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COARSE SHEAF COHOMOLOGY FOR FOLIATIONS

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ABSTRACT. We show that coarse cohomology for foliations can be defined as a sheaf cohomology and use this fact to compute two examples.

1. Introduction

In [HH], we introduced a new invariant of a foliation of a compact manifold, its coarse cohomology. This cohomology combines the usual cohomology of the ambient manifold with the coarse cohomology (due to Roe [R]) of the holonomy covers of the leaves of the foliation. It is expected that this theory will play an important role in the theory of foliations. In particular, secondary classes, the coarse Chern character, higher torsion invariants, spectral invariants of leafwise operators, and the coarse index approach to the Miscenko-Kasparov theory and its applications to the Novikov Conjecture should all have natural expressions in terms of coarse cohomology. For a more complete discussion of this, see the introduction of [HH].

In this paper, we show that this cohomology is in fact a sheaf cohomology. The immediate advantage of this fact is a spectral sequence which converges to the coarse cohomology. We then use this spectral sequence to compute the coarse cohomology of two important foliations, the double Reeb foliation and the Sullivan foliation.

2. Coarse sheaf cohomology

In this section we define the differential sheaf over a compact foliated manifold whose associated cochain complex of continuous sections is the coarse de Rham complex of the foliation. We will freely use the notation and results of [HH].

Let F be a codimension q foliation of a compact n dimensional manifold M without boundary. Recall the *holonomy groupoid* $\mathcal{G}_F$ of F. A point $y = [\gamma] \in \mathcal{G}_F$ is the equivalence class of a path $\gamma\colon [0,1] \mapsto M$ whose image is contained in a single leaf L. Two such leafwise paths γ_1 and γ_2 are equivalent provided $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$ and the holonomy along the two paths is the same on some transversal containing $\gamma_1(0)$. There are natural maps $s, r\colon \mathcal{G}_F \mapsto M$ defined by $s(y) = \gamma(0)$, $r(y) = \gamma(1)$. $\mathcal{G}_F$ is a (generally non-Hausdorff) $2n - q$ dimensional

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manifold with the local charts given as follows. Let U and V be foliation charts of $s(y)$ and $r(y)$ respectively and choose $\gamma \in y$. Then the local chart (U, γ, V) consists of all equivalence classes of leafwise paths which start in U, end in V and which are homotopic to γ through a homotopy of leafwise paths whose end points remain in U and V respectively. It is easy to see that if $U, V \simeq \boldsymbol{R}^{n-q} \times \boldsymbol{R}^q$, then $(U, \gamma, V) \simeq \boldsymbol{R}^{n-q} \times \boldsymbol{R}^{n-q} \times \boldsymbol{R}^q$. In order to apply the results of [HH], we assume that $\mathcal{G}_F$ is Hausdorff.

We now recall the coarse de Rham cohomology for F. For each $x \in M$, $s^{-1}(x) \simeq \widetilde{L}_x$, the holonomy cover of the leaf L_x containing x. Denote by $\mathcal{G}_\ell$ the submanifold of $\times_\ell \mathcal{G}_F$ consisting of those points $(y_1, \dots, y_\ell)$ with $s(y_1) = s(y_j)$ for $j = 2, \dots, \ell$. We also denote by s the map $s\colon \mathcal{G}_\ell \mapsto M$ given by $s(y_1, \dots, y_\ell) = s(y_1)$. Note that $s^{-1}(x) \simeq \times_\ell \widetilde{L}_x$. Choose a metric on M. This induces a metric on each leaf L and so also on $\widetilde{L}$ and $\times_\ell \widetilde{L}$ which makes them complete Riemannian manifolds. Their quasi isometry types are independent of the choice of metric on M since M is compact. Denote the metric on $s^{-1}(x)$ by D_x. Given $A \subseteq \mathcal{G}_\ell$ and $r > 0$, define

$$\begin{aligned}\operatorname{Pen}(A, r) &= \{(y_1', \dots, y_\ell') \in \mathcal{G}_\ell \mid \exists\ (y_1, \dots, y_\ell) \in A \text{ with} \\ s(y_i) &= s(y_i') \text{ and } D_{s(y_i)}(y_i, y_i') < r \text{ for } i = 1, \dots, \ell\ \}.\end{aligned}$$

