MONOTONE BEHAVIOR OF COHOMOLOGY GROUPS UNDER PROPER MAPPINGS[1]

BY
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Introduction

The algebraic varieties which occur in this paper are defined over an algebraically closed groundfield of arbitrary characteristic. We use the terms algebraic variety, quasi-projective and projective variety, regular and proper mappings, coherent sheaves, etc., as defined in [1] and [2].

If $f:X \to Y$ is a continuous mapping from a topological space X into a topological space Y, and F a sheaf of abelian groups over X, the sheaves $R^qf(F)$ over Y are well defined for all $q \geqq 0$; see [3], Section 3.7. (We write $f(F)$ instead of $R^0f(F)$ for the direct image of F.) A spectral sequence is associated with f and F whose initial term is $E_2^{p,q}(F) = H^p(Y, R^qf(F))$ and whose final term is $E^n(F) = H^n(X, F)$; see [3], Theorem 3.7.3. Consequently, there exists a natural homomorphism $\alpha^n:H^n(Y, f(F)) \to H^n(X, F)$ for all $n \geqq 0$, which is a monomorphism for $n = 1$. The main theorem of this paper states:

THEOREM 1. *Let $f:X \to Y$ be a proper mapping from an irreducible, quasi-projective variety X onto an algebraic variety Y of the same dimension r. Then, if F is a coherent sheaf over X, the natural homomorphism*

$$\alpha^r:H^r(Y, f(F)) \to H^r(X, F)$$

is an epimorphism.

We observe that, since is X is irreducible and f is onto, Y is irreducible. We do not know whether the theorem remains correct if we assume only that X is an irreducible, algebraic variety which is not necessarily quasi-projective.

Many conclusions can be drawn from Theorem 1. For example, if we assume that the Y in that theorem is normal and the f is birational, $f(O_X) = O_Y$; O_X and O_Y denote the sheaves of local rings of, respectively, X and Y. Theorem 1 then states that $\alpha^r:H^r(Y, O_Y) \to H^r(X, O_X)$ is an epimorphism which, in the special case that X and Y are projective, was proved by much more complicated methods on page 94 of [4]. If furthermore $r = 2$ and X is projective (in which case Y is necessarily complete, and all the cohomology groups under investigation are finite-dimensional vectorspaces over k), the epimorphism α^2, together with the monomorphism

$$\alpha^1:H^1(Y, O_Y) \to H^1(X, O_X),$$

Received January 20, 1961.
[1] This research was supported in part by a National Science Foundation grant.

gives us that $\chi(Y, O_Y) \geqq \chi(X, O_X)$. (We use here the notation of Section 79 of [1] to denote the Euler-Poincaré characteristic of a variety with values in a sheaf.) If Y is projective, this inequality becomes Theorem 4 of [5]. The author has not been able to squeeze the monotone behavior of the arithmetic genus for normal, projective varieties of dimension $\geqq 3$, described on page 83 of [6], out of this material.

The proof of Theorem 1 is based on three lemmas which we now discuss.

Let $f:X \to Y$ be a proper mapping from a quasi-projective variety X onto an algebraic variety Y. We do not assume that X is irreducible or that X and Y have the same dimension. For each $y \in Y$, $f^{-1}(y)$ is a closed subvariety of X, whence it is a quasi-projective variety; we denote its dimension by $\dim(f^{-1}(y))$. Since f is onto, $\dim(f^{-1}(y)) \geqq 0$. We will use the following notations:

(*a*) $s = \text{Minimum } (\dim(f^{-1}(y)))$, where y runs through Y.

(*b*) For each integer $q \geqq 0$, Y_q is the subset of Y which consists of those points $y \in Y$ for which $\dim(f^{-1}(y)) \geqq q$.

We observe that $s \geqq 0$, and that $Y_q \supset Y_p$ if $q \leqq p$. Of course, for large enough q, $Y_q = \emptyset$. Finally, $Y_0 = Y$.

LEMMA 1. *Y_q is a closed subset of Y, for all $q \geqq 0$.*

LEMMA 2. *If F is a coherent sheaf over X,* Sup $(R^q f(F)) \subset Y_q$, *for all $q \geqq 0$.* (Sup *stands for Support.*)

LEMMA 3. $\dim(X) - \dim(Y) \geqq s$.

In Section 1, we prove Theorem 1 as a formal consequence of these three lemmas, by the use of spectral sequences. Lemmas 1, 2, and 3 are proved in, respectively, Sections 3, 4, and 5. In Section 2, we identify proper mappings with regular mappings which can be "completed," a result we need for the proofs of the lemmas.

Lemma 3 will be applied (in Section 1) to the restriction $f_q:f^{-1}(Y_q) \to Y_q$ of f to $f^{-1}(Y_q)$. Since $f^{-1}(Y_q)$ may very well be reducible, even if Y_q is irreducible, it is important that we do not assume that X is irreducible in Lemma 3. If X is irreducible, the inequality of Lemma 3 may be replaced by equality, as shown in Section 6. The example of Section 5 (see Remark 5.1) shows that, if X is reducible, $\dim(X) - \dim(Y)$ may be greater than s.

Sections 3, 4, and 5 begin with "topological preliminaries" which require no algebraic geometry for their reading. These preliminaries have been carefully marked off and comprise Propositions 3.1–3.3, 4.1–4.4, and 5.1. They are dependent on one another, but not on the remainder of the paper.

Matsumura proved Theorem 1 in the special case that $f:X \to Y$ is birational; see [13], Proposition 5.1. The proof method of Matsumura's paper is similar to the present one: The special case of Lemma 2 where X (and hence Y) is irreducible, occurs as Proposition 4.3 in [13].

The key to the present generalization of Matsumura's work lies in the "topological preliminaries," mentioned above, and Lemma 3.

1. The proof of Theorem 1

Let $f:X \to Y$ be a proper mapping from a quasi-projective variety X onto an algebraic variety Y. We do not assume that X is irreducible or that X and Y have the same dimension. *We assume that the three lemmas of the introduction have been proved.*

PROPOSITION 1.1. $\dim(f^{-1}(Y_q)) \geqq q + \dim(Y_q)$, *for all* $q \geqq 0$ *for which* $Y_q \neq \emptyset$.

