# BIG PROJECTIVE MODULES ARE FREE[1]

BY

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## 1. Introduction

Finitely generated projective modules arise significantly in certain geometric and arithmetic questions. We shall show here that nonfinitely generated projective modules, in contrast, invite little interest; for we show that an obviously necessary "connectedness" condition for such a module to be free is also sufficient.

More precisely, call an $R$-module $P$ *uniformly* $\aleph$*-big*, where $\aleph$ is an infinite cardinal, if (i) $P$ can be generated by $\aleph$ elements, and (ii) $P/\mathfrak{A}P$ requires $\aleph$ generators for all two-sided ideals $\mathfrak{A}$ ($\neq R$). A free module with a basis of $\aleph$ elements is manifestly uniformly $\aleph$-big. Our main result (Corollary 3.2) asserts, conversely, that, with suitable chain conditions on $R$, *a uniformly big projective $R$-module is free.*

Finally, we ask, for what $R$ are all nonfinitely generated projective modules uniformly big? For commutative rings, the answer is quite satisfactory; with mild assumptions one requires only that spec $(R)$ be connected (i.e., that there exist no nontrivial idempotents). For $R = Z\pi$, with $\pi$ a finite group, Swan (unpublished) has established this conclusion when $\pi$ is solvable, and it is undoubtedly true in general.

Our method relies on two basic tools. One, naturally enough, is Kaplansky's remarkable theorem [2, Theorem 1] which asserts that every projective module is a direct sum of countably generated modules. The second is an elegant little swindle, observed several years ago by Eilenberg, and which might well have sprung from the brow of Barry Mazur. It is this result, recorded below, which permits us to waive the delicate arithmetic questions which plague the finitely generated case.

EILENBERG'S LEMMA. *If $P \oplus Q = F$ with $F$ a nonfinitely generated free module, then $P \oplus F \cong F$.*

*Proof.*

$$F \cong F \oplus F \oplus \cdots = P \oplus Q \oplus P \oplus Q \oplus \cdots$$
$$\cong P \oplus F \oplus F \oplus \cdots \cong P \oplus F.$$

Finally, I wish to thank Peter Freyd for several helpful conversations, and, in particular, for pointing out the method in §2.

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## 2. Very big projectives: Freyd's theorem

*Notation.* We denote by $J(R)$ the Jacobson radical of $R$. If $P$ is an $R$-module, then $P^* = \mathrm{Hom}_R(P, R)$, and we denote by $\tau_R(P)$ (or simply $\tau(P)$) the image of the natural pairing $P^* \otimes P \to R$; $\tau(P)$ is a two-sided ideal. When $P$ is projective, one knows (or easily verifies) that for any two-sided ideal $\mathfrak{A}$, $\tau_{R/\mathfrak{A}}(P/\mathfrak{A}P)$ is the image in $R/\mathfrak{A}$ of $\tau_R(P)$. It follows that *if $\tau_R(P) = R$, then $P/\mathfrak{A}P \neq 0$ for all $\mathfrak{A}$.* We shall say that $R$ is *p-connected* if $\tau_R(P) = R$ for all nonzero projective $R$-modules $P$.

COROLLARY 2.1. *If $R$ is p-connected, then any infinite direct sum of nonzero projective $R$-modules, in particular any noncountably generated one (by Kaplansky's theorem), is uniformly big.*

*Proof.* By Kaplansky's theorem we can reduce the problem to consideration of a countably infinite direct sum, $P$, of nonzero countably generated projective modules. We must show that $P/\mathfrak{A}P$ is never finitely generated. But this is evident from the remark above since an infinite direct sum cannot be finitely generated.

THEOREM 2.2. *Suppose $R/J(R)$ is left or right Noetherian, and let $\aleph$ be an uncountable cardinal. Then a uniformly $\aleph$-big projective $R$-module $P$ is free.*

*Proof.* By Kaplansky's theorem, $P = \sum \oplus_{\gamma\in\Gamma} P_\gamma$ with each $P_\gamma$ countably generated; hence card $\Gamma = \aleph$. Note first that

(∗) if card $\Gamma' < \aleph$, then $\sum \oplus_{\gamma\notin\Gamma'} P_\gamma$ is still uniformly $\aleph$-big.

