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# RATIOS OF VOLUMES AND FACTORIZATION THROUGH $\ell_\infty$

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## Introduction

The projection constant $\lambda(X)$ of a finite dimensional normed space $X$ is often difficult to compute, but it plays an important role in the classical and in the local theory of Banach spaces. We extend its definition in a natural way so as to include the class of quasi-normed spaces as well, and we present a new method for getting a lower bound for $\lambda(X)$ in terms of ratios of volumes. This bound allows us for example to to easily obtain the right asymptotic estimate for $\lambda(\ell_p^n)$ in the case $0 < p < 1$ and dispenses with the logarithmic factor in the estimate obtained by Peck [Pe] who used some involved probabilistic method. The method applies also for the Schatten classes $s_p^n$ $(0 < p \le 1)$ of operators on $\ell_2^n$.

Given a centrally symmetric body $K$ in $\mathbb{R}^n$ we can endow $\mathbb{R}^n$ with the quasi-norm defined by

$$\|x\| = \inf\{a > 0;\ x \in aK\},$$

and let $E = (\mathbb{R}^n, \|.\|)$ be the n-dimensional quasi-normed space with $K$ as its unit ball. Let $B = B_X$ be the unit ball of a given Banach space X. We define the volume ratio $\mathrm{vr}(E, X)$, also denoted by $\mathrm{vr}(K, B)$, to be

$$\mathrm{vr}(E, X) = \inf\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(T(B))}\right)^{1/n}$$

where the infimum ranges over all onto linear maps $T : X \to \mathbb{R}^n$ satisfying $T(B) \subset K$. We define the external volume ratio $\mathrm{evr}(E, X)$, also denoted by $\mathrm{evr}(K, B)$, to be

$$\mathrm{evr}(E, X) = \inf\left(\frac{\mathrm{vol}_n(T(B))}{\mathrm{vol}_n(K)}\right)^{1/n}$$

where the infimum ranges over all onto linear maps $T : X \to \mathbb{R}^n$ such that $T(B) \supset K$. For $0 < p \le \infty$, let $\ell_n^p$ be the space $\mathbb{R}^n$ equipped with the quasi-norm

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

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and let $B_p^n$ be its unit ball:

$$B_p^n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \,;\ \sum_{i=1}^n |x_i|^p \le 1 \right\} .$$

Let $Q_n = [-1, 1]^n = B_\infty^n$ be the unit cube of $\mathbb{R}^n$ and $\mathbf{C}_n = B_1^n$ be its polar body:

$$\mathbf{C}_n = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n;\ \sum_{i=1}^n |x_i| \le 1 \right\} .$$

The ratio $\operatorname{evr}(K, B_\infty^n) = \operatorname{evr}(K, Q_n)$, known also as the *cubic ratio* of $K$, was studied by various authors (see [B1], [Ge], [PS]) in relation to the classical *volume ratio* $\operatorname{vr}(K, B_n^2)$. The *zonoid ratio*, which in our notation is $\operatorname{vr}(K, B_{\ell_\infty})$, was also studied in [B1].

We prove that if $K$ is the unit ball of a quasi-normed space $X$, then

$$\operatorname{evr}(K, Q_n) \operatorname{vr}(K, B_{\ell_\infty}) \le \lambda(X);$$

geometrically this means that there exist a parallelotope $P$ and a zonoid $Z$ such that $Z \subset K \subset P$, and $(\frac{\operatorname{vol}_n(P)}{\operatorname{vol}_n(Z)})^{1/n} \le \lambda(X)$, and we study various relations among the above mentioned quantities and other parameters associated with centrally symmetric, and not necessarily convex, bodies $K$.

## Notation

If $I$ is a finite set, we shall denote by $|I|$ its cardinality, and by $\mathcal{M}_{n,m}$ the set of all matrices $A = [a_{ij}]_{i=1,\dots,n,\, j=1,\dots,m}$ with real entries consisting of $n$ rows and $m$ columns. For $I \subset \{1, 2, \dots, n\}$ and $J \subset \{1, 2, \dots, m\}$, let $A_{IJ} = [a_{ij}]_{i\in I,\, j\in J}$. We will make use of the following Cauchy-Binet formula: If $A \in \mathcal{M}_{n,m}$ and $B \in \mathcal{M}_{m,n}$, and if $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$, $1 \le n \le m$, then

$$\det(AB) = \sum_{I \subset M,\, |I| = n} \det(A_{NI}) \det(B_{IN}) .$$

Let $v_n = \pi^{\frac{n}{2}} / \Gamma(1 + \frac{n}{2})$ denote the volume of the Euclidean ball $B_2^n$ of $\mathbb{R}^n$; then $v_n^{1/n} \sim \sqrt{\frac{2\pi e}{n}}$. If $K$ is a centrally symmetric body (not necessarily convex) in $\mathbb{R}^n$, we note that by our definition

$$\operatorname{vr}(K, \ell_1^n) = \min\left\{ \left(\frac{\operatorname{vol}_n(K)}{\operatorname{vol}_n(\mathbf{C})}\right)^{1/n} ;\ \mathbf{C} \subset K \text{ is the symmetric convex hull of } n \text{ points} \right\},$$

and

$$\operatorname{vr}(K, B_{\ell_\infty}) = \min\left\{ \left(\frac{\operatorname{vol}_n(K)}{\operatorname{vol}_n(Z)}\right)^{1/n} ;\ Z \subset K \text{ is a zonoid} \right\} .$$

The cubic ratio of $K$ is

$$\mathrm{evr}(K, Q_n) = \min\left\{\left(\frac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(K)}\right)^{1/n} ;\ P \text{ is a parallelotope containing } K\right\},$$

and the classical volume ratio of $K$ is

$$\mathrm{vr}(K, B_2^n) = \min\left\{\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(D)}\right)^{1/n} ;\ D \subset K \text{ is an ellipsoid}\right\}.$$

Since $B_2^n$ is a zonoid and $\ell_1^n$ has uniformly bounded volume ratio, it is clear that

$$\mathrm{vr}(E, \ell_\infty) \le \mathrm{vr}(E, \ell_2^n) \le \mathrm{vr}(E, \ell_1^n)\, \mathrm{vr}(\ell_1^n, \ell_2^n) \sim \sqrt{\frac{2e}{\pi}}\, \mathrm{vr}(E, \ell_1^n).$$

## Ratios of volumes and factorization through $l_\infty$

The following proposition is essentially known [B1], [Ge], [PS].