Denote by $A_c^{k,\ell}(F)$ the space of k forms ω on $\mathcal{G}_{\ell+1}$ such that for all $r > 0$, $\operatorname{sup}(\omega) \cap \operatorname{Pen}(\Delta_{\ell+1}, r)$ is relatively compact, where $\Delta_{\ell+1}$ is the diagonal of $\mathcal{G}_{\ell+1}$. We have two differentials defined on $A_c^{k,\ell}(F)$, the usual exterior derivative $d\colon A_c^{k,\ell}(F) \mapsto A_c^{k+1,\ell}(F)$, and $\delta\colon A_c^{k,\ell}(F) \mapsto A_c^{k,\ell+1}(F)$ given by

$$\delta\omega = \sum_{j=1}^{\ell+2} (-1)^{j+1} \pi_j^* \omega$$

where the $\pi_j\colon \mathcal{G}_{\ell+2} \mapsto \mathcal{G}_{\ell+1}$ deletes the j th entry. The cohomology of the bicomplex $\{A_c^{*,*}(F), \delta, d\}$ is the coarse de Rham cohomology $HX^*(F)$ of F.

Definition 1. The *coarse presheaf* L^* of F is the differential presheaf which associates to each open set $U \subset M$ and each non-negative integer q, the space

$$L^q(U) = \sum_{k+\ell=q} A_c^{k,\ell}(U) \quad \text{where} \quad A_c^{k,\ell}(U) = \{\omega\,|_{s^{-1}(U)} \mid \omega \in A_c^{k,\ell}(F)\}.$$

The differential $D\colon L^q(U) \mapsto L^{q+1}(U)$ is given by $D \mid A_c^{k,\ell}(U) = d + (-1)^k \delta$. The *coarse sheaf* $\mathcal{L}^*$ of F is the differential sheaf associated to the differential presheaf L^*.

For each q, $\mathcal{L}^q$ is a fine sheaf, so the Čech bicomplex associated to $\mathcal{L}^*$ computes the coarse de Rham cohomology of F. Theorem 2.1, page 132 of [B] gives a spectral sequence which converges to $HX^*(F)$. Its E_2 term is

$$E_2^{p,q} = H^p(M; \mathcal{H}^q(\mathcal{L}^*))$$

where $\mathcal{H}^q(\mathcal{L}^*)$ is the homology sheaf of the differential sheaf $\mathcal{L}^*$. In particular,

$$\mathcal{H}^q(\mathcal{L}^*) = Ker(D\colon \mathcal{L}^q \mapsto \mathcal{L}^{q+1})/Im(D\colon \mathcal{L}^{q-1} \mapsto \mathcal{L}^q).$$

3. Two examples

In this section, we compute the coarse cohomology of the double Reeb foliation and the Sullivan foliation using the above spectral sequence.

1. The double Reeb foliation.

The double Reeb foliation is the foliation F of $M = S^1 \times S^2 = S^1 \times D^2 \cup S^1 \times D^2$ where each copy of $S^1 \times D^2$ is a Reeb component [MS, p. 41]. This foliation has a single compact leaf which is diffeomorphic to T^2 and all the other leaves are diffeomorphic to $\boldsymbol{R}^2$. For each $x \in T^2$, the holonomy cover $\widetilde{L}_x$ is quasi isometric to $S^1 \times \boldsymbol{R}$ with the usual metric which makes it coarsely equivalent to $\boldsymbol{R}$ with the usual metric. For $x \in M - T^2$, $\widetilde{L}_x$ is isometric to $\boldsymbol{R}^2$ with a metric making it coarsely equivalent to $[0, \infty)$ with the usual metric.

PROPOSITION 2. *Let $\mathcal{R}_{T^2}$ be the direct image under the inclusion $T^2 \hookrightarrow M$ of the constant sheaf $T^2 \times \boldsymbol{R}$. The coarse cohomology of the double Reeb foliation F is*

$$HX^*(F) \simeq H^{*-1}(M; \mathcal{R}_{T^2}).$$

Proof. We need only show that $\mathcal{H}^q(\mathcal{L}^*) = \mathbf{0}$ for $q \neq 1$ and $\mathcal{H}^1(\mathcal{L}^*) = \mathcal{R}_{T^2}$.

