ON THE HOMOLOGY DECOMPOSITION OF POLYHEDRA[1]

BY

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Introduction. Summary

Let X_1 be a simply connected polyhedron (i.e., a simply connected finite CW-complex). According to Eckmann-Hilton [6]—see also Brown-Copeland [3]—it is homotopy-equivalent to a polyhedron X built up by subcomplexes X_i, $X_2 \subset X_3 \subset \cdots \subset X_N = X$, where X_r is constructed out of X_{r-1} in a very perspicuous way by means of the r^{th} integer homology group of X and an element in a homotopy group of X_{r-1}. Following [6] we call $X = \{X_r\}$ a normal polyhedron, and the collection $\{X_r\}$ of the X_r a homology decomposition of X.

It is the purpose of this note to exemplify our opinion that the concept of the homology decomposition can be used profitably to study homotopy sets $\Pi(X, Y)$ of the maps of a space X into a space Y.

All considerations rely on Proposition 2.2 which describes the circumstances under which a map $f : X \to Y$ of the normal polyhedra $X = \{X_r\}$, $Y = \{Y_r\}$ induces a map $f_r : X_r \to Y_r$ compatible with f. Proposition 2.2 follows from Proposition 2.1, which generalizes the Blakers-Massey theorem on relative homotopy groups [2, p. 198].

Section 3 contains the first example of an application of the homology decomposition. Proposition 3.3 is a powerful lemma of Thom [10, p. 59], for which we give a new proof. The idea of our proof is to climb up a homology decomposition, using at each step known facts about homotopy groups of spheres.

From Section 4 on, we restrict our attention to "selfmaps" $f : X \to X$ of a simply connected polyhedron X. The composition of maps defines in the homotopy set $\Pi(X, X)$ a multiplication turning $\Pi(X, X)$ into a monoid. Denote by $T(X)$ the homotopy set of all selfmaps of X which induce the trivial endomorphism of $\oplus_k H^k(X; H_k(X))$. It is a multiplicatively closed subset of $\Pi(X, X)$. Theorem 4.2 states that $T(X)$ is nilpotent. The order $t(X)$ of nilpotency of $T(X)$ is a homotopy invariant of X which, by appealing to a theorem of Novikov [8], can be shown to assume any given value for an appropriate X (Proposition 4.5).

An endomorphism Φ of $\oplus_k H^k(X; H_k(X))$ induced by a map $f : X \to X$ satisfies necessarily a certain relation, and such a Φ will be called admissible (Definition 4.6, Lemma 4.7). The question for which spaces X every admissible endomorphism of $\oplus_k H^k(X; H_k(X))$ can be realized by a selfmap of X is dealt with in Theorem 4.9: For 2-connected X this is the case if and

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only if X is homotopy-equivalent to a wedge of Moore-spaces (polyhedra with exactly one nonvanishing integer homology group) and

$$\mathrm{Ext}(H_r, H_{r+1}) = 0$$

for all r.

In Section 5 we consider selfmaps of a suspension ΣX. Its natural comultiplication defines in $\Pi(\Sigma X, \Sigma X)$ a group operation in addition to the multiplication by composition of maps, and $\Pi(\Sigma X, \Sigma X)$ is a near-ring (i.e., a set with two binary operations coupled by only one law of distributivity; see e.g. [1]). By using the distributor series of a near-ring (Definition 5.1, compare [7]) Theorem 5.3 measures to what extent $\Pi(\Sigma X, \Sigma X)$ fails to be a ring. It furthermore suggests the introduction of an integer-valued homotopy invariant $d(X)$ of X which is equal to one in case $\Pi(\Sigma X, \Sigma X)$ is a ring (Definition 5.2).

In the sense of the Eckmann-Hilton duality [4] everything carries over under considerable simplifications to spaces with a finite number of homotopy groups.

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1. Notations. Definitions

(a) All spaces considered are polyhedra and have a basepoint 0 which is respected by maps $f, g, \cdots$ and their homotopies $f \sim g, \cdots$. The trivial map $X \to 0 \in Y$ is also denoted by 0. A map $f : X \to Y$ and its homotopy class in the homotopy set $\Pi(X, Y)$ will usually not be distinguished. Σ stands for the suspension operators for spaces, maps, and homotopy sets. The space obtained by attaching the cone CA over A to X by means of the map (or class) f is denoted by $CA \cup_f X$.

(b) For Moore-spaces (polyhedra with exactly one nontrivial integer homology group in dimension $r > 1$) consult e.g. [9]. We denote them by $K'(G, r)$ or simply K'_r, L'_r. Their groups $G, H, \cdots$ as well as all coefficient groups considered will be supposed to be finitely generated. The "homotopy groups of X with coefficients G" are defined as $\Pi(K'(G, r), X)$ and are related to the ordinary homotopy groups $\pi_r(X)$ by an exact sequence [5], "the coefficient formula for homotopy groups". For $\Pi(X, K'(G, r))$, the cohomotopy groups with coefficients G, and their coefficient formula, see [9].

(c) The inclusion

$$i : X_r \to X = CK'(G, r) \cup_\alpha X_r$$

is called an elementary cofibration (with attaching class or map α and projection $\phi : X \to X/X_r = K'(G, r + 1))$ if

(1) X_r is simply connected;
(2) $\dim X_r \leqq r + 1$ and $H_{r+1}(X_r) = 0$;
(3) $i_* : H_k(X_r) \cong H_k(X)$ for $k \leqq r$.