PROPOSITION 1. *Let $K$ be a centrally symmetric convex body in $\mathbb{R}^n$ and $K^o$ be its polar body with respect to the ordinary scalar product denoted by $<,>$. Suppose that $u_i \in R^n$, $< u_i, u_i >= 1$ and $c_i > 0$, $i = 1, \ldots, m$ satisfy*

$$\sum_{i=1}^m c_i < u_i, x > u_i = x$$

*for every $x \in \mathbb{R}^n$. Then:*

(i) *If $u_i \in K^o$, $i = 1 \ldots, m$, there exists a parallelotope $P$ such that $K \subset P$ and $(\mathrm{vol}_n(P))^{1/n} \le \sqrt{e}(\mathrm{vol}_n(Q_n))^{1/n} = 2\sqrt{e}$.*
(ii) *If $u_i \in K$, $i = 1 \ldots, m$, there exists a cross-polytope $\mathbf{C}$ such that $C \subset K$ and $(\mathrm{vol}_n(\mathbf{C}))^{1/n} \ge \frac{1}{\sqrt{e}}(\mathrm{vol}_n(\mathbf{C}_n))^{1/n} \sim 2\frac{\sqrt{e}}{n}$.*

*Proof.* If $C \in \mathcal{M}_{n,m}$ is the matrix whose columns are the coordinates of the vectors $\sqrt{c_i}\, u_i$, $1 \le i \le m$, in the canonical basis of $\mathbb{R}^n$, we have $CC^* = I_n$, where $I_n$ denotes the identity on $\mathbb{R}^n$ and thus $\sum_{i=1}^m c_i = n$. It follows from the Cauchy-Binet identity that

$$\begin{aligned} 1 &= \sum_{I\subset\{1,\ldots,m\},\,|I|=n} \left(\prod_{i\in I} c_i\right) (\det(u_i)_{i\in I})^2 \\ &\le \binom{m}{n} \max_{I\subset\{1,\ldots,m\},|I|=n} (\det(u_i)_{i\in I})^2 \left(\frac{\sum_{|I|=n}\prod_{i\in I} c_i}{\binom{m}{n}}\right) \end{aligned}$$

Now, since $\sum_{i=1}^m c_i = n$, by Newton's inequality we get

$$1 \le \binom{m}{n}\left(\frac{n}{m}\right)^n \max_{I\subset\{1,\dots,m\},\ |I|=n} (\det(u_i)_{i\in I})^2 .$$

It follows that

$$\max_{I\subset\{1,\dots,m\},|I|=n} |\det_{i\in I}(u_i)|^{1/n} \ge \frac{1}{\sqrt{e}} .$$

In case (i) for some $I \subset \{1,\dots,m\}$, $|I| = n$, the parallelotope $P = \{x \in \mathbb{R}^n;\ |<x, u_i>| \le 1$ for every $i \in I\}$ satisfies the required properties. Case (ii) follows from (i) by replacing $K^o$ with $K$ and taking $\mathbf{C} = P^o$. □

The following results relate $\mathrm{vr}(K, B_2^n)$ to $\mathrm{evr}(K, Q_n)$. It is a direct consequence of the preceding proposition.

COROLLARY 2. ([B2], [Ge], [PS]) *Let $K$ be a convex symmetric convex body. Then*

$$\mathrm{vr}(Q_n, B_2^n) \le \mathrm{evr}(K, Q_n)\,\mathrm{vr}(K, B_2^n) \le \sqrt{e}\ \mathrm{vr}(Q_n, B_2^n)$$

*where* $\mathrm{vr}(Q_n, B_2^n) = \frac{2}{v_n^{1/n}} \sim \sqrt{\frac{2n}{\pi e}}$.

*Proof.* The left-hand side inequality follows from the definition. For the right hand-side, we may suppose that the Euclidean ball is the maximal volume ellipsoid inside $K$, that is the John ellipsoid of $K$; then by [J], both assumptions (i), (ii) of Proposition 1 are satisfied. The result follows. □

The next corollary is an improvement of an estimate due independently to many authors ([BF], [BP], [C], [Gl], and B. Maurey in [Pi1]).

COROLLARY 3. *There exists a constant $c > 0$ such that if $x_1, \dots, x_m \in \mathbb{R}^n$, $m \ge n$ and $K = \mathrm{conv}(\pm x_i, 1 \le i \le m)$, then*

$$(\mathrm{vol}_n(K))^{1/n} \le c\sqrt{\ln\left(\frac{2m}{n}\right)} \max_{I\subset\{1,\dots,m\},|I|=n} (\mathrm{vol}_n(\mathrm{conv}(\pm x_i, i \in I)))^{1/n} .$$

*Proof.* Let $A \in \mathcal{M}_{n,n}$ be a matrix such that the minimal volume ellipsoid containing the body $K' = AK$ is the Euclidean ball. Then again the assumptions of Proposition 1,(ii) are satisfied. Thus there exists a cross-polytope $\mathbf{C} \subset K'$ such that

$(\mathrm{vol}_n(\mathbf{C}))^{1/n} \geq \frac{1}{\sqrt{e}}((\mathrm{vol}_n(\mathbf{C}_n))^{1/n}$. It follows that

$$\begin{aligned}
&\left(\frac{\mathrm{vol}_n(K)}{\max_{y_1,\dots,y_n\in K}\mathrm{vol}_n\left(\mathrm{conv}(\pm y_i,\ 1\leq i\leq n)\right)}\right)^{1/n}\\
&\quad=\left(\frac{\mathrm{vol}_n(K')}{\max_{z_1,\dots,z_n\in K'}\mathrm{vol}_n\left(\mathrm{conv}(\pm z_i,\ 1\leq i\leq n)\right)}\right)^{1/n}\\
&\quad\leq\left(\frac{\mathrm{vol}_n(K')}{\mathrm{vol}_n(\mathbf{C})}\right)^{1/n}\leq\sqrt{e}\left(\frac{\mathrm{vol}_n(K')}{\mathrm{vol}_n(\mathbf{C}_n)}\right)^{1/n}.
\end{aligned}$$

