THE SET OF ZEROS OF A SEMISTABLE PROCESS

BY
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Introduction

Let $Y(t)$ be a stable process of index α, $1 < \alpha \leqq 2$, such that $Y(0) = 0$, $EY(t) = 0$, and $Y(t)$ is continuous from the right and has limits from the left. Then

$$\log E\{\exp(i\phi Y(t))\} = -\zeta t \mid \phi \mid^{\alpha}(1 + i\theta(\phi/\mid \phi \mid)\tan \tfrac{1}{2}\pi\alpha), \qquad -\infty < \phi < \infty;$$

where θ and ζ are constants such that $-1 \leqq \theta \leqq 1$ and $\zeta > 0$.

In his 1962 Princeton thesis E. Boylan, in generalizing a theorem of H. Trotter [12], has demonstrated the existence and smoothness of local time for a large class of Markov processes, including the above stable processes.

Let M denote Lebesgue measure.

THEOREM (Boylan). *With probability* 1, *there is a function* $L(t, x)$ *jointly continuous in* t *and* x *such that for any Borel set* A

$$M\{\tau \mid 0 \leqq \tau \leqq t \text{ and } Y(\tau) \in A\} = \int_A L(t, x)\, dx.$$

The random variables $Y(t)$ and $L(t, x)$ are defined on the same probability space. We remove from this space the set of probability zero where the statement of Boylan's theorem fails to hold.

Let $m(x)$, $-\infty < x < \infty$, be a right-continuous strictly increasing function. Let

$$S(t) = \int_{-\infty}^{\infty} L(t, x)\, dm(x),$$

$S^{-1}(t) = \sup\,[\tau \mid S(\tau) \leqq t]$, and $X(t) = Y(S^{-1}(t))$. $S(t)$ and $S^{-1}(t)$ are continuous and strictly increasing, and $X(t)$ is continuous from the right and has limits from the left. If $m(x)$ is absolutely continuous, then

$$S(t) = \int_0^t m'(Y(\tau))\, d\tau.$$

$X(t)$ is a stationary strong Markov process. If $m'(x) = 1$, then $X(t) = Y(t)$. If $\alpha = 2$, $X(t)$ is a diffusion process with infinitesimal generator $\zeta\, D_m D_x$ (cf. [5]).

This method of construction has been fundamental in the theory of diffusion processes as developed by Itô and McKean [5]. In Section 1 we shall see that some of the results for diffusion processes carry over to these more general processes.

Received June 29, 1962.

In Section 2 we consider $X(t)$ corresponding to $m'(x)$ of a specific form. Many of the classical properties of Brownian motion in Lévy [8] are shown to hold also for these processes.

As a special case of this form, $m'(x) = 1$, so that the results of Section 2 are valid for the above stable processes. Some of these results for the symmetric stable processes ($\theta = 0$) are in papers by Blumenthal and Getoor [2] and Getoor [4].

In the case $\alpha = 2$, the results of Section 2 are all used in [11] to obtain limit theorems for the processes being considered there. Under the further mild restriction $\rho_1 = \rho_2$, most of these results have been obtained by Itô and McKean [5] and Lamperti [6].

The processes in Section 2 satisfy the following property (see Theorem 2): for any $c > 0$, $c^{-1/\alpha+\beta}X(ct)$ has the same finite-dimensional distributions as $X(t)$. Lamperti [7] calls any process satisfying such a property "semi-stable." He shows that semistable processes arise naturally in certain limit theorems (as in [11]). He also investigates the set of zeros of such a process.