Each $x \in M - T^2$ has a neighborhood U so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the metric family $\{U \times [0, \infty), d_U, \pi, U\}$. See [HH]. For each $u \in U$, d is the induced metric on $\widetilde{L}_u = s^{-1}(u)$, and d_U is the usual metric on $\{u\} \times [0, \infty)$. As they are coarsely equivalent, these two families have the same coarse cohomology. But the coarse cohomology of $[0, \infty)$ is trivial, so the spectral sequence of [HH] gives that the coarse cohomology of $\{U \times [0, \infty), d_U, \pi, U\}$ is trivial. Thus the coarse cohomology of $\{s^{-1}(U), d, s, U\}$ is also trivial, so for all $x \in M - T^2$ and *all* q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.

Each neighborhood of $x \in T^2$ contains a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the metric family $\mathcal{F}_x = \{\mathcal{U}, d_U, \pi, U\}$, where $\mathcal{U} \subset D^3 \times \boldsymbol{R}$ is the set

$$\mathcal{U} = \{(u_1, u_2, u_3, t) \mid t \geq -u_1^{-2}\},$$

$\pi\colon \mathcal{U} \to U$ is the natural projection, and for each $u \in U$, the metric d_U on $\pi^{-1}(u) = [-u_1^{-2}, \infty)$ is the usual metric. We use the convention that if $u_1 = 0$, then $t > -\infty$.

Let

$$\mathcal{U}_{\ell+1} = \{(u_1, u_2, u_3, t_0, \dots, t_\ell) \in U \times \boldsymbol{R}^{\ell+1} \mid t_i \geq -u_1^{-2},\ i = 0, \dots, \ell\}.$$

Then the coarse cohomology of the metric family $\mathcal{F}_x$ may be computed just as the de Rham coarse cohomology for a foliation is computed, with $\mathcal{U}_{\ell+1}$ substituted for $\mathcal{G}_{\ell+1}$. In particular, we have the spaces $A_c^{k,\ell}(\mathcal{F}_x)$ of k forms ω on the space $\mathcal{U}_{\ell+1}$ such that for all $r > 0$, $\sup(\omega) \cap \mathrm{Pen}(\Delta_{\ell+1}, r)$ is relatively compact. Also d: $A_c^{k,\ell}(\mathcal{F}_x) \mapsto A_c^{k+1,\ell}(\mathcal{F}_x)$ and δ: $A_c^{k,\ell}(\mathcal{F}_x) \mapsto A_c^{k,\ell+1}(\mathcal{F}_x)$ are as above, and the coarse cohomology of $\mathcal{F}_x$ is the cohomology of the bicomplex $\{A_c^{*,*}(\mathcal{F}_x), \delta, d\}$.

$HX^0(\mathcal{F}_x)$ consists of constant functions on $\mathcal{U}_1$ with compact support. As $\mathcal{U}_1$ is not compact, $HX^0(\mathcal{F}_x) = 0$ which implies $\mathcal{H}^0(\mathcal{L}^*)_x = 0$.

To compute the higher coarse cohomology of the metric family $\mathcal{F}_x$, we need the following lemma. Denote by d^h: $A_c^{k,\ell}(\mathcal{F}_x) \mapsto A_c^{k+1,\ell}(\mathcal{F}_x)$ the exterior derivative with respect to the u coordinates only.

LEMMA 3 (CONTROLLED POINCARÉ LEMMA). *Suppose $k > 0$ and that each term of $\omega_{k,\ell} \in A_c^{k,\ell}(\mathcal{F}_x)$ contains at least one du_i. Further suppose that $d^h(\omega_{k,\ell}) = 0$. Then there is $\omega_{k-1,\ell} \in A_c^{k-1,\ell}(\mathcal{F}_x)$ such that $d^h(\omega_{k-1,\ell}) = \omega_{k,\ell}$.*

Proof. For a monomial $\omega_{k,\ell} = f(u,t)du_{i_1} \wedge \cdots \wedge du_{i_r} \wedge dt_{j_1} \wedge \cdots \wedge dt_{j_{k-(r+1)}} \in A_c^{k-1,\ell}(\mathcal{F}_x)$, set

$$\phi(\omega)(u,t) = \left(\int_0^1 s^{r-1} f(su,t)ds\right) du_{i_1} \wedge \cdots \wedge du_{i_r} \wedge dt_{j_1} \wedge \cdots \wedge dt_{j_{k-(r+1)}}.$$