Suppose we can find a single direct summand of $P$ isomorphic to $R$. Since this requires only a finite number of the $P_\gamma$'s, we can remove them, and then, by (∗), find another such direct summand. Apply Zorn's lemma now, and observe, again by (∗), that we could meet no obstruction before removing $\aleph$ direct summands isomorphic to $R$. But then, by Eilenberg's lemma, we are done. Hence, we need only find one such direct summand.

Since $P$ is uniformly big, certainly $\tau(P) = R$. Hence if $\mathfrak{A}$ is any proper ideal (left or right), we can find $f : P \to R$ such that $f(P) \not\subset \mathfrak{A}$. Therefore, $f(P_\gamma) \not\subset \mathfrak{A}$ for some $\gamma$. It follows that we can inductively construct

$$f_i : P_i \to R, \qquad i = 1, 2, \cdots,$$

so that $f_n(P_n) \not\subset \sum_{i<n} f_i(P_i) + J(R)$. If $R$ is left Noetherian, we are forced to stop, but this means that $\sum_{i\leqq n} f_i(P_i) + J(R) = R$; hence, by Nakayama's lemma, $\sum_{i\leqq n} f_i(P_i) = R$. But then $\sum_{i\leqq n} f_i : \sum \oplus_{i\leqq n} P_i \to R$ splits off the desired direct summand.

If, on the other hand, we must use the right chain condition, we simply replace $f_i(P_i)$ by $f_i(a_i)R$ for suitable $a_i \in P_i$, in the above argument, and proceed in the same way.

COROLLARY 2.3. *If $R/J(R)$ is left or right Noetherian, then a uniformly $\aleph$-big projective R-module P is a direct sum of uniformly $\aleph_0$-big modules.*

*Proof.* If $\aleph = \aleph_0$, there is nothing to prove, and if $\aleph > \aleph_0$, then $P$ is free by the theorem above.

A cheap extension of the above argument even requires no chain condition.

PROPOSITION 2.4. *Suppose R is p-connected and P is a projective R-module. Then the countable direct sum $P \oplus P \oplus \cdots$ is free.*

*Proof.* By Kaplansky's theorem we reduce our problem to the case: $P$ is countably generated and nonzero. Since $\tau(P) = R$, we have $f_i \in P^*$, $i = 1, \cdots, n$, such that $\sum_i f_i(P) = R$; hence $P \oplus \cdots \oplus P$ ($n$ times) has a direct summand isomorphic to $R$, so $P \oplus P \oplus \cdots$ has a free direct summand of infinite rank; now use Eilenberg's lemma.

## 3. Reasonably big projectives

Corollary 2.3 reduces our problem to countably generated modules, and here we must do some rather tedious and uninspired juggling with infinite matrices.

THEOREM 3.1. *Suppose $R/J(R)$ is left Noetherian. Then a uniformly $\aleph_0$-big projective R-module is free.*

COROLLARY 3.2 (Main Theorem). *If $R/J(R)$ is left Noetherian, any uniformly big projective R-module is free.*

*Proof of* 3.1. We are given a uniformly $\aleph_0$-big projective module $P$; say $P \oplus Q = F$ with $F$ a free module whose elements we shall identify with (ultimately zero) sequences of elements of $R$. Say $P$ is generated by $\alpha_1$, $\alpha_2, \cdots, \alpha_n, \cdots$, and $\alpha_n = (a_{n1}, a_{n2}, \cdots, a_{nk}, \cdots)$.

Consider the row finite matrix

$$A = (a_{ij}), \qquad i, j = 1, 2, 3, \cdots .$$

The elements of the first column generate a left ideal which, by hypothesis, is finitely generated mod $J(R)$, say by $a_{11}, \cdots, a_{n_1 1}$. Then, after subtracting linear combinations of $\alpha_1, \cdots, \alpha_{n_1}$ from the remaining $\alpha$'s, we may assume $a_{i1} \in J(R)$ for $i > n_1$. Now apply the same procedure to the elements of the second column *beyond the* ${n_1}^{\text{th}}$. Then, for suitable $n_2$, we render the second column zero mod $J(R)$ beyond the $(n_1 + n_2)^{\text{th}}$ row. We continue this way, considering next the elements of the third column beyond the $(n_1 + n_2)^{\text{th}}$, etc. Since this process alters a given $\alpha$ only finitely often, it defines a legitimate change of generators for $P$, the result being that we may assume *$A$ is also column finite* mod $J(R)$.

