ON THE STRUCTURE OF STOCHASTIC INDEPENDENCE

BY

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Introduction and summary

The object of this note is to complete the solution to several problems posed by S. Banach [5] and by E. Marczewski [12]. These problems concern relations between various conditions related to stochastic independence. The problems can be summarized as follows.

If $\{(Y, \mathfrak{M}_t, \mu_t)\}$ $(t \in \mathfrak{H})$ is an arbitrary at-least-countable family of probability σ-measure spaces, what are the relations between the four conditions below?

(C_0) The $\{\mathfrak{M}_t\}$ are *σ-independent*, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$ whenever $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i and $t_i \neq t_j$ for $i \neq j$.

(C_1) The $\{\mathfrak{M}_t\}$ are *stochastically independent* with respect to the μ_t, i.e., there exists a σ-measure μ (called the stochastic extension of the μ_t) on $\mathcal{S}(\bigcup_t \mathfrak{M}_t)$ such that $\mu(\bigcap_1^n A_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i})$ whenever $A_{t_i} \in \mathfrak{M}_{t_i}$ for $1 \leqq i \leqq n$ and $t_i \neq t_j$ when $i \neq j$.

(C_2) The $\{\mathfrak{M}_t\}$ are *almost σ-independent* with respect to the $\{\mu_t\}$, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$, whenever $\mu_{t_i}(A_{t_i}) \neq 0$ and $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i; and $t_i \neq t_j$ for $i \neq j$.

(C_3) The $\{\mathfrak{M}_t\}$ are *quasi-σ-independent* with respect to the $\{\mu_t\}$, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$, whenever $\prod_1^\infty \mu_{t_i}(A_{t_i}) \neq 0$, and $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i; and $t_i \neq t_j$ for $i \neq j$.

From the definitions it is easily seen that $(C_0) \rightarrow (C_2) \rightarrow (C_3)$. Further, since any set of positive measure is nonempty and each σ-measure is continuous from above, it follows that $(C_1) \rightarrow (C_3)$. Finally, Banach [5], Sikorski [21], Sherman [16] and the author [4], pp. 66–68, have demonstrated that $(C_0) \rightarrow (C_1)$.

Therefore, in order to complete the solution to the original problem one must answer the following two questions.

(A) What is the relationship between (C_1) and (C_2)?

(B) Are any two of the conditions equivalent?

The answers to these and related questions lie in the product σ-measure space $(Z, \mathfrak{S}, \nu) = \times_t (Y, \mathfrak{M}_t, u_t)$. It is seen (Theorem 1) that (C_1) is satisfied if and only if the $\{(Y, \mathfrak{M}_t, u_t)\}$ generate a σ-measure space which is the measure σ-homomorphic image of $(Z, \mathfrak{S}, \nu)$; and (Theorem 2) that it is possible to construct the desired σ-homomorphism whenever (C_2) is satisfied. The fact that no two of the conditions are equivalent is demonstrated by a set of examples.

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The results can be summarized as follows.

$$(\mathrm{C}_2) \to (\mathrm{C}_1) \quad [\text{Theorem 2}],$$

$$(\mathrm{C}_3) \not\to (\mathrm{C}_1) \quad [\text{Example 5}], \qquad (\mathrm{C}_1) \not\to (\mathrm{C}_2) \quad [\text{Example 4}],$$

$$(\mathrm{C}_2) \not\to (\mathrm{C}_0) \quad [\text{Example 3}].$$

The paper is divided into four sections. Examples of the various types of independence are presented in Section 1. In Section 2 the terminology, notation, and preliminaries are introduced. The product representation is constructed in Section 3; and Section 4 contains the main theorem.

1. Examples of independence

By virtue of their construction, product spaces provide the best examples of independence. In fact, it can be shown that any space exhibiting independence is in some way an image of a product space.

Example 1. Let $\{(Y_t, \mathcal{S}_t, u_t)\}$ $(t \in \mathfrak{H})$ be any at-least-countable family of probability σ-measure spaces. (See Section 2β). Form the product space, i.e., let $Z = \times_t Y_t$; $\tilde{B}_{t_0} = \times_t A_t$, where

$$A_t = \begin{cases} Y_t & \text{for } t \neq t_0 \\ B_{t_0} & \text{for } t = t_0 \end{cases} \qquad \text{for all } B_{t_0} \in \mathcal{S}_{t_0} \text{ and all } t_0 \in \hat{\mathfrak{H}};$$

$\mathfrak{S}_t = \{\tilde{B}_t \mid B_t \in \mathcal{S}_t\}$ for all $t \in \hat{\mathfrak{H}}$; and $\nu_t(\tilde{B}_t) = \mu_t(B_t)$ for all $B_t \in \mathcal{S}_t$ and all $t \in \hat{\mathfrak{H}}$.

Then $(Z, \mathcal{S}_t, \nu_t)$ is a well-defined probability σ-measure space for all $t \in \hat{\mathfrak{H}}$. Further, it is well known (e.g. [2], pp. 90–92) that there exists a unique probability σ-measure ν on $\mathfrak{S} = \mathcal{S}(