FUNCTIONS WHOSE PARTIAL DERIVATIVES ARE MEASURES

BY
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Let f denote a real-valued function on euclidean N-space such that the gradient grad f in the sense of Schwartz distribution theory is a vector-valued measure [21, p. 37]. In distribution theory functions differing in zero Lebesgue N-measure are equivalent. We intend to show that, at least in two important cases, there is a function $\bar{f}$ equivalent to f which is determined more precisely than f and is in a natural sense nicer. The function $\bar{f}$ is defined as the limit in a suitable sense of a sequence of "elementary" functions. It turns out that $\bar{f}$ is determined up to Hausdorff $(N - 1)$-measure 0.

In Part I this is done when grad f is itself a function, which is the same as to say that f is equivalent to a function locally absolutely continuous in G. C. Evans's sense. Part II concerns the more difficult case when f is an integer-valued function, dual to a current c of dimension N such that both c and its boundary bc have finite mass. In Part III we apply these results to the study of sets with finite perimeter in the sense of Caccioppoli and De Giorgi. In the special case when the boundary of a set E is a compact $(N - 1)$-manifold X such that X occupies zero Lebesgue N-measure, the perimeter of E is shown to agree with the integralgeometric $(N - 1)$-area of the inclusion map i_X.

PART I

1. Introduction

We adopt the following notation throughout: $x = (x^1, \cdots, x^N)$ is a generic point of euclidean N-space R^N $(N \geqq 2)$. For $k \leqq N$ let m_k denote Hausdorff k-dimensional measure in R^N $(=$ Lebesgue measure for $k = N)$. We use fr E, cl E for the frontier, closure of a set E, respectively. For $s > 0$, $[E]_s$ will denote the s-neighborhood of E. We write spt f for the support of f. $\mathfrak{D}$ is Schwartz's space of all infinitely differentiable functions $f(x)$ with compact support. Let BV stand for the space of all locally summable functions f such that, for each $i = 1, \cdots, N$, the i^{th} distribution theory partial derivative of f is a measure μ_i. The notation is justified by the result proved independently by Krickeberg [17] and Federer [Bull. Amer. Math. Soc., vol. 60 (1954), Abstract no. 407, p. 339] that $f \in BV$ if and only if f is locally of bounded variation in the sense of Cesari and Tonelli. We write grad f for the vector-valued measure $(\mu_1, \cdots, \mu_N)$. The total variation measure of a real- or vector-valued measure μ will be denoted by $|\mu|$; see, for ex-

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ample, Whitney [22, Chapter 11]. For $f \in BV$ and E any Borel set, we write

$$L_1(f, E) = \int_E |f| \, dm_N, \qquad I(f, E) = |\operatorname{grad} f| \, (E),$$

finite or $+\infty$. For $E = R^N$ we write merely $L_1(f)$, $I(f)$.

Let AC denote the set of all $f \in BV$ such that grad f is a function (i.e., grad f is m_N-absolutely continuous). Calkin and Morrey showed [4, Lemma 3.2, Theorem 6.3] that for $f \in AC$ there exists $\bar{f}$ equivalent to f such that $\bar{f}$ is locally absolutely continuous in Evans's sense relative to every choice of coordinates in R^N, and

$$(1) \qquad I(f, E) = \int_E |\operatorname{Grad} \bar{f}| \, dm_N, \qquad \text{all } E,$$

where Grad $\bar{f}$ is the ordinary gradient of $\bar{f}$. They obtained $\bar{f}$ as follows. Consider a suitable approximate identity ϕ_n; and let $f_n = f * \phi_n$, $n = 1, 2, \cdots$, where $*$ denotes convolution. Then every f_n is continuously differentiable, and $\lim I(f_n - f, K) = 0$ for any compact K. Let

$$(2) \qquad \bar{f}(x) = \lim_{n \to \infty} f_n(x)$$

for all x such that the limit exists and is finite.

If the partial derivatives of f are functions locally of class L_p for $p > 1$, then more is known. For $p = 2$, the case which arises in Newtonian potential theory, $\bar{f}$ can be determined up to a set of zero outer capacity of order 2; see [2], [8], or [15]. Fuglede showed [15, Chapter III] that for any $p > 1$, $\bar{f}$ can be determined up to a set E on which the M. Riesz potential of order p of a nonnegative function of class L_p can be $+\infty$. Such a set E has a capacitary dimension $N - p$; for $p > N$, $\bar{f}$ is continuous. These results are best possible. For $p = 1$, there are partial results in terms of Riesz potentials; see [15] and references cited there.

In §5 we show that the limit (2) exists in γ-measure where γ is a new set function introduced in §4. According to the theory of functional completion of Aronszajn and Smith [2], applied to the space $\mathfrak{D}$ in the I-norm, this result is best possible. The exceptional sets in the perfect functional completion are precisely the γ-null sets. Using an inequality recently proved by Gustin [16], it is then shown that the γ-null sets are precisely the m_{N-1}-null sets (Theorem 4.3).

2. Preliminary lemmas

In this section we collect some lemmas, mostly well known, about functions $f \in BV$.

Regularization [21, p. 21]. For $n = 1, 2, \cdots$, choose $\phi_n \in \mathfrak{D}$ such that $\phi_n \geqq 0$, $L_1(\phi_n) = 1$, and spt $\phi_n \subset$ (n^{-1}-neighborhood of 0). Write f_n for the convolution:

$$(1) \qquad f_n(x) = f * \phi_n(x) = \int_{R^N} f(y)\phi_n(x - y) \, dm_N(y), \qquad n = 1, 2, \cdots.$$

From standard reasoning and the formula

$$\text{(2)} \qquad \operatorname{grad} f_n = (\operatorname{grad} f) * \phi_n = f * (\operatorname{Grad} \phi_n) \qquad [21, \text{p. } 16]$$

one obtains

2.1. *For any locally summable* f,

(a) f_n *is infinitely differentiable,* $n = 1, 2, \cdots$;

(b) $\operatorname{spt} f_n \subset [\operatorname{spt} f]_{1/n}$, $n = 1, 2, \cdots$;

(c) *if* $f(x) \geqq a$ *for all* $x \epsilon [E]_{1/n}$, *then* $f_n(x) \geqq a$ *for all* $x \epsilon E$;

(d) *if* f *is uniformly continuous in* $[E]_{1/n}$ *for some* n, *then* f_n *tends to* f *uniformly on* E;

(e) *for* K *compact,* $\lim_n L_1(f_n - f, K) = 0$; *and if* $f \epsilon AC$, $\lim_n I(f_n - f, K) = 0$;

(f) *if* $f \epsilon BV$, *then* $I(f_n, E) \leqq I(f, [E]_{1/n})$, $n = 1, 2, \cdots$.

Part (*f*) follows from (2) and the principle that regularization does not increase mass.

2.2. *Let* f_n *be any sequence in* BV *such that* $f_n(x)$ *tends to a finite limit* $f(x)$ m_N-*almost everywhere and* $I(f_n, K)$ *is bounded for every compact set* K. *Then* $f \epsilon BV$, *and there is a subsequence of* n *such that* grad f_n *tends to* grad f *weakly,*

$$\text{(3)} \qquad \lim_n L_1(f_n - f, K) = 0, \qquad \textit{any compact } K,$$

$$\text{(4)} \qquad I(f) \leqq \liminf_n I(f_n).$$

Proof. From 2.1 it is enough to consider the case when f_n is infinitely differentiable. Let K_0 be any N-cube in R^N. Choose constants c_n such that

$$\int_{K_0} (f_n - c_n)\, dm_N = 0, \qquad n = 1, 2, \cdots.$$

By an elementary lemma [7b, Lemma I], $L_1(f_n - c_n, K_0)$ is bounded, together with $I(f_n - c_n, K_0) = I(f_n, K_0)$. There exist a subsequence of n and f_0 such that

$$\lim_n L_1(f_n - c_n - f_0, K_0) = 0, \qquad \lim_n [f_n(x) - c_n] = f_0(x),$$

m_N-almost everywhere in K_0, by a compactness criterion [21, p. 44]. Since $f_n(x)$ tends to $f(x)$ m_N-almost everywhere in K_0, we find that c_n tends to a finite limit c and $f = f_0 + c$. Diagonalizing, we take the subsequence independent of K_0. Then (3) is immediate; and the remaining assertions are then well known consequences of weak convergence.

2.3. *For* $f \epsilon BV$, $I(|f|) \leqq I(f)$, *with equality if* $f \epsilon AC$.

For $f \epsilon AC$ this was proved in [8, p. 316]; and the general case follows by regularization of f and (4).

3. More background material

Suppose that $f_E \in BV$ is the characteristic function of a set E. Following De Giorgi [7] we call *perimeter of* E the quantity

$$P(E) = I(f_E).$$

According to [7a, Theorem VI], if E has finite perimeter, then either E or $R^N - E$ has finite m_N-measure. The following notion of exterior normal is due to Federer [11a, p. 48]. Let E be m_N-measurable. Then E has the unit vector $n(x)$ as exterior normal at x if, letting

$$S(x, r) = [y: | y - x | < r],$$

$$S_+(x, r) = [y \in S(x, r): (y - x) \cdot n(x) \geqq 0],$$

$$S_-(x, r) = [y \in S(x, r): (y - x) \cdot n(x) \leqq 0],$$

we have

$$\lim_{r \to 0^+} \frac{m_N\{S_-(x, r) \cap E\}}{m_N\{S_-(x, r)\}} = 1, \qquad \lim_{r \to 0^+} \frac{m_N\{S_+(x, r) \cap E\}}{m_N\{S_+(x, r)\}} = 0.$$

The following is a refinement by Federer [11d] of one of De Giorgi's main theorems [7b, Theorem III]:

3.1 THEOREM [7b][11d]. *Let E have finite perimeter $P(E)$. Let B denote the set of x for which $n(x)$ exists. Then*

(1) $P(E) = m_{N-1}(B)$;

(2) *the Gauss-Green theorem holds.*

Let us call, with De Giorgi, B the *reduced boundary* of E. Since $B \subset \mathrm{fr}\, E$ we have from (1)

$$P(E) \leqq m_{N-1}(\mathrm{fr}\, E). \tag{3}$$

We also need the following formulas for $I(f)$, used in elementary cases by De Giorgi. Formula (4) is a special case of a co-area formula for Lipschitz mappings from subsets of R^k into R^l $(l \leqq k)$ due independently to Federer [11e, Theorem 3.1] and Young [23b, Theorem 4], while (5) was proved by Rishel and the author [13].

