma \times \tau} E)$ is an oriented orthonormal basis of T_pM such that $\sigma = s_1$, $\tau = s_2$ and $\sigma \times \tau = s_3$ where s_1, s_2, s_3 are defined by (2.1). Since $R(X \wedge Y)\sigma$ is a vertical vector at σ, one has, by (2.3),

$$
\| R(X \wedge Y)\sigma \|^2 = g(\mathscr{R}(\tau), X \wedge Y)^2 + g(\mathscr{R}(\sigma \times \tau), X \wedge Y)^2
\tag{3.3}
$$

Let

$$
V_i = X \wedge E_i - K_\sigma X \wedge K_\sigma E_i; \quad \overline{V}_i = X \wedge E_i + K_\sigma X \wedge K_\sigma E_i, \quad i = 1, \dots, 4.
$$

Then (3.2) and (3.3) give

$$
\begin{aligned}
\frac{4}{3t} g(\mathscr{R}(V_i), \overline{V}_i) &= g(\mathscr{R}(\tau), V_i) g(\mathscr{R}(\tau), \overline{V}_i) \\
&\quad + g(\mathscr{R}(\sigma \times \tau), V_i) g(\mathscr{R}(\sigma \times \tau), \overline{V}_i), \quad i = 1, \dots, 4.
\end{aligned}
\tag{3.4}
$$

If $X = \sum_{i=1}^4 \lambda_i E_i$, then

$$
\begin{aligned}
V_1 &= -\lambda_3 s_2 - \lambda_4 s_3, & \overline{V}_1 &= -\lambda_2(\bar{s}_1 + s_1) - \lambda_3 \bar{s}_2 - \lambda_4 \bar{s}_3, \\
V_2 &= \lambda_3 s_3 - \lambda_4 s_2, & \overline{V}_2 &= \lambda_1(\bar{s}_1 + s_1) - \lambda_3 \bar{s}_3 + \lambda_4 \bar{s}_2, \\
V_3 &= \lambda_1 s_2 - \lambda_2 s_3, & \overline{V}_3 &= -\lambda_4(\bar{s}_1 - s_1) + \lambda_1 \bar{s}_2 + \lambda_2 \bar{s}_3, \\
V_4 &= \lambda_1 s_3 + \lambda_2 s_2, & \overline{V}_4 &= \lambda_3(\bar{s}_1 - s_1) - \lambda_2 \bar{s}_2 + \lambda_1 \bar{s}_3.
\end{aligned}
\tag{3.5}
$$

Substituting (3.5) into (3.4) and then varying $(\lambda_1, \ldots, \lambda_4)$ one sees that the identity (3.4) implies

$$(3.6)\quad \frac{4}{3t} g(\mathscr{R}(\tau), \bar{s}_k) = g(\mathscr{R}(\tau), \tau) g(\mathscr{R}(\tau), \bar{s}_k) + g(\mathscr{R}(\sigma \times \tau), \tau) g(\mathscr{R}(\sigma \times \tau), \bar{s}_k), \quad k=1,2,3.$$

It follows from the curvature identity defining the class $\mathscr{AH}_3$ that the Ricci tensor of $(\mathscr{Z}, h_t)$ is J_n-Hermitian, $n = 1$ or 2. Then, by [4, formula (3.1)] one has

$$\left(12 - ts(p) + 6tg(\mathscr{W}_-(\sigma), \sigma)\right)\mathscr{B}(\sigma) = 0$$

where s is the scalar curvature of M. This implies that either $\mathscr{B}_p \equiv 0$ or

$$12 - ts(p) + 6tg(\mathscr{W}_-(\sigma), \sigma) = 0 \quad \text{for all } \sigma \in \mathscr{Z}_p.$$