1. Results for general $m(x)$

Let $L_X(t, x) = L(S^{-1}(t), x)$. Then $L_X(t, x)$ is jointly continuous in t and x. For any Borel set A

$$M\{\tau \mid 0 \leqq \tau \leqq t \text{ and } X(\tau) \in A\} = \int_A L_X(t, x)\, dm(x).$$

Proof. Let $f(x)$ be a continuous function. Suppose first that $m(x)$ is absolutely continuous. Then

$$\begin{aligned}\int_0^{S(t)} f(X(\tau))\, d\tau &= \int_0^{S(t)} f(Y(S^{-1}(\tau)))\, d\tau \\ &= \int_0^t f(Y(\tau))\, d_\tau S(\tau) = \int_0^t f(Y(\tau))\, d_\tau \int_0^\tau m'(Y(s))\, ds \\ &= \int_0^t f(Y(\tau))m'(Y(\tau))\, d\tau = \int_{-\infty}^{\infty} f(x)L(t, x)\, dm(x).\end{aligned}$$

To prove that

$$\int_0^{S(t)} f(Y(S^{-1}(\tau)))\, d\tau = \int_{-\infty}^{\infty} f(x)L(t, x)\, dm(x)$$

for general $m(x)$, it suffices to approximate $m(x)$ by absolutely continuous functions and pass to the limit on both sides. By another extension we can show that the above equation is valid whenever $f(x)$ is the characteristic function of a Borel set A. This completes the proof.

Let $L(t) = L_X(t, 0)$ and $L^{-1}(t) = \sup[\tau \mid L(\tau) \leqq t]$. Then[1] $L^{-1}(0) = 0$;

[1] By Theorem 1 the statements in this paragraph involving $L^{-1}(t)$ hold with probability 1.

$L^{-1}(t)$ is strictly increasing and continuous from the right and has limits from the left.

$$L(t) = \sup [\tau \mid L^{-1}(\tau) \leqq t] = \inf [\tau \mid L^{-1}(\tau) \geqq t].$$

Let $Z(t) = \{\tau \mid \tau \leqq t \text{ and } X(\tau) = 0\}$, and let $R(t)$ be the set of points of increase of $L(t)$ up to time t or equivalently the closure of the intersection of $[0, t]$ with the range of $L^{-1}(t)$.

THEOREM 1. *With probability* 1

(1) $Z(t) = \{\tau \mid \tau \leqq t \text{ and } X(\tau-) = 0\}$;

(2) $L(t) > 0 \quad \text{for} \quad t > 0$;

(3) $Z(t) = R(t)$;

(4) $\inf [t \mid t > 0 \text{ and } X(t) = 0] = 0$;

(5) $\sup [t \mid X(t) = 0] = +\infty$;

(6) $L(t) \to +\infty \quad \text{as} \quad t \to +\infty$;

(7) $L^{-1}(t)$ *is an additive process.*

Proof. We first prove (1)–(6) for the process $Y(t)$, which corresponds to $m(x) = x$. In this case we have $S(t) = S^{-1}(t) = t$, $X(t) = Y(t)$, and $L(t) = L_X(t, 0) = L(t, 0)$.

Choose $\varepsilon > 0$. With probability 1, $J_\varepsilon = \{t : | Y(t) - Y(t-)| > \varepsilon\}$ is a countable set. By the independence properties of the jumps of $Y(t)$ (cf. Loève [9, p. 550])

$$P\{\exists\, t \text{ in } J_\varepsilon : Y(t-) = 0 \quad \text{or} \quad Y(t) = 0\} = 0.$$

Since ε can be made arbitrarily small, this proves (1).

Suppose (2) is false. Then, with probability 1, $L(t, 0) = 0$ for $t > 0$. This follows from the Blumenthal zero-one law, the continuity of $L(t, 0)$, and the strong Markov property of $Y(t)$. Using the homogeneity in space of $Y(t)$, we obtain that if $\{x_n\}$ is a countable dense set of $(-\infty, \infty)$, then with probability 1, $L(t, x_n) = 0$ for all t and x_n. Since $L(t, x)$ is jointly continuous in t and x, it follows that with probability 1, $L(t, x) = 0$ for all t and x. But this is impossible since

$$\int_{-\infty}^{\infty} L(t, x)\, dx = t.$$

It is now easy to show that $Z(t)$ and $R(t)$ are both closed and each is dense in the other, so that $Z(t) = R(t)$. (4) follows trivially from (2) and (3). (5) and (6) follow from the interval recurrence of $Y(t)$.