It follows from (3) that

$$\phi_* : H_{r+1}(X) \cong H_{r+1}(K'(G, r + 1)) \cong G.$$

A normal polyhedron $X = \{X_r\}$ is one obtained by a sequence

$$X_2 \to X_3 \to \cdots \to X_N = X$$

of elementary cofibrations. The groups of the cofibers X_r/X_{r-1} are the integer homology groups of X; if $H_s(X) = 0$ for some s, we insert a term $X_s = X_{s-1}$ so that we have $X_r/X_{r-1} = K'(H_r(X), r)$ for all r. Denote by $i_r : X_r \to X$ the imbedding; $i_r^* : H^k(X; G) \to H^k(X_r ; G)$, G arbitrary, is isomorphic for $k \leqq r$ and epimorphic for $k \geqq r + 1$. The r^{th} fundamental class $h^r \in H^r(X; H_r(X))$ of the normal polyhedron $X = \{X_r\}$ is defined as

$$h^r = i_r^{*-1}\phi_r^* h_1^r ,$$

where h_1^r is the fundamental class of $H^r(K'(H_r(X), r); H_r(X))$, and ϕ_r the projection $X_r \to K'(H_r(X), r)$; for arbitrary G the induced

$$\phi_r^* : H^r(K'(H_r(X), r); G) \to H^r(X_r ; G)$$

is monomorphic. Distinguish X_r and the r-skeleton X^r.

2. Compression problems in cofibrations

For $X = S^N$ the following proposition reduces to Theorem II of [2].

PROPOSITION 2.1. *Let X be an N-dimensional polyhedron, and $j : A \to Y$ a cofibration with projection $\psi : Y \to Y/A$ such that*

$$Y \text{ is } (N - r)\text{-connected},$$
$$Y/A \text{ is } r\text{-connected},$$
$$N - r \leqq r.$$

Then the sequence of homotopy sets

$$\Pi(X, A) \xrightarrow{j_*} \Pi(X, Y) \xrightarrow{\psi_*} \Pi(X, Y/A)$$

is exact: If $\psi_* f = 0$, then f is compressible into A.

Proof.[2] Replace ψ by a fibration $p : E \to Z_\psi$, where ψ is the mapping

[2] A proof is also possible by methods developed by I. Namioka in *Maps of pairs in homotopy theory,* Proc. London Math. Soc. (3), vol. 12 (1962), pp. 725–738.

cylinder of ψ, and E the space of all paths in Z_ψ which start in

$$Y = Y \times 0 \subset Z_\psi .$$

The fibre F of p consists of the paths in E ending at the basepoint of Z_ψ. Let $\tau : Y/A \to Z_\psi$ be the imbedding, and $\sigma : Y \to E$ the map defined by $\sigma y = y \times I \in E$; σ induces a map $\rho : A \to F$. Consider the homotopy sequences of the pairs (Y, A) and (E, F):

$$\begin{array}{ccccccccccc}
\to \pi_k(Y) & \to & \pi_k(Y, A) & \to & \pi_{k-1}(A) & \to & \pi_{k-1}(Y) & \to & \pi_{k-1}(Y, A) & \to \\
\downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle (\sigma, \rho)_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle (\sigma, \rho)_*} & \\
\to \pi_k(E) & \to & \pi_k(E, F) & \to & \pi_{k-1}(F) & \to & \pi_{k-1}(E) & \to & \pi_{k-1}(E, F) & \to .
\end{array}$$

In view of $\pi_k(E, F) \cong \pi_k(Z_\psi)$ and the Blakers-Massey theorem, $(\sigma, \rho)_*$ is isomorphic for $k \leqq N$ and epimorphic for $k = N + 1$. By the five-lemma ρ_* is epimorphic on $\pi_N(A)$ and isomorphic in the lower dimensions. Hence the map ρ is N-connected:

$$\to \pi_N(A) \xrightarrow{\rho_*} \pi_N(F) \to \pi_N(\rho) \to \pi_{N-1}(A) \xrightarrow{\rho_*} \pi_{N-1}(F) \to .$$

Replace ρ by the inclusion of A into the mapping cylinder Z_ρ to infer that under these circumstances any map $f : X \to F$ can be factored up to homotopy through ρ if $\dim X \leqq N$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & F \\ & A \arrow[u, "\rho"'] \end{tikzcd} \qquad f \sim \rho g.$$

This means that ρ_* is epimorphic on $\Pi(X, A)$:

$$\begin{array}{ccccc}
\Pi(X, A) & \xrightarrow{j_*} & \Pi(X, Y) & \xrightarrow{\psi_*} & \Pi(X, Y/A) \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \tau_*} \\
\Pi(X, F) & \longrightarrow & \Pi(X, E) & \longrightarrow & \Pi(X, Z_\psi).
\end{array}$$

The lower line of this diagram is exact; σ_* and τ_* are one-to-one correspondences, and ρ_* is epimorphic. Therefore the upper line is also exact.

We now apply Proposition 2.1 to maps of normal polyhedra.

PROPOSITION 2.2. *Let $f : X \to Y$ be a map of the normal polyhedra $X = \{X_r\}$, $Y = \{Y_r\}$, and let $h^{r+1} \in H^{r+1}(Y;\, H_{r+1}(Y))$ be the $(r + 1)^{st}$ fundamental class of $Y = \{Y_r\}$ (see Section 1(c)). There exists a map $f_r : X_r \to Y_r$ compatible with f, $f i_r \sim j_r f_r$:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_r \arrow[u, "i_r"] \arrow[r, "f_r"] & Y_r \arrow[u, "j_r"'] \end{tikzcd}$$

if and only if

$$i_r^* f^* h^{r+1} = 0 \in H^{r+1}(X_r\,;H_{r+1}(Y)) \cong \operatorname{Ext}(H_r(X), H_{r+1}(Y)).$$

Proof. (a) Suppose there is a map f_r with $fi_r \sim j_r f_r$. Write

$$L'_{r+1} = K'(H_{r+1}(Y), r+1),$$

and pass to cohomology with $H_{r+1}(Y)$ as coefficients:

$$\begin{tikzcd} H^{r+1}(X) \arrow[dd, "i_r^*"'] & H^{r+1}(Y) \arrow[l, "f^*"'] \arrow[d, "j_{r+1}^*"] & \\ & H^{r+1}(Y_{r+1}) \arrow[d, "j_{r,r+1}^*"] & H^{r+1}(L'_{r+1}). \arrow[l, "\psi_r"'] \\ H^{r+1}(X_r) & H^{r+1}(Y_r) \arrow[l, "f_r^*"'] & \end{tikzcd}$$

By definition $h^{r+1} = j_{r+1}^{*-1}\psi_r^* h_1^{r+1}$, where h_1^{r+1} denotes the fundamental class of $H^{r+1}(L'_{r+1})$. Together with $i_r^* f^* j_{r+1}^{*-1} = f_r^* j_{r,r+1}^*$ we have

$$i_r^* f^* h^{r+1} = i_r^* f^* j_{r+1}^{*-1} \psi_r^* h_1^{r+1} = f_r^* j_{r,r+1}^* \psi_r^* h_1^{r+1} = 0$$

because of $\psi_r j_{r,r+1} = 0$.