Denote by i_X interior multiplication by the vector field $X = u_1\partial/\partial u_1 + u_2\partial/\partial u_2 + u_3\partial/\partial u_3$ on $\mathcal{U}_{\ell+1}$. Then the proof of the usual Poincaré lemma [W, page 155] shows that the form $\omega_{k-1,\ell} = \phi(i_X\omega_{k,\ell})$ satisfies the conclusion of the lemma. Extend to general $\omega_{k,\ell}$ by linearity. □

Let $\alpha = [\sum_{i+j=k} \omega_{i,j}]$ be a $k > 0$ dimensional coarse class for $\mathcal{F}_x$, where $\omega_{i,j} \in A_c^{i,j}(\mathcal{F}_x)$. Then $\omega_{k,0}$ is a closed k form on $\mathcal{U}_1$. We may write $\omega_{k,0} = \omega_{k,0}^h + \omega_{k,0}^f$ where $\omega_{k,0}^h$ is a k form involving only the du_i and $\omega_{k,0}^f$ is a k form with each term having a dt_0. The fact that $d(\omega_{k,0}) = 0$ implies that $d^h(\omega_{k,0}^h) = 0$. The Controlled Poincaré Lemma gives a $k-1$ form $\omega_{k-1,0} \in A_c^{k-1,0}(\mathcal{F}_x)$ such that $d^h(\omega_{k-1,0}) = \omega_{k,0}^h$. Recall that the differential of the bicomplex $\{A_c^{*,*}(\mathcal{F}_x), \delta, d\}$ is $D = d + (-1)^k\delta$. As $\alpha = [\sum_{i+j=k} \omega_{i,j} - D(\omega_{k-1,0})]$, we may assume that $\omega_{k,0}^h = 0$. Now $\omega_{k,0}^f = \omega_{k-1,0}^f \wedge dt_0$ for some element $\omega_{k-1,0}^f \in A_c^{k-1,0}(\mathcal{F}_x)$, and $\omega_{k-1,0}^f$ has no terms involving a dt_0. In addition, as $d(\omega_{k,0}^f) = d^h(\omega_{k,0}^f) = 0$, we have $d^h(\omega_{k-1,0}^f) = 0$ also. If $k > 1$, the Controlled Poincaré Lemma again implies that there is an element $\omega_{k-2,0} \in A_c^{k-2,0}(\mathcal{F}_x)$ with $d^h(\omega_{k-2,0}) = \omega_{k-1,0}^f$. Thus $d(\omega_{k-2,0} \wedge dt_0) = \omega_{k-1,0}^f \wedge dt_0 = \omega_{k,0}^f$. As $\alpha = [\sum_{i+j=k} \omega_{i,j} - D(\omega_{k-2,0} \wedge dt_0)]$, we may assume that $\omega_{k,0}^f$ and so $\omega_{k,0}$ is 0. If $k = 1$, $\omega_{k,0}$ is a form of pure fiber type, i.e., it contains no du_i. As $d(\omega_{k,0}) = 0$, it must in fact be entirely independent of u. Thus, in both the $k > 1$ and $k = 1$ cases, we may assume that $\omega_{k,0}$ is completely independent of u, and we may regard it as an element of $A_c^{k,0}(\boldsymbol{R})$, i.e., as a coarse cochain on $\boldsymbol{R}$. Here we use the extended coarse cohomology of [HH] to compute the coarse cohomology of $\boldsymbol{R}$.

Now consider $\omega_{k-1,1}$, a $k-1$ form on $\mathcal{U}_2$. As $d(\omega_{k-1,1}) = \delta(\omega_{k,0})$, which is independent of u, we have $d^h(\omega_{k-1,1}) = 0$ and proceeding as above, we may assume $\omega_{k-1,1}$ contains no forms which contain no dt_j. As $d^h(\omega_{k-1,1})$ is still zero, proceeding as above, we may assume that it contains no forms which contain one dt_j, and then if $k > 2$ that it contains no forms which contain $dt_0 \wedge dt_1$. If $k = 2$, the fact that $\omega_{k-1,1} = \phi(u,t)dt_0 \wedge dt_1$ and $d^h(\omega_{k-1,1}) = 0$ means that we may assume that $\omega_{k-1,1}$ is entirely independent of u, and so may be regarded as an element of $A_c^{k-1,1}(\boldsymbol{R})$.