This completes the proof of (1)–(6) for $Y(t)$. These results can be immediately extended to $X(t)$.

We now prove (7) of Theorem 1 for general $X(t)$.

$$L^{-1}(t-) = \inf [\tau \mid L(\tau) = t]$$

is a Markov time, and with probability 1, $X(L^{-1}(t-)) = 0$. Thus with probability 1, $L(\tau) > t$ for $\tau > L^{-1}(t-)$ and $L^{-1}(t) = L^{-1}(t-)$; hence $L^{-1}(t)$ is a Markov time. Therefore, for $0 \leqq t_1 < t_2$

$$L^{-1}(t_2) - L^{-1}(t_1) = \sup [\tau \mid L(\tau + L^{-1}(t_1)) - L(L^{-1}(t_1)) \leqq t_2 - t_1]$$

is independent of $X(\tau)$ for $0 \leqq \tau \leqq L^{-1}(t)$ and consequently of $L^{-1}(\tau)$ for $0 \leqq \tau \leqq t$. This completes the proof of (7).

2. Semistable processes

In this section we assume that $m(x)$ is absolutely continuous and $m'(x)$ is of the specific form

$$\begin{aligned} m'(x) &= \rho_1(-x)^\beta \quad \text{if} \quad -\infty < x < 0, \\ &= \rho_2\, x^\beta \quad\ \ \text{if} \quad 0 < x < \infty, \end{aligned}$$

where $\beta + 1$, ρ_1, and ρ_2 are positive constants.

Let $\nu = \alpha - 1/\alpha + \beta$. We make the notational convention that

$$A + B/C + D = (A + B)/(C + D).$$

Let

$$E_\nu(\lambda) = \textstyle\sum_{n=0}^{\infty} (-\lambda)^n/\Gamma(n\nu + 1).$$

$E_\nu(\lambda)$ is the Laplace transform of the Mittag-Leffler distribution, which has density zero for $x \leqq 0$ and

$$(1/\pi\nu) \textstyle\sum_{n=1}^{\infty} ((-1)^{n-1}/n!) \sin \pi\nu n \Gamma(n\nu + 1) x^{n-1}, \qquad x \geqq 0.$$

Given two stochastic processes, $X_1(t)$ and $X_2(t)$, whose paths are continuous from the right and have limits from the left, we say they are equivalent ($X_1(t) \equiv X_2(t)$) if they have the same finite-dimensional distributions. Two such equivalent processes assign the same measure to the space of path functions.

We now eliminate the exceptional set of probability zero where the statements of Theorem 1 fail to hold. Given $t > 0$, let

$$t' = \inf [\tau \mid \tau \geqq t \quad \text{and} \quad X(\tau) = 0].$$

Then $Z(t')$ is a closed set. Its complement in $[0, t']$ is an open set and hence the union of a countable number of disjoint open intervals. Let $N(t, \varepsilon)$ be the number of these intervals of length greater than ε. Let

$$N(t) = \lim_{\varepsilon \to 0} \varepsilon^\nu N(t, \varepsilon)$$

provided that this limit exists.

THEOREM 2. *If $c > 0$, then*

$$X(ct) \equiv c^{1/\alpha+\beta} X(t), \qquad L(ct) \equiv c^\nu L(t), \quad \textit{and} \quad L^{-1}(ct) \equiv c^{1/\nu} L^{-1}(t).$$

$L^{-1}(t)$ is an increasing stable process of order ν. For $t > 0$ and $\lambda > 0$

$$\log E\{\exp(-\lambda \delta^{1/\nu} t^{-1/\nu} L^{-1}(t))\} = -\lambda^\nu,$$

and

$$E\{\exp(-\lambda\delta^{-1}t^{-\nu}L(t))\} = E_\nu(\lambda),$$

where δ *is a positive constant independent of* t *and* λ. *For* $0 < t_0 < t_1$

$$P\{\exists\, t \text{ in } (t_0, t_1) : X(t) = 0\} = (\Gamma(\nu)\Gamma(1-\nu))^{-1}\int_0^{t_1 - t_0/t_0} s^{-\nu}(1+s)^{-1}\, ds$$