But since $K' \subset B_2^n$ is the convex hull of $Ax_1, \dots, Ax_m$, it follows from [BP], [BF], [C], [Gl] or [Pi1] that for some constant $d > 0$, independent of $n$ and $m$, we have

$$\left(\mathrm{vol}_n(K')\right)^{1/n} \leq d\sqrt{\ln\left(\frac{2m}{n}\right)}\,(\mathrm{vol}_n(\mathbf{C}_n))^{1/n}.$$

Combining the preceding inequalities, we get our estimate. □

*Remark.* If the convex body $K$ is not supposed to be centrally symmetric, then Proposition 1 can be generalized in both cases if we replace the parallelotope $P$ and the cross-polytope $\mathbf{C}$ by a simplex, and $Q_n$ and $\mathbf{C}_n$ by the regular simplices circumscribed to $B_2^n$ and inscribed in $B_2^n$. Observe also that Corollary 3 can be generalized as follows: if $K = \mathrm{conv}(x_1, \dots, x_m)$, then

$$(\mathrm{vol}_n(K))^{1/n} \leq d\sqrt{\ln\left(\frac{2m}{n}\right)}\ \max\{(\mathrm{vol}_n(\Delta))^{1/n}\ ;\ \Delta \text{ simplex },\ \Delta\subset K\},$$

for some constant $d > 0$ independent of $n$, $m > n+1$ and $x_1, \dots, x_m \in \mathbb{R}^n$.

If $E$ is a subspace of $\mathbb{R}^n$, then $P_E$ will denote the orthogonal projection onto $E$.

LEMMA 4. *Let $B$ be a symmetric body in $\mathbb{R}^n$ (not necessarily convex). Let $1 \leq k \leq n \leq m$ and $T = VU$, with $T \in \mathcal{M}_{n,n}$, $V \in \mathcal{M}_{n,m}$ and $U \in \mathcal{M}_{m,n}$,* $\mathrm{rank}(T) = k$ *and $UB \subset \|U\|Q_m$, where $\|U\| > 0$. Then*

$$\begin{aligned}
\frac{\mathrm{vol}_k(TB)}{\mathrm{vol}_k(P_{\ker(T)^{\perp}}B)} &\leq \frac{k!}{4^k}\sqrt{\binom{n}{k}}\,\lambda_k(B^o)\|U\|^k\max_{\dim(E)=k}\mathrm{vol}_k(P_E VQ_m)\\
&= \sqrt{\binom{n}{k}}\,\|U\|^k\max_{\dim(E)=k}\mathrm{vol}_k(P_E VQ_m)\\
&\quad\times\left(\min_{\dim(E)=k}\left(\mathrm{evr}(P_E B, Q_k)^k\,\mathrm{vol}_k(P_E B)\right)\right)^{-1}.
\end{aligned}$$

*where $B^o = \{y \in \mathbb{R}^n;\ \langle x, y\rangle \leq 1,$ for all $x \in B\}$ and for a subset $C$ of $\mathbb{R}^n$, $\lambda_k(C) =$:* $\max\{\mathrm{vol}_k(\mathrm{conv}(\pm x_1, \dots, \pm x_k));\ x_1, \dots, x_k \in C\}$.

*Proof.* By standard linear algebra methods, we have

$$\mathrm{vol}_k(TB) = \left(\sum_{|I|=|J|=k} (\det(T_{IJ}))^2\right)^{1/2} \mathrm{vol}_k(P_{\ker(T)^\perp} B).$$

For $I, J \subset \{1, \dots, n\}$ with $|I| = |J| = k$ define $t_{IJ} = \det(T_{IJ})$ and similarly for $u_{IJ}$ and $v_{IJ}$. Then

$$t_{IJ} = \sum_{K\subset\{1,\dots,m\},|K|=k} v_{IK} u_{KJ},$$

so that

$$\begin{aligned}\sum_{|I|=|J|=k} t_{IJ}^2 &= \sum_{|I|=|J|=k} \left(\sum_{|K|=|L|=k} v_{IK}u_{KJ}v_{IL}u_{LJ}\right)\\ &= \sum_I \left(\sum_{K,L} v_{IK}v_{IL}\left(\sum_J u_{KJ}u_{LJ}\right)\right)\\ &\le \left(\max_K \sum_J u_{KJ}^2\right)\left(\sum_I\left(\sum_K |v_{IK}|\right)^2\right)\end{aligned}$$

For $I \subset \{1, \dots, n\}$, $\mathrm{card}(I) = k$, let $\Pi_I : \mathbb{R}^n \to \mathbb{R}^I$ denote the orthogonal projection; we have

$$\sum_{|K|=k} |v_{IK}| = 2^{-k}\, \mathrm{vol}_k(\Pi_I V Q_m).$$

Indeed, $Q_m = \sum_{j=1}^m [-e_j, e_j]$. Hence $\Pi_I V Q_m = \sum_{j=1}^m [-\Pi_I v_j, \Pi_I v_j] \subset \Pi_I(\mathbb{R}^n) = \mathbb{R}^k$, where $\{v_j\}_{j=1}^m$ denote the columns of the matrix $V$; and now it is well known [Mc] that if $Z = \sum_{j=1}^m [-z_j, z_j]$ is a zonotope in $\mathbb{R}^k$ then

$$\mathrm{vol}_k(Z) = 2^k \sum_{J\subset\{1,\dots,m\},|J|=k} |\det(z_j;\, j \in J)|.$$

It follows that

$$\sum_I \left(\sum_K |v_{IK}|\right)^2 \le \binom{n}{k}\left(2^{-k} \max_{\dim(E)=k} \mathrm{vol}_k(P_E V Q_m)\right)^2.$$

Observe also that since $U(B) \subset \|U\|\, Q_m$, the rows $U_1, \dots, U_m$ of the matrix $U$ are vectors of $\mathbb{R}^n$ which satisfy $U_i \in \|U\| B^o$ for $1 \le i \le m$. Then, for $K \subset \{1, \dots, m\}$, $|K| = k$, we have

$$\left(\sum_{J\subset\{1,\dots,n\},|J|=k} u_{KJ}^2\right)^{1/2} = \frac{k!}{2^k}\, \mathrm{vol}_k(\mathrm{conv}(\pm U_i;\, i \in K)).$$