Continuing in this manner, we may assume that $\sum_{i+j=k} \omega_{i,j}$ is entirely independent of u and so α may be viewed as a k dimensional coarse class on $\boldsymbol{R}$. Now the coarse cohomology of $\boldsymbol{R}$ is zero in all dimensions except in dimension one where it is $\boldsymbol{R}$. In fact, one can show that $\alpha \mapsto \int_{-\infty}^{\infty} \omega_{1,0}^f$ gives an isomorphism between $\mathcal{H}^1(\mathcal{L}^*)_x$ and $\boldsymbol{R}$.

Thus for $q \neq 1$, we have $\mathcal{H}^q(\mathcal{L}^*) = \boldsymbol{0}$. For $q = 1$ and $x \in T^2$, $\mathcal{H}^1(\mathcal{L}^*)_x = \boldsymbol{R}$, and for $x \in M - T^2$, $\mathcal{H}^1(\mathcal{L}^*)_x = 0$. It is not difficult to see that there is no twisting in the sheaf $\mathcal{H}^1(\mathcal{L}^*)$ so it is in fact $\boldsymbol{R}_{T^2}$. □

The point here is that the metric family $\mathcal{F}_x$ is sufficiently similar to the trivial metric family $(U \times \boldsymbol{R}, d, \pi, U)$ that they have the same coarse cohomology. Thus for the double Reeb foliation we have $\mathcal{H}^q(\mathcal{L}^*) = \mathcal{H}\mathcal{X}^q(\widetilde{L})$ where $\mathcal{H}\mathcal{X}^q(\widetilde{L})$ is the sheaf which assigns $HX^q(\widetilde{L}_x)$ to each x. Below, we will see that the Sullivan foliation does not satisfy this property.

To finish the computation of the coarse cohomology of the double Reeb foliation, we compute $H^*(M;\mathcal{R}_{T^2})$. To do so, we tensor $\mathcal{R}_{T^2}$ with the fine, torsionless resolution of the constant $\boldsymbol{R}$ sheaf over M given by $\boldsymbol{R} \mapsto \mathcal{C}^0 \mapsto \mathcal{C}^1 \mapsto \mathcal{C}^2 \mapsto \cdots$ where $\mathcal{C}^k$ is the sheaf of germs of smooth k forms on M. This gives us the differential sheaf $\mathcal{R}_{T^2} \otimes \mathcal{C}^*$ on M, whose stalk at $x \in M - T^2$ is 0, and at $x \in T^2$ is $\mathcal{C}_x^*$. The set of continuous sections $\Gamma(\mathcal{R}_{T^2} \otimes \mathcal{C}^k)$ of $\mathcal{R}_{T^2} \otimes \mathcal{C}^k$ is thus the germs on T^2 of smooth k forms on M, and the cohomology of the cochain complex $\Gamma(\mathcal{R}_{T^2} \otimes \mathcal{C}^*)$ is $H^*(M;\mathcal{R}_{T^2})$. Now the normal bundle of T^2 in M is trivial, so the cohomology of this cochain complex is the same as the de Rham cohomology of $T^2 \times D^1$, which is the de Rham cohomology of T^2. Thus we have the following.

THEOREM 4. *Let F be the double Reeb foliation of $S^1 \times S^2$. Then*

$$HX^*(F) \simeq H^{*-1}(T^2).$$

The isomorphism is effected as follows. Let (θ, x_1, x_2, x_3) be coordinates on $M = S^1 \times S^2$. Then T^2 is given by $T^2 = \{(\theta, 0, x_2, x_3)\}$. Note that $\mathcal{G}_F$ is diffeomorphic to $(M \times \boldsymbol{R}^2) - (T^2 \times \{(0,0)\})$. Define $\phi_1 \colon \mathcal{G}_F \mapsto \boldsymbol{R}$ by