Proof. Let $f_q:f^{-1}(Y_q) \to Y_q$ be the restriction of f to $f^{-1}(Y_q)$. It follows from Lemma 1 that $f^{-1}(Y_q)$ is a closed subset of X, and hence, since $f^{-1}(Y_q) \neq \emptyset$, f_q is a proper mapping. (See [2], Section 2, Proposition 4.) Consequently, we may apply Lemma 3 to f_q, and Proposition 1.1 follows immediately. Done.

We observe that Proposition 1.1 is obviously false if $Y_q = \emptyset$.

PROPOSITION 1.2. *If F is a coherent sheaf over X, the initial term* $E_2^{p,q}(F) = 0$ *for* $p > \dim(Y_q)$.

Proof. $E_2^{p,q}(F) = H^p(Y, R^qf(F))$, and we conclude from Lemmas 1 and 2 that $H^p(Y, R^qf(F)) \simeq H^p(Y_q, R^qf(F) \mid Y^q)$; here, $R^qf(F) \mid Y_q$ denotes the restriction of $R^qf(F)$ to Y_q, and we are using Proposition 8 of Section 26 of [1]. We conclude from Theorem 3.6.5 of [3] (even though $R^qf(F) \mid Y_q$ may not be coherent over Y_q) that $H^p(Y_q, R^qf(F) \mid Y_q) = 0$ for $p > \dim(Y_q)$. Done.

PROPOSITION 1.3. *If F is a coherent sheaf over X, the initial term* $E_2^{p,q}(F) = 0$ *for* $p + q > \dim(f^{-1}(Y_q))$.

Proof. If $p + q > \dim(f^{-1}(Y_q))$ and $Y_q \neq \emptyset$, we conclude first from Proposition 1.1 that $p > \dim(Y_q)$, and then from Proposition 1.2 that $E_2^{p,q}(F) = 0$. If $Y_q = \emptyset$, we conclude from Lemma 2 that $E_2^{p,q}(F) = 0$ for all $p \geqq 0$. Done.

Proof of Theorem 1. We assume that the hypotheses of Theorem 1 are satisfied. Since $\dim(X) = \dim(Y)$, we conclude from Lemma 3 that $s = 0$. Hence, if $q > 0$, $Y_q \neq Y$ whence, since f is onto, $f^{-1}(Y_q) \neq X$. We conclude from Lemma 1 that $f^{-1}(Y_q)$ is a closed subvariety of X whence, since X is irreducible, $\dim(f^{-1}(Y_q)) \leqq r - 1$. Proposition 1.3 then tells us that $E_2^{p,q}(F) = 0$ for $p + q > r - 1$ and $q > 0$. We now apply the theory of spectral sequences. The natural homomorphism $\alpha^r:E_2^{r,0}(F) \to E^r(F)$ is the compositum of the two homomorphisms

$$E_2^{r,0}(F) \xrightarrow{\beta} E_\infty^{r,0}(F) \xrightarrow{\gamma} E^r(F).$$

Here, β is always an epimorphism, and, since $E_2^{p,q}(F) = 0$ for $p + q = r$ and $q > 0, \gamma$ is an isomorphism. Hence, α^r is an epimorphism. Done.

Remark 1.1. Let the hypotheses of Theorem 1 be satisfied. We showed, above, that then $\dim(f^{-1}(Y_q)) \leqq r - 1$ for $q > 0$. Hence, applying Proposition 1.1 for $q = 1$, we find that $\dim(Y_1) \leqq r - 2$, if $Y_1 \neq \emptyset$; if $Y_1 = \emptyset$, this inequality is trivially correct, as long as $r \geqq 1$. We observe that, if Y is normal and f is birational, Y_1 is "the fundamental variety of f on Y." The inequality, $\dim(Y_1) \leqq r - 2$, for the fundamental variety occurs as a corollary on page 514 of [7], in the special case that X and Y are projective.

2. Proper mappings and completions of mappings

We begin with two preliminary propositions which are needed for the sequel.

PROPOSITION 2.1. *Let $f: X \to Y$ be a closed mapping from a topological space X into a topological space Y. Let Y' be a subspace of Y and denote $f^{-1}(Y') = X'$. Then, the mapping $f \mid X': X' \to Y'$ is closed. (f does not have to be continuous.)*

Proof. Let A be a closed subset of X'; we have to show that $f(A)$ is closed in Y'. Hereto, let $A = X' \cap B$, where B is closed in X. Then, $f(A) = f(X') \cap f(B)$, because $X' = f^{-1}(Y')$. We conclude that $f(A)$ is closed in $f(X')$. The fact that $f(X') = Y' \cap f(X)$ and that $f(X)$ is closed in Y, shows that $f(X')$ is closed in Y'. Hence, $f(A)$ is closed in Y'. Done.

PROPOSITION 2.2. *Let $f: X \to Y$ be a proper mapping from an algebraic variety X into an algebraic variety Y. Let Y' be a locally closed subvariety of Y, and denote $f^{-1}(Y') = X'$. Then,*

(a) *$f \mid X': X' \to Y'$ is a proper mapping.*

(b) *If Y' is complete, X' is complete.*

Proof. (a) Let Z be an algebraic variety. We are given that the mapping $f \times 1: X \times Z \to Y \times Z$ is closed, and we have to show that the mapping $f' \times 1: X' \times Z \to Y' \times Z$ is closed, where we denoted $f \mid X'$ by f'. This however follows from Proposition 2.1, since $(f \times 1)^{-1}(Y' \times Z) = X' \times Z$ and $(f \times 1) \mid (X' \times Z) = f' \times 1$.

(b) We continue the notation of (a). We are given that the projection $p: Y' \times Z \to Z$ is a closed mapping, and we have to show that the projection $q: X' \times Z \to Z$ is a closed mapping. Since $q = p(f' \times 1)$, this follows immediately from the proof of (a). Done.

We now come to the main part of this section.