This may require an explanation: Let $\{z_i\}_{i=1}^k$ be $k$ vectors in $\mathbb{R}^n$ and $C = \operatorname{conv}(\pm z_i;\ 1 \le i \le k)$. Denote by $Z \in \mathcal{M}_{k,n}$ the matrix with $z_i$'s as rows. Then there is an orthogonal matrix $\Lambda \in \mathcal{M}_{n,n}$ such that $Z\Lambda = [W, 0]$, where $W \in \mathcal{M}_{k,k}$ is a the matrix with rows $\{w_i\}_{i=1}^k$ and 0 denotes the zero matrix in $\mathcal{M}_{k,n-k}$. Obviously, $WW^* = ZZ^*$, and we obtain

$$\begin{aligned}\operatorname{vol}_k(C) &= \operatorname{vol}_k(\operatorname{conv}(\pm w_i,\ 1 \le i \le k)) = \operatorname{vol}_k(W(\mathbf{C}_k)) = \frac{2^k}{k!}|\det(W)| \\ &= \frac{2^k}{k!}\sqrt{\det(WW^*)} = \frac{2^k}{k!}\sqrt{\det(ZZ^*)} \\ &= \frac{2^k}{k!}\left(\sum_{J\subset\{1,\dots,n\},\,|J|=k} (\det(Z_{KJ}))^2\right)^{1/2},\end{aligned}$$

where $K = \{1, \dots, k\}$.
It follows that

$$\left(\max_K \sum_J u_{KJ}^2\right)^{1/2} \le \frac{k!}{2^k}\|U\|^k \lambda_k(B^o).$$

Finally, by duality we have

$$\frac{k!}{2^k}\lambda_k(B^o) = 2^k\left(\min_{\dim(E)=k}\left(\operatorname{evr}(P_E B,\, Q_k)^k \operatorname{vol}_k(P_E B)\right)\right)^{-1}.$$

The lemma follows. □

LEMMA 5. *Let $K$ be a symmetric body in $\mathbb{R}^n$ (not necessarily convex). Let $1 \le n \le m$ and $T = VU$, with $T \in \mathcal{M}_{n,n}$, $V \in \mathcal{M}_{n,m}$ and $U \in \mathcal{M}_{m,n}$, with* $\operatorname{rank}(T) = n$ *and $U(K) \subset \|U\| Q_m$, where $\|U\| > 0$. Then*

$$\operatorname{evr}(K, Q_n)\,|\det(T)|^{1/n} \le \|U\| \left(\frac{\operatorname{vol}_n(V(Q_m))}{\operatorname{vol}_n(K)}\right)^{1/n}.$$

*Proof.* Apply Lemma 4 with $k = n$. □

Now let $E$ and $F$ be two $n$-dimensional quasi-normed spaces with unit balls $B_E$ and $B_F$ respectively. For $T \in L(E, F)$ define

$$\gamma_\infty(T) = \inf\{\|U\|\|V\|\}$$

where the infimum is taken over all the factorizations $T = VU$, $U \in L(E, \ell_\infty)$, $V \in L(\ell_\infty, F)$, and if $B_\infty$ denotes the unit ball of $\ell_\infty$, $\|U\| = \inf\{a > 0;\ U(B_E) \subset aB_\infty\}$ and $\|V\| = \inf\{b > 0;\ V(B_\infty) \subset bB_F\}$.

For a centrally symmetric body $K$ in $\mathbb{R}^n$, if $E$ is the quasi-normed space such that $B_E = K$, we define the projection constant of $E$ or, of $K$, to be

$$\lambda(K) = \lambda(E) = \gamma_\infty(I)$$

where $I : E \to E$ denotes the identity.

THEOREM 6. *Let $T \in L(E, F)$, where $E = (\mathbb{R}^n, \|\ \|_E)$, and $F = (\mathbb{R}^n, \|\ \|_F)$ are $n$-dimensional quasi-normed spaces. Then,*

$$\operatorname{evr}(E, \ell_\infty^n)\ \operatorname{vr}(F, \ell_\infty)|\det T|^{1/n} \le \gamma_\infty(T)\left(\frac{\operatorname{vol}_n(B_F)}{\operatorname{vol}_n(B_E)}\right)^{1/n}.$$

*Proof.* By Lemma 5, if we let $K = B_E$ then for any factorization $T = VU$ through $\ell_\infty^m$,

$$\operatorname{evr}(E, \ell_\infty^n)\,|\det(T)|^{1/n} \le \|U\|\left(\frac{\operatorname{vol}_n(V(Q_m))}{\operatorname{vol}_n(B_E)}\right)^{1/n}.$$

Thus, if $Z$ is the zonoid $\|V\|^{-1}V(Q_m)$, then $Z \subset B_F$. □

COROLLARY 7. *If $E$ is an $n$- dimensional quasi-normed space, with unit ball $B_E$, then*

$$\begin{aligned}\lambda(E) &\ge \operatorname{evr}(E, \ell_\infty^n)\ \operatorname{vr}(E, \ell_\infty)\\ &= \max\left\{\left(\frac{\operatorname{vol}_n(P)}{\operatorname{vol}_n(Z)}\right)^{1/n};\ Z \text{ zonoid, } P \text{ parallelotope, } Z \subset B_E \subset P\right\}.\end{aligned}$$

*Remarks.* (1) It is easy to prove that, under the hypothesis of the preceding corollary, we have

$$\operatorname{evr}(E, \ell_\infty^n) \le \inf\left\{\|U\|\left(\frac{\operatorname{vol}_n(V(B_\infty))}{\operatorname{vol}_n(B_E)}\right)^{1/n}\right\}$$

where the infimum is taken over all linear operators $U : X \to \ell_\infty$ and $V : \ell_\infty \to X$ such that $VU$ is the identity on $\mathbb{R}^n$.