$$(= (2/\pi)\cos^{-1}(t_0/t_1)^{1/2} \quad \text{for } \nu = \tfrac{1}{2}).$$

With probability 1, *for* $t > 0$, $Z(t)$ *has Hausdorff-Besicovitch dimension* ν, $N(t)$ *exists, and* $\delta\Gamma(1-\nu)N(t) = L(t)$, *and hence*

$$E\{\exp(-\lambda\Gamma(1-\nu)t^{-\nu}N(t))\} = E_\nu(\lambda).$$

Proof. Observing the expression for $\log E\{\exp(i\phi Y(t))\}$, we see that $Y(ct) \equiv c^{1/\alpha}Y(t)$ for $c > 0$. In general

$$\begin{aligned}
X(ct) &= Y(S^{-1}(ct)) = Y\left(\sup\left[\tau \,\middle|\, \int_0^\tau m'(Y(s))\,ds \leqq ct\right]\right)\\
&= Y\left(c^{\alpha/\alpha+\beta}\sup\left[\tau \,\middle|\, \int_0^{\tau c^{\alpha/\alpha+\beta}} m'(Y(s))\,ds \leqq ct\right]\right)\\
&= Y\left(c^{\alpha/\alpha+\beta}\sup\left[\tau \,\middle|\, c^{-\beta/\alpha+\beta}\int_0^{\tau} m'(Y(c^{\alpha/\alpha+\beta}s))\,ds \leqq t\right]\right)\\
&= Y\left(c^{\alpha/\alpha+\beta}\sup\left[\tau \,\middle|\, \int_0^{\tau} m'(c^{-1/\alpha+\beta}Y(c^{\alpha/\alpha+\beta}s))\,ds \leqq t\right]\right)\\
&\equiv c^{1/\alpha+\beta}Y\left(\sup\left[\tau \,\middle|\, \int_0^{\tau} m'(Y(s))\,ds \leqq t\right]\right) = c^{1/\alpha+\beta}X(t).
\end{aligned}$$

Let $\kappa = \beta + 1/\rho_1 + \rho_2$. Then

$$\begin{aligned}
L(ct) &= \lim_{\varepsilon\downarrow 0}\kappa\varepsilon^{-\beta-1}M\{\tau \mid 0 \leqq \tau \leqq ct \quad\text{and}\quad |X(\tau)| \leqq \varepsilon\}\\
&= c^{-\beta-1/\alpha+\beta}\lim_{\varepsilon\downarrow 0}\kappa\varepsilon^{-\beta-1}M\{\tau \mid 0 \leqq \tau \leqq ct \quad\text{and}\quad |X(\tau)| \leqq c^{1/\alpha+\beta}\varepsilon\}\\
&= c^{\alpha-1/\alpha+\beta}\lim_{\varepsilon\downarrow 0}\kappa\varepsilon^{-\beta-1}M\{\tau \mid 0 \leqq \tau \leqq t \quad\text{and}\quad |X(c\tau)| \leqq c^{1/\alpha+\beta}\varepsilon\}\\
&\equiv c^{\nu}\lim_{\varepsilon\downarrow 0}\kappa\varepsilon^{-\beta-1}M\{\tau \mid 0 \leqq \tau \leqq t \quad\text{and}\quad |X(\tau)| \leqq \varepsilon\} = c^{\nu}L(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
L^{-1}(ct) &= \sup[\tau \mid L(\tau) \leqq ct] = c^{1/\nu}\sup[\tau \mid L(c^{1/\nu}\tau) \leqq ct]\\
&\equiv c^{1/\nu}\sup[\tau \mid L(\tau) \leqq t] = c^{1/\nu}L^{-1}(t).
\end{aligned}$$

Therefore, by (7) of Theorem 1, $L^{-1}(t)$ is an increasing stable process of order ν, and its Laplace transform is of the desired form. Since

$$\delta^{-1}t^{-\nu}L(t) = L(\delta^{-1/\nu}) = \sup[\tau \mid \delta^{1/\nu}L^{-1}(\tau) \leqq 1],$$

the Laplace transform of $\delta^{-1}t^{-\nu}L(t)$ is also of the desired form (cf. Pollard

[10]). By (3) of Theorem 1, the next two statements of Theorem 2 are equivalent to analogous statements for the range of an increasing stable process, for proof of which see [1] and [2].