Before continuing it will be convenient to introduce some provisional notation. If $\beta = (b_1, b_2, \cdots) \in F$, we call $S(\beta) = \{i \mid b_i \notin J(R)\}$ the "mod $J$ support" of $\beta$. Moreover, we write

$(\beta)$ = the two-sided ideal generated by those $b_i \notin J(R)$,
$(\beta)'$ = the two-sided ideal generated by the $b_i$ for all $i$,
$[\beta)$ = the right ideal generated by those $b_i \notin J(R)$.

We propose now to construct a linear combination, $\beta$, of the $\alpha$'s such that $\beta$ generates a free direct summand of $F$. Specifically, we shall construct a sequence $n_1 < n_2 < \cdots < n_i < \cdots$ and homomorphisms $f_i : F \to R$ such that

(i) the $n_i$ may be taken to exceed any prescribed bound;
(ii) if $i \neq j$, then $S(\alpha_{n_i}) \cap S(\alpha_{n_j}) = \emptyset$;
(iii) $f_i$ is a (right) linear combination of coordinate projections on $S(\alpha_{n_i})$; and
(iv) the sequence of left ideals $\sum_{i \leq j} Rf_i(\alpha_{n_i})$ strictly increases mod $J(R)$.

Suppose for the moment that this is possible. Then, since $R/J(R)$ is left Noetherian, we are compelled by (iv) to stop at some $k$, at which point $\sum_{i \leq k} Rf_i(\alpha_{n_i}) = R$. Hence we can write $1 = \sum_{i \leq k} r_i f_i(\alpha_{n_i})$ for suitable $r_i \in R$. Let $f = \sum_{i \leq k} f_i$. Then by (ii) and (iii) we see that

$$f(\alpha_{n_i}) \equiv f_i(\alpha_{n_i}) \mod J(R)$$

for all $i$. Hence, if $\beta = \sum r_i \alpha_{n_i}$, then $f(\beta) \equiv 1 \mod J(R)$, so $f(\beta)$ is a unit, and $\beta$ splits off the desired direct summand.

Now since $A$ is column finite mod $J(R)$, $S(\alpha_n)$ is disjoint from $S(\alpha_{n_i})$ for sufficiently large $n$. By condition (i) we may construct a second sequence as above with these large $n$, and produce another element, say $\beta_2$, generating a free direct summand of $F$. The restriction on $n$ guarantees that $S(\beta) \cap S(\beta_2) = \emptyset$. Continuing in this way we get a third such element, a fourth, etc. We see thus that once the above construction is accomplished, then we will be able to produce a sequence $\beta_1, \beta_2, \beta_3, \cdots$ such that

(1) each $\beta_n$ is in $P$ (i.e., is a linear combination of the $\alpha$'s),
(2) $S(\beta_n) \cap S(\beta_m) = \emptyset$ for $n \neq m$, and
(3) $[\beta_n) = R$ for all $n$. (For since $\beta_n$ generates a free direct summand of $F$, its coordinates generate the unit right ideal, and we may then disregard coordinates in $J(R)$.)

Before going further with these $\beta$'s let us return to the construction from which they are to be derived. We shall proceed by induction, so assume $n_1 < \cdots < n_j$ and $f_1, \cdots, f_j$ have been constructed satisfying (i), (ii), (iii), and (iv). We are interested in continuing only if