(2) By Lemma 5, if $E$, $F$ are $n$-dimensional, then

$$\sup_{T:E\to F, T\neq 0}\frac{|\det(T)|^{1/n}}{\gamma_\infty(T)} \le \frac{1}{\operatorname{evr}(E, \ell_\infty^n)\ \operatorname{vr}(F, \ell_\infty)}\left(\frac{\operatorname{vol}_n(B_F)}{\operatorname{vol}_n(B_E)}\right)^{1/n}.$$

In particular it follows that

$$\operatorname{evr}(E, \ell_\infty^n)\operatorname{evr}(F, \ell_\infty^n)\operatorname{vr}(E, \ell_\infty)\ \operatorname{vr}(F, \ell_\infty) \le \inf_{T\in L(E,F)}\left\{\gamma_\infty(T)\gamma_\infty(T^{-1})\right\}.$$

Let us recall now some definitions; if $X$ is a normed space, we define the following quantities:

(1) The Gordon-Lewis constant $gl_2(X)$ (according to G. Pisier [Pi1]) is the least constant $C$ such that for any operator $T \in L(X, l_2)$,

$$\gamma_1(T) \leq C\pi_1(T).$$

(2) The Gordon-Lewis constant $gl(X)$ of $X$ (see [GL]), is the least constant $C$ such that for any normed space Y and any operator $T \in L(X, Y)$,

$$\gamma_1(T) \leq C\pi_1(T).$$

(3) The weak Gordon-Lewis constant $wrgl_2(X)$ of an $n$-dimensional space $X$ (according to K. Ball [B1], a different definition was introduced in [Pi3]) is the least constant K such that for any $T \in L(X, l_2)$

$$(\mathrm{vol}_n\,(T(B_X)))^{1/n} \leq \frac{2K}{n}\pi_1(T).$$

(For the definition of the ideal norm $\pi_p(T)$ the reader may refer to the books [Kö], [Pie], [Pi2], [Tj].) It was proved in [B2] that for some constant $c > 0$, independent of $X$,

$$(*) \qquad wrgl_2(X) \leq c\, \min\{gl_2(X), \mathrm{vr}(X, B_2^{\dim(X)})\}.$$

and moreover, if $X$ is finite dimensional,

$$wrgl_2(X) \sim \mathrm{vr}(X, \ell_\infty)$$

in the sense that there exist absolute constants $c_1$ and $c_2 > 0$ such that

$$(**) \qquad c_1\, \mathrm{vr}(X, \ell_\infty) \leq wrgl_2(X) \leq c_2\, \mathrm{vr}(X, \ell_\infty).$$

To better see how these numbers are related, let us also define the local unconditional constant of $X$, $\chi_u(X)$ ([GL]): this is the least constant $C$ such that for any finite-dimensional subspace $F \subset X$ there exists a Banach space $U$ with a finite unconditional basis constant $\chi(U)$, and operators $A \in L(F, U)$ and $B \in L(U, X)$ with $BA = i_F$ (the inclusion of F into X), and satisfying

$$\|A\|\|B\|\chi(U) \leq C.$$

Of course if E is finite-dimensional, $\chi_u(E) = \chi_u(E^*)$, and clearly subspaces of $L_1$, and quotients of $L_\infty$, have finite $gl_2$ constants. We see then, by results of [GL] and inequalities $(*)$ and $(**)$ that for some absolute constants $c$ and $d > 0$, we have

$$(***) \qquad c \leq d\, wrgl_2(X) \leq gl_2(X) \leq gl(X) \leq \chi_u(X) \leq \chi(X) \leq \sqrt{\dim(X)}.$$

The following result improves inequality $(*)$:

COROLLARY 8. *There is an absolute positive constant c such that for every finite dimensional normed space $F$, if $F^*$ denotes the dual of $F$, we have*

$$\mathrm{vr}(F,\ell_\infty)\ \mathrm{vr}(F^*,\ell_\infty) \sim wrgl_2(F)\ wrgl_2(F^*) \leq c\ \min(gl_2(F), gl_2(F^*)).$$

*Proof.* Take $E$ in Theorem 6 to be the space $l_2^n$, then $\mathrm{evr}(E,\ell_\infty^n) = \left(\frac{2^n}{\mathrm{vol}_n(B_E)}\right)^{1/n} \sim \sqrt{n}$. Now, multiplying the inequality of Theorem 6 by $(\mathrm{vol}_n(B_{F^*}))^{1/n}$, and using Santalò's inequality, we have that

$$wrgl_2(F)\left(\mathrm{vol}_n(T^*B_{F^*})\right)^{1/n} \leq \frac{c}{n}\gamma_\infty(T).$$

The inequality

$$wrgl_2(F)\,wrgl_2(F^*) \leq c\ gl_2(F)$$

follows now immediately from the fact that $T^* \in L(F^*, l_2^n)$, and that $\gamma_\infty(T) = \gamma_1(T^*) \leq gl_2(F^*)\pi_1(T^*)$, and the definition of $wrgl_2(F^*)$. For the second inequality replace $F$ by $F^*$. We use then (**). □

*Remarks.* (a) In particular, since $gl_2(F^*) \leq \sqrt{\dim(F)}$ and $\lambda(F^*) \leq \sqrt{\dim(F)}$, it follows from Corollaries 7 and 8 that if $wrgl_2(F^*) \sim \sqrt{\dim(F)}$, then $\mathrm{evr}(F^*, \ell_\infty^{\dim(F)}) \sim 1$. In other words there is a cross-polytope $C$ contained in $B_F$ (see the comments before Proposition 10) such that $\left(\frac{\mathrm{vol}_n(B_F)}{\mathrm{vol}_n(C)}\right)^{1/n} \sim 1$; in 'volume sense' $B_F$ is equivalent to a cross-polytope; moreover both $\lambda(F^*)$, $gl_2(F^*)$ and $gl_2(F)$ are then asymptotically equivalent to $\sqrt{\dim(F)}$.

(b) If $gl_2(F) \sim 1$, which happens for example when $F$ is 'well' complemented in a Banach lattice, then both $B_F$ and $B_{F^*}$ are in 'volume sense' equivalent to zonoids.

As we shall see, the estimate given by Corollary 7 for $\lambda(K)$ can provide good information about its real value; however, we ignore whether it is a sharp estimate. The weaker estimate, $\mathrm{evr}(F, \ell_\infty^{\dim(F)}) \leq \lambda(F)$, is not sharp, as it is shown by the following example.