(b) By the cellular approximation theorem there exists a map $g_r : X_r \to Y_{r+1}$ with $fi_r \sim j_{r+1} g_r$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y & \\ & Y_{r+1} \arrow[u, "j_{r+1}"'] \arrow[r, "\psi_r"] & L'_{r+1}\,. \\ X_r \arrow[uu, "i_r"] \arrow[ur, "g_r"] & Y_r \arrow[u, "j_{r,r+1}"'] & \end{tikzcd}$$

By hypothesis $i_r^* f^* h^{r+1} = 0$. Therefore, with h_1^{r+1} as in (a), we have

$$g_r^* \psi_r^* h_1^{r+1} = i_r^* f^* j_{r+1}^{*-1} \psi_r^* h_1^{r+1} = 0.$$

Because any map $k : X_r \to L'_{r+1}$ is homotopic to zero if and only if $k^* h_1^{r+1} = 0$, we conclude $\psi_r g_r \sim 0$. By Proposition 2.1 it follows from $\psi_r g_r \sim 0$ that g_r is compressible into Y_r, i.e., that there exists a map $f_r : X_r \to Y_r$ with $g_r \sim j_{r,r+1} f_r$, and the proof of Proposition 2.2 is complete.

COROLLARY 2.3. *Let $f : X \to Y$ be a map of the normal polyhedra $X = \{X_r\}$, $Y = \{Y_r\}$. Suppose that f induces the trivial homomorphism on $\oplus_k H^k(Y; H_k(Y))$. Then there exist maps $f_r : X_r \to Y_r$ and $f'_r : K'_r \to L'_r$ such that*

$$f_r\, i_{r-1,r} \sim j_{r-1,r}\, f_{r-1} \quad \text{and} \quad f'_r\, \phi_r \sim \psi_r\, f_r$$

for all elementary cofibrations $i_{r-1,r}$ of X and $j_{r-1,r}$ of Y:

$$\begin{array}{ccc} K'_r & \xrightarrow{f'_r} & L'_r \\ \phi_r \Big\uparrow & & \Big\uparrow \psi_r \\ X_r & \xrightarrow{f_r} & Y_r \\ i_{r-1,r} \Big\uparrow & & \Big\uparrow j_{r-1,r} \\ X_{r-1} & \xrightarrow{f_{r-1}} & Y_{r-1}\,. \end{array}$$

Proof. Since $i_r^* f^* h^{r+1} = 0$ for all r, there are, by Proposition 2.2, maps $f_r : X_r \to Y_r$ for all r. By carrying out the compressions from the cofibration at the "top" on downwards, the map f_{r-1} will be compatible with f_r. In view of $\psi_r f_r\, i_{r-1,r} \sim \psi_r\, j_{r-1,r} f_{r-1} = 0$ and the homotopy extension property of $i_{r-1,r}$, there exists a map f'_r with $f'_r \phi_r \sim \psi_r f_r$.

3. A lemma of Thom

The following Proposition 3.3 is a lemma of Thom [10, p. 59] for which we give a proof by induction on homology decompositions. Since the lemma deals with the homology homomorphisms induced by maps, we first consider Lemmata 3.1 and 3.2 which follow directly from the coefficient formulae for cohomology and homotopy groups (for the latter see [5]). Note that any induced homomorphism is always an element in an abelian group Hom(,).

LEMMA 3.1. *Let $f : \Sigma X \to Y$ be a map. Suppose $f_* \mid H_k(\Sigma X)$ of finite order for $k = r - 1, r$. Then $f^* \mid H^r(Y; G)$ is also of finite order for all coefficients G.*

LEMMA 3.2. *Consider a map f of Moore-spaces, $f : K'(G, r) \to K'(H, r)$. Then we have*

(a) $f^* \mid H^r(K'(H, r); H) \equiv 0$ *if and only if* $f_* \mid H_r(K'(G, r)) \equiv 0$.

(b) *If $f_* \mid H_r(K'(G, r)) \equiv 0$, then there exists an integer m such that $mf \sim 0$.*

PROPOSITION 3.3 (Thom). *Let X, Y be polyhedra, X of dimension*

$\leqq 2n - 2$ *and* Y *at least* $(n-1)$*-connected. Suppose* $f_* \mid H_*(X)$ *of finite order. Then* f *itself is of finite order in the group*

$$\Pi(X, Y) \cong \Pi(\Sigma X, \Sigma Y) \cong \cdots .$$

Proof. (a) It is no restriction to assume Y simply connected and $X = \Sigma^2 X'$. Let $\{X_r\}$ be the suspension of a homology decomposition of $\Sigma X'$, and $\{Y_r\}$ any decomposition of Y. All X_r are suspensions, and all

$$\phi_r : X_r \rightarrow K'(H_r(X), r)$$

are suspended maps. If for $f, g : X \rightarrow Y$ there exist maps

$$f_r, g_r : X_r \rightarrow Y_r ,$$

then this is also the case for $f + g$, and we can take $(f+g)_r = f_r + g_r$.

(b) Let $j : Y_{r-1} \rightarrow Y_r$ with projection $\psi : Y_r \rightarrow L'_r$ be part of $\{Y_r\}$, and K'_s one of the cofibers of $\{X_r\}$. Then we have

(1) The sequence $\Pi(K'_s, Y_{r-1}) \xrightarrow{j_*} \Pi(K'_s, Y_r) \xrightarrow{\psi_*} \Pi(K'_s, L'_r)$ is exact.
(2) The group $\Pi(K'_s, Y_r)$ is finite for $s > r$.

(1) follows from the Blakers-Massey theorem in [2, p. 198]; (2) is an easy consequence of (1), the fact that $\pi_r(S^m)$ is finite for $r \leqq 2m - 2$, and the coefficient formulae for homotopy and cohomotopy groups.