$$\mathfrak{A} = \sum_{i \leq j} Rf_i(\alpha_{n_i}) + J(R) \neq R.$$

Now since the matrix $A$ is column finite mod $J(R)$, we can choose an $N \gg n_j$ so that if $n \geq N$, then $S(\alpha_n) \cap S(\alpha_m) = \emptyset$ for all $m \leq n_j$. I claim further that we can choose $n \geq N$ such that $(\alpha_n)' \not\subset \mathfrak{A}$. For otherwise, if $\mathfrak{B} = \sum_{n \geq N} (\alpha_n)'$, then $\mathfrak{B} \neq R$ and $P/\mathfrak{B}P$ is generated by the images of $\alpha_1, \cdots, \alpha_N$, contrary to our assumption that $P$ is uniformly big. We may therefore choose $n_{j+1} \geq N$ so that $(\alpha_{n_{j+1}})' \not\subset \mathfrak{A}$. It is clear then that we have satisfied (i) and (ii). We must now construct $f_{j+1}$. Since $\mathfrak{A}$ is a left ideal containing $J(R)$,

$$(\alpha_{n_{j+1}})' \not\subset \mathfrak{A} \Rightarrow (\alpha_{n_{j+1}}) \not\subset \mathfrak{A} \Rightarrow [\alpha_{n_{j+1}}) \not\subset \mathfrak{A}.$$

The last relation means that there is a right linear combination of the coordinates of $\alpha_{n_{j+1}}$ in $S(\alpha_{n_{j+1}})$ which does not lie in $\mathfrak{A}$. This same right linear combination of coordinate projections on $S(\alpha_{n_{j+1}})$ then defines a functional $f_{j+1} : F \to R$ such that $f_{j+1}(\alpha_{n_{j+1}}) \notin \mathfrak{A}$. We have thus constructed $f_{j+1}$ satisfying (iii) and (iv).

As always our strategy will be to find in $P$ a large free direct summand and then invoke Eilenberg's lemma. This will be accomplished once we show that, if $\beta_1, \beta_2, \cdots$ is a sequence of elements of $F$ satisfying (2) and (3) above, then the submodule generated by the $\beta$'s contains a free direct summand of $F$ of infinite rank.

To prove the above statement there is no harm in adding to $F$ a large free direct summand to guarantee that there are infinitely many coordinates at which all the $\beta_n$ have zero projection. Choose such a free coordinate for each $n$, and let $S'(\beta_n)$ denote the result of adjoining it to $S(\beta_n)$. Since $[\beta_n) = R$, there is an elementary change of basis involving the coordinates in only $S'(\beta_n)$, which puts a 1 in the free coordinate and zeros elsewhere for $\beta_n$ in the $S'(\beta_n)$ coordinates. Since $S'(\beta_n) \cap S'(\beta_m) = \emptyset$ for $n \neq m$, these transformations define a global change of basis. After this change each $\beta_n$ has a distinguished coordinate where it has a unit, and all other coordinates are in $J(R)$. Moreover the distinguished coordinates are different for different $n$. Hence, if we list the distinguished coordinates first, in sequence, and then the remaining ones, the coordinate matrix for the $\beta$'s assumes the form

$$\left(\begin{array}{cccccc|c} u_1 & & & & & & \\ & u_2 & & & & & \\ & & \ddots & & \in J(R) & & \in J(R) \\ & \in J(R) & & u_n & & & \\ & & & & \ddots & & \end{array}\right).$$

This matrix is row finite (by definition). By subtracting multiples of $\beta_1$ from $\beta_2, \beta_3, \cdots$, multiples of $\beta_2$ from $\beta_3, \beta_4, \cdots$, etc., we may further render the matrix *zero below the main diagonal.* It is important to note that these subtractions leave unchanged the format of the matrix, in particular leave units on the diagonal, in order to justify continuing from one column to the next.

After altering the $\beta$'s in this way, it is clear that we can choose a subsequence $\gamma_1 = \beta_{n_1}, \gamma_2 = \beta_{n_2}, \cdots$ so that if $i \neq j$, $\gamma_i$ and $\gamma_j$ do not simultaneously have nonzero projection on any of the first set of coordinates.

If we list the coordinates now according to the distinguished coordinates of the $\gamma$'s, then the coordinate matrix of the $\gamma$'s takes the form

$$\left(\begin{array}{cccccc|c} v_1 & & & & & & \\ & v_2 & & & 0 & & \\ & & \ddots & & & & \in J(R) \\ & 0 & & v_n & & & \\ & & & & \ddots & & \end{array}\right).$$

The $\gamma$'s evidently generate the desired free direct summand of $F$.