*Example.* There is an $n$-dimensional subspace $F$ of $\ell_\infty^{2n}$ such that

$$\lambda(F) \sim \sqrt{n} \text{ and } \mathrm{evr}(F,\ell_\infty^n) \leq \sqrt{2}e$$

(for another example of the same type, see [B1]). In order to show this we first prove:

(a) Let $E$ be a $n$-dimensional subspace of $\ell_\infty^m$ with $B_E$ as unit ball. Then

$$\mathrm{evr}(B_E, Q_n) \leq \left(\sqrt{\binom{m}{n}}\right)^{1/n} \leq \sqrt{\frac{me}{n}}.$$

Let $P_E$ be the orthogonal projection of $\mathbb{R}^m$ onto $E$. Then $B_E$ can be described as follows:

$$B_E = \{x \in E;\ | < x, P_E e_i > | \le 1,\ \text{for } 1 \le i \le m\}\,.$$

Let $u_1, \ldots, u_n$ be an orthonormal basis of $E$, and set $P_E = \sum_{i=1}^m \sum_{j=1}^n \mu_{ij} e_i \otimes u_j$. Denote by $M \in \mathcal{M}_{m,n}$ the matrix $(\mu_{ij})_{1\le i\le m,\,1\le j\le n}$ which represents $P_E$. Since $P_E$ is an orthogonal projection, the matrix $M^*M \in \mathcal{M}_{n,n}$ is the identity on $E$. Therefore by the Cauchy-Binet formula,

$$1 = \det(M^*M) = \sum_{I\subset\{1,\ldots,m\},\,|I|=n} (\det(M_{IN}))^2\,,$$

where $N = \{1, \ldots, n\}$. The matrix $M_{IN}$ represents the operator $P_E|\operatorname{span}\{e_i, i \in I\}$ which maps $e_i$ to $P_E(e_i) = \sum_{j=1}^n \mu_{ij} u_j$ for every $i \in I$. Hence, denoting by $\mathbf{C}^I$ the cross-polytope $\operatorname{conv}(\pm P_E(e_i), i \in I)$, we obtain

$$\operatorname{vol}_n(\mathbf{C}^I) = \frac{2^n}{n!}|\det(M_{IN})|\,.$$

It follows that

$$\sum_{|I|=n} \left(\frac{n!}{2^n} \operatorname{vol}_n(\mathbf{C}^I)\right)^2 = 1\,.$$

Therefore there exists $J \subset \{1, \ldots, m\}$, $|J| = n$ such that

$$\operatorname{vol}_n(\mathbf{C}^J) \ge \frac{2^n}{n!\sqrt{\binom{m}{n}}}\,.$$

Now the parallelotope $Q^J = \{x \in E;\ | < x, P_E e_i > | \le 1 \text{ for } i \in J\}$ contains $B_E$, and moreover since $(Q^J)^o = \mathbf{C}^J$, we have $\operatorname{vol}_n(Q^J)\operatorname{vol}_n(\mathbf{C}^J) = 4^n/n!$, from which it follows that

$$\left(\operatorname{vol}_n(Q^J)\right)^{1/n} \le 2\left(\sqrt{\binom{m}{n}}\right)^{1/n} \le 2\sqrt{\frac{me}{n}}\,.$$

But it follows from [V] that $\operatorname{vol}_n(B_E) \ge \operatorname{vol}_n(Q_n) = 2^n$. Therefore we have

$$\operatorname{evr}(B_E, Q_n) \le \sqrt{\frac{me}{n}}\,.$$

(b) Under the same hypothesis as in (a), if $i : \ell_2^m \to \ell_\infty^m$ denotes the identity map, and if $i_E$ denotes its restriction to $E$, we have

$$\|i_E\| \ge \sqrt{\frac{n}{m}}\,.$$

In fact, if we set $u_i = \sum_{j=1}^m u_{ij} e_j$, $1 \le i \le n$, then

$$i_E = \sum_{i=1}^n u_i \otimes u_i = \sum_{i=1}^n \sum_{j=1}^m u_{ij} u_i \otimes e_j ,$$

so that

$$\|i_E\|^2 = \max_{a_1^2+\ldots+a_n^2\le 1} \left( \max_{j=1,\ldots,m} \left( \sum_{i=1}^n a_i u_{ij} \right)^2 \right) = \max_{j=1,\ldots,m} \left( \max_{a_1^2+\ldots+a_n^2\le 1} \left( \sum_{i=1}^n a_i u_{ij} \right)^2 \right)$$
$$= \max_{j=1,\ldots,m} \sum_{i=1}^n u_{ij}^2 \ge \frac{1}{m} \sum_{j=1}^n \sum_{i=1}^m u_{ij}^2 = \frac{n}{m} .$$

(c) Now let $F$ be an $n$-dimensional subspace of $\ell_\infty^m$ and let $q_F : \ell_\infty^m \to \ell_\infty^m / F$ be the quotient mapping. If $P$ is any linear projection of $\ell_\infty^m$ onto $F$, set $E = i^{-1}(\ker P)$ and $Q = I_\infty - P$, where $I_\infty$ denotes the identity mapping on $\ell_\infty^m$. If we set

$$r_Q(q_F y) = Qy \text{ for every } y \in \ell_\infty^m ,$$

we get a mapping $r_Q : \ell_\infty^m / F \to \ker P$ such that $r_Q q_F$ is the identity on $\ker P$, hence $i_E = r_Q\, q_F\, i_E$, and moreover $\|r_Q\| = \|Q\| = \|I_\infty - P\|$, so that

$$\|i_E\| \le \|I_\infty - P\|\, \|q_F\, i_E\|.$$

Let now $m = 2n$; then it follows from a result of Kashin [K1] that for some constant $c > 0$, independent of $n$, there exists an $n$-dimensional subspace $F$ of $\ell_\infty^{2n}$ so that, with the previous notation,

$$\|q_F\, i\| \le \frac{c}{\sqrt{n}}.$$

Applying (b), we get

$$\frac{1}{\sqrt{2}} = \sqrt{\frac{n}{2n}} \le \|i_E\| \le \|I_\infty - P\| \|q_F\, i_E\| \le \|I_\infty - P\| \|q_F\, i\| \le (1 + \|P\|) \frac{c}{\sqrt{n}}$$

for every projection $P : \ell_\infty^m \to F$ , with $i^{-1}(\ker P) = E$. It follows that

$$\lambda(F) \ge \frac{\sqrt{n}}{c\sqrt{2}} - 1 ,$$

but by (a) applied to $F$, we have

$$\operatorname{evr}(B_F, Q_n) \le \sqrt{2e} .$$

THEOREM 9. *For every* $0 < p \leq 1$ *there exists a constant* $c(p) > 0$ *such that for every integer* $n$, *we have*

$$c(p)\, n^{\frac{1}{p}-\frac{1}{2}} \leq \mathrm{evr}(\ell_p^n, \ell_\infty^n) \leq \lambda(\ell_p^n) \leq n^{\frac{1}{p}-\frac{1}{2}}.$$