3.2 THEOREM [11e][23b]. *Let f satisfy a Lipschitz condition. For real z, let $B_z = [x:f(x) = z]$. Then*

$$I(f) = \int_{-\infty}^{\infty} m_{N-1}(B_z)\, dz. \tag{4}$$

3.3 THEOREM [13]. *For any $f \in BV$ and real z, let $E_z = [x:f(x) > z]$. Then*

$$I(f) = \int_{-\infty}^{\infty} P(E_z)\, dz. \tag{5}$$

3.4 LEMMA. *Let* $f \in BV$, $f_n \in BV$ *be integrable, integer-valued functions such that* $\lim_n I(f_n - f) = 0$. *Then* $\lim_n L_1(f_n - f) = 0$.

Proof of 3.4. For $i = 0, 1, 2, \cdots$, $n = 1, 2, \cdots$, let

$$E_{in} = [x : |f_n(x) - f(x)| > i].$$

Then

$$L_1(f_n - f) = \sum_{i=0}^{\infty} m_N(E_{in}), \qquad n = 1, 2, \cdots.$$

By 3.3 and 2.3

$$\sum_{i=0}^{\infty} P(E_{in}) = I(|f_n - f|) \leqq I(f_n - f), \quad n = 1, 2, \cdots.$$

Since $m_N(E_{in})$ is finite, we have by [7a, Theorem VI]

$$m_N(E_{in}) \leqq [P(E_{in})]^{N/N-1};$$

and the conclusion of 3.4 follows.

4. The set function γ

In this section we replace $\mathfrak{D}$ by the slightly larger class $\mathfrak{D}_1$ of all ϕ with compact support satisfying a Lipschitz condition. Then $|\phi| \in \mathfrak{D}_1$ if $\phi \in \mathfrak{D}_1$. For any compact set K, let

$$\mathfrak{D}^+(K) = [\phi \in \mathfrak{D}_1 : \phi \geqq 0 \quad \text{and} \quad \phi(x) \geqq 1, \quad \text{all } x \in K].$$

In the present setting $I(\phi)$ is a reasonable measurement of the size of $\phi \in \mathfrak{D}_1$. To get a corresponding measurement for compact sets, we define

$$\gamma(K) = \inf_{\phi \in \mathfrak{D}^+(K)} I(\phi), \qquad K \text{ compact}.$$

Our first step is to get a pair of useful alternative formulas for $\gamma(K)$.

4.1 LEMMA. *For any compact set* K,

$$\gamma(K) = \inf_{K \subset G} m_{N-1}(\text{fr } G), \qquad G \textit{ open and bounded}. \tag{1}$$

Proof. Let $\gamma_1(K)$ denote the right side of (1). For $\varepsilon > 0$ arbitrary, choose G open and bounded such that $K \subset G$ and

$$m_{N-1}(\text{fr } G) < \gamma_1(K) + \varepsilon.$$

Consider regularizations f_n of the characteristic function f_G. By 2.1, $f_n \in \mathfrak{D}^+(K)$ for large n and $I(f_n) \leqq I(f_G) = P(G)$. Then by using §3, (3),

$$\gamma(K) \leqq I(f_n) \leqq m_{N-1}(\text{fr } G).$$

Since $\varepsilon > 0$ is arbitrary, $\gamma \leqq \gamma_1$.

To prove the reverse inequality, given ε choose $f \in \mathfrak{D}^+(K)$ so that $I(f) \leqq \gamma(K) + \varepsilon$. By using 3.2,

$$I(f) = \int_0^{\infty} m_{N-1}(B_z)\, dz \geqq \int_0^1 m_{N-1}(B_z)\, dz.$$

Let $G = [x:f(x) > z_0]$ where $0 < z_0 < 1$ and

$$m_{N-1}(\text{fr } G) \leqq m_{N-1}(B_{z_0}) \leqq I(f).$$

Then G is open, bounded, $K \subset G$, and

$$\gamma_1(K) \leqq m_{N-1}(\text{fr } G) < \gamma(K) + \varepsilon,$$

from which $\gamma_1 \leqq \gamma$.

An inspection of the proof reveals that we have also established

$$\gamma(K) = \inf_{K \subset G} P(G), \qquad G \text{ open and bounded.} \tag{2}$$

4.2 COROLLARY. *For any compact set K,*

$$\gamma(K) = \gamma(\text{fr } K). \tag{3}$$

For, if G is open and fr $K \subset G$, then $G \cup K$ is open, and fr $(G \cup K) \subset$ fr G.

The definition of γ is extended to arbitrary sets E by

$$\gamma(E) = \inf_{E \subset \cup K_i} \textstyle\sum_{i=1}^{\infty} \gamma(K_i), \qquad K_i \text{ compact.} \tag{4}$$

It is immediate that γ is monotone and countably subadditive. From (1), γ is right continuous on compact sets; i.e., for every K compact and $\varepsilon > 0$ there exist G open, $K \subset G$, such that $K' \subset G$ implies $\gamma(K') < \gamma(K) + \varepsilon$. This implies that γ is outer regular; i.e., $\gamma(E) = \inf \gamma(G)$, $E \subset G$, G open. One can easily show (see Choquet [6a, p. 202]) that

$$\gamma(K_1 \cup K_2) + \gamma(K_1 \cap K_2) \leqq \gamma(K_1) + \gamma(K_2), \quad K_1, K_2 \text{ compact.}$$

An inequality of this type holds for classical capacity of various orders and has a prominent place in Choquet's general theory of capacities [6a]. If one knew that $\gamma(G) = \sup \gamma(K)$, $K \subset G$, for G open, it would then follow from Choquet's results on capacitability [6a, §30] that for any analytic set E with $\gamma(E)$ finite and any $\varepsilon > 0$ there exist K_ε and G_ε with $K_\varepsilon \subset E \subset G_\varepsilon$ and $\gamma(G_\varepsilon) - \gamma(K_\varepsilon) < \varepsilon$. The author believes that this is true but has not proved it.

In the terminology of Aronszajn and Smith [2], γ is a capacity generated by the functional space $\mathfrak{D}_1$ normed by I. By 2.3 the norm I has the strong majorization property [2, p. 153]. Hence $\mathfrak{D}_1$ has a perfect functional completion, and the class of exceptional sets for it is the class of all γ-null sets. For this class one has the remarkably simple characterization:

4.3 THEOREM. $\gamma(E) = 0$ *if and only if* $m_{N-1}(E) = 0$.

Proof. It follows from the definition of Hausdorff $(N-1)$-measure m_{N-1} that there is a constant a, depending only on the dimension N, such that for every set E and $\varepsilon > 0$ there exist open N-cubes $q_1, q_2, \cdots$ of diameter $< \varepsilon$ such that

$$\textstyle\sum_{i=1}^{\infty} m_{N-1}(\text{fr } q_i) \leqq a m_{N-1}(E) + \varepsilon, \qquad E \subset \bigcup_{i=1}^{\infty} q_i . \tag{5}$$

(The constant factor a appears first because, although $m_{N-1}(\text{fr } q_i)$ is a constant factor times $(\text{diam } q_i)^{N-1}$, the constant is not the one appearing in the definition of m_{N-1}, and second because we cover with cubes rather than arbitrary sets.) From (5) we have

$$\gamma(E) \leqq am_{N-1}(E), \qquad \text{all } E. \tag{6}$$

In particular, $m_{N-1}(E) = 0$ implies $\gamma(E) = 0$. The converse rests on the following inequality, posed as a question by the author and proved by W. Gustin [16]. A shorter variant of Gustin's proof appears in [11f].

4.4 BOXING INEQUALITY [16]. *Any compact set $K \subset R^N$ with $m_{N-1}(\text{fr } K) > 0$ can be covered by a finite number of N-cubes $q_1, \cdots, q_n$, such that*

$$\textstyle\sum_{i=1}^n m_{N-1}(\text{fr } q_i) \leqq \sigma m_{N-1}(\text{fr } K),$$

where σ is a constant depending only on the dimension N.

Gustin assumed that K is itself a finite union of cubes. The general case reduces to this one by covering fr K with cubes $q_1', \cdots, q_r'$ such that

$$\textstyle\sum m_{N-1}(\text{fr } q_i') \leqq a' m_{N-1}(\text{fr } K),$$

where a' is any constant larger than a. Now suppose $\gamma(E) = 0$. Then by (4), (1), and 4.4, given $\varepsilon > 0$ there exist cubes $q_1, q_2, \cdots$ such that

$$\textstyle\sum_{i=1}^\infty m_{N-1}(\text{fr } q_i) < \varepsilon, \qquad E \subset \bigcup_{i=1}^\infty q_i .$$

This implies that $m_{N-1}(E) = 0$, by definition of m_{N-1}.

5. Functions AC, made precise

In this section we encounter little additional difficulty in treating functions f defined in an open set G, rather than all of R^N, for which grad f is a function. This class is called AC_G.

5.1 DEFINITION. A function f is termed *precise* if there exists a sequence of functions f_n defined in G such that

(i) f_n is continuously differentiable in G (or equally well, infinitely differentiable in G);

(ii) $f(x) = \lim_n f_n(x)$ for γ-almost all $x \in G$; and

(iii) for every compact $K \subset G$, $\lim_n I(f_n - f, K) = 0$.

The term precise is borrowed from Deny and Lions [8].

5.2 THEOREM. *Let $f \in AC_G$. Then f is m_N-almost everywhere equal to a precise function $\bar{f}$. Any two such precise functions $\bar{f}$, $\bar{F}$ agree γ-almost everywhere.*

5.3 THEOREM. *Let $f \in AC_G$ be precise. Then*

(a) *f is locally absolutely continuous in Evans's sense*; *and*

(b) *given $\varepsilon > 0$ there is a set $E_\varepsilon \subset G$ such that $\gamma(E_\varepsilon) < \varepsilon$ and the restriction of f to $G - E_\varepsilon$ is continuous.*

5.4 THEOREM. *Let G be connected and f, f_n precise in G for $n = 1, 2, \cdots$, such that* $\lim_n I(f_n - f, K) = 0$, *all compact $K \subset G$. Then there are a sequence of constants c_n and a subsequence of n such that* $f(x) = \lim_n [f_n(x) + c_n]$ *for γ-almost all $x \in G$.*

These theorems can be proved by straightforward modifications of methods used in [2, §§5 and 6] and [8], together with [7b, Lemma I].

PART II

We turn to study the following class:

$$\mathfrak{F} = [f \in BV : f \text{ is integer valued, } L_1(f) + I(f) < \infty].$$

The role played by the space $\mathfrak{D}$ in Part I is now taken by the class $\mathfrak{F}_1$ of all functions W such that $W(x)$ is the winding number about x of an $(N - 1)$-cycle given by a Lipschitz mapping from a closed $(N - 1)$-polyhedron. Theorem 5.2 has an analogue (10.1) in the class $\mathfrak{F}$. The proof involves the study of Lipschitz chains of dimensions N and $N - 1$, and of countable sums of chains (termed σ-Lipschitz chains). We use the result of De Giorgi [7b] that the reduced boundary of a set with finite perimeter defines a σ-Lipschitz chain. We also need further estimates for $\gamma(E)$, first when E is compact with $(N - 1)$-rectifiable frontier (9.1), and then for open sets E (9.3).