(c) Let $f'_* \mid H_*(X)$ be of finite order. By Lemma 3.1 there exists an integer q such that $(qf')^* = f^* = 0$ on $H^k(Y; H_k(Y))$ for all k. By Corollary 2.3 there are maps $f_r : X_r \rightarrow Y_r$ compatible with each other for all r.

To anchor the induction let X be a Moore-space, $X = K'(G, s)$. There are two possibilities (1), (2) for the corresponding Y_s:

(1) $Y_s = K'(H, s)$. We know that $f_s^* \mid H^s(K'(H, s); H) = 0$. Apply Lemma 3.2(a) and (b) to conclude that $mf \sim 0$.

(2) Y_s has several homology groups (write $\psi = \psi_s$, $j = j_{s-1,s}$):

$$\begin{tikzcd} & Y & L'_s \\ K'(G,s) \arrow[ur, "f"] \arrow[r, "f_s"] \arrow[dr, dashed, "g"'] & Y_s \arrow[u] \arrow[ur, "\psi"'] & \\ & Y_{s-1} \arrow[u, "j"'] & \end{tikzcd} .$$

In view of $(\psi f_s)^* \mid H^s(L'_s; H_s(Y)) \equiv 0$ and Lemma 3.2(a) and (b) there is an integer m such that $m(\psi f_s) = m(\psi_* f_s) = \psi_* mf_s = 0$. By remark (1) of (b) we infer $mf_s = j_* g$, $g \in \Pi(K'(G, s), Y_{s-1})$, and this element g is of finite order by remark (2) of (b). Thus $mf_s = j_* g$ and, consequently, mf are also of finite order.

(d) For the induction step we have to prove: Suppose $f_r : X_r \to Y_r$ of finite order, and, of course, $f^*_{r+1} \mid H^{r+1}(Y_{r+1}, H_{r+1}(Y)) \equiv 0$. Then f_{r+1} is also of finite order (write $\phi = \phi_r$, $i_{r,r+1} = i$, etc.):

$$\begin{tikzcd}
K'_{r+1} \arrow[dr, dashed, "g"] \arrow[ddr, dashed, "h"'] & L'_{r+1} \\
X_{r+1} \arrow[u, "\phi"] \arrow[r, "f_{r+1}"] & Y_{r+1} \arrow[u, "\psi"'] \\
X_r \arrow[u, "i"] \arrow[r, "f_r"] & Y_r . \arrow[u, "j"']
\end{tikzcd}$$

We can assume $f_r \sim 0$. Therefore $f_{r+1} \sim g\phi$ (see diagram) because of the homotopy extension property of i and the relation $f_{r+1}\, i \sim jf_r \sim 0$. Since ϕ^* is monomorphic on the $(r + 1)$-dimensional cohomology, we conclude $(\psi g)^* \mid H^{r+1}(L'_{r+1}\, ; H_{r+1}(Y)) \equiv 0$, and by Lemma 3.2 also

$$m(\psi g) = m(\psi_* g) = \psi_*(mg) = 0.$$

Again by (b) we have $mg = j_* h$ with $h \in \Pi(K'_{r+1}, Y_r)$ of finite order. Since ϕ is a suspended map, ϕ^* is homomorphic. Therefore $mf_{r+1} = \phi^*(mg) = \phi^* j_* h$, which means that mf_{r+1} is of finite order as the homomorphic image of h, and the proof of Proposition 3.3 is complete.

4. Selfmaps of polyhedra. The invariant $t(X)$

The composition of maps defines in $\Pi(X, X)$ the structure of a monoid with identity and zero element. Consider the cohomology functor $\mathcal{H}^*$ which maps the monoid $\Pi(X, X)$ homomorphically into the multiplicative structure of the ring of endomorphisms of $\oplus_k H^k(X; H_k)$, $H_k = H_k(X)$. The set $\mathcal{H}^{*-1}(0)$ of all "cohomologically trivial" selfmaps is multiplicatively closed in $\Pi(X, X)$ and will be denoted by $T(X)$, $T(X) \subset \Pi(X, X)$. In the first part of this section we study $T(X)$ and define an integer-valued homotopy invariant $t(X)$ of X; in the second part (Definition 4.6 et seq.) we discuss the case when $\mathcal{H}^*$ is epimorphic.

The following proposition prepares for Theorem 4.2.

PROPOSITION 4.1. *Consider a Moore-space $K'(G, r)$. If G has no 2-torsion, then $T(K'(G, n)) = 0$. Otherwise $T(K'(G, r))$ is nilpotent of order $\leqq 2$.*

Proof. (a) Write $K' = K'(G, r)$, $G = H_r(K')$. By Lemma 3.2(a) we know that $f \in T(K')$ if and only if $f_* \mid G \equiv 0$. Consider the coefficient formula for homotopy groups in (b) below. Since $\pi_{r+1}(K') \cong G \otimes Z_2$, obviously Ext $(G, G \otimes Z_2) = 0$ if G has no 2-torsion. Therefore, if G has no 2-torsion, any $f : K' \to K'$ with $f_* \mid G \equiv 0$ is homotopic to zero.

(b) Let G be arbitrary and $f_* \mid G \equiv g_* \mid G \equiv 0$. The coefficient formula is natural with respect to covariant maps:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ext}(G, \pi_{r+1}(K')) & \xrightarrow{\alpha} & \Pi(K', K') & \xrightarrow{\beta} & \operatorname{Hom}(G, G) & \to 0 \\ & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_2} & \\ 0 \to & \operatorname{Ext}(G, \pi_{r+1}(K')) & \xrightarrow{\alpha} & \Pi(K', K') & \xrightarrow{\beta} & \operatorname{Hom}(G, G) & \to 0, \end{array}$$

where f_i are induced by $f_0 : \pi_k(K') \to \pi_k(K')$. By hypothesis $\beta g = 0$. Therefore $g = \alpha g_1$ and $f_* g = f_* \alpha g_1 = \alpha f_1 g_1$. In (c) below, we show that $f_1 = 0$, and the proof of Proposition 4.1 will be complete.