*Proof.* Observe that a parallelotope contains the unit ball $B_p^n$ of $\ell_p^n$, $0 < p \leq 1$, if and only if it contains $C_n = B_1^n$. Therefore, since $\mathrm{vr}(C_n, B_2^n)$ is bounded and hence from Corollary 2, $\mathrm{evr}(C_n, Q_n) \sim \sqrt{n}$, it follows from Corollary 7 that

$$\begin{aligned}\lambda(\ell_p^n) &\geq \mathrm{evr}(B_p^n, Q_n) = \mathrm{evr}(C_n, Q_n)\left(\frac{\mathrm{vol}_n(C_n)}{\mathrm{vol}_n(B_p^n)}\right)^{1/n} \\ &\geq c(p)\sqrt{n}\, n^{-1+\frac{1}{p}} = c(p)\, n^{\frac{1}{p}-\frac{1}{2}}.\end{aligned}$$

The upper estimate is trivial, since the distance between $\ell_p^n$ and $\ell_1^n$ is $n^{1/p-1}$, and $\lambda(\ell_1^n) < \sqrt{n}$. □

*Remark.* The preceding lower estimate for $\lambda(\ell_p^n)$ has been obtained in [Pe], with an extra multiplicative $\ln(n)$, using a much more involved proof.
We also observe that easy calculation of $\mathrm{evr}(\ell_p^n, Q_n)$ yields the known asymptotic estimates of the projection constants for all values of $p \geq 1$.

For $0 < p \leq +\infty$, let $s_p^n$ be the $n^2$-dimensional space of all real $[n \times n]$ matrices $A$ equipped with the quasi-norm

$$\|A\|_p = (\sum_{i=1}^{n} \lambda_i^p)^{1/p},$$

where $(\lambda_1, \dots, \lambda_n)$ are the eigenvalues of $(A^*A)^{1/2}$, and let $S_p^n$ be the unit ball of $s_p^n$.

THEOREM 10. *For every* $0 < p \leq 1$, *there exists a positive constant* $a(p)$ *such that*

$$a(p)\, n^{1/p} \leq \mathrm{evr}(S_p^n, Q_{n^2}) \leq \lambda(s_p^n) \leq n^{1/p}.$$

*Proof.* By Corollary 8 of [S], for some constant $d(p) > 0$, $(\mathrm{vol}_{n^2}(S_p^n))^{1/n^2} \sim d(p) n^{-(\frac{1}{2}+\frac{1}{p})}$ (the proof of [S] considers only the case $p \geq 1$, but it is easily seen that it yields this estimate for $0 < p \leq 1$). As in Theorem 9, we have

$$\mathrm{evr}(S_p^n, Q_{n^2}) = \mathrm{evr}(S_1^n, Q_{n^2})\left(\frac{\mathrm{vol}_{n^2}(S_1^n)}{\mathrm{vol}_{n^2}(S_p^n)}\right)^{1/n^2} \geq d(p)\,\mathrm{evr}(S_1^n, Q_{n^2}) n^{-1+\frac{1}{p}}.$$

But $S_1^n \subset S_2^n \subset \sqrt{n}\, S_1^n$ and $\mathrm{vr}(S_1^n, B_2^{n^2}) \leq \left(\mathrm{vol}_{n^2}(S_1^n)/\,\mathrm{vol}_{n^2}(S_2^n)\right)^{1/n^2} \sqrt{n} \leq c_1$. Hence, by Corollary 2, $\mathrm{evr}(S_1^n, Q_{n^2}) \geq c_2 n$ , from which the lower estimates follows. For the upper estimate, observe that

$$\lambda(s_p^n) \leq \lambda(s_1^n)\,\mathrm{d}(s_p^n, s_1^n) \leq n . n^{\frac{1}{p}-1} = n^{\frac{1}{p}}$$

where $\mathrm{d}(.,.)$ denotes here the Banach-Mazur distance. □

*Remark.* If $K$ is a centrally symmetric convex body in $\mathbb{R}^n$, let

$$K^o = \{x \in \mathbb{R}^n;\ < x, y > \leq 1 \text{ for every } x \in K\}$$

be its polar body. Then for some absolute constant $c > 0$,

$$\frac{2}{\pi} \leq \frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq c.$$

Indeed,

$$\begin{aligned}\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} &= \inf_{P \subset K}\ \sup_{C \subset K^o} \left(\frac{\mathrm{vol}_n(P)\,\mathrm{vol}_n(\mathrm{C})}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\right)^{1/n}\\ &\geq \inf_{P \supset K} \left(\frac{\mathrm{vol}_n(P)\,\mathrm{vol}_n(P^o)}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\right)^{1/n}\\ &= \left(\frac{4^n}{n!\,\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\right)^{1/n},\end{aligned}$$

and by Santalò's inequality we get

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \geq \left(\frac{4^n}{n! v_n^2}\right)^{1/n} \geq \frac{2}{\pi}.$$

On the other hand, by the inverse Santalò's inequality (see [BM] or [Pi2])

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq \sup_{C \subset K^o} \left(\frac{\mathrm{vol}_n(\mathrm{C})\,\mathrm{vol}_n(\mathrm{C}^o)}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\right)^{1/n} \leq c.$$

If we suppose that $K$ or $K^o$ is a zonoid, using [R] or [GMR] we have

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o) \geq \frac{4^n}{n!},$$

so that

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq 1.$$

PROPOSITION 11. *Let $Z$ be a zonoid in $\mathbb{R}^n$. Then* $\mathrm{vr}(Z, B_1^n) \geq \mathrm{vr}(Q_n, C_n) \geq \frac{\sqrt{n}}{e}$.