Let $M(s, \varepsilon) = N(L^{-1}(s), \varepsilon)$. Then $N(t, \varepsilon) = M(L(t), \varepsilon)$. $M(s, \varepsilon)$ is the number of jumps of $L^{-1}(\tau)$ of length greater than ε up to and including time s. $M(s, \varepsilon)$ has a Poisson distribution (cf. [9, pp. 329–330 and 550]) with mean

$$-s(\Gamma(-\nu))^{-1} \int_{\delta^{1/\nu}\varepsilon}^{\infty} x^{-\nu-1}\, dx = s(\delta\Gamma(1-\nu))^{-1}\varepsilon^{-\nu}.$$

Since the number of jumps of $L^{-1}(\tau)$ of lengths belonging to disjoint intervals are mutually independent, we can apply the strong law of large numbers. We obtain that with probability 1

$$\delta\Gamma(1-\nu)\lim_{\varepsilon\to 0} \varepsilon^{\nu} M(s, \varepsilon) = s.$$

With probability 1, this limit holds simultaneously for all $s \geqq 0$. Thus we can let $s = L(t)$ and obtain

$$\delta\Gamma(1-\nu)\lim_{\varepsilon\to 0} \varepsilon^{\nu} N(t, \varepsilon) = L(t).$$

This completes the proof of Theorem 2.

Finally, we shall consider the computation of δ. We note first that

$$E\{L(t)\} = -\delta t^{\nu} E_{\nu}'(0) = \delta t^{\nu}(\Gamma(\nu+1))^{-1}.$$

Suppose that the distribution of $X(t)$, $t > 0$, has a density $p(t, x)$ with respect to the measure $m(dx)$, and that this density is jointly continuous in t and x. Under these conditions it is not hard to show that

$$E\{L(t)\} = \int_0^t p(\tau, 0)\, d\tau.$$

Consequently

$$\delta = \nu^{-1}\Gamma(\nu+1)t^{1-\nu}\frac{d}{dt}E\{L(t)\} = \Gamma(\nu)p(1, 0),$$

or alternatively

$$\delta = \int_0^{\infty} e^{-t} E\{L(t)\}\, dt = \int_0^{\infty} e^{-t} p(t, 0)\, dt.$$

In the special case $m'(x) = 1$, $X(t) = Y(t)$, and the above supposition is valid. Using the Fourier inversion formula and recalling that $\nu = \alpha - 1/\alpha$ in this case, we obtain

$$\begin{aligned}\delta &= \Gamma\left(1-\frac{1}{\alpha}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\zeta\,|\,\phi\,|^{\alpha}\left(1+i\theta\frac{\phi}{|\,\phi\,|}\tan\frac{\pi\alpha}{2}\right)\right)d\phi \\ &= \pi^{-1}\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)\zeta^{-1/\alpha}\operatorname{Re}\left(\left(1+i\theta\tan\frac{\pi\alpha}{2}\right)^{-1/\alpha}\right).\end{aligned}$$

If we specialize still further by setting $\alpha = 2$, $\theta = 0$, and $\zeta = \frac{1}{2}$, then $\delta = 2^{-1/2}$.

$Y(t)$ is now Brownian motion, and the results of Theorem 2 agree with those found in Lévy [8, pp. 209–241].

Let $m'(x)$ be as defined at the beginning of this section, but choose $\alpha = 2$ and $\zeta = 1$. Then the above supposition is again valid. We can obtain

$$\delta = \int_0^\infty e^{-t} p(t, 0)\, dt = \frac{\Gamma(1 + \nu)}{\Gamma(1 - \nu)\nu^{2\nu}(\rho_1^\nu + \rho_2^\nu)} .$$

This computation is given in [11].

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