(c) To compute $\pi_{r+1}(K')$, inspect, according to Peterson [9], the first derived homotopy exact couple of Massey (see Fig. 1 in [9]) which is natural with respect to maps f:

$$\begin{array}{ccccc} \pi_{r+1}(K') & \xleftarrow{\cong} \Gamma \xrightarrow{\cong} & H_r(K'; Z_2) & \cong G \otimes Z_2 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{**}} & \\ \pi_{r+1}(K') & \xleftarrow{\cong} \Gamma \xrightarrow{\cong} & H_r(K'; Z_2) & \cong G \otimes Z_2 . \end{array}$$

In view of $f_* \mid G \equiv 0$ we have $f_{**} \mid G \otimes Z_2 \equiv 0$. Therefore $f_0 = 0$, and $f_1 \mid \operatorname{Ext}(G, \pi_{r+1}(K')) \equiv 0$ as was to be proved.

THEOREM 4.2. *Let X be a simply connected polyhedron with q nontrivial homology groups, q_1 of which have 2-torsion. Then any product of $q + q_1$ self-maps which induce the trivial endomorphism of $\oplus_k H^k(X; H_k(X))$ vanishes. In other words, $T(X)$ is nilpotent of order $\leqq q + q_1$.*

Proof. Apply Proposition 4.1 if $X = K'(G, r)$. Suppose for induction that for any Y with s homology groups $T(Y)$ is nilpotent of order t. Consider a homology decomposition $\{X_r\}$ of an X with $s + 1$ homology groups, and let p be such that X_p has s homology groups. Let $f, g, h_1, \cdots, h_t$ be elements of $T(X)$, and recall Corollary 2.3:

$$\begin{tikzcd} K'_r \arrow[r, "f'"] & K'_r \arrow[r, "g'"] & K'_r \arrow[dr, "h"] & \\ X \arrow[u, "\phi"] \arrow[r, "f"] & X \arrow[u] \arrow[r, "g"] & X \arrow[u] \arrow[r, "\prod h_j"] & X \\ X_p \arrow[u, "i"] \arrow[r] & X_p \arrow[u] \arrow[r] & X_p \arrow[u] \arrow[r] & X_p . \arrow[u] \end{tikzcd}$$

By hypothesis $\prod_{j=1}^{t} h_j$ induces on X_p the trivial map. Therefore $\prod h_j \sim h\phi$.

If $H_r(X)$ has 2-torsion, then $g'f' \sim 0$ by Proposition 4.1, and

$$\left(\prod h_j\right)gf \sim hg'f'\phi \sim 0;$$

in the opposite case $g' \sim f' \sim 0$, and already $\left(\prod h_j\right)g \sim 0$. Thus $T(X)$ is nilpotent of order $\leqq t + 2$ or $\leqq t + 1$ according to the presence or absence of 2-torsion in $H_r(X)$.

COROLLARY 4.3. *Let X be a simply connected polyhedron with q nontrivial homology groups, q_1 of which have 2-torsion. Denote by $T_0(X) \subset \Pi(X, X)$ the set of all selfmaps of X which induce the trivial endomorphism of $H_*(X)$. Then $T_0(X)$ is nilpotent of order $\leqq 2q + 2q_1$.*

Proof. Write out three times the coefficient formula for cohomology groups to verify that $[T_0(X)]^2 \subset T(X)$. Therefore $[T_0(X)]^{2(q+q_1)} = 0$.

Examples of applications of Theorem 4.2. Let X be a simply connected polyhedron.

(1) The endomorphism of $\pi_r(X)$ induced by a selfmap f such that $f_* \mid H_*(X) \equiv 0$ is nilpotent for all r.

(2) An idempotent element $f \in \Pi(X, X)$, $f^2 = f$, cannot satisfy the condition $f_* \mid H_*(X) \equiv 0$ unless it is equal to zero.

(3) If $f \in \Pi(X, X)$ suspends trivially, $\Sigma f = 0$, then f is nilpotent (consider the induced homomorphisms of $\Pi(X, X)$ and $\Pi(\Sigma X, \Sigma X)$).

Theorem 4.2 suggests Definition 4.4 which in its most general form reads as follows:

DEFINITION 4.4. Let X be a topological space, and H a homology theory. Define $t(H; X)$ to be the order of nilpotency of the multiplicatively closed subset $T(H; X)$ of $\Pi(X, X)$ consisting of all maps which induce the trivial endomorphism of $\oplus_k H^k(X; H_k(X))$.

Considering the integer homology one defines analogously $t_0(H; X)$. For polyhedra $t(H; X)$, $t_0(H; X)$ depend only on the homotopy type of X. In this case we write $t(H; X) = t(X)$, $t_0(H; X) = t_0(X)$ and, as we have already done, $T(H; X) = T(X)$, $T_0(H; X) = T_0(X)$. To determine the range of values of $t(X)$ considered as a function of X, we invoke the following theorem of Novikov.

THEOREM OF NOVIKOV [8]. *For any given integer $r > 0$ there exist integers $k_1 > k_2 > \cdots > k_r$ and maps f_s of spheres, $f_s : S^{k_s} \to S^{k_{s+1}}$, such that the composition*

$$S^{k_1} \xrightarrow{f_1} S^{k_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{r-1}} S^{k_r}$$

is not homotopic to zero.

PROPOSITION 4.5. 1. *For any given integer $r > 0$ there exists a simply connected polyhedron X with r nontrivial homology groups and $t(X) = r$.*

2. *For any given integer $r > 0$ there exists a simply connected polyhedron X with r nontrivial homology groups and $t(X) = 1$.*

Proof. 1. Consider $X = S^{k_1} \vee \cdots \vee S^{k_r}$ and $f : X \to X$ defined by $f \mid S^{k_s} = f_s : S^{k_s} \to S^{k_{s+1}}$, the k_s and f_s being the integers and maps of Novikov's theorem. Obviously $f \in T_0(X)$. Let $i : S^{k_1} \to X$ and $p : X \to S^{k_r}$ be the injection of the first and the projection onto the last factor of the wedge X. Since $pff \cdots fi = pf^{r-1}i = f_{r-1}f_{r-2} \cdots f_1$ is not homotopic to zero, f^{r-1} is not trivial either. This means $t_0(X) \geqq r$. Because all homology groups of X are infinite cyclic, we have $t_0(X) = t(X)$. By Theorem 4.1, $t(X) \leqq r$. Therefore $t(X) = r$.