DEFINITION 2.1. *Let $f: X \to Y$ be a regular mapping from an algebraic variety X into an algebraic variety Y. We say that f can be completed if the following conditions are satisfied:*

(a) *X is a locally closed subvariety of a complete variety X^* and Y is a locally closed subvariety of a complete variety Y^*.*

(b) *There exists a regular (and hence proper) mapping $g: X^* \to Y^*$.*

(c) $g \mid X = f$ *and* $g^{-1}(Y) = X$. *In that case, we call* $g\colon X^* \to Y^*$ *a completion of* f.

We call an algebraic variety *quasi-complete*, if it is a locally closed subvariety of a complete variety. The question, whether every algebraic variety is quasi-complete, is unsolved.

PROPOSITION 2.3. *Let* $f\colon X \to Y$ *be a regular mapping from a quasi-complete variety* X *into a quasi-complete variety* Y. *Then,* f *can be completed if and only if* f *is proper. If* f *is proper and* X *and* Y *are quasi-projective, there exists a completion* $g\colon X^* \to Y^*$ *of* f *such that* X^* *and* Y^* *are projective varieties.*

Proof. Suppose that f can be completed, and let $g\colon X^* \to Y^*$ be a completion of f. According to Proposition 2.2 (a), $g \mid g^{-1}(Y)\colon g^{-1}(Y) \to Y$ is then a proper mapping. Since $g^{-1}(Y) = X$ and $g \mid X = f$, this shows that f is then proper. Conversely, assume that f is proper. Let X (respectively Y) be a locally closed subvariety of a complete variety $\bar{X}$ (respectively $\bar{Y}$). We denote the graph of $f\colon X \to Y$ by G, and the closure of G in $\bar{X} \times \bar{Y}$ by $\bar{G}$. Since f is proper, G is closed in $\bar{X} \times Y$, i.e., $G = \bar{G} \cap (\bar{X} \times Y)$. We have the commutative diagram:

$$\begin{array}{ccccc} \bar{G} & \xrightarrow{\bar{\imath}} & \bar{X} \times \bar{Y} & \xrightarrow{\bar{\pi}} & \bar{Y} \\ \uparrow & & & & \uparrow \\ G & \xrightarrow{i} & X \times Y & \xrightarrow{\pi} & Y. \end{array}$$

Here, i, $\bar{\imath}$, and the vertical arrows are inclusion mappings, while π and $\bar{\pi}$ are projections. In order to show that f can be completed, we replace X by G as usual, and prove that $\bar{\pi}\bar{\imath}\colon \bar{G} \to \bar{Y}$ is a completion of $\pi i\colon G \to Y$. All we have to check on is condition (c) of Definition 2.1, i.e., that $\bar{\pi}\bar{\imath} \mid G = \pi i$ and that $(\bar{\pi}\bar{\imath})^{-1}(Y) = G$. The first equality is equivalent to saying that the above diagram is commutative. The second equality follows from the fact that $(\bar{\pi}\bar{\imath})^{-1}(Y) = \bar{G} \cap (\bar{X} \times Y)$ and the above remark that $G = \bar{G} \cap (\bar{X} \times Y)$. Finally, if X and Y are quasi-projective, we may choose projective varieties for $\bar{X}$ and $\bar{Y}$, in which case $\bar{G}$ is necessarily projective. Done.

3. The proof of Lemma 1

We begin with some topological preliminaries.

PROPOSITION 3.1. *Let* $f\colon X \to Y$ *be a continuous, closed mapping from a topological space* X *into a topological space* Y. *Let* $y \in Y$. *Then, the set* $\{f^{-1}(U) \mid U$ *is an open neighborhood of* $y\}$ *is a fundamental system of neighborhoods of* $f^{-1}(y)$.

Proof. Let V be an open subset of X which contains $f^{-1}(y)$. We have to find an open neighborhood U of y such that $f^{-1}(U) \subset V$. We denote $Y - f(X - V) = U$ and prove that this U has the required properties. Since f is closed, U is open in Y. Furthermore, $f^{-1}(y) \subset V$ means that $y \notin f(X - V)$, i.e., that $y \in U$. Finally, if $x \in f^{-1}(U)$, $f(x) \in U$, i.e.,

$f(x) \notin f(X - V)$. This shows that $x \notin X - V$, i.e., that $x \in V$. Hence, $f^{-1}(U) \subset V$. Done.

If X is a topological space, the largest integer n, such that there exists a properly descending sequence $F_0 \supset F_1 \supset \cdots \supset F_n$ of closed, irreducible subspaces of X, is called *the combinatorial dimension of* X. See [8], pages 21–23, for irreducible spaces; every irreducible space is nonempty by definition. *All dimensions of topological spaces which occur in this paper are combinatorial dimensions.* We denote the dimension of a topological space X by $\dim(X)$. Of course, $\dim(X)$ may be infinite. If $X = \emptyset$, $\dim(X) = -1$. If X is an algebraic variety, $\dim(X)$ is equal to its dimension as an algebraic variety.

DEFINITION 3.1. *A topological space X is called a D-space, if the following condition is satisfied. Every closed, finite-dimensional subspace W of X possesses an open neighborhood U with the property that, if $Z \subset U$ and Z is closed in X, then* $\dim(Z) \leqq \dim(W)$.

We chose the letter D, because it is the first letter of Dimension. Projective varieties are examples of D-spaces. (See Proposition 3.4.)

PROPOSITION 3.2. *A closed subspace of a D-space is a D-space.*

Proof. Let X be a D-space, and Y a closed subspace of X. If W is a closed, finite-dimensional subspace of Y (and hence of X), W possesses an open neighborhood U relative to X which has, relative to X, the property formulated in Definition 3.1. Then, $Y \cap U$ is an open neighborhood of W relative to Y which has, relative to Y, the property formulated in Definition 3.1. Done.

PROPOSITION 3.3. *Let $f\colon X \to Y$ be a continuous, closed mapping from a D-space X into a T_1-space Y. For each integer $q \geqq -1$, let*

$$Y_q = \{y \mid y \in Y \text{ and } \dim(f^{-1}(y)) \geqq q\}.$$

Then, Y_q is closed in Y, for all $q \geqq -1$.