$$\phi_1(\theta, x_1, x_2, x_3, t_1, t_2) = \ln(t_1^2 + t_2^2 + e^{-x_1^{-2}})$$

and ϕ: $\mathcal{G}_F \mapsto M \times \boldsymbol{R}$ by $\phi(y) = (s(y), \phi_1(y))$. The image of ϕ is $\{(x,t) \mid t \geq -x_1^{-2}\}$ where we have the convention that if $x_1 = 0$, then $t > -\infty$. The map $\Phi = (\phi, \mathrm{Id})$ is a leafwise map [HH] between the metric families $\{\mathcal{G}_F, d, s, M\}$ and $\mathcal{M} = \{M \times \boldsymbol{R}, d^r, \pi, M\}$ where d^r is the usual metric on each $\{x\} \times \boldsymbol{R}$, and π: $M \times \boldsymbol{R} \mapsto M$ is the projection. The induced map Φ^* on *coarse cochains* is the map of interest to us. Let ψ: $\boldsymbol{R} \mapsto \boldsymbol{R}$ be a smooth, non-negative function with non-empty support contained in $[-2, -4]$. The closed coarse one cochain $\omega = \omega_{1,0} + \omega_{0,1}$, where $\omega_{1,0} = \psi(t)dt$ and $\omega_{0,1}(t_0, t_1) = \int_{t_0}^{\infty} \psi(t)dt - \int_{t_1}^{\infty} \psi(t)dt$, defines a non-zero class in $HX^1(\mathcal{M})$. Set $M_2 = \{(\theta, x_1, x_2, x_3) \mid x_1^2 > 1/2\}$. Note that on $s^{-1}(M_2) \subset \mathcal{G}_1$, $\Phi^*(\omega_{1,0}) = 0$, and on $s^{-1}(M_2) \subset \mathcal{G}_2$, $\Phi^*(\omega_{0,1}) = 0$. Consider the forms on M, $\alpha_1 = d\theta$, and $\alpha_2 = \rho^*(d\theta)$, where ρ: $M \mapsto S^1$ is the map $(\theta, x_1, x_2, x_3) \mapsto (x_2^2 + x_3^2)^{-1/2}(x_2, x_3)$. Now the map ρ is not defined on $S^1 \times \{(\pm 1, 0, 0)\} \subset M_2$, so α_2 is not defined on all of M. However, $s^*(\alpha_2) \wedge \Phi^*(\omega_{0,1})$ is a globally well-defined one form on $\mathcal{G}_2$ and $s^*(\alpha_2) \wedge \Phi^*(\omega_{1,0})$ is also a globally well-defined two form on $\mathcal{G}_1$. Thus $s^*(\alpha_2) \wedge \Phi^*(\omega)$ is a well-defined coarse one cochain for F. We leave it to the reader to check that $D(s^*(\alpha_2) \wedge \Phi^*(\omega)) = 0$, and that $D(s^*(\alpha_1) \wedge \Phi^*(\omega)) = 0$. The cohomology of T^2 is generated by the classes of the forms $1, d\theta_1, d\theta_2$ and $d\theta_1 \wedge d\theta_2$. Under the isomorphism $H^{*-1}(T^2) \simeq HX^*(F)$, these classes map to the classes of $\Phi^*(\omega)$, $s^*(\alpha_1) \wedge \Phi^*(\omega)$, $s^*(\alpha_2) \wedge \Phi^*(\omega)$, and $s^*(\alpha_1 \wedge \alpha_2) \wedge \Phi^*(\omega)$.

2. The Sullivan foliation.

The Sullivan foliation F is a one dimensional foliation of the compact manifold $M = T^1S^2 \times S^2$ with all leaves compact, but with the volumes of the leaves unbounded. See [S]. Here T^1S^2 is the unit tangent bundle of S^2. For $-1 \leq t \leq 1$ set $T_t = T^1S^2 \times \{(x, y, z) \mid z = t\}$, where (x, y, z) are global coordinates on $S^2 \subset \boldsymbol{R}^3$. Then the foliation on $T_{\pm 1}$ is given by the fibers of π: $T^1S^2 \mapsto S^2$. For $-1 < t < 1$, T_t is saturated by the leaves of F and the length of each leaf in T_t is $f(t)$ where f: $(-1, 1) \mapsto \boldsymbol{R}$ is a smooth positive function with $\lim_{t \to \pm 1} f(t) = \infty$. Set $T = T_1 \cup T_{-1}$. Then $H^*(M, T; \boldsymbol{R})$ is the cohomology of $\Lambda^*(M, T)$, the forms on M which are zero on some neighborhood of T, and s: $\mathcal{G}_F \mapsto M$ induces a well-defined map s^*: $\Lambda^*(M, T) \mapsto A_c^{*,0}(F)$. Recall the maps π_1, π_2: $\mathcal{G}_2 \to \mathcal{G}_1$ and note that since $\pi_1 \circ s = \pi_2 \circ s$, $\delta \circ s^* = 0$. In addition, $d \circ s^* = s^* \circ d$, so we have a well-defined natural map s^*: $H^*(M, T; \boldsymbol{R}) \mapsto HX^*(F)$.