COROLLARY 3.3 (Kaplansky). *If $R$ is local, all projective $R$-modules are free.*

*Proof.* One need only show that if $P$ is projective and not finitely generated then $P/J(R)P$ is not finitely generated. This follows from Nakayama's lemma and the fact that $P \neq J(R)P$ for any nonzero projective $P$ over any ring [1, Proposition 2.7].

Now from Corollary 2.1 we get

COROLLARY 3.4. *If $R$ is p-connected and $R/J(R)$ is left Noetherian, then any infinite direct sum $P$ of nonzero projective $R$-modules is free. In particular, this is so if $P$ is not countably generated.*

## 4. Commutative rings

In this section $R$ is commutative and spec $(R)$ is its space of prime ideals (Zariski topology). We shall (abusively) call $R$ *connected* if spec $(R)$ is connected. One checks easily that $R$ is connected if and only if $R$ has no nontrivial idempotents. Moreover, it is evident that $R$ is connected if it is $p$-connected.

We shall find it convenient to introduce a further notion suggested by Serre's treatment in [3, §§2–3]. Let $\sim$ denote the equivalence relation on spec $(R)$ generated by inclusion $(\subset)$. We call the $\sim$ equivalence classes the *S-components* of spec $(R)$, and we call $R$ *S-connected* if there is a unique $S$-component.

Let $P$ be a projective $R$-module and $\mathfrak{p} \in \operatorname{spec}(R)$. Then $P_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank $\rho_P(\mathfrak{p})$. The virtue of $S$-components is that $\rho_P$ is clearly constant on each of them.

PROPOSITION 4.1. *Every nonfinitely generated projective $R$-module is uniformly big if and only if*

(a) *$\rho_P$ is constant on* spec $(R)$ *for all projective $P$, and*

(b) *if a projective module $P$ is locally finitely generated, then $P$ itself is finitely generated.*

*Proof.* Suppose every nonfinitely generated $P$ is uniformly big; then surely $R$ must be connected, so (a) holds for finitely generated $P$. Hence, to prove (a) and (b) we may, by Kaplansky's theorem, assume $P$ is countably, but not finitely, generated. Then $P_{\mathfrak{M}}/\mathfrak{M}P_{\mathfrak{M}} = P/\mathfrak{M}P$ is free of rank $\aleph_0$ over $R/\mathfrak{M}$ for every maximal ideal $\mathfrak{M}$, since $P$ is uniformly big. It is then clear that $\rho_P(\mathfrak{p}) = \aleph_0$ for all $\mathfrak{p}$, and this proves (a) and (b).

Conversely, assuming (a) and (b) we want to show that $P$ is uniformly big, where, again, we may assume $P$ is countably, but not finitely, generated. If $P/\mathfrak{M}P$ is finitely generated for some maximal $\mathfrak{M}$, then (a) implies this is so for all $\mathfrak{M}$, so $P$ is locally finitely generated, contradicting (b).

PROPOSITION 4.2. *Suppose $R$ has only finitely many primes minimal above 0. Then*

(1) *The $S$-components agree with the components, and $R$ is a finite direct product of $S$-connected rings.*

(2) *If $P$ is a locally finitely generated projective $R$-module, then $P$ itself is finitely generated.*

*Proof.* (1) Let $\mathfrak{S}_{ij}$, $i = 1, \cdots, n, j = 1, \cdots, h_i$, be the primes minimal above 0, with $\mathfrak{S}_{i'j'} \sim \mathfrak{S}_{ij}$ if and only if $i = i'$. Set $\mathfrak{S}_i = \bigcap_j \mathfrak{S}_{ij}$; then $\mathfrak{S} = \bigcap_i \mathfrak{S}_i$ is the nil radical of $R$. Moreover, since for $i \neq i'$

$$\mathfrak{S}_{ij} + \mathfrak{S}_{i'j'} = R \qquad \text{for all } j, j',$$

it follows that $\mathfrak{S}_i + \mathfrak{S}_{i'} = R$. Hence $R/\mathfrak{S}$ is the direct product of the $R/\mathfrak{S}_i$ (Chinese Remainder Theorem). Since idempotents can be lifted modulo a nil ideal, it follows that $R$ itself splits into a direct product of $n$ rings, each of which is clearly $S$-connected; this establishes (1).