Proof. Let $q \geqq -1$ be an integer, and let y belong to the closure of Y_q in Y. We have to show that then $\dim(f^{-1}(y)) \geqq q$. Suppose that $\dim(f^{-1}(y)) < q$. Since Y is a T_1-space, $f^{-1}(y)$ is closed in X, whence, since X is a D-space, $f^{-1}(y)$ possesses an open neighborhood U which satisfies the condition of Definition 3.1. We conclude from Proposition 3.1 that y possesses an open neighborhood V such that $f^{-1}(V) \subset U$. There exists some $z \in V \cap Y_q$. Clearly, $f^{-1}(z)$ is a subset of U which is closed in X, and hence $\dim(f^{-1}(z)) \leqq \dim(f^{-1}(y)) < q$. This contradicts the assumption that $z \in Y_q$. Done.

This finishes the topological preliminaries, and we now apply them to algebraic geometry.

PROPOSITION 3.4. *A projective variety is a D-space.*

Proof. We see from Proposition 3.2 that all we have to show is that a projective space P is a D-space. Let $\dim(P) = n$, and let W be a closed subspace of P. We have to show that W possesses an open neighborhood U which satisfies the condition of Definition 3.1. If $W = P$, choose $U = P$. If $W \neq P$, $\dim(W) < n$, and there exists an $(n-\dim(W)-1)$-dimensional hyperplane H in P, such that $W \cap H = \emptyset$. In order to show that the open neighborhood $U = P - H$ of W has the required property, select $Z \subset U$, where Z is closed in P. Then, $Z \cap H = \emptyset$, whence $\dim(Z) + \dim(H) < n$. (See [9], Chapter II, Section 7.) This means precisely that $\dim(Z) \leqq \dim(W)$. Done.

Remark 3.1. Although we do not need it for this paper, it is interesting to observe that Proposition 3.4 can be sharpened as follows: *Every locally closed subspace W of a projective variety X possesses an open neighborhood U with the property that, if $Z \subset U$ and Z is closed in X, then* $\dim(Z) \leqq \dim(W)$. This follows immediately from Proposition 3.4, together with the fact that, if W is locally closed in a projective variety X, then $\dim(W) = \dim(\bar{W})$, where $\bar{W}$ denotes the closure of W in X.

Proof of Lemma 1. Let $f: X \to Y$ be a proper mapping from a quasi-projective variety X into an algebriac variety Y. Let Y_q, for $q \geqq -1$, have the same meaning as in Proposition 3.3. We have to show that Y_q is a closed subset of Y, for all $q \geqq -1$. (Hence, for Lemma 1, it is not necessary that f is onto.)

Case 1. *X is projective.* X is then a D-space (see Proposition 3.4), Y is a T_1-space, and f is continuous and closed. Hence, Lemma 1 then follows from Proposition 3.3.

Case 2. *X and Y are quasi-projective.* Let $g: X^* \to Y^*$ be a completion of f, where X^* and Y^* are projective; the existence of g is guaranteed by Proposition 2.3. It follows from Definition 2.1 that $f^{-1}(y) = g^{-1}(y)$, for all $y \in Y$. Hence, $Y_q = Y_q^* \cap Y$, where $Y_q^* = \{y \mid y \in Y^* \text{ and } \dim(g^{-1}(y)) \geqq q\}$. We conclude from Case 1 that Y_q^* is closed in Y^*, whence Y_q is closed in Y.

Case 3. *X is quasi-projective, and Y is an arbitrary algebraic variety.* Let $Y = Y_1 \cup \cdots \cup Y_n$, where each Y_i is an open, affine subvariety of Y. We denote $f^{-1}(Y_i) = X_i$ and $f \mid X_i = f_i$. We conclude from Proposition 2.2 (a) that $f_i : X_i \to Y_i$ is a proper mapping, for $i = 1, \cdots, n$. Since X_i and Y_i are both quasi-projective, we conclude from Case 2 that the set $Y_q^{(i)} = \{y \mid y \in Y_i \text{ and } \dim(f_i^{-1}(y)) \geqq q\}$ is closed in Y_i. Since $Y_q^{(i)} = Y_q \cap Y_i$, it follows that Y_q is closed in Y. Done.

We have not been able to prove Lemma 1 under the weaker assumption that X is an arbitrary algebraic variety.

4. The proof of Lemma 2

We begin with some topological preliminaries.

A topological space is called a Zariski space if its open subsets satisfy the ascending chain condition. (See [8], page 23, where the term "Noetherian space" is used, instead.) Zariski spaces are not necessarily finite-dimensional. (See [3], page 171.)

We say that a collection $\mathfrak{B}$ of open subsets of a topological space X is *finitary* if the following condition is satisfied: *For every finite subset Y of X, the neighborhoods of Y which belong to $\mathfrak{B}$ form a fundamental system of neighborhoods of Y.* Observe that $\mathfrak{B}$ is necessarily a base for the open sets of X.

PROPOSITION 4.1. *Let X be a T_0-space, and $\mathfrak{B}$ a finitary collection of open subsets of X. Let W be a finite-dimensional, Zariski subspace of X; we denote* $\dim(W) = c$. *Then, if U is an open neighborhood of W, there exists $B_1, \cdots, B_{c+1} \in \mathfrak{B}$, such that $W \subset B_1 \cup \cdots \cup B_{c+1} \subset U$.*

Proof. A Zariski space which is 0-dimensional and a T_0-space consists of a finite number of points. Hence, if $c = 0$, Proposition 4.1 follows immediately from the definition of "finitary." Let $c > 0$, and let U be an open neighborhood of W; we proceed by induction on c. W has only a finite number of irreducible components, say $W_1, \cdots, W_s$. We choose a point $x_i \in W_i$, for $i = 1, \cdots, s$, and select $B_1 \in \mathfrak{B}$, such that $\{x_1, \cdots, x_s\} \subset B_1 \subset U$. Since $\dim(W - B_1) < c$, there exists (by induction on c) $B_2, \cdots, B_{c+1} \in \mathfrak{B}$, such that $W - B_1 \subset B_2 \cup \cdots \cup B_{c+1} \subset U$. It follows that

$$W \subset B_1 \cup \cdots \cup B_{c+1} \subset U.$$

Done.