THEOREM 5. *For the Sullivan foliation F, the natural map*

$$s^*\colon H^*(M, T; \boldsymbol{R}) \mapsto HX^*(F)$$

is an isomorphism.

Proof. Note that on $M - T$ the length of the leaves changes smoothly, which implies that there is no leafwise holonomy there and so the graph of F over $M - T$,

$\mathcal{G}_{M-T}$ is locally a product bundle with fiber S^1. In particular, if $-1 < r_1 < r_2 < 1$ then $s^{-1}(\cup_{r_1 \leq t \leq r_2} T_t) \subset \mathcal{G}_F = \mathcal{G}_1$ is compact. Using this fact, the argument after the proof of Proposition 4.2 of [HH] shows that for $x \in M - T$ and $q > 0$, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$. We will show below that for $x \in T$, and *all* q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$. Thus for $q > 0$, $\mathcal{H}^q(\mathcal{L}^*) = 0$.

The argument of [HH] also shows that the sheaf $\mathcal{H}^0(\mathcal{L}^*)$ restricted to $M - T$ consists of germs of constant functions on $\mathcal{G}_{M-T}$, so it is the trivial $\boldsymbol{R}$ sheaf. Proposition 6 below gives $\mathcal{H}^0(\mathcal{L}^*)_x = 0$ for $x \in T$. As the codimension of T in M is two, any twisting in the global sheaf $\mathcal{H}^0(\mathcal{L}^*)$ must take place in $M - T$, so there is no twisting and $\mathcal{H}^0(\mathcal{L}^*)$ is the direct image of the trivial $\boldsymbol{R}$ sheaf over $M - T$.

Let $\mathcal{M}^*$ be the differential sheaf over M of germs of differential forms on M which are zero on some neighborhood of T. The usual Poincaré lemma shows that for $x \in M - T$, $\mathcal{H}^0(\mathcal{M}^*)_x = \boldsymbol{R}$ and for $q > 0$, $\mathcal{H}^q(\mathcal{M}^*)_x = 0$. Note that $\mathcal{H}^0(\mathcal{M}^*)_x$ is just germs of constant functions on $M - T$. For $x \in T$, $\mathcal{M}^*_x = 0$, so for $x \in T$ and all q, $\mathcal{H}^q(\mathcal{M}^*)_x = 0$. As above, $\mathcal{H}^0(\mathcal{M}^*)$ is the direct image of the trivial $\boldsymbol{R}$ sheaf over $M - T$.

It is now obvious that the map of differential sheaves $s^*\colon \mathcal{M}^* \mapsto \mathcal{L}^*$ obtained from $s^*\colon \Lambda^*(M, T) \mapsto A_c^{*,0}(F)$ induces an isomorphism $s^*\colon \mathcal{H}^*(\mathcal{M}^*) \mapsto \mathcal{H}^*(\mathcal{L}^*)$. As both $\mathcal{L}^*$ and $\mathcal{M}^*$ are fine sheaves, Theorem 2.2 of [B, page 132] then gives the theorem. $\square$

Note that $H^p(M, T; \boldsymbol{R}) \simeq H_c^p(M - T; \boldsymbol{R}) \simeq H^{p-1}(T^1S^2 \times S^1; \boldsymbol{R})$. As $T^1S^2 \simeq SO_3$, $HX^p(F) \simeq \boldsymbol{R}$ for $p = 1, 2, 4$ and 5 and is zero otherwise.

To finish proof of the theorem, we have the following.

PROPOSITION 6. *For all* $x \in T$, *and* all q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$

Proof. We may assume that $x \in T^1S^2 \times \{(0, 0, 1)\}$. It has a neighborhood $U \simeq D^5$ with coordinates $u_1, \dots, u_5$ so that $T \cap U = \{u \mid u_4 = u_5 = 0\}$ and if $T_t \cap U \neq \emptyset$,then $T_t \cap U = \{u \mid u_4^2 + u_5^2 = 1 - t^2\}$. Sullivan's example is actually given by the flow Φ_r of a unit vector field on M, so $s^{-1}(U)$, the graph of F over U, admits a surjection $U \times \boldsymbol{R} \mapsto s^{-1}(U)$. This surjection is given by mapping the point (u, r_0) to the class of the curve $r \mapsto \Phi_r(u)$ where the domain of the curve is $[0, r_0]$. Consider the metric family

$$\mathcal{F}_x = \{U \times \boldsymbol{R}, d_U, \pi, U\}$$

where $\pi\colon U \times \boldsymbol{R} \to U$ is the natural projection and for each $u \in U$, d_U is the usual metric on $\{u\} \times \boldsymbol{R}$. The metric family $\mathcal{F}_x$ surjects onto the metric family $\{s^{-1}(U), d, s, U\}$ where the map is given by identifying each $(u, r) \in U \times \boldsymbol{R}$ with $(u, r + f(1 - u_4^2 - u_5^2))$. If $u_4^2 + u_5^2 = 0$ we make no identifications.