(2) First suppose $P$ is finitely generated modulo each $\mathfrak{p}_{ij}$. Then clearly $P$ is finitely generated modulo $\mathfrak{S}$. Hence there is a finitely generated $Q \subset P$ such that $Q + \mathfrak{S}P = P$. If $\mathfrak{M}$ is any maximal ideal, then $\mathfrak{S} \subset \mathfrak{M}$ and $P_{\mathfrak{M}}$ is free of finite rank (by hypothesis), so, by Nakayama's lemma, $Q_{\mathfrak{M}} = P_{\mathfrak{M}}$. Hence $Q = P$ since $(P/Q)_{\mathfrak{M}} = 0$ for all $\mathfrak{M}$.

This permits us to assume (passing to $R/\mathfrak{p}_{ij}$) that $R$ is an integral domain and $P$ is a projective module of finite rank, say $n$. Let $M \subset P$ be free of rank $n$, so $P/M$ is torsion. Write $P \oplus Q = F$ with $F$ free, and say $M \subset F_0$ with $F_0$ generated by a finite number of basis elements of $F$; $F = F_0 \oplus F_1$. Since the torsion submodule of $F/M$ is contained in $F_0/M$, it follows that $P \subset F_0$, and hence $P$ is finitely generated (being a direct summand of $F_0$).

THEOREM 4.3. *Assume* (i) *$R$ has only finitely many primes minimal above* 0, (ii) *$R$ is connected, and* (iii) *$R/J(R)$ is Noetherian. Then every nonfinitely generated projective $R$-module $P$ is free.*

*Proof.* We need only verify that $P$ is uniformly big, by (iii) and Corollary 3.2; hence we must establish (a) and (b) of Proposition 4.1. But (ii) and

Proposition 4.2 tell us, first, that $R$ is $S$-connected, which secures (a), and secondly affirms (b).

COROLLARY 4.4. *If $R$ has only finitely many primes minimal above zero and $R/J(R)$ is Noetherian, then for every projective $R$-module $P$ we can write $1 = e_1 + \cdots + e_r$ as an orthogonal sum of idempotents so that $e_i P$ is either finitely generated or free as an $e_i R$-module.*

COROLLARY 4.5. *If $R$ is connected and Noetherian, every nonfinitely generated projective $R$-module is free.*

COROLLARY 4.6. *With $R$ as in Theorem* 4.3, *and $X$ an indeterminate, every projective $R[X]$-module is free as an $R$-module.*

We conclude this section with a couple of examples to show how Propositions 4.1 and 4.2 can be disturbed.

(1) If $P$ is the $Z$-submodule of $Q$ generated by $\{p^{-1} \mid p \text{ is a prime}\}$, then $P$ is locally free of rank one, but neither projective nor finitely generated.

(2) Let $R = C(I)$, the ring of continuous functions on $I = [0, 1]$, let $\mathfrak{M}$ be the maximal ideal of all functions vanishing at some $x \in I$, and let $P$ be the ideal of all functions vanishing in some neighborhood of $x$. Then $R$ is connected; indeed every finitely generated projective $R$-module is free (since $I$ is contractible). However, $P$ is a countably, but not finitely, generated, indecomposable, faithful projective $R$-module such that $\tau(P) = P$. It follows that if $\mathfrak{p}$ is any prime ideal, then either (i) $\mathfrak{p} \not\subset \mathfrak{M}$, $P_\mathfrak{p} = R_\mathfrak{p}$, and even $\mathfrak{p} + P = R$; or (ii) $\mathfrak{p} \subset \mathfrak{M}$, $P_\mathfrak{p} = 0$, and $P \subset \mathfrak{p}$. Thus $R$ is neither $p$-connected nor $S$-connected. This example, which seems to be well known, was first pointed out to me by Kaplansky.

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