2. Let $M_{(r)}$ be the complex projective space of r complex dimensions. Since $\dim M_{(r)} = 2r$ and $\pi_i(M_{(r)}) = 0$ for $i < 2$ and $3 < i < 2r$, a map $f : M_{(r)} \to M_{(r)}$ is homotopic to zero if and only if its induced homomorphism on $H_2(M_{(r)})$ is trivial. This means that $f_* = 0$ implies $f \sim 0$. Hence $t_0(M_{(r)}) = 1$. But again $t_0(M_{(r)}) = t(M_{(r)})$ because all homology groups of $M_{(r)}$ are infinite cyclic.

We now turn to the question which endomorphisms of $\oplus_k H^k(X; H_k)$, $H_k = H_k(X)$, can be realized by a selfmap $f : X \to X$.

Consider f^* induced by $f : X \to X$:

$$\begin{array}{ccccccccc}
0 \to \operatorname{Ext}(H_{r-1}, H_r) & \xrightarrow{\alpha} & H^r(X; H_r) & \xrightarrow{\beta} & \operatorname{Hom}(H_r, H_r) \to 0 \\
\downarrow f_1 & & \downarrow f^* & & \downarrow f_2 \\
0 \to \operatorname{Ext}(H_{r-1}, H_r) & \xrightarrow{\alpha} & H^r(X; H_r) & \xrightarrow{\beta} & \operatorname{Hom}(H_r, H_r) \to 0,
\end{array}$$

where f_k is induced by $f_* \mid H_*$. The natural ring structure of $\operatorname{Hom}(H_r, H_r)$ defines a ring structure in $H^r(X; H_r)/\alpha \operatorname{Ext}(H_{r-1}, H_r)$, and f^* induces an endomorphism $f^\#$ of the additive group, (or, as we shall say, an additive endomorphism) of the ring $H^r(X; H_r)/\alpha \operatorname{Ext}(H_{r-1}, H_r)$. Since $f^\#$ is essentially f_2 induced by $f_* \mid H_r$, we have $f^\#(ab) = af^\#(b)$ in the ring

$$H^r(X; H_r)/\alpha \operatorname{Ext}(H_{r-1}, H_r).$$

We summarize these considerations in Definition 4.6 and Lemma 4.7.

Definition 4.6. The endomorphism $\Phi = \oplus_k \Phi_k$ of $\oplus_k H^k(X; H_k)$ is called *admissible* if Φ_k induces for all k an additive endomorphism $\Phi_k^\#$ of the ring $H^k(X; H_k)/\alpha \operatorname{Ext}(H_{k-1}, H_k)$ such that $\Phi_k^\#(ab) = a\Phi_k^\#(b)$.

Lemma 4.7. *The endomorphism f^* of $\oplus_k H^k(X; H_k)$ induced by $f : X \to X$ is admissible in the sense of Definition* 4.6.

To study the problem under which circumstances every admissible endomorphism of $\oplus_k H^k(X; H_k)$ can be realized by a selfmap, we first have to consider Proposition 4.8.

Proposition 4.8. *Let X be a simply connected polyhedron. Consider the following admissible endomorphism $\Phi = \oplus_k \Phi_k$ of $\oplus_k H^k(X; H_k)$:*

$$\Phi_k^{(r)} \equiv \mathrm{Id} \quad \textit{for} \quad k \leqq r,$$
$$\equiv 0 \quad \textit{for} \quad k > r.$$

Suppose that $\Phi^{(r)}$ *can be realized by a map* $f : X \to X$, $\Phi^{(r)} = f^*$. *Then* $\mathrm{Ext}(H_r, H_{r+1}) = 0$. *If furthermore* X *is 2-connected, then in any homology decomposition* $\{X_n\}$ *of* X *the* r^{th} *attaching map* $\alpha_r : K'(H_{r+1}, r) \to X_r$ *vanishes.*

Proof. (a) Consider a homology decomposition $\{X_n\}$ of X. By Proposition 2.2 and the homotopy extension property of $i = i_{r,r+1}$, there exist maps f_r, f_{r+1}, f' compatible with each other (write $\phi = \phi_{r+1}$, $K'_{r+1} = K'(H_{r+1}, r+1)$):

$$\begin{array}{ccc} K'_{r+1} & \xrightarrow{f'} & K'_{r+1} \\ \phi\uparrow & & \uparrow\phi \\ X_{r+1} & \xrightarrow{f_{r+1}} & X_{r+1} \\ i\uparrow & & \uparrow i \\ X_r & \xrightarrow{f_r} & X_r . \end{array}$$

It follows from the coefficient formula for cohomology groups and the fact that an epimorphic endomorphism of a finitely generated group is isomorphic that f_* is an automorphism of H_k for $k \leqq r$. Therefore f_r is a homotopy equivalence. Consider on the other hand cohomology with H_{r+1} as coefficients:

$$\begin{array}{ccc} H^{r+1}(X) & \xleftarrow{f^* = 0} & H^{r+1}(X) \\ i_r^*\downarrow & & \downarrow i_r^* \\ H^{r+1}(X_r) & \xleftarrow{f_r^*} & H^{r+1}(X_r). \end{array}$$

Since i_r^* is epimorphic and $f^* = 0$, we have $f_r^* = 0$. But f_r was seen to be an equivalence. Therefore $H^{r+1}(X_r) \cong \mathrm{Ext}(H_r, H_{r+1}) = 0$.