6. Currents of finite mass

For $0 \leqq k \leqq N$ let $R_{[k]}$ and $R^{[k]}$ denote the spaces of k-vectors ξ and k-covectors ω of N-space R^N [22, Chapter I]. Let e_λ and e^λ, $\lambda = (\lambda_1, \cdots, \lambda_k)$, $\lambda_1 < \cdots < \lambda_k$, be dual bases for $R_{[k]}$ and $R^{[k]}$, corresponding to dual orthonormal bases $e_1, \cdots, e_N$ and $e^1, \cdots, e^N$ for R^N and its conjugate space. In terms of these bases, $\xi = \sum_\lambda \xi^\lambda e_\lambda$, $\omega = \sum_\lambda \omega_\lambda e^\lambda$. For scalar product and norm we write, respectively,

$$\omega \cdot \xi = \sum\nolimits_\lambda \omega_\lambda \xi^\lambda, \qquad |\xi| = [\sum\nolimits_\lambda (\xi^\lambda)^2]^{1/2}.$$

Let Ω_k denote the space of all k-forms ω with coefficients $\omega_\lambda(x) \in \mathfrak{D}$. We norm Ω_k by comass [22, p. 52],

$$\| \omega \| = \sup\nolimits_{x, |\xi|=1} | \omega \cdot \xi |, \qquad \xi \text{ simple},$$

where ξ simple means that ξ is an exterior product of vectors $x_1, \cdots, x_k$.

A current T is a linear functional on Ω_k, continuous in the topology imposed by Schwartz and De Rham (in De Rham's book [9, Chapter III], T would be called current homogeneous of dimension k and odd kind). The *mass* of T is the quantity

$$\| T \| = \sup\nolimits_{\| \omega \| \leqq 1} | T(\omega) |.$$

It is known that T has finite mass if and only if there is a k-vector-valued measure Ψ with finite total variation such that

$$(1) \qquad T(\omega) = \int_{R^N} \omega \cdot d\Psi, \qquad \text{all } \omega \in \Omega_k.$$

The mass $\| T \|$ is the total variation of Ψ [22, Chapter XI, especially p. 326], provided $R_{[k]}$ is normed by mass [22, p. 52] rather than $|\ \ |$. For the extreme dimensions $k = 0, 1, N - 1, N$ with which we shall be principally concerned, all k-vectors are simple, and $| \xi |$ equals the mass of ξ. The boundary bT is defined as in [9, p. 53] by the formula $bT(\omega) = T(d\omega)$.

For any distribution T let $*T$ denote the current of dimension N dual to T, defined by

$$*T(fe^{1\cdots N}) = T(f), \qquad \text{all } f \in \mathfrak{D}. \tag{2}$$

Suppose T and $b(*T)$ have finite mass. For each $(N - 1)$-form

$$\omega = \textstyle\sum_{i=1}^N \omega_{\bar{i}} e^{\bar{i}}, \qquad \bar{i} = (1, \cdots, i - 1, i + 1, \cdots, N),$$

one finds by direct calculation and (2) that

$$b(*T)(\omega) = \sum_{i=1}^{N} (-1)^i \frac{\partial T}{\partial x^i} (\omega_{\bar{i}}). \tag{3}$$

From (3) the finiteness of $\| b(*T) \|$ is equivalent to the fact that the partial derivatives of T are totally finite measures, whence T is a function $f \in BV$ by [21, p. 41]. Moreover,

$$L_1(f) = \| f \| = \| *f \|, \qquad I(f) = \| b(*f) \|. \tag{4}$$

We say *polyhedral k-chain* for a current which corresponds to a finite linear combination of oriented k-simplexes, with real coefficients. Given k $(1 \leqq k \leqq N)$ and an open set G, let V denote the set of all polyhedral k-chains with support in G. V is a vector space [22, p. 152]. For $c \in V$, let

$$\mu_0(c) = \inf \| c' \|, \qquad c' \in V, \quad bc' = bc;$$

$$\mu(c) = \inf \| T \|, \qquad bT = bc,$$

where spt T is at positive distance from $R^N - G$.

6.1 LEMMA. *$\mu_0(c) = \mu(c)$, all $c \in V$.*

Using methods of [14a], one can prove more; namely, each such T is the weak limit of a sequence $c_n \in V$ with $bc_n = bT = bc$, $n = 1, 2, \cdots$, and $\lim \| c_n \| = \| T \|$. Instead we give a short proof of 6.1 making use of Whitney's notions of flat chain and cochain in the open set G [22, p. 232]. To begin with, we observe that while the flat norm of c is usually less than $\| c \|$, we do have

$$\mu_0(c) = \inf (\text{flat norm of } c'), \qquad c' \in V, \quad bc' = bc. \tag{5}$$

Clearly $\mu(c) \leqq \mu_0(c)$. Suppose there exist c_0 and T_0 with $bT_0 = bc_0$ such that $\| T_0 \| < \mu_0(c_0)$. Now μ_0 is a seminorm on V depending only on bc. By the Hahn-Banach theorem [3, p. 29] applied to the vector space bV, there is a linear functional l on V, depending only on bc, such that

$$| l(c) | \leqq \mu_0(c), \quad \text{all } c \in V, \qquad l(c_0) = \mu_0(c_0). \tag{6}$$

By (5) and (6), l is bounded in the flat norm, and so defines a flat cochain in G. There exist regularizations ω_n of l [22, p. 176], ω_n defined in $G_n = G - [\text{fr } G]_{1/n}$, such that ω_n is a k-form with infinitely differentiable coefficients and

$$(7) \qquad c(\omega_n) = 0 \quad \text{if} \quad c \in V, \quad bc = 0, \quad \text{spt } c \subset G_n ,$$

$$(8) \qquad \lim_n c(\omega_n) = l(c), \qquad \text{all } c \in V,$$

$$(9) \qquad |\sigma(\omega_n)| \leqq \| \sigma \|, \qquad \text{any } k\text{-simplex } \sigma \subset G_n .$$

From (9), $\| \omega_n \| \leqq 1$. From (7) it follows by theorems of De Rham [9, pp. 94, 114, 117] that $\omega_n = d\phi_n$, ϕ_n a $(k - 1)$-form with infinitely differentiable coefficients. Hence $T_0(\omega_n) = c_0(\omega_n)$; and by (8)

$$\| T_0 \| < \mu_0(c_0) = l(c_0) = \lim T_0(\omega_n) \leqq T_0 ,$$

a contradiction. This proves 6.1.

Now suppose bc is a $(k - 1)$-cycle with *integer* coefficients. Let

$$\mu_1(c) = \inf \| c' \|, \qquad bc' = bc, \quad c' \in V \text{ with integer coefficients.}$$

6.2 LEMMA. *Let c be a polyhedral k-chain with real coefficients, where $k = 1$, $N - 1$, or N, such that bc has integer coefficients. Then c is a convex combination $c = \sum a_i c_i$, where every c_i has integer coefficients, $bc_i = bc$,* spt $c_i \subset$ spt c, *and $\| c \| = \sum a_i \| c_i \|$.*

This can be proved by purely formal changes in the reasoning used in [14a, 3.3 and §8]. Except for the requirement that the masses add, it is a well known consequence of the absence of torsion $(k - 1)$-cycles with the indicated values of k for complexes embedded in R^N [1, p. 390]. From 6.1 and 6.2 we have

6.3 THEOREM. *For $k = 1, N - 1$, or N and any open set G, $\mu_1(c) = \mu_0(c) = \mu(c)$ for all $c \in V$ such that bc has integer coefficients.*

It is an interesting question whether $\mu_1 = \mu_0$ for arbitrary k, or perhaps $\mu_1(c)$ is no more than a constant multiple of $\mu_0(c)$. There is some evidence suggesting that $\mu_0(c)$ and $\mu_1(c)$ may not always agree [14b, Theorem 3].

7. σ-Lipschitz chains

Let g be a Lipschitz mapping from an m_k-measurable set $A \subset R^k$ into R^N. A set D such that $D = g(A)$ for some such pair (g, A) is termed a *k-rectifiable* set if A is bounded, *countably k-rectifiable* in the general case [11b, p. 126]. A countably k-rectifiable set D has an m_k-approximate tangent k-plane $\pi(x)$ at x for m_k-almost all $x \in D$.

Such a Lipschitz mapping g has a Lipschitz extension to R^N with the same Lipschitz constant [18]. For m_k-almost all $u \in A$ the Jacobian k-vector

$$\theta(u) = \sum_\lambda \theta^\lambda(u) e_\lambda , \qquad \theta^\lambda(u) = \frac{\partial(x^{\lambda_1}, \cdots, x^{\lambda_k})}{\partial(u^1, \cdots, u^k)}$$

exists and does not depend on the particular Lipschitz extension of g to $R^N - A$. Let

$$a(g, A) = \int_A |\theta(u)| \, dm_k(u),$$

which is the classical formula for k-area. If $a(g, A)$ is finite, g defines a current c by

$$c(\omega) = \int_A \omega[g(u)]\cdot\theta(u) \, dm_k(u), \qquad \text{all } \omega \in \Omega_k. \tag{1}$$

Since $\theta(u)$ is a simple k-vector, we have from the definition of $\| \omega \|$ and $\| c \|$

$$\| c \| \leqq a(g, A). \tag{2}$$

A current c will be called a *σ-Lipschitz k-chain of finite mass* if c has a representation (1). Let $\mathfrak{L}_k$ denote the class of all σ-Lipschitz k-chains of finite mass.