*Proof.* Let us observe first that

$$\mathrm{vr}(Q_n, C_n) = \left(\frac{n!}{\max\{|\det(\theta_{ij},\ 1 \leq i, j \leq n)|;\ |\theta_{ij}| \leq 1\}}\right)^{1/n} \geq \frac{\sqrt{n}}{e}.$$

Indeed, any cross-polytope $\mathbf{C} \subset Q_n$ has the form $\mathrm{conv}(\pm \sum_{j=1}^n \theta_{ij} e_j,\ 1 \leq i \leq n)$, for some choice of the $n \times n$ matrix $\Theta = (\theta_{ij})$ and clearly

$$\mathrm{vol}_n(\mathbf{C}) = \frac{2^n}{n!}|\det(\Theta)| = \frac{\mathrm{vol}_n(Q_n)}{n!}|\det(\Theta)|.$$

By Hadamard's inequality, $|\det(\Theta)| \leq \Pi_{i=1}^n \left(\sum_{j=1}^n \theta_{ij}{}^2\right)^{1/2} \leq n^{n/2}$, hence

$$\mathrm{vr}(Q_n, C_n) \geq \left(\frac{n!}{n^{n/2}}\right)^{1/n} \geq \frac{\sqrt{n}}{e}.$$

Since a zonoid can be approximated by zonotopes in the Hausdorff metric, we may reduce to the latter case and suppose that $Z = \sum_{j=1}^m [-z_j, z_j]$ for some $z_j \in \mathbb{R}^n$, $1 \leq j \leq m$ and $n \leq m$.

Let $A \in \mathcal{M}_{n,m}$ be the matrix with the coordinates of $z_j$, $1 \leq j \leq m$ in the canonical basis of $\mathbb{R}^n$ as columns. If $x_1, \ldots, x_n$ are points in $Z$, then they have the form $x_i = \sum_{j=1}^m \theta_{ij} z_j$, with $\theta_{ij} \in [-1, 1]$, so letting $\mathbf{C} = \mathrm{conv}(\pm x_1, \ldots, \pm x_n) \subset Z$, and denoting by $L = [x_1, \ldots, x_n] \in \mathcal{M}_{n,n}$ the corresponding matrix, and by $\Theta \in \mathcal{M}_{m,n}$ the matrix with entries $\theta_{ji}$ in the $i$-th row and $j$-th column for $1 \leq i \leq m$, $1 \leq j \leq n$, we have $L = A\Theta$. By the Cauchy-Binet formula,

$$\det(L) = \sum_{I \subset \{1,\ldots,m\},\ |I|=n} \det(A_{NI}) \det(\Theta_{IN}).$$

But $\mathrm{vol}_n(Z) = 2^n \left(\sum_{I \subset \{1,\ldots,m\}, |I|=n} |\det(A_{NI})|\right)$, and hence

$$2^n |\det(L)| \leq \mathrm{vol}_n(Z) \max_{|I|=n} |\det(\Theta_{IN})| \leq (n!) \frac{\mathrm{vol}_n(Z)}{(\mathrm{vr}(Q_n, C_n))^n}.$$

Therefore

$$\left(\frac{\mathrm{vol}_n(Z)}{\mathrm{vol}_n(\mathbf{C})}\right)^{1/n} = \left(\frac{\mathrm{vol}_n(Z)}{\frac{2^n}{n!}|\det(L)|}\right)^{1/n} \geq \mathrm{vr}(Q_n, C_n) \geq \frac{\sqrt{n}}{e}. \qquad \square$$

*Remarks.* (1) The estimate $\mathrm{vr}(Q_n, C_n) \geq \frac{(n!)^{1/n}}{\sqrt{n}}$ is sharp in the case when $n = 2^k$, $k = 1, 2, \ldots$ (use Walsh matrix). For an upper estimate of $\mathrm{vr}(K, C_n)$, valid for every convex symmetric body $K$, observe that the quantity

$$\max_{x_1,\ldots,x_n \in K} \det(x_1, \ldots, x_n)$$

decreases under Steiner symmetrization of $K$ (see [M] for instance ). It follows that

$$\mathrm{vr}(K, C_n) \leq \mathrm{vr}(B_2^n, C_n) = \left(\frac{v_n n!}{2^n}\right)^{1/n} \sim \sqrt{\frac{\pi n}{2e}}.$$

This estimate was proved in [K2], up to a multiplicative constant.

(2) Proposition 11 allows us to give an easy geometric proof of the following result, which is also a consequence of the fact, originally due to Bourgain and Milman [BM], that the finite-dimensional subspaces $\{F\}$ of an infinite-dimensional normed space of cotype 2, have uniformly bounded volume ratios $\mathrm{vr}(F, \ell_2^{\dim(F)})$ (see also [Pi2], [Tj], and [GK] for the general quasi-normed case). This applies in particular for $\ell_1$ which has cotype 2: In this case, we see that every zonoid $Z$ in $\mathbb{R}^n$ satisfies $\mathrm{vr}(Z^o, B_2^n) \leq e\sqrt{\frac{\pi}{2}}$. Indeed, by Corollary 2, the remarks preceding Proposition 11 and Proposition 11 itself, we have successively

$$\mathrm{vr}(Z^o, B_2^n) \leq \sqrt{e}\sqrt{\frac{2n}{\pi e}}.\frac{1}{\mathrm{evr}(Z^o, Q_n)} \leq \sqrt{\frac{2n}{\pi}}.\frac{\pi/2}{\mathrm{vr}(Z, C_n)} \leq \sqrt{\frac{\pi n}{2}}.\frac{e}{\sqrt{n}} \leq e\sqrt{\frac{\pi}{2}}.$$

It was proved by K. Ball [B2], using more involved arguments, that if $Z$ is a zonoid in $\mathbb{R}^n$, one always has

$$\mathrm{vr}(Z^o, B_2^n) \leq \mathrm{vr}(\mathbf{C}_n, B_2^n) \sim \sqrt{\frac{2e}{\pi}}.$$

It may be observed that finding the exact maximum of $\mathrm{evr}(K, Q_n)$ over all the centrally symmetric convex bodies $K$ in $\mathbb{R}^n$ is still an open problem for $n \geq 3$ (see [Ba], where it is solved for $n = 2$).

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