(b) For the map $f' : K'_{r+1} \to K'_{r+1}$ induced by f_{r+1} and f_r (see diagram of (a)) we have $f'^* \mid H^{r+1}(K'_{r+1}; H_{r+1}) \equiv 0$. Therefore $f'f' \sim 0$ by Proposition 4.1. This implies $\phi f_{r+1} f_{r+1} \sim f'f'\phi \sim 0$. If X is 2-connected, it follows from Proposition 2.1 that $f_{r+1} f_{r+1}$ is compressible into X_r, $f_{r+1} f_{r+1} \sim ig$:

$$\begin{tikzcd} & X_{r+1} \arrow[r, "f_{r+1} f_{r+1}"] \arrow[dr, "g"] & X_{r+1} \\ K'(H_{r+1}, r) \arrow[r, "\alpha_r"] & X_r \arrow[u, "i"] \arrow[r, "f_r f_r"] & X_r . \arrow[u, "i"'] \end{tikzcd}$$

In view of $if_r f_r \sim f_{r+1} f_{r+1} i \sim igi$, we have $i_* g_* i_* = i_*(f_r f_r)_*$ for the homology groups; i_* is isomorphic because i is an elementary cofibration, and $(f_r f_r)_*$ is isomorphic because f_r is a homotopy equivalence of X_r. Therefore $gi : X_r \to X_r$ is also an equivalence. On the other hand, $gi\alpha_r \sim 0$ because g maps $X_{r+1} = CK'(H_{r+1}, r) \cup_{\alpha_r} X_r$ into X_r. Therefore $\alpha_r \sim 0$, and the proof of Proposition 4.8 is complete.

THEOREM 4.9. 1. *Let X be homotopy-equivalent to a wedge of Moore-spaces. Suppose* $\mathrm{Ext}(H_r, H_{r+1}) = 0$ *for all r*, $H_r = H_r(X)$. *Then every admissible endomorphism (see Lemma 4.7) of $\oplus_k H^k(X; H_k)$ can be realized by a selfmap $f : X \to X$.*

2. *Let X be an m-connected polyhedron, $m \geqq 1$, with the property that every admissible endomorphism of $\oplus_k H^k(X; H_k)$ can be realized by a selfmap $f : X \to X$. Then* $\mathrm{Ext}(H_r, H_{r+1}) = 0$ *for all r. Furthermore,*

(a) *If $m \geqq 2$, then X is homotopy-equivalent to a wedge of Moore-spaces.*

(b) *If $m = 1$, then all attaching maps of any homology decomposition of X suspend trivially (i.e., ΣX is homotopy-equivalent to a wedge of Moore-spaces).*

Proof. 1. In view of $\mathrm{Ext}(H_r, H_{r+1}) = 0$ for all r, it suffices to prove the following statement (the application of which to all Moore-spaces of the wedge X will prove part 1 of Theorem 4.9): Every admissible automorphism Φ of $H^r(K'(G, r); G)$ can be realized by a selfmap $f : K'(G, r) \to K'(G, r)$.

Let $\Phi : H^r(K'(G, r); G) \to H^r(K'(G, r); G)$ be admissible. Define $\Phi' : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, G)$ by $\Phi' = \beta\Phi\beta^{-1}$:

$$\begin{array}{ccccccc} 0 \to & H^r(K'(G,r); G) & \xrightarrow{\beta} & \mathrm{Hom}(G,G) & \to 0 \\ & \Big\downarrow \Phi & & \Big\downarrow \Phi' & \\ 0 \to & H^r(K'(G,r); G) & \xrightarrow{\beta} & \mathrm{Hom}(G,G) & \to 0. \end{array}$$

Since Φ is admissible, so is $\Phi' : \Phi'(ab) = a\Phi'(b)$; in particular, $\Phi'(a) = a\Phi'(\mathrm{Id})$ for all $a \in \mathrm{Hom}(G, G)$. Realize $\Phi'(\mathrm{Id}) : G \to G$ by a map

$$f : K'(G, r) \to K'(G, r)$$

with the aid of the coefficient formula for homotopy groups. Obviously $f^* = \Phi$, and part 1 of Theorem 4.9 is proved.

2(a). By hypothesis the admissible endomorphisms $\Phi^{(r)}$ of Proposition 4.8 can be realized by maps for all r. It follows from the same Proposition 4.8 that $\mathrm{Ext}(H_r, H_{r+1}) = 0$, and that all attaching maps of any homology decomposition of X vanish provided X is 2-connected.

2(b). Identify $\oplus_k H^k(X; H_k(X)) = \oplus_k H^k(\Sigma X; H_k(\Sigma X)) = H^*$, and denote by $\mathrm{Hom}^*(H^*, H^*)$ the subgroup of admissible endomorphisms of H^*.

The meaning of $\mathcal{H}_1^*$, $\mathcal{H}_2^*$ in the following commutative diagram is obvious:

$$\begin{tikzcd} \Pi(X, X) \arrow[rr, "\Sigma"] \arrow[dr, "\mathcal{H}_1^*"'] & & \Pi(\Sigma X, \Sigma X) \arrow[dl, "\mathcal{H}_2^*"] \\ & \mathrm{Hom}^*(H^*, H^*). & \end{tikzcd}$$

If $\mathcal{H}_1^*$ is epimorphic, so is $\mathcal{H}_2^*$. Apply 2(a) to ΣX to conclude that ΣX is homotopy-equivalent to a wedge of Moore-spaces, and the proof of Theorem 4.9 is complete.

5. Selfmaps of suspensions. The invariant $d(X)$

The natural comultiplication of ΣX defines in $\Pi(\Sigma X, \Sigma X)$ a group operation which will be written as addition though the group $\Pi(\Sigma X, \Sigma X)^+$ may be nonabelian. The addition is connected with the monoid-multiplication by only one law of distributivity in general: $f(a + b) = fa + fb$. It is customary to call a set P with two binary operations $+$ and $\cdot$ (called addition and multiplication) a near-ring if (1) $(P, +)$ is a group, (2) $(P, \cdot)$ is a monoid, and (3) multiplication is left distributive with respect to addition (for near-rings see e.g. [1]). Since the composition of maps is associative, $\Pi(\Sigma X, \Sigma X)$ is an associative near-ring with an identity. Call $g \in \Pi(\Sigma X, \Sigma X)$ distributive if $(a + b)g = ag + bg$ for all a, b; e.g. any suspended map $g = \Sigma g'$ is distributive. If $\Sigma : \Pi(X, X) \to \Pi(\Sigma X, \Sigma X)$ is epimorphic for some reason, then $\Pi(\Sigma X, \Sigma X)$ is a ring.