In the second case, $ts(p) = 12$ since $\operatorname{Trace} \mathscr{W}_- = 0$. Therefore $(\mathscr{W}_-)_p = 0$. Suppose that $\mathscr{B}_p \neq 0$. Then (3.6) becomes

$$(8 - ts(p)) g(\mathscr{B}(\tau), \bar{s}_k) = 0, \quad k = 1,2,3.$$

Hence $g(\mathscr{B}(\tau), \bar{s}_k) = 0$, $k = 1,2,3$, since $ts(p) = 12$. It follows that $\mathscr{B}_p = 0$, a contradiction. Thus $\mathscr{B} \equiv 0$ and the arguments in [4] show that $\mathscr{W}_- = 0$. In fact, consider $\mathscr{W}_-$ as a self-adjoint endomorphism of $\Lambda^2_- T_pM$, $p \in M$, and denote by μ_1, μ_2, μ_3 its eigenvalues. Since $\mathscr{R}(\sigma) = (s/6)\sigma + \mathscr{W}_-(\sigma)$ for $\sigma \in \Lambda^2_- T_pM$ and $\|\mathscr{R}(\cdot)\| = \text{const}$ on every fibre of $\mathscr{Z}$ [4, formula (3.2)] we have $|\mu_1 + s/6| = |\mu_2 + s/6| = |\mu_3 + s/6|$. Moreover, $\mu_1 + \mu_2 + \mu_3 = \operatorname{trace} \mathscr{W}_- = 0$. Hence either $\mu_1 = \mu_2 = \mu_3 = 0$ or $\{\mu_1, \mu_2, \mu_3\} = \{s/3, s/3, -2s/3\}$. It follows that either $\|\mathscr{W}_-\| \equiv 0$ or $\|\mathscr{W}_-\|^2 \equiv 2s^2/3$. So we have to consider only the case when $\|\mathscr{W}_-\|^2 \equiv 2s^2/3$. Since M is Einstein, $\delta\mathscr{W}_- = 0$ (for example, see [2, §16.5]) and Proposition 5.(iii) of [5] gives $\nabla\mathscr{W}_- = 0$. For every oriented Riemannian 4-manifold with $\delta\mathscr{W}_- = 0$, one has (see [2, §16.73])

$$\Delta\|\mathscr{W}_-\|^2 = -s\|\mathscr{W}_-\|^2 + 18 \det \mathscr{W}_- - 2\|\nabla\mathscr{W}_-\|^2$$

which implies in our case $s = 0$. Hence $\mathscr{W}_- = 0$.

Now let M be Einstein and self-dual. Then $\mathscr{R}|\Lambda^2_- TM = s/6 \operatorname{Id}$. Note also that

$$K_\sigma X \wedge K_\sigma Y - X \wedge Y \in \Lambda^2_- T_pM$$

for each $X, Y \in T_pM$, $p = \pi(\sigma)$. Using (3.1) and the well-known expression of the Riemannian curvature tensor by means of sectional curvatures (for example, see [11]), a direct computation shows that the twistor space $(\mathscr{Z}, h_t, J_n)$ is of class $\mathscr{AH}_2$.

Thus the statement (i) is proved.

To prove (ii) and (iii) assume first that $(\mathscr{Z}, h_t, J_n) \in \mathscr{AH}_1$. Since $\mathscr{AH}_1 \subset \mathscr{AH}_3$ it follows from (i) that the base manifold M is Einstein and self-dual. Using (3.1) one sees that the Kähler curvature identity $\mathscr{AH}_1$ holds for the horizontal vectors of $\mathscr{Z}$ iff

$$g(X \wedge Y, \mathscr{R}(Z \wedge T - K_\sigma Z \wedge K_\sigma T)) \\ - 8t(s/24)^2 g(X \wedge Y, Z \wedge T - K_\sigma Z \wedge K_\sigma T) = 0$$

for every $\sigma \in \mathscr{Z}$ and $X, Y, Z, T \in T_pM$, $p = \pi(\sigma)$. Since $Z \wedge T - K_\sigma Z \wedge K_\sigma T \in \Lambda^2_- T_pM$ and $\mathscr{R}|\Lambda^2_- T_pM = s/6\,\mathrm{Id}$, the above identity implies that either $s = 0$ or $st = 12$.

Now suppose M is Einstein and self-dual. Then a direct computation involving (3.1) shows that if $s = 0$, both almost-complex structures J_1 and J_2 satisfy the Kähler curvature identity $\mathscr{AH}_1$; if $st = 12$, this identity is satisfied only by J_1.

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Bulgarian Academy of Sciences
Sofia, Bulgaria

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Sofia, Bulgaria

University of Sofia
Sofia, Bulgaria