7.1. LEMMA. *$c \in \mathfrak{L}_k$ if and only if there exist a countably k-rectifiable set $D \subset R^N$ and m_k-measurable functions $\Theta(x)$, $M(x)$ defined on D, such that*
(i) *$\Theta(x)$ is an m_k-approximate tangent k-vector to D at x, and $| \Theta(x) | = 1$;*
(ii) *$M(x)$ is integer valued and m_k-integrable on D;*
(iii) *$c(\omega) = \int_D M(\omega\cdot\Theta) \, dm_k$, all $\omega \in \Omega_k$; and*
(iv) *$\| c \| = \int_D | M | \, dm_k$.*
Every $c \in \mathfrak{L}_k$ has a Lipschitz representation (1) *such that equality holds in* (2). *$\mathfrak{L}_k$ is a group under addition, complete in the norm $\| c \|$.*

Remark. The nonnegative number $a(g, A) - \| c \|$ appears several times in later sections. It measures, roughly speaking, how much the mapping g leads to tangent k-planes "back to back," i.e., with opposite orientations. Equality in (2) means that this occurs m_k-almost nowhere in R^N. By 7.1 and [11b, 5.9], by taking $D = g(A)$,

$$m_k\{[x \in g(A) : M(x) = 0]\} \leqq a(g, A) - \| c \|.$$

If D_1 and D_2 are countably k-rectifiable, then so is $D_1 \cup D_2$ and the m_k-approximate tangent k-planes agree m_k-almost everywhere in $D_1 \cap D_2$. Then if $c_1, c_2 \in \mathfrak{L}_k$, one finds Θ, M_1, M_2 defined on $D = D_1 \cup D_2$ with $M_1(x) = 0$ for $x \in D_2 - D_1$, $M_2(x) = 0$ for $x \in D_1 - D_2$, and

$$(c_1 \pm c_2)(\omega) = \int_D (M_1 \pm M_2)(\omega\cdot\Theta) \, dm_k, \qquad \text{all } \omega \in \Omega_k. \tag{3}$$

From this we get, letting $\triangle$ denote symmetric difference

7.2 LEMMA. *For $i = 1, 2$ let c_i have representation (g_i, A_i). Then*

$$m_k\{g_1(A_1) \triangle g_2(A_2)\} \leqq \| c_1 - c_2 \| + \textstyle\sum_{i=1}^2 \{a(g_i, A_i) - \| c_i \|\}.$$

7.3 LEMMA. *For every* $f \in \mathfrak{F}$, $b(*f) \in \mathfrak{L}_{N-1}$.

A stronger result (8.5) is proved later.

Lemma 7.1 can be proved by known techniques [11b, §5], [23a, Appendix B]. If f is a characteristic function, 7.3 follows from 7.1 and De Giorgi [7b, Theorem III]. The general case can then be obtained by using 3.3. Lemmas 7.3 and 8.4 below can also be gotten as consequences of general theorems for k-dimensional currents in R^N, obtained by different methods, to appear in a forthcoming paper by Federer and the author, "Normal and Integral Currents." In that paper the elements of $\mathfrak{L}_k$ are termed rectifiable currents of dimension k.

8. Special chains

Besides the notion of Lipschitz chain defined in §7 in terms of currents, we need to consider a more classical one. Let P be an oriented polyhedron in some euclidean space which is the union of finitely many k-simplexes. Let C be a simplicial complex subdividing P, and $\sigma_1, \cdots, \sigma_n$ the k-simplexes of C oriented consistently with P. Let us assume that the chain $\sum \sigma_i$, regarded as having integer coefficients, has boundary of the special form $\sum \tau_j$, where the τ_j are distinct $(k-1)$-simplexes. We write ∂P for its carrier with the induced orientation. (We could equally well consider arbitrary chains over C with integer coefficients, but this would only serve to complicate the discussion to follow.)

Let h be a Lipschitz mapping from P into R^N. We call the pair (h, P) a *special k-chain, special k-cycle* if ∂P is void. Let $A \subset R^k$ be the union of disjoint k-simplexes $s_1, \cdots, s_n$, and ψ a mapping from A onto P taking each s_i affinely onto σ_i so that the orientation induced on σ_i agrees with the one given. To (h, P) corresponds an element c of $\mathfrak{L}_k$ represented (§7, (1)) by $(h \circ \psi, A)$. We write $a(h, P)$ for the k-area integral $a(h \circ \psi, A)$. By [22, p. 298(8)], $bc \in \mathfrak{L}_{k-1}$ and corresponds to $(h, \partial P)$.

Now let $k = N$, and let $(\hat{h}, \hat{P})$ be a special N-chain. For $x \notin \hat{h}(\partial\hat{P})$, let $W(x)$ denote the local degree at x of $(\hat{h}, \hat{P})$ as defined in [1, p. 474]. For $x \in \hat{h}(\partial\hat{P})$ we set $W(x) = 0$. Then W is well defined at every point x, which is important since this is the class $\mathfrak{F}_1$ of functions to be used in making precise functions of $\mathfrak{F}$. When $(\hat{h}, \hat{P})$ carries a subscript n, the corresponding $\hat{c}$ and W carry that same subscript.

8.1 LEMMA. *For any special N-chain* $(\hat{h}, \hat{P})$, $*W = \hat{c}$.

Proof. Let $\hat{h}_n$ be a sequence of simplexwise affine approximations to $\hat{h}$ with uniformly bounded Lipschitz constants, such that $\hat{h}_n$ tends uniformly to $\hat{h}$ and the corresponding Jacobians θ_n tend boundedly to θ [22, pp. 289–294]. The validity of 8.1 for $(\hat{h}_n, \hat{P})$ is an easy consequence of the definitions. Now $\hat{c}_n$ tends weakly to $\hat{c}$, $W_n(x)$ tends to $W(x)$ for $x \notin \hat{h}(\partial\hat{P})$ by Rouché's theorem [1, p. 459], and $m_N\{\hat{h}(\partial\hat{P})\} = 0$ since $\hat{h}$ is Lipschitz. By §7, (1) and §6, (4)

$$I(W_n) = \| b\hat{c}_n \| \leqq a(\hat{h}_n, \partial\hat{P}),$$

which is bounded. The functions W_n have uniformly bounded supports. By 2.2, $\lim L_1(W - W_n) = 0$, from which 8.1 follows.

Let (h, P) be a special $(N - 1)$-cycle, and c the corresponding element of $\mathcal{L}_{N-1}$. By cone construction there is a polyhedron $\hat{P}$ with boundary P. Let $\hat{h}$ be a Lipschitz extension of h to $\hat{P}$. Now $W(x)$ equals the order (or winding number) of (h, P) about x [1, p. 474, Satz V]. Hence $W(x)$ does not depend on the particular choice of $\hat{P}$ and $\hat{h}$. Since $b\hat{c} = c$, we have from 8.1

8.2 LEMMA. *For any special $(N - 1)$-cycle (h, P), $b(*W) = c$.*

This is a way of saying the Gauss-Green theorem for special $(N - 1)$-cycles.

Remark. In area theory one considers sequences h_n of simplexwise affine mappings (h_n, P) from a closed $(N - 1)$-polyhedron P into R^N such that h_n tends uniformly to a limit h (h merely assumed continuous) and the $(N - 1)$-areas $a(h_n, P)$ are bounded. Then $\| c_n \|$ is bounded, from which c_n tends weakly to a limit c for a subsequence of n. The above reasoning shows that if $m_N\{h(P)\} = 0$, then $b(*W) = c$. By 7.3, $c \in \mathcal{L}_{N-1}$. The problem is to relate c to the mapping h. There are several results known in this direction; see Cecconi [5] for $N = 3$, Michael [19], and Federer [Notices Amer. Math. Soc., vol. 6 (1959), Abstracts No. 558–33, 558–34, pp. 381, 382].

8.3 LEMMA. *Let q and q' be concentric k-cubes in R^k $(1 \leqq k \leqq N)$ with $q \subset q'$. Let g be a Lipschitz mapping from a set $A \subset q$ into R^N, and t the Lipschitz constant of g. Then g has a Lipschitz extension g' to q', such that* (a) *g' has Lipschitz constant $t' \leqq rt$, where r depends only on k and N;* (b) *there is a subdivision of* fr q' *on each $(k - 1)$-simplex of which g' is one-one and affine.*

Proof. First, g has some Lipschitz extension g_1 to q' with Lipschitz constant t [18]. Let δ be the distance between q and fr q'. By [22, p. 290] there is a constant $r > 1$ and a simplexwise affine mapping g_2 of fr q' into R^N, such that $| g_2(u) - g_1(u)| < (r - 1)t\delta$, $u \in \text{fr } q'$, and g_2 has Lipschitz constant $\leqq rt$. By small modifications of g_2 if necessary, we arrange that (b) holds, upon setting

$$g'(u) = \begin{cases} g(u), & u \in A \\ g_2(u), & u \in \text{fr } q'. \end{cases}$$

Then on $A \cup \text{fr } q'$, g' has Lipschitz constant $\leqq rt$. We extend g' to all of q' with this same Lipschitz constant.

8.4 LEMMA. *Let G be open; let $c_0 \in \mathcal{L}_{N-1}$ be such that $bc_0 = 0$ and* spt c_0 *has positive distance from $R^N - G$. Then given $\eta > 0$ there exists a special $(N - 1)$-*

cycle (h, P) *with* $c \in \mathfrak{L}_{N-1}$ *corresponding to* (h, P), *such that*

(i) $h(P) \subset G$;

(ii) $\| c_0 - c \| < \eta$;

(iii) $\| c \| \leqq a(h, P) \leqq \| c \| + \eta$.

Proof. By 7.1 there is a Lipschitz representation (g, A) of c_0 such that $a(g, A) = \| c_0 \|$ and $g(A) \subset \operatorname{spt} c_0$. Let F be a figure which is the union of disjoint $(N - 1)$-cubes $q_1, \cdots, q_s$ chosen so that

$$a(g, A - F) < \eta/4, \qquad m_{N-1}(F - A) < \eta/8(rt)^{N-1},$$

where t is the Lipschitz constant of g, and r is as in 8.3. Let $F' = \bigcup q_i'$ where q_i' is concentric with q_i, $q_i \subset q_i'$, $q_1', \cdots, q_s'$ are disjoint, and

$$m_{N-1}(F' - F) < \eta/8(rt)^{N-1}.$$

Let g' be an extension of g from $A \cap F$ to F' with the properties in 8.3 on each q_i'. For small η, $g'(F') \subset G$. Writing c' for the element of $\mathfrak{L}_{N-1}$ corresponding to (g', F'), we find that

$$\| c_0 - c' \| \leqq a(g, A - F) + a(g', F' - (A \cap F))$$
$$< \eta/4 + (rt)^{N-1} m_{N-1}(F' - (A \cap F)) < \eta/2,$$

$$a(g', F') \leqq a(g, A \cap F) + a(g', F' - (A \cap F)) \leqq \| c_0 \| + \eta/4.$$

Now $b(c_0 - c') = -bc'$ is polyhedral with integer coefficients, and we apply 6.3. Let c_1 be a polyhedral $(N - 1)$-chain with integer coefficients, $\operatorname{spt} c_1 \subset G$, such that

$$bc_1 = -bc', \qquad \| c_1 \| < \eta/2.$$

By an elementary construction we arrange that each $(N - 1)$-simplex of c_1 is counted exactly once, without changing bc_1.