To measure the deviation of $\Pi(\Sigma X, \Sigma X)$ from being a ring we use the following definition of the distributor series $D^r(P)$ of a near-ring P. It is slightly different from that given in [7].

DEFINITION 5.1. Let P be a near-ring.

(1) Define $D^0(P) = P$.

(2) Call $[a, b, f] \equiv (a + b)f - bf - af$ with $a, b, f \in P$ a 1-distributor of P, and denote the subgroup of P^+ generated by all 1-distributors by $D^1(P)$.

(3) $D^n(P)$, $n \geqq 2$, is defined as the subgroup of P^+ generated by all elements of the form $[a, b, f]$ with $f \in P$ and $a, b \in D^{n-1}(P)$. The generators of $D^n(P)$ are called n-distributors of P.

It is obvious that $D^{n+k}(P) = 0$ for $k > 0$ if $D^n(P) = 0$. Thus it makes sense to write $d(P) = n$ if $D^{n-1}(P) \neq 0$, $D^n(P) = 0$ such that $d(P) = 1$ if P is a ring. If $D^k(P) \neq 0$ for all k, we write $d(P) = \infty$. This obviously suggests the following definition.

DEFINITION 5.2. Let X be a topological space. Call $d(X) = d(\Pi(\Sigma X, \Sigma X))$ the order of distributivity of X.

The integer $d(X)$ is an invariant of the homotopy type of X but not of ΣX in general. If e.g. X is an $(n-1)$-connected polyhedron of dimension $\leqq 2n-1$, then $d(X)=1$.

THEOREM 5.3. *Let X be a simply connected polyhedron with n nontrivial homology groups. Then $d(X) \leqq n$ (i.e., all n-distributors of $\Pi(\Sigma X, \Sigma X)$ vanish).*

Proof. (a) Write $\Sigma X = Y$. Consider a homology decomposition $\{Y_r\}$ of Y obtained by suspending one of X. By induction we shall prove in (b), (c) below the following statement which obviously includes Theorem 5.3: Let Y_r have s nontrivial homology groups, and let $a^{(s-1)}$, $b^{(s-1)}$ be two $(s-1)$-distributors of $\Pi(Y, Y)$. Then for any $f: Y_r \to Y$ we have

$$[a^{(s-1)}, b^{(s-1)}, f] \equiv (a^{(s-1)} + b^{(s-1)})f - b^{(s-1)}f - a^{(s-1)}f = 0.$$

(b) Let $s=1$, i.e., Y_r is a Moore-space K'; $a, b \in \Pi(Y, Y)$.

$$\begin{tikzcd} & Y \arrow[r, "{a, b}"] & Y \\ Y_r = K' \arrow[ur, "f"] & & \end{tikzcd}$$

In all possible cases considered $\Sigma: \Pi(\Sigma^{-1}K', X) \to \Pi(K', Y)$, $Y = \Sigma X$, is epimorphic. Therefore $f: K' \to Y$ is a suspended map. Consequently $[a, b, f] = 0$.

(c) We assume the theorem for $s = q$. If $a^{(q)}$, $b^{(q)}$ are q-distributors of $\Pi(Y, Y)$ and $f: Y_r \to Y$ is any map, we have to prove that $[a^{(q)}, b^{(q)}, f] = 0$ if Y_r has $q+1$ homology groups:

$$\begin{tikzcd} & Y \arrow[r, "{a^{(q)}, b^{(q)}}"] & Y \\ Y_r \arrow[ur, "f"] & & \end{tikzcd}$$

By definition $a^{(q)} = [a_1, a_2, f']$, where a_i are $(q-1)$-distributors of $\Pi(Y, Y)$ and $f': Y \to Y$ is any map.

Let Y_t, $t < r$, have q nontrivial homology groups. Since $i: Y_t \to Y$ is a suspended map, we have

$$a^{(q)}i = [a_1, a_2, f']i = [a_1, a_2, f'i]; \qquad f'i: Y_t \to Y \to Y.$$

By induction hypothesis, $[a_1, a_2, f'i] = 0$. Therefore we can find maps $\bar{a}$, $\bar{b}$ with $\bar{a}\phi \sim a^{(q)}$, $\bar{b}\phi \sim b^{(q)}$:

$$\begin{tikzcd}
 & Y/Y_t \arrow[dr, "\bar{a}, \bar{b}"] & \\
 & Y \arrow[u, "\phi"] \arrow[r, "a^{(q)}, b^{(q)}"] & Y \\
Y_r \arrow[ur, "f"] & Y_t \arrow[u, "i"'] &
\end{tikzcd}$$

Now use the fact that ϕ and ϕf are suspended maps:

$$(a^{(q)} + b^{(q)})f \sim (\bar{a}\phi + \bar{b}\phi)f \sim ((\bar{a} + \bar{b})\phi)f \sim (\bar{a} + \bar{b})(\phi f)$$
$$\sim \bar{a}\phi f + \bar{b}\phi f \sim a^{(q)}f + b^{(q)}f,$$

which means $[a^{(q)}, b^{(q)}, f] = 0$.

Example. The following example of a space X with three nontrivial homology groups and $d(X) = 3$ is due to P. J. Hilton.

Take

$$X = \bigvee_{k=1}^{4} S_k^1 \vee S^2 \vee S^5, \qquad \Sigma X = \bigvee_{k=1}^{4} S_k^2 \vee S^3 \vee S^6.$$

Consider the identity maps $i_k : S_1^2 \to S_k^2$, the Hopf map $\gamma : S^3 \to S_1^2$, the generator $\alpha : S^6 \to S^3$ of $\pi_6(S_3)$. Then $[i_1, i_2, \gamma] = \delta_1 \neq 0$, $[i_3, i_4, \gamma] = \delta_2 \neq 0$, $[\delta_1, \delta_2, \alpha] \neq 0$. Therefore $d(X) \geqq 3$. But $d(X) \leqq 3$ by a simple direct argument. Hence $d(X) = 3$.

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