The zero dimensional coarse cohomology of $\{s^{-1}(U), d, s, U\}$ consists of constant functions on $s^{-1}(U)$ which have compact support. As $s^{-1}(U)$ is non-compact, this cohomology group is trivial so for all $x \in T$, $\mathcal{H}^0(\mathcal{L}^*)_x = 0$.

Now suppose $\alpha = [\sum_{i+j=k} \omega_{i,j}]$ is a $k > 0$ dimensional coarse class for $\{s^{-1}(U), d, s, U\}$. Using the above surjection, we may regard $\omega_{i,j}$ as a form on $U \times \boldsymbol{R}^{j+1}$ whose restriction to $\{u \mid u_4^2 + u_5^2 = 1 - t^2\} \times \boldsymbol{R}^{j+1}$ is periodic in each $\boldsymbol{R}$ variable with period $f(t)$. Note that $\omega_{i,j}$ is not necessarily a coarse cochain for $\mathcal{F}_x$, since in general it will not satisfy the coarse support condition. We now proceed just as we did in the case of the double Reeb foliation to show that we may assume $\sum_{i+j=q} \omega_{i,j}$ is completely independent of u. However, we must take care that the construction we use in the Controlled Poincaré Lemma respects the identification.The flow of the vector field $X = u_1\partial/\partial u_1 + u_2\partial/\partial u_2 + u_3\partial/\partial u_3$ on $U \times \boldsymbol{R}^{j+1}$ preserves the set $\{u \mid u_4^2 + u_5^2 = 1 - t^2\} \times \boldsymbol{R}^{j+1}$, so if we integrate along the flow lines of X, the form $\omega_{i-1,j}$ we construct with $d^h(\omega_{i-1,j}) = \omega_{i,j}$ will be a form on $U \times \boldsymbol{R}^{j+1}$ whose restriction to $\{u \mid u_4^2 + u_5^2 = 1 - t^2\} \times \boldsymbol{R}^{j+1}$ is also periodic in each $\boldsymbol{R}$ variable with period $f(t)$, i.e., $\omega_{i-1,j}$ will define a coarse cochain for $\{s^{-1}(U), d, s, U\}$. More specifically, in the definition of $\phi(\omega)(u, t)$ in the Controlled Poincaré Lemma, replace the integral $\left(\int_0^1 s^{r-1} f(su, t)ds\right)$ by the integral $\left(\int_0^1 s^{r-1} f(su_1, su_2, su_3, u_4, u_5, t)ds\right)$. Then we may assume that α is represented by a cochain $\sum_{i+j=k} \omega_{i,j}$ which is completely independent of u. In particular each $\omega_{i,j}$ is a form on $U \times \boldsymbol{R}^{j+1}$ which does satisfies the coarse support condition since it is independent of u and it must do so at $u_4^2 + u_5^2 = 0$. At the same time it must be periodic in each $\boldsymbol{R}$ variable of period $f(t)$ for $t \in (s, 1)$ for some $s < 1$. It is clear that the zero form is the only such form, so $\alpha = 0$. □

REFERENCES

[B] Glen E. Bredon. *Sheaf theory*, McGraw-Hill, New York, 1967.

[HH] James L. Heitsch and Steven Hurder, *Coarse cohomology for families*, Illinois J. of Math., to appear.

[MS] Calvin C. Moore and Claude Schochet, *Global analysis on foliated spaces*. MSRI Pub. Vol. 9, Springer-Verlag, New York, 1988.

[R] John Roe, Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc., no. 104, 1993.

[S] Dennis Sullivan, *A counterexample to the periodic orbit conjecture*, Inst. Hautes Études Sci. Publ. Math. **46** (1976), 5–14.

[W] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*. Scott, Foresman, Glenview, Illinois, 1971.

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