Let $c = c' + c_1$. Then (ii) holds, and it remains to find a special $(N - 1)$-cycle (h, P) representing c such that (i) and (iii) hold. Now $\operatorname{spt} bc_1 = g'(\operatorname{fr} F')$ is an $(N - 2)$-polyhedron Q, and g' is simplicial from fr F' onto Q relative to suitable subdivisions K', K of fr F' and Q, respectively. For each simplex $s \in K'$ we construct (abstractly) a cylinder Z_s with bases s and $g'(s)$. Then F', spt c_1, and the cylinders Z_s define a closed oriented $(N - 1)$-polyhedron P, which we may embed in some euclidean space. Define the mapping h to agree with g' on F', with the inclusion map on spt c_1, and to be constant on each line joining $u \in \operatorname{fr} F'$ with $g(u)$ in a cylinder Z_s containing them.

8.5 THEOREM. $f \in \mathfrak{F}$ *if and only if* $*f \in \mathfrak{L}_N$ *and* $b(*f) \in \mathfrak{L}_{N-1}$. *The class* $\mathfrak{F}_1$ *is dense in* $\mathfrak{F}$ *in the norm* $L_1(f) + I(f)$. *In fact, given* $f \in \mathfrak{F}$ *there exist special* $(N - 1)$-*cycles* (h_n, P_n), *with corresponding* $W_n(x)$, *such that*

(a) $\lim_n [I(f - W_n) + L_1(f - W_n)] = 0$,

(b) $I(W_n) \leqq a(h_n, P_n) \leqq I(W_n) + n^{-1}, \quad n = 1, 2, \cdots.$

Proof. Suppose $*f \in \mathfrak{L}_N$ and $b(*f) \in \mathfrak{L}_{N-1}$. Then f is integer valued (m_N-almost everywhere) by 7.1. By §6,(4), $L_1(f) + I(f)$ is finite. Conversely, let $f \in \mathfrak{F}$. Apply 7.3 and then 8.4 with $G = R^N$, $\eta = n^{-1}$, $c_0 = b(*f)$. Then (b) holds, and $I(f - W_n)$ tends to 0, from which we get (a) using 3.4. Since $*W_n \in \mathfrak{L}_N$ and $b(*W_n) \in \mathfrak{L}_{N-1}$ for every n, the same holds for f by completeness of $\mathfrak{L}_N$ and $\mathfrak{L}_{N-1}$.

9. Further estimates for γ

According to 4.2 and §4,(6), for every compact K, $\gamma(K) \leqq a m_{N-1}(\text{fr } K)$. If fr K is $(N-1)$-rectifiable, a more precise result is

9.1 THEOREM. *Let K be a compact set with $(N-1)$-rectifiable frontier* fr K. *Let B be the reduced boundary of K. Then*

$$\gamma(K) \leqq m_{N-1}(B) + 2m_{N-1}(\text{fr } K - B).$$

Proof. Recall the notation of §3. Consider first the set B. By [7b, Theorem III] there exists $B_1 \subset B$ with $m_{N-1}(B - B_1) = 0$ such that, for $x \in B_1$, the m_{N-1}-density of B at x is 1 and $\pi(x)$ is an m_{N-1}-approximate tangent plane to B at x. For $s > 0$ let $\sum(x, s)$ denote the set of all points distant $< s$ from $\pi(x)$. Let α denote the volume of the unit $(N-1)$-ball. Fix $\varepsilon > 0$ and $x \in B_1$. Then, for small r,

(a) $m_N\{S_-(x, r) - K\} < \varepsilon^2 \alpha r^N$

(b) $|m_{N-1}\{B \cap S(x, r)\} - \alpha r^{N-1}| < \varepsilon \alpha r^{N-1}$

(c) $m_{N-1}\{B \cap [S(x, r) - \sum(x, \varepsilon r)]\} < \varepsilon \alpha r^{N-1}$.

Let $I(s) = (S_-(x, r) - K) \cap \text{fr} \sum(x, s)$. Then

(d) $\displaystyle\int_{\varepsilon r}^{2\varepsilon r} m_{N-1}\{I(s)\}\, ds \leqq \int_0^r m_{N-1}\{I(s)\}\, ds = m_N\{S_-(x, r) - K\}$.

By (a) there exists s_0, $\varepsilon r < s_0 < 2\varepsilon r$, such that

$$m_{N-1}\{I(s_0)\} < \varepsilon \alpha r^{N-1}.$$

Let $G = \sum(x, s_0) \cap S(x, r)$. Then

$$m_{N-1}\{\text{fr } G - K\} \leqq m_{N-1}\{I(s_0)\} + m_{N-1}\{\text{fr } S(x, r) \cap \textstyle\sum(x, s_0)\} + m_{N-1}\{\text{fr} \textstyle\sum(x, s_0) \cap S_+(x, r)\}.$$

The middle term on the right has order $2s_0\, \alpha(N-1)r^{N-2}$, and is less than $(4N-1)\varepsilon \alpha r^{N-1}$ for any sufficiently small ε. The last term is no more than αr^{N-1}. Thus, for small r,

$$m_{N-1}(\text{fr } G - K) \leqq (1 + 4N\varepsilon)\alpha r^{N-1} < \frac{1 + 4N\varepsilon}{1 - 2\varepsilon} m_{N-1}(B \cap G).$$

Since ε is still fixed, the sets G do not get too "thin" as x and r vary. Each $x \in B_1$ is the intersection of a family of sets G shrinking to x. Since (b) holds, we may apply a covering theorem [20, 4.1] to find a disjointed family of such sets $G_1, \cdots, G_p$, such that

$$(1) \qquad \sum_{i=1}^{p} m_{N-1}(\text{fr } G_i - K) < \frac{1 + 4N\varepsilon}{1 - 2\varepsilon} m_{N-1}(B),$$

$$(2) \qquad m_{N-1}(B - \cup_{i=1}^{p} G_i) < \varepsilon.$$

Consider now the set $D = \text{fr } K - B$. By a simplification of the previous reasoning, omitting all reference to exterior normals, one finds disjoint open sets $G_1', \cdots, G_q'$ such that

$$(3) \qquad \sum_{j=1}^{q} m_{N-1}(\text{fr } G_j') < 2 \frac{1 + N\varepsilon}{1 - 2\varepsilon} m_{N-1}(D),$$

$$(4) \qquad m_{N-1}(D - \cup_{j=1}^{q} G_j') < \varepsilon.$$

The part E of fr K still not covered has m_{N-1}-measure $< 2\varepsilon$ by (2) and (4). By §4, (6) there exists a countable set of open sets G_l'', such that

$$(5) \qquad \sum_{l=1}^{\infty} m_{N-1}(\text{fr } G_l'') < 2a\varepsilon,$$

$$(6) \qquad \text{fr } K < (\cup G_i) \cup (\cup G_j') \cup (\cup G_l'').$$

Since fr K is compact, it is covered by a finite number of these open sets. The union of these finitely many open sets and K is an open set G_0. By (1), (3), and (5) there is a constant w, such that

$$m_{N-1}(\text{fr } G_0) < m_{N-1}(B) + 2m_{N-1}(D) + w\varepsilon.$$

This proves 9.1. Using similar reasoning, together with right continuity of γ on compact sets, we can show

9.2 LEMMA. a. *For any countably $(N - 1)$-rectifiable set D, $\gamma(D) \leqq 2m_{N-1}(D)$.* b. *For any open set G, compact $(N - 1)$-rectifiable set $K \subset G$, and $\delta > 0$, there exists an open bounded set G', such that*

$$K \subset G' \subset \text{cl } G' \subset G, \qquad \gamma(\text{cl } G') \leqq 2m_{N-1}(K) + \delta.$$

9.3. THEOREM. *Let E be an open set such that both $P(E)$ and $m_N(E)$ are finite. Then $\gamma(E) \leqq P(E)$.*

Proof. We shall construct a certain sequence of special $(N - 1)$-cycles (h_n, P_n), with W_n corresponding to (h_n, P_n) as in §8. Let H_n be the 2^{-n}-neighborhood of fr E. Choose (h_1, P_1) by 8.4 with $c_0 = b(*f_E)$, $G = H_1$, η to be determined later. Then by using §6, (4) and 8.2,

$$I(f_E - W_1) = \| b(*f_E) - b(*W_1) \| < \eta.$$

Let $E_1 = [x : W_1(x) \neq 0]$, and let B_1 be its reduced boundary. Then by §3, (3), $P(E_1) \leqq m_{N-1}(\text{fr } E_1)$. But fr $E_1 \subset h_1(P_1)$, from which by [11b, 5.9], $m_{N-1}(\text{fr } E_1) \leqq a(h_1, P_1)$. Thus

$$m_{N-1}(B_1) = P(E_1) \leqq m_{N-1}(\text{fr } E_1) < P(E) + 2\eta.$$

On the other hand, by 3.4 and 2.2, $\liminf_{\eta\to 0} P(E_1) \geqq P(E)$. Hence, given any $\varepsilon > 0$ we may choose η such that $\eta < \varepsilon/16$ and

$$m_{N-1}(B_1) + 2m_{N-1}(\mathrm{fr}\, E_1 - B_1) < P(E) + \varepsilon/2.$$

By 9.1 and right continuity of γ for compact sets, there is an open bounded set G_1' with

$$\mathrm{cl}\, E_1 \subset G_1', \qquad \gamma(\mathrm{cl}\, G_1') < P(E) + \varepsilon/2.$$

We next define inductively, for $n = 2, 3, \cdots$, (h_n, P_n), $E_n = [x: W_n(x) \neq 0]$, and G_n open, G_n' open and bounded, such that

(i) $G_n = H_n \cup G_1' \cup \cdots \cup G_{n-1}'$;

(ii) $\mathrm{fr}\, E_n \subset G_n' \subset \mathrm{cl}\, G_n' \subset G_n$, from which $\mathrm{spt}\, \mathrm{b}(*W_n) \subset \mathrm{fr}\, E_n \subset G_n'$;

(iii) $I(f_E - \sum_{i=1}^n W_i) < \varepsilon/2^{n+3}$; and

(iv) $\gamma(K_n) < \varepsilon/2^n$, $K_n = \mathrm{cl}\,(E_n \cup G_n')$.

At each step we apply 8.4 with $c_0 = b(*f_E) - b(*\sum_{i=1}^{n-1} W_i)$, $G = G_n$, $\eta = \varepsilon/2^{n+3}$. Call the special $(N-1)$-cycle obtained (h_n, P_n). Then (iii) is immediate. We have

$$m_{N-1}(\mathrm{fr}\, E_n) \leqq a(h_n, P_n) < \varepsilon/2^{n+2} + 2\eta < \varepsilon/2^{n+1}.$$

By 9.2b there is a set G_n' such that (ii) holds and $\gamma(\mathrm{cl}\, G_n') < \varepsilon/2^n$. Since $\mathrm{fr}\, E_n \subset G_n'$, $\mathrm{fr}\,(E_n \cup G_n') = \mathrm{fr}\, G_n'$. By 4.2, $\gamma(K_n) = \gamma(\mathrm{fr}\, G_n') = \gamma(\mathrm{cl}\, G_n')$.