Let F be a sheaf of abelian groups over a topological space X as base space. We say that a collection $\mathfrak{A}$ of subsets of X is *F-acyclic* if the following condition is satisfied: *If $A_1, \cdots, A_n \in \mathfrak{A}$, where $1 \leqq n < \infty$, then $H^q(A_1 \cap \cdots \cap A_n, F) = 0$, for $q > 0$.* This condition occurs in Corollary 4 on page 176 of [3].

Consider now a subset W of our topological space X. The direct limit, $\varinjlim H^q(U, F)$, where U runs through the directed set of open neighborhoods of W, is well defined for all $q \geqq 0$. We denote this group by $H_0^q(W, F)$. Needless to say that $H_0^q(W, F)$ and $H^q(W, F)$ should not be confused.

PROPOSITION 4.2. *Let X be a T_0-space, and F a sheaf of abelian groups over X as base space. Let $\mathfrak{B}$ be a finitary collection of open subsets of X, which is F-acyclic. Then, for every finite-dimensional, Zariski subspace W of X, $H_0^q(W, F) = 0$ for $q > \dim(W)$.*

Proof. Denote $\dim(W) = c$, and let U be an open neighborhood of W. We will show that there exists an open neighborhood V of W, such that $V \subset U$ and $H^q(V, F) = 0$, for $q > c$; Proposition 4.2 is, of course, an immediate corollary of this. Hereto, select $B_1, \cdots, B_{c+1} \in \mathfrak{B}$, such that $W \subset B_1 \cup \cdots \cup B_{c+1} \subset U$; this can be done, according to Proposition 4.1. We denote $B_1 \cup \cdots \cup B_{c+1} = V$, and all there remains to be proved is that $H^q(V, F) = 0$, for $q > c$. The sets $B_1, \cdots, B_{c+1}$ form a covering $\mathfrak{B}$ of V, and it is trivial that $H^q(\mathfrak{B}, F) = 0$, for $q > c$. We conclude from [3], Corollary 1, page 175, that $H^q(\mathfrak{B}, F) = H^q(V, F)$, for all $q \geqq 0$. Done.

PROPOSITION 4.3. *Let f: $X \to Y$ be a continuous, closed mapping from a* *topological space X into a topological space Y. Let F be a sheaf of abelian groups over X as base space, and let $y \in Y$. Then, $(R^q f(F))_y = H_0^q(f^{-1}(y), F)$, for all $q \geqq 0$.*

Proof. $(R^q f(F))_y = \varinjlim H^q(f^{-1}(U), F)$, where U runs through the open neighborhoods of y. Hence, Proposition 4.3 is an immediate corollary of Proposition 3.1 and the definition of $H_0^q(f^{-1}(y), F)$. Done.

The following proposition interrelates the support of $R^q f(F)$ with the set $Y_q = \{y \mid y \in Y \text{ and } \dim(f^{-1}(y)) \geqq q\}$, where $q \geqq 0$.

PROPOSITION 4.4. *Let $f: X \to Y$ be a continuous, closed mapping from a T_0-space X into a topological space Y, and let F be a sheaf of abelian groups over X as base space. We assume furthermore that X is a Zariski space and possesses a finitary collection of open subsets, which is F-acyclic. Then,* $\operatorname{Sup}(R^q f(F)) \subset Y_q$, *for all $q \geqq 0$.*

Proof. Let $q \geqq 0$, and let $y \in Y$ be such that $\dim(f^{-1}(y)) < q$. We have to show that then $(R^q f(F))_y = 0$. We conclude from Proposition 4.3 that $(R^q f(F))_y = H_0^q(f^{-1}(y), F)$, and from Proposition 4.2 that $H_0^q(f^{-1}(y), F) = 0$. Done.

This finishes the topological preliminaries, and we now apply them to algebraic geometry.

Proof of Lemma 2. Let X be an algebraic variety, and F a coherent sheaf over X. The collection $\mathfrak{B}$ of open, affine subvarieties of X is F-acyclic as follows from [1], Proposition 1, page 234, and Corollary 1, page 239. This collection $\mathfrak{B}$ is definitely not finitary, if X is an arbitrary algebraic variety as follows from [10], Theorem 3. (The points P and P', which occur in that theorem, are not contained in any open, affine subvariety of the surface V of that theorem.) It is well known, however, and also follows easily from [1], Lemma 1, page 244, that $\mathfrak{B}$ is finitary if X is quasi-projective. Hence, the following proposition is an immediate corollary of Proposition 4.4.

PROPOSITION 4.5. *Let f: $X \to Y$ be a continuous, closed mapping from a quasi-projective variety X into a topological space Y, and let F be a coherent sheaf over X. Then,* $\operatorname{Sup}(R^q f(F)) \subset Y_q$, *for $q \geqq 0$.*

Since proper mappings are continuous and closed, Lemma 2 has now of course been proved. We see that the assumption that Y is algebraic (or that f is onto) is irrelevant for that lemma.

Remark 4.1. Let the conditions of Proposition 4.5 be satisfied, and suppose furthermore that $f^{-1}(y)$ consists of a finite number of points, for all $y \in Y$. We conclude from Proposition 4.5 that then $R^q f(F) = 0$, for $q > 0$ and, consequently, that $H^q(X, F) \simeq H^q(Y, f(F))$, for all $q \geqq 0$. This situation occurs, for instance, in the following two cases.

(a) Y is an irreducible, projective variety, and f: $X \to Y$ is the usual, birational correspondence from a derived, normal model X of Y, onto Y.