Let $F = \mathrm{cl}\, G_1' \cup K_2 \cup K_3 \cup \cdots$. Suppose $x_0 \in E - F$. Then $x_0 \notin \mathrm{fr}\, F$ since $\mathrm{fr}\, F \subset F \cup \mathrm{fr}\, E$ by construction. Hence there is a neighborhood U of x_0 with $U \subset E - F$. Then $W_n(x) = 0$ for $x \in U$ and $n = 1, 2, \cdots$. Since $L_1(f_E - \sum_{i=1}^n W_i)$ tends to 0, by 3.4 and (iii), and $f_E(x) = 1$ for $x \in E$, this is impossible. Therefore $E \subset F$, from which

$$\gamma(E) \leqq \gamma(\mathrm{cl}\, G_1') + \sum_{n=2}^\infty \gamma(K_n) < P(E) + \varepsilon.$$

This proves 9.3.

Remark. For bounded open sets E, 9.3 is much better than 9.1 applied to $\mathrm{cl}\, E$. According to well known examples in area theory, $P(E)$ may be finite while $m_{N-1}(\mathrm{fr}\, E)$ is infinite. In §12 we show that if $\mathrm{fr}\, E$ is a compact manifold of zero m_N-measure, then $P(E)$ equals the integralgeometric area.

9.4 Corollary. *For every set* E,

$$\gamma(E) = \inf \sum_{i=1}^\infty P(G_i), \qquad E \subset \bigcup_{i=1}^\infty G_i, \qquad G_i \textit{ open, bounded.}$$

Proof. From the definition of γ and §4, (2) the left side is not less than the right, and the opposite inequality follows from 9.3.

9.5 Lemma. *Let* (h_1, P_1) *and* (h_2, P_2) *be special* $(N-1)$*-cycles, and let* $E = [x: W_1(x) \neq W_2(x)]$. *Then*

$$\gamma(E) \leqq 3I(W_1 - W_2) + 2\sum_{i=1}^2 [a(h_i, P_i) - I(W_i)].$$

Proof. $E = E' \cup E''$, where

$$E' = E - [h_1(P_1) \cup h_2(P_2)], \qquad E'' = E \cap [h_1(P_1) \triangle h_2(P_2)],$$

$\triangle$ denoting symmetric difference. E' is open, and by 9.3, 3.3, and 2.3

$$\gamma(E') \leqq P(E') = P(E) \leqq I(|W_1 - W_2|) \leqq I(W_1 - W_2).$$

To complete the proof, we apply 7.2, 9.2a, and the fact that

$$\gamma(E) \leqq \gamma(E') + \gamma(E'') \leqq \gamma(E') + \gamma[h_1(P_1) \triangle h_2(P_2)].$$

10. Precise functions in $\mathfrak{F}$

With the results of §§6 through 9 we can now prove the following analogue of Theorem 5.2:

10.1 THEOREM. *Given $f \in \mathfrak{F}$ there is a sequence of special $(N - 1)$-cycles with properties* (a), (b) *of* 8.5 *and in addition a function $\bar{f}$, such that*

(c) $\bar{f}(x) = f(x)$, *m_N-almost all x;*

(d) $\lim_n W_n(x) = \bar{f}(x)$, *$\gamma$-almost all x.*

The conditions (a)–(d) *determine $\bar{f}$ uniquely up to γ-measure* 0.

Proof. (a) and (d) imply (c). To prove (d), let $E_{nm} = [x: W_n(x) \neq W_m(x)]$. By 9.5, (a), and (b), $\gamma(E_{mn})$ tends to 0 as m, n tend to ∞. Thus, the sequence $W_n(x)$ is fundamental in γ-measure, and converges to a finite limit $\bar{f}(x)$ for a subsequence of n, which we take as our new sequence.

To prove uniqueness, let (h_n', P_n') be another such sequence. Then $\lim I(W_n - W_n') = 0$ by (a). Let $E_n = [x: W_n(x) \neq W_n'(x)]$. By 9.5, $\gamma(E_n)$ tends to 0, from which $\bar{f}(x) = \bar{f}'(x)$ γ-almost everywhere.

To correspond to 5.1, we term $\bar{f}$ precise.

Densities (*Added November 11, 1959*). Federer pointed out that the following is a corollary of 10.1 and his proof [11f] of inequality 4.4. *For any $f \in \mathfrak{F}$ the spherical N-density*

$$\hat{f}(x) = \lim_{r \to 0^+} L_1[f, S(x, r)]/m_N[S(x, r)]$$

exists and is finite for m_{N-1}-almost all $x \in R^N$. If D is a countably $(N - 1)$-rectifiable set on which $b(*f)$ is represented according to 7.1, *then $\hat{f}(x) = \bar{f}(x)$ for m_{N-1}-almost all $x \in R^N - D$.*

From [11f, Lemma, §2] there exist absolute positive constants λ and μ such that, if G is any open set and $x \in G$, then

$$(1) \qquad \lambda m_N[S(x, r) \cap (R^N - G)]/r^N \leqq m_{N-1}[S(x, r) \cap \operatorname{fr} G]/r^{N-1}$$

in any interval $0 < r < \delta$ in which the right side of (1) is no more than μ. For m_{N-1}-almost every $x \in R^N - D$ there exists n_0 such that (a) $W_n(x) = \bar{f}(x)$, $n \geqq n_0$; (b) $x \notin h_n(P_n)$, $n \geqq n_0$, where h_n and P_n are as in 8.5; and (c) $D \cup \bigcup_{n=n_0}^{\infty} h_n(P_n)$ has spherical $(N - 1)$-density 0 at x (cf. [11b, 3.2]).

For any such x apply (1) with

$$G = [y \notin h_n(P_n) : W_n(y) = \bar{f}(x)], \qquad n \geqq n_0 .$$

Given $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $n \geqq n_0$ and $0 < r < \delta$,

$$m_N[y \in S(x, r) : W_n(y) \neq \bar{f}(x)] \leqq \varepsilon m_N[S(x, r)].$$

By letting n tend to ∞, the same inequality holds with $f(y)$ in place of $W_n(y)$. This proves that $\hat{f}(x)$ exists and equals $\bar{f}(x)$.

If $f \in \mathfrak{F}$ is a characteristic function, then by 3.1, $\hat{f}(x) = \frac{1}{2}$ for m_{N-1}-almost all $x \in D$. For any $f \in \mathfrak{F}$ one finds, using 3.3 for every integer i, a set E_i whose characteristic function f_i belongs to $\mathfrak{F}$, such that if B_i is the reduced boundary of E_i

$$f = \cdots + f_2 + f_1 + f_0 - f_{-1} - f_{-2} - \cdots ,$$

$$I(f) = \sum m_{N-1}(B_i), \qquad m_{N-1}[D \bigtriangleup (\bigcup B_i)] = 0.$$

By the preceding argument, for any j, $(f - \sum_{|i| \leqq j} f_i)^{\wedge}(x)$ exists and is an integer, for m_{N-1}-almost all $x \notin D_j = \bigcup_{|i|>j} B_i$. It follows that $\hat{f}(x)$ exists and is half an integer for m_{N-1}-almost all $x \notin D_j$, and hence, since j is arbitrary, for m_{N-1}-almost all $x \in R^N$.

If, in §8, $W_n(x)$ had been defined to be $\hat{W}_n(x)$ for $x \in h_n(P_n)$, then $\bar{f}(x)$ would agree with $\hat{f}(x)$ m_{N-1}-almost everywhere in R^N.

Part III

11. Sets with finite perimeter

Even in case f is a characteristic function f_E, the approximating functions W_n in Theorem 10.1 need not be characteristic functions. We shall give a modified version of 10.1 in terms of characteristic functions, i.e., in terms of sets with finite perimeter.

11.1 Definition. A set E is termed *elementary* if E is open and bounded and fr E is $(N - 1)$-rectifiable.

The requirement that E be open, rather than closed, is arbitrary but agrees with the convention in §8 that the winding number $W(x)$ is 0 on the support of the defining special $(N - 1)$-cycle. In place of 9.5 we need

11.2 Lemma. *Let E_1 and E_2 be elementary sets with respective reduced boundaries B_1 and B_2. Then*

$$\gamma(E_1 \bigtriangleup E_2) \leqq 3P(E_1 \bigtriangleup E_2) + 2m_{N-1}\{(\text{fr } E_1 - B_1) \cup (\text{fr } E_2 - B_2)\},$$

where $\bigtriangleup$ denotes symmetric difference.

Proof. $E_1 \bigtriangleup E_2 \subset E' \cup E'' \cup E'''$, where

$$E' = (E_1 \bigtriangleup E_2) - (\text{fr } E_1 \cup \text{fr } E_2),$$

$$E'' = (B_1 - \text{fr } E_2) \cup (B_2 - \text{fr } E_1),$$

$$E''' = (\text{fr } E_1 - B_1) \cup (\text{fr } E_2 - B_2).$$

Since E' is open, $\gamma(E') \leqq P(E') = P(E_1 \triangle E_2)$ by 9.3. If $x \in E''$, say $x \in B_1 - \mathrm{fr}\, E_2$, then a neighborhood of x lies entirely in E_2 or in $R^N - \mathrm{cl}\, E_2$. Since E_1 has an exterior normal at x, so does $E_1 \triangle E_2$. Therefore, E'' is contained in the reduced boundary of $E_1 \triangle E_2$, from which by 9.2a and 3.1(1), $\gamma(E'') \leqq 2m_{N-1}(E'') \leqq 2P(E_1 \triangle E_2)$. Finally, $\gamma(E''') \leqq 2m_{N-1}(E''')$ by 9.2a.

11.3 THEOREM. *Let E be a set with finite perimeter $P(E)$ and finite m_N-measure. Then there exist a sequence E_n of elementary sets and a set $\bar{E}$, such that*
(a) $\lim_n[P(E_n \triangle E) + m_N(E_n \triangle E)] = 0$;
(b) $\lim_n m_{N-1}(\mathrm{fr}\, E_n - B_n) = 0$;
(c) $m_N(E \triangle \bar{E}) = 0$;
(d) $\lim_n \gamma(E_n \triangle \bar{E}) = 0$.
The conditions (a)–(d) *determine $\bar{E}$ uniquely up to γ-measure* 0.

Proof. Let f be the characteristic function of E, let (h_n, P_n) be as in 8.5, and let

$$E_n = [x : W_n(x) > 0].$$

Write f_n for the characteristic function of E_n. By 3.3

$$I(W_n) = I(f_n) + I(W_n - f_n).$$

Now $I(W_n)$ tends to $I(f)$, and f_n tends to f m_N-almost everywhere, by 10.1. By 2.2, $I(f) \leqq \liminf I(f_n)$. Therefore, $I(W_n - f_n)$ tends to 0. Then so does $P(E_n \triangle E)$, since

$$P(E_n \triangle E) = I(\,|f_n - f|\,) \leqq I(f_n - f) \leqq I(f_n - W_n) + I(W_n - f).$$

By 8.5(a), $m_N(E_n \triangle E)$ tends to 0, which proves (a). Now

$$m_{N-1}(\mathrm{fr}\, E_n - B_n) = m_{N-1}(\mathrm{fr}\, E_n) - m_{N-1}(B_n) \leqq a(h_n, P_n) - m_{N-1}(B_n).$$

Both $a(h_n, P_n) - I(W_n)$ and $I(W_n) - I(f_n) = I(W_n) - m_{N-1}(B_n)$ tend to 0, which proves (b). The proof now proceeds as for 10.1, by using 11.2 in place of 9.5.

12. A connection with area theory

One may ask whether in case $\mathrm{fr}\, E$ is a compact $(N - 1)$-manifold X the perimeter $P(E)$ agrees with the $(N - 1)$-area of X according to some reasonable definition of area. Let i_X denote the inclusion map of X into R^N. If X is finitely triangulable, we may consider the Lebesgue area $L(i_X)$. Without this triangulability assumption one may still consider areas defined in terms of suitable multiplicity functions associated with projections of X on hyperplanes.

Let p denote projection of $(x^1, \cdots, x^N)$ onto $(x^1, \cdots, x^{N-1})$, $\mathfrak{O}$ the group of orthogonal transformations ρ of R^N, and μ Haar measure on $\mathfrak{O}$, $\mu(\mathfrak{O}) = 1$. For any mapping g from X into R^{N-1}, and $y \in R^{N-1}$, let $M(g, y)$ denote Federer's combinatorial multiplicity [11c, p. 336]. Then the integralgeometric $(N-1)$-area $M(i_X)$ is defined by

$$M(i_X) = \frac{1}{\beta}\int_{\Theta}\int_{R^{N-1}} M(p \circ \rho \circ i_X, y)\, dm_{N-1}(y)\, d\mu(\rho), \tag{1}$$

where β is that number such that

$$|x| = \frac{1}{\beta}\int_{\Theta} |(p \circ \rho)(x)|\, d\mu(\rho), \qquad \text{all } x \in R^N \tag{2}$$

[11b, p. 120].

12.1 THEOREM. *Let E be a set such that* fr E *is a compact $(N-1)$-manifold X with $m_N(X) = 0$. Then $P(E) = M(i_X)$.*

The condition $m_N(X) = 0$ is to some extent natural, since otherwise one must say what part of X is to be included in E. There are nevertheless some interesting unsolved problems for the case $m_N(X) > 0$, which we shall mention below.

Federer has shown that for any mapping g from a finitely triangulable subset of a k-manifold into R^N $(k \leqq N)$ such that the range of g has m_{k+1}-measure 0, the Lebesgue and integralgeometric areas of g are equal [11c, 7.8] [Notices Amer. Math. Soc., vol. 6 (1959), Abstract No. 560–44, p. 619]. From this result and 12.1 we deduce

12.2 THEOREM. *Besides the assumptions of* 12.1, *suppose X finitely triangulable. Then $P(E) = L(i_X)$.*

The author's original proof of 12.1 was much more complicated than the one to be given, and was written out for $N = 3$ only. The present proof came about after several helpful suggestions from Federer. He also first posed 12.2 as a problem [11d, p. 451].

Let us write $x = (y, z)$, where $y = (x^1, \cdots, x^{N-1})$, $z = x^N$. Given $f(y, z)$ with compact support and y, let $\nu(f, y)$ denote the essential total variation of f as a function of z; i.e., $\nu(f, y)$ is the total variation calculated using only intervals on the z-axis at whose endpoints f is m_1-approximately continuous as a function of z. The main step in the proof of 12.1 is

12.3 LEMMA. *For every $\rho \in \Theta$ and $y \in R^{N-1}$, $\nu(f_E \circ \rho, y) \geqq M(p \circ \rho \circ i_X, y)$. If $m_1\{X \cap (p \circ \rho)^{-1}(y)\} = 0$, then equality holds.*

Before proving this lemma, let us show that 12.1 follows from it. For $f \in BV$, let $I_j(f) = |\partial f/\partial x^j|(R^N)$, $j = 1, \cdots, N$. Choose $\rho_1 =$ identity, $\rho_2, \cdots, \rho_N$ such that $I_j(f) = I_1(f \circ \rho_j)$. By [17, p. 117], $I(f)$ is finite if and only if $\nu(f \circ \rho_j, y)$ is an m_{N-1}-integrable function of y for $j = 1, \cdots, N$. If $I(f)$ is finite, then [17, p. 117, 5.1],

$$I_j(f) = \int_{R^{N-1}} \nu(f \circ \rho_j, y)\, dm_{N-1}(y), \qquad j = 1, \cdots, N; \tag{3}$$

and also

(4) $$I(f) = \frac{1}{\beta}\int_{\Theta} I_1(f \circ \rho)\, d\mu(\rho).$$

If f is, say, continuously differentiable with $I(f)$ finite, then (4) follows from (2) applied to Grad f and Fubini's theorem. In the general case let f_n be regularizations of f (§2). Then $\lim I(f_n) = I(f)$, $\lim I_1(f_n \circ \rho) = I_1(f \circ \rho)$, and $I_1(f_n \circ \rho) \leqq I(f_n) \leqq I(f)$. We apply Lebesgue's convergence theorem.

Since $m_N(X) = 0$, by 12.3 we have for any $\rho \in \Theta$

(5) $$\nu(f_E \circ \rho, y) = M(p \circ \rho \circ i_X, y), \qquad m_{N-1}\text{-almost all } y.$$

Suppose $P(E) = I(f_E)$ is finite. From (4) with $f = f_E$, (3) with $j = 1$ and $f = f_E \circ \rho$, (5), and (1), $P(E) = M(i_X)$. Suppose $P(E) = +\infty$. Then, for any $\rho \in \Theta$, $I(f_E \circ \rho) = +\infty$ from which $\nu(f \circ \rho \circ \rho_j, y)$ is not m_{N-1}-integrable in y for at least one j. It follows from (1) and (5) that $M(i_X) = +\infty$.

Remark. In case $m_N(X) > 0$ the first part of 12.3 still yields $P(E) \geqq M(i_X)$, E the bounded component of $R^N - X$ (or equally well, the unbounded component of $R^N - X$). Federer pointed out that if $N \leqq 3$ and both components of $R^N - X$ have finite perimeter, then $m_N(X) = 0$. Is this true if $N > 3$?

Proof of 12.3. It suffices to consider the case $\rho =$ identity. For brevity write $M(y) = M(p \circ i_X, y)$, $\nu(y) = \nu(f_E, y)$. For $y \in R^{N-1}$ and $r > 0$, let

$$K(y, r) = [y' \in R^{N-1}: |y' - y| < r]$$

and let $C(y, r)$ be the open cylinder $K(y, r) \times R^1$. Let $F(y, r)$ denote the set of components V of $X \cap C(y, r)$, and for $(y, z) \notin X$, $F(y, z, r)$ the set of $V \in F(y, r)$ such that V does not meet the negative half line whose points are (y, z'), $z' < z$. For each V the mapping $p \circ i_X$ induces a homomorphism of the Čech cohomology groups with integer coefficients:

$$(p \circ i_X)^*: H^{N-1}\{\text{cl } K(y, r), \text{fr } K(y, r)\} \to H^{N-1}(\text{cl } V, \text{fr } V).$$

Both of these groups are infinite cyclic, and $(p \circ i_X)^*$ maps a generator of the first onto an integral multiple σ_V of a generator of the second. We suppose X and R^{N-1} oriented, and the generators chosen to agree with the assigned orientations. By definition of M,

(6) $$M(y) = \lim_{r \to 0^+} \textstyle\sum_{V \in F(y,r)} |\sigma_V|.$$

For $(y, z) \notin X$ the order (or winding number) $W(y, z)$ can be defined by

(7) $$W(y, z) = \textstyle\sum_{V \in F(y,z,r)} \sigma_V$$

for any r small enough that no V meets both the positive and negative half lines through (y, z). W is constant on each of the two components of $R^N - X$, and $W = 0$ on the unbounded component [19, §2]. It is well known that $W = \pm f_E$, depending on the orientations chosen; but this also comes out in the proof below.

Given $V \in F(y, r)$; $(y', z') \in C(y, r) - V$, and r' small enough that $C(y', r') \subset C(y, r)$, we shall also need to consider the set $F_V(y', z', r')$ of all components of $V \cap C(y', r')$ not meeting the negative half line through (y', z'), and the function

$$(8) \qquad W_V(y', z') = \textstyle\sum_{U \in F(y', z', r')} \sigma_U ,$$

for r' small enough that no U meets both positive and negative half lines through (y', z'). $W_V(y', z')$ is constant on each component of $C(y, r) - V$, and is 0 for z' near $+\infty$. For z' near $-\infty$, $W_V(y', z') = \sigma_V$ since this is clearly true if $y' = y, r' = r$. For each V, $C(y, r) - V$ has exactly two components. Indeed, let S^N be an N-sphere, q_0 a point of S^N, and η a relative homeomorphism of (cl C, fr C) onto $(S^N, \{q_0\})$. Then $(\eta \circ i_{\text{cl } V})^*$ gives an isomorphism from $H^{N-1}(\eta(\text{cl } V), \{q_0\})$ onto $H^{N-1}(\text{cl } V, \text{fr } V)$ [10, p. 266]; and the assertion follows from Alexander's duality theorem, since $H^{N-1}(\eta(\text{cl } V))$ and $H^{N-1}(\eta(\text{cl } V), \{q_0\})$ are isomorphic for $N \geqq 2$.