See [7], Theorem 7, page 511. As an example $f(O_X)$, where O_X is the sheaf of local rings of X, is now "the sheaf of the integral closures of the local rings of Y." Hence, the sheaf of local rings O_Y of Y is a subsheaf of $f(O_X)$, and the support of the quotient-sheaf $f(O_X)/O_Y$ is the conductor variety Δ of Y. Denoting $\dim(Y) = r$, we know that $\dim(\Delta) < r$, whence the exact sequence $0 \to O_Y \to f(O_X) \to f(O_X)/O_Y \to 0$ gives rise to the exact sequence $H^r(Y, O_Y) \to H^r(Y, f(O_X)) \to 0$. Consequently, we conclude from the isomorphism $H^r(Y, f(O_X)) \simeq H^r(X, O_X)$, that *there exists an epimorphism* $H^r(Y, O_Y) \to H^r(X, O_X)$. Further conclusions can of course be drawn if $\dim(\Delta) < r - 1$.

(b) X and Y are irreducible, projective varieties of the same dimension, $f: X \to Y$ is a regular, rational mapping from X onto Y, and "f^{-1} has no fundamental points on Y." See [11], page 6.

5. The proof of Lemma 3

We begin with some topological preliminaries.

Let $f: X \to Y$ be a continuous, closed mapping from a finite-dimensional Zariski space X *onto* a finite-dimensional space Y. (Since a continuous image of a Zariski space is a Zariski space, Y is necessarily a Zariski space.) We put $s = \text{Minimum}(\dim(f^{-1}(y)))$, where y runs through Y.

PROPOSITION 5.1. *If X is a D-space and Y is a T_1-space,*

$$\dim(X) - \dim(Y) \geqq s.$$

Proof. Denote $\dim(Y) = t$. If $t = 0$, Proposition 5.1 follows from the fact that the dimension of a topological space is greater than or equal to that of any of its subspaces. We may hence make the induction hypothesis that Proposition 5.1 has been proved for $\dim(Y) = 0, 1, \cdots, t - 1$ and that $t \geqq 1$.

Case (a). *X is irreducible.* Let $F_0 \supset F_1 \supset \cdots \supset F_t$ be a properly descending chain of closed, irreducible subspaces of Y. Clearly, $\dim(F_1) = t - 1$, and we consider the mapping $f \mid f^{-1}(F_1) : f^{-1}(F_1) \to F_1$. This mapping is still continuous, closed, and onto, and we conclude from Proposition 3.2 that $f^{-1}(F_1)$ is a D-space. Hence, we may infer from the induction hypothesis that $\dim(f^{-1}(F_1)) \geqq t - 1 + s_1$; here, $s_1 = \text{Minimum}(\dim(f^{-1}(y)))$, where y runs through F_1. The fact that $F_1 \neq Y$ and that $f: X \to Y$ is an epimorphism, shows that $f^{-1}(F_1) \neq X$. Consequently, since X is irreducible and $f^{-1}(F_1)$ is a closed subset of X, $\dim(f^{-1}(F_1)) \leqq \dim(X) - 1$, which proves that $\dim(X) - 1 \geqq t - 1 + s_1$. It is obvious that $s_1 \geqq s$, and we are done with Case (a).

Case (b). *Y is irreducible.* Let $X = X_1 \cup \cdots \cup X_n$ be the decomposition of X into irreducible components. Then $Y = f(X_1) \cup \cdots \cup f(X_n)$, and hence, since Y is irreducible and f is closed, $Y = f(X_i)$ for at least one i, $1 \leqq i \leqq n$. We assume that the enumeration was such that $f(X_1) = \cdots =$

$f(X_m) = Y$ and $f(X_{m+i}) \neq Y$ for $i = 1, \cdots, n - m$. It may of course happen that $m = n$.

We denote the mapping $f \mid X_i : X_i \to Y$ by f_i and the Minimum$(\dim(f_i^{-1}(y)))$, where y runs through Y, by s_i for $i = 1, \cdots, m$. We conclude from Proposition 3.2 that X_i is a D-space, whence Case (a) may be applied to f_i. It follows that $\dim(X_i) \geqq t + s_i$, for $i = 1, \cdots, m$. Since $\dim(X) \geqq \dim(X_i)$, we will be done as soon as we have shown that $s = \text{Maximum}(s_1, \cdots, s_m)$.

Hereto, we denote $Z = \{y \mid y \in Y \text{ and } \dim(f^{-1}(y)) = s\}$ and $Z_i = \{y \mid y \in Y$ and $\dim(f_i^{-1}(y)) = s_i\}$ for $i = 1, \cdots, m$. Since Z is the complement of the set $Y_{s+1} = \{y \mid y \in Y \text{ and } \dim(f^{-1}(y)) \geqq s + 1\}$, we conclude from Proposition 3.3 that Z is an open subset of Y. The same proposition may also be applied to the mappings f_i, whence Z_i is an open subset of Y for $i = 1, \cdots, m$. It is obvious that $Z, Z_1, \cdots, Z_m$ are all nonempty. The fact that f is closed and Y is irreducible implies that $f(X_{m+1}) \cup \cdots \cup f(X_n) \neq Y$, and we denote the open, nonempty complement of $f(X_{m+1}) \cup \cdots \cup f(X_n)$ by V. (If $m = n$, we put $V = Y$.)

Since Y is irreducible, we can select a point $y \in V \cap Z \cap Z_1 \cap \cdots \cap Z_m$. We conclude from $y \in V$ that $f^{-1}(y) \subset X_1 \cup \cdots \cup X_m$ and hence that $f^{-1}(y) = \bigcup_{i=1}^m (f^{-1}(y) \cap X_i) = \bigcup_{i=1}^m f_i^{-1}(y)$. Each $f_i^{-1}(y)$ is closed in $f^{-1}(y)$, whence $\dim(f^{-1}(y)) = \text{Maximum}(\dim(f_i^{-1}(y)))$, where $i = 1, \cdots, m$. The fact that $y \in Z$ shows that $\dim(f^{-1}(y)) = s$; the fact that $y \in Z_i$ shows that $\dim(f_i^{-1}(y)) = s_i$, and we are done with Case (b).

Case (c). *Y is reducible.* Let G be an irreducible component of dimension t of Y. We denote the mapping $f \mid f^{-1}(G) \colon f^{-1}(G) \to G$ by g and the Minimum$(\dim(g^{-1}(y)))$, where y runs through G, by s_0. Proposition 3.2 shows that $f^{-1}(G)$ is a D-space, whence we conclude from Case (b) that $\dim(f^{-1}(G)) \geqq t + s_0$. Clearly, $g^{-1}(y) = f^{-1}(y)$ for all $y \in G$, and hence $s_0 \geqq s$. It is obvious that $\dim(f^{-1}(G)) \leqq \dim(X)$, and hence we are done.