Suppose that $\sigma_V \neq 0$. Let

$$C = K(y, r) \times [-b, b], \qquad C^* = \{\text{fr } K(y, r)\} \times [-b, b],$$

with b large enough that $|z| < b$ for $(y, z) \in X$. Consider the triad (cl C; cl V, C^*), whose cohomology sequence

$$\xrightarrow{j^*} H^{N-1}(\text{cl } C, C^*) \xrightarrow{i^*} H^{N-1}(\text{cl } V, \text{fr } V) \xrightarrow{\delta} H^N(\text{cl } C, \text{cl } V \cup C^*) \xrightarrow{j^*} 0$$

is exact [10, p. 37, p. 257]. Now p induces an isomorphism of $H^{N-1}(\text{cl } K, \text{fr } K)$ onto $H^{N-1}(\text{cl } C, C^*)$. Let u and v be generators for the infinite cyclic groups $H^{N-1}(\text{cl } C, C^*)$ and $H^{N-1}(\text{cl } V, \text{fr } V)$, respectively. Then $i^*u = \pm \sigma_V v$, and $\delta i^* u = 0$ by exactness. Thus $\delta(\sigma_V v) = \sigma_V \delta v = 0$, from which $\delta v = 0$ since $H^N(\text{cl } C, \text{cl } V \cup C^*)$ is free abelian. But δ is onto, which implies $H^N(\text{cl } C, \text{cl } V \cup C^*) = 0$. Then i^* is onto, whence

$$(9) \qquad |\sigma_V| = 1.$$

Write l for the line $p^{-1}(y)$, and let

$$0_+ = [(y, z) \in l : W_V(y, z) = 0]$$
$$0_- = [(y, z) \in l : W_V(y, z) = \sigma_V].$$

Then $l = 0_+ \cup 0_- \cup (V \cap l)$. Therefore, there is a component ζ of $V \cap l$ which meets both cl 0_+ and cl 0_-; if not, l would split into disjoint, nonempty, closed sets. Since X is locally connected, there is an open set G containing ζ such that $G \cap X = G \cap V$. Choose $(y, z') \in 0_+$ and $(y, z'') \in 0_-$ such that the segment joining (y, z') and (y, z'') lies in G. Then

$$W(y, z') - W(y, z'') = W_V(y, z') - W_V(y, z'').$$

For if $z' > z''$, then each side equals $\sum \sigma_U$, $U \in F_V(y, z'', r') - F_V(y, z', r')$ for small r', and the case $z'' > z'$ goes the same way. Hence

$$(10) \qquad |W(y, z') - W(y, z'')| = |\sigma_V| = 1.$$

Now let n be any integer $\leqq M(y)$. For small r there exist n distinct components $V_1, \cdots, V_n$ of the sort just considered. Choose pairs of points (y, z_j'), (y, z_j'') as above, $j = 1, \cdots, n$, such that the segments joining each pair are disjoint. From (10), $\nu(y) \geqq n$. Hence $\nu(y) \geqq M(y)$, completing the first part of the proof.

Consider $z_1 < z_2 < \cdots < z_{m+1}$ such that $(x, z_i) \notin X$ for every i. From (7), for small r

$$| W(y, z_{i+1}) - W(y, z_i) | \leqq \sum_{V \in F(y, z_i, r) - F(y, z_{i+1}, r)} | \sigma_V | .$$

Then from (6)

$$\sum_{i=1}^{m} | W(y, z_{i+1}) - W(y, z_i) | \leqq M(y).$$

But if $m_1\{X \cap p^{-1}(y)\} = 0$, $\nu(y)$ is the supremum of such sums. Then $\nu(y) \leqq M(y)$, completing the proof of 12.3.

13. Extremal elements

Let K be an N-cube in R^N, and a real, positive. The set Γ of all f with compact support contained in K and $I(f) \leqq a$ is convex and compact in the L_1-norm. Let Γ_e denote the set of extreme points of Γ. It was shown in [12, pp. 99–100] that if $f \in \Gamma_e$ there exists a set $E \subset K$, such that

$$f_E = \pm a^{-1} P(E) f; \tag{1}$$

if $N = 2$, for $f \in \Gamma_e$ it is necessary and sufficient that E be equivalent to the region inside a simple closed planar curve in K. To treat the case $N \geqq 3$, we need

13.1 Definition. Write $B' \dot{\subset} B$ to mean $m_{N-1}(B - B') = 0$. *A set E with finite perimeter has indecomposable reduced boundary B if, for any set E' with reduced boundary $B' \dot{\subset} B$, E' is equivalent either to E or to $R^N - E$.*

Since $B \subset \mathrm{fr}\, E$, the following gives a sufficient condition for indecomposability. It covers the situation in §12. It would be interesting to know whether 13.2 remains true if $m_N(\mathrm{fr}\, E) > 0$.

13.2 Theorem. *Let E be an open set such that* (1) *$P(E)$ is finite;* (2) *E and $R^N - \mathrm{cl}\, E$ are connected;* (3) *$m_N(\mathrm{fr}\, E) = 0$. Let E' have reduced boundary $B' \dot{\subset} \mathrm{fr}\, E$. Then E' is equivalent to either E or $R^N - \mathrm{cl}\, E$.*

Proof. We may assume that $m_N(E)$ and $m_N(E')$ are finite, since either these sets or their complements have this property. By 8.4 there is a sequence of special $(N - 1)$-cycles (h_n, P_n) with

$$h_n(P_n) \subset [\mathrm{fr}\, E]_{1/n}, \qquad \lim_n I(W_n - f_{E'}) = 0.$$

By 3.4, $\lim_n L_1(W_n - f_{E'}) = 0$. Let $x', x'' \in E$. Since E is open and connected, x' and x'' can be joined by an arc not meeting $h_n(P_n)$ for large n. Then $W_n(x') = W_n(x'')$, for large n, which implies that $f_{E'}$ is m_N-almost everywhere constant on E. Similarly $f_{E'}$ is m_N-almost everywhere constant on $R^N - \text{cl}\, E$. Then $m_N(E \triangle E') = 0$; i.e., E and E' are equivalent.

13.3 LEMMA. *Let E_1 and E_2 have finite perimeter and finite m_N-measure. Write B_1, B_2, B', B'' for the reduced boundary of E_1, E_2, $E_1 \cup E_2$, $E_1 \cap E_2$, respectively. Then $B' \cup B'' \dot{\subset} B_1 \cup B_2$.*

Proof. Write f_1, f_2, f', f'' for the respective characteristic functions. Then $f_1 + f_2 = f' + f''$, from which grad $(f_1 + f_2) = \text{grad}\, f' + \text{grad}\, f''$. The measure $|\text{grad}\ (f_1 + f_2)|$ is 0 outside $B_1 \cup B_2$. By 3.3, $I(f_1 + f_2) = I(f') + I(f'')$, which may be rewritten

$$|\text{grad}\ (f_1 + f_2)|\ (B_1 \cup B_2) = |\text{grad}\, f'|\ (R^N) + |\text{grad}\, f''|\ (R^N).$$

Then $|\text{grad}\, f'|$ and $|\text{grad}\, f''|$ are 0 outside $B_1 \cup B_2$, from which the conclusion follows.

13.4 LEMMA. *Suppose that E and E' have finite perimeter with $B' \dot{\subset} B$ and either $E \subset E'$* or *$E' \subset E$. Let f, f', f'' denote characteristic functions of E, E', $E \triangle E'$, respectively. Then $I(f) = I(f') + I(f'')$.*

Proof. Consider the case $E \subset E'$, the other case being similar. Since $f'' = f' - f$, grad f'' is 0 outside $B' \cup B$. Thus $B'' \dot{\subset} B' \cup B \dot{\subset} B$. From the definition of exterior normal, $B'' \cap B' \cap B$ is void. Hence $B'' \dot{\subset} B - B'$. Then

$$m_{N-1}(B) = I(f) \leqq I(f') + I(f'') = m_{N-1}(B') + m_{N-1}(B'') \leqq m_{N-1}(B),$$

from which the conclusion follows.

13.5 THEOREM. *The set Γ_e of extreme points of Γ consists of all f for which there exists a set $E \subset K$ with positive finite perimeter and indecomposable reduced boundary such that* (1) *holds.*

Proof. Let $f \in \Gamma_e$. Then E exists such that (1) holds [12]. Consider any E' with $B' \dot{\subset} B$. We may assume $m_N(E')$ is finite, from which $E' \subset K$ (except in m_N-measure 0). Write f for f_E, f' for $f_{E \cup E'}$, and $f'' = f' - f$. By 13.3 and 13.4, $I(f) = I(f') + I(f'')$. Suppose that f'' is not equivalent to 0. Then $I(f'') > 0$. Let

$$F' = \frac{I(f)}{I(f')} f', \qquad F'' = -\frac{I(f)}{I(f'')} f'', \qquad \lambda = \frac{I(f')}{I(f)}.$$

Then $F', F'' \in \Gamma$, $0 < \lambda < 1$, and $f = \lambda F' + (1 - \lambda) F''$. Since $f \in \Gamma_e$, $F' = F'' = f$, which is impossible. Therefore f'' is equivalent to 0, from which E' is equivalent to a subset of E. Then by a variation of the argument just made, E' is equivalent to E. Thus B is indecomposable.

To prove the converse, we appeal to a theorem of Choquet [6b]. Let f be given by (1) with B indecomposable. By Choquet's theorem there is a measure μ on Γ such that open sets in the L_1-topology are μ-measurable,

$\mu(\Gamma) = 1$, $\mu(\Gamma - \Gamma_e) = 0$, and for all bounded measurable functions ϕ

$$\int_{R^N} f(x)\phi(x)\, dm_N(x) = \int_{\Gamma_e}\int_{R^N} F(x)\phi(x)\, dm_N(x)\, d\mu(F).$$

Then

$$(\text{grad } f)\cdot\omega = \int_{\Gamma_e} (\text{grad } F)\cdot\omega\, d\mu(F), \qquad \text{all } \omega \in \Omega_1. \tag{2}$$

By Lebesgue's convergence theorem the class of ω with Borel measurable coefficients and $\|\omega\| \leqq 1$ such that (2) holds is closed under pointwise convergence. Hence (2) is true for all such ω.

By [11a, 4.5], B is a Borel set. Now $|\text{grad } f|$ is 0 outside B, and $|\text{grad } f|(B) = I(f) = a$. By [22, p. 319, 5B] there exists ω_f with Borel measurable coefficients such that $\|\omega_f\| = 1$, ω_f is 0 outside B, and

$$|\text{grad } f|(A) = \int_A \omega_f \cdot d(\text{grad } f), \qquad \text{all Borel sets } A.$$

Then

$$\text{grad } F\cdot\omega_f \leqq |\text{grad } F|(R^N) = I(F) \leqq a;$$

and since $I(F) \leqq a$, equality implies that $|\text{grad } F|$ is 0 outside B. But

$$a = (\text{grad } f)\cdot\omega_f = \int_{\Gamma_e} (\text{grad } F\cdot\omega_f)\, d\mu(F) \leqq a\mu(\Gamma) = a,$$

from which $|\text{grad } F|$ is 0 outside B for μ-almost all F. In other words, by writing E_F for the set corresponding to $F \in \Gamma_e$, $B_F \mathrel{\dot{\subset}} B$ for μ-almost all F. Since B is indecomposable, $m_N(E_F \triangle E) = 0$, from which F and f are equivalent, for μ-almost all F. This proves that $f \in \Gamma_e$.

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