This finishes the topological preliminaries, and we now apply them to algebraic geometry.

Proof of Lemma 3. Let $f \colon X \to Y$ be a proper mapping from an algebraic variety X *onto* an algebraic variety Y. Let $s = \text{Minimum}(\dim(f^{-1}(y)))$, where y runs through Y. Lemma 3 is contained in the following proposition. We see that the assumption that X is quasi-projective is irrelevant for that lemma.

PROPOSITION 5.2. $\dim(X) - \dim(Y) \geqq s$.

Proof. Case (a). *X and Y are both quasi-projective.* According to Proposition 2.3, there exists a completion $g \colon X^* \to Y^*$ of f, where X^* and Y^* are projective varieties. We denote the closure of X in X^* (respectively of Y in Y^*) by $\bar{X}$ (respectively $\bar{Y}$). Since g is a closed mapping and f is an epimorphism, $g(\bar{X}) = \bar{Y}$, and we denote $g \mid \bar{X} \colon \bar{X} \to \bar{Y}$ by h. Evidently, h is also a completion of f, while $\bar{X}$ and $\bar{Y}$ are projective varieties.

Since h is an epimorphism and $\bar{X}$ is a D-space (see Proposition 3.4), we may apply Proposition 5.1 to h, which gives $\dim(\bar{X}) \geqq \dim(\bar{Y}) + \bar{s}$; here, $\bar{s} = \text{Minimum}(\dim(h^{-1}(y)))$ where y runs through $\bar{Y}$. We conclude from $\dim(\bar{X}) = \dim(X)$ and $\dim(\bar{Y}) = \dim(Y)$ that $\dim(X) \geqq \dim(Y) + \bar{s}$. We now proceed in two steps.

Step (a.1). *Y is irreducible.* We shall show that then $\bar{s} = s$. Since h is a completion of f, $h^{-1}(y) = f^{-1}(y)$ for all $y \in Y$. Hence, all we have to show is that $\dim(h^{-1}(y)) = \bar{s}$ for some $y \in Y$. Hereto, let

$$Z = \{y \mid y \in \bar{Y} \text{ and } \dim(h^{-1}(y)) = \bar{s}\}.$$

We may apply Proposition 3.3 to h, whence Z is an open subset of $\bar{Y}$. (Namely, Z is the complement of the set $\{y \mid y \in \bar{Y} \text{ and } \dim(h^{-1}(y)) \geqq \bar{s} + 1\}$.) Hence, Z and Y are open, nonempty subsets of the necessarily irreducible space $\bar{Y}$, whence $Z \cap Y \neq \emptyset$. Done with Step (a.1).

Step (a.2). *Y is reducible.* Let W be an irreducible component of Y, such that $\dim(W) = \dim(Y)$. The result of Step (a.1) may be applied to the mapping $f \mid f^{-1}(W): f^{-1}(W) \to W$, whence $\dim(f^{-1}(W)) \geqq \dim(Y) + s'$; here, $s' = \text{Minimum}(\dim(f^{-1}(y)))$ where y runs through W. Since $\dim(f^{-1}(W)) \leqq \dim(X)$ and $s' \geqq s$, we are done with Case (a).

Case (b). *X is quasi-projective and Y is an arbitrary algebraic variety.* Let W be an irreducible component of Y, such that $\dim(W) = \dim(Y)$. Let V be an open, affine subvariety of W. Then, $\dim(V) = \dim(Y)$, and we conclude from Proposition 2.2 that the mapping $f \mid f^{-1}(V): f^{-1}(V) \to V$ is proper. Case (a) may hence be applied to this mapping, whence $\dim(f^{-1}(V)) \geqq \dim(Y) + s'$; here, $s' = \text{Minimum}(\dim(f^{-1}(y)))$ where y runs through V. It is obvious that $\dim(f^{-1}(V)) \leqq \dim(X)$ and that $s' \geqq s$; done with Case (b).

Case (c). *X and Y are arbitrary algebraic varieties.* Let V be a quasi-projective variety of the same dimension as X, and $g: V \to X$ a proper mapping from V onto X. The existence of V and g is assured by the Chow Lemma; see [12], page 123. We may apply Case (b) to the mapping $fg: V \to Y$, and we conclude that $\dim(X) \geqq \dim(Y) + s'$; here, $s' = \text{Minimum}(\dim((fg)^{-1}(y)))$ where y runs through Y. All there remains to be shown is that $s' \geqq s$.

Hereto, select $z \in Y$ such that $\dim((fg)^{-1}(z)) = s'$. Since $(fg)^{-1}(z) = g^{-1}(f^{-1}(z))$, and since Case (b) may be applied to the mapping $g': g^{-1}(f^{-1}(z)) \to f^{-1}(z)$, where g' is the restriction of g to $g^{-1}(f^{-1}(z))$, we can certainly assert that $\dim((fg)^{-1}(z)) \geqq \dim(f^{-1}(z))$. It is trivial that $\dim(f^{-1}(z)) \geqq s$, and we are done.

Remark 5.1. We shall see in the next section that, if X is irreducible and quasi-projective, $\dim(X) - \dim(Y) = s$.

If X is reducible, it may very well happen that $\dim(X) - \dim(Y) > s$, even when X and Y are projective varieties and Y is irreducible. For example, let $X = X_1 \cup X_2$ where X_1 is an irreducible, projective variety of dimension 1, X_2 is an irreducible, projective variety of dimension 2, and $X_1 \cap X_2 = \emptyset$.

We choose $Y = X_1$ and define $f: X \to Y$ as follows: On X_1, f is the identity mapping, and f maps all of X_2 on a point of X_1. Then $\dim(X) = 2$, $\dim(Y) = 1$, $s = 0$, and hence $\dim(X) - \dim(Y) > s$.

6. The equality $\dim(X) - \dim(Y) = s$

Let $f: X \to Y$ be a proper mapping from an algebraic variety X *onto* an algebraic variety Y. Let $s = \text{Minimum}(\dim(f^{-1}(y)))$, where y runs through Y. It is the purpose of this section to prove

PROPOSITION 6.1. *If X is quasi-projective and irreducible,*

$$\dim(X) - \dim(Y) = s.$$

In order to do this, we first prove the "dimension theorem" (see [9], page 38) in the following form.

An open, nonempty subset U of an algebraic variety X with sheaf of local rings O_X is itself an algebraic variety with sheaf of local rings $O_X \mid U$. We say that "X is locally an affine space," if every point of X is contained in an open subset which is biregularly isomorphic with an affine space. Examples of such varieties are affine spaces, projective spaces, and products of varieties which are themselves locally affine spaces.

THE DIMENSION THEOREM. *Let X be an irreducible, algebraic variety which is locally an affine space. Let V and W be closed, irreducible subsets of X, and C an irreducible component of $V \cap W$. (Irreducible spaces are nonempty by definition; see* [8], *page* 21.) *Then,* $\dim(C) \geqq \dim(V) + \dim(W) - \dim(X)$.

Proof. Select $x \in C$ and an open neighborhood U of x which is biregularly isomorphic with an affine space. Since $V \cap W \cap U$ is an open subset of $V \cap W$, we conclude from [8], page 22, (2.1.6), that $C \cap U$ is an irreducible component of $V \cap W \cap U$. We may apply the dimension theorem for affine spaces (see [9], page 38, the Corollary) to the closed, irreducible subsets $V \cap U$ and $W \cap U$ of U, from which it follows that $\dim(C \cap U) \geqq \dim(V \cap U) + \dim(W \cap U) - \dim(U)$. Evidently, $\dim(C \cap U) = \dim(C)$, $\dim(V \cap U) = \dim(V)$, $\dim(W \cap U) = \dim(W)$, and $\dim(U) = \dim(X)$. Done.

Proof of Proposition 6.1. We denote $\dim(X) = r$, $\dim(Y) = t$ and conclude from Proposition 5.2 that $r - t \geqq s$. All there remains to be shown is that $r - t \leqq s$.

Case (a). *Y is an affine space.* X is a locally closed subvariety of an n-dimensional projective space P, and the graph G of $f: X \to Y$ is a closed subvariety of $P \times Y$ (since f is proper). We replace f by πi, where $i: G \to P \times Y$ is the inclusion mapping and $\pi: P \times Y \to Y$ is the natural projection. We select $y \in Y$ and have to show that $r - t \leqq \dim((\pi i)^{-1}(y))$. Since f is onto, $(\pi i)^{-1}(y) \neq \emptyset$, and we can select an irreducible component C of $(\pi i)^{-1}(y)$. We will show that $\dim(C) \geqq r - t$, which is more than is required.

Hereto, we observe that $P \times Y$ is locally an affine space, that $(\pi i)^{-1}(y) = G \cap (P \times y)$ and apply the Dimension Theorem to the closed, irreducible subsets G and $P \times y$ of $P \times Y$. It follows that

$$\dim(C) \geqq \dim(G) + n - (n + t) = r - t.$$

Done.

Case (b). *Y is an affine variety.* Y is necessarily irreducible, and we may apply the Noether normalization theorem. In geometric form this theorem states that there exists a proper mapping $g: Y \rightarrow S$ from Y onto an affine space S of the same dimension t, such that $g^{-1}(z)$ consists of a finite number of points, for all $z \in S$. We denote $s' = \text{Minimum}(\dim((gf)^{-1}(z)))$, where z runs through S, and apply Case (a) to the proper mapping $gf: X \rightarrow S$. We conclude that $r - t \leqq s'$, whence all there remains to be shown is that $s' = s$. It is obvious that $s \leqq s'$, and we now show that $s' \leqq s$.

Let $Y_{s+1} = \{y \mid y \in Y \text{ and } \dim(f^{-1}(y)) \geqq s + 1\}$. Evidently, $Y_{s+1} \neq Y$, and we conclude from Lemma 1 that Y_{s+1} is closed in Y, whence $\dim(Y_{s+1}) < t$. If $Y_{s+1} \neq \emptyset$, we apply Proposition 5.2 to the proper mapping $g \mid Y_{s+1} : Y_{s+1} \rightarrow g(Y_{s+1})$, which shows that $\dim(g(Y_{s+1})) < t$, and hence that $g(Y_{s+1}) \neq S$; if $Y_{s+1} = \emptyset$, it is trivial that $g(Y_{s+1}) \neq S$. We can consequently select a point $z \in S$ such that $z \notin g(Y_{s+1})$, i.e., such that $g^{-1}(z) \cap Y_{s+1} = \emptyset$. This means that, for all $y \in g^{-1}(z)$, $\dim(f^{-1}(y)) = s$. Since $g^{-1}(z)$ consists of a finite number of points, it follows that $\dim(f^{-1}(g^{-1}(z))) = s$, and hence that $s' \leqq s$. Done.

Case (c). *Y is any* (*necessarily irreducible*) *algebraic variety.* Select an open, affine subvariety U of Y, and consider the mapping

$$f \mid f^{-1}(U): f^{-1}(U) \rightarrow U.$$

This mapping is proper, according to Proposition 2.2, and we may apply Case (b) to it. Hence, if $m = \text{Minimum}(\dim(f^{-1}(y)))$ where y runs through U, then $\dim(f^{-1}(U)) - \dim(U) \leqq m$. Since $\dim(f^{-1}(U)) = r$ and $\dim(U) = t$, all there remains to be shown is that $m = s$.

For this purpose, all we have to prove is that there exists a point $y \in U$ such that $\dim(f^{-1}(y)) = s$. Let Y_{s+1} have the same meaning as under (b). We conclude, as under (b), that the complement Z of Y_{s+1} is an open, non-empty subset of Y, whence $U \cap Z \neq \emptyset$. For all $y \in U \cap Z$, $\dim(f^{-1}(y)) = s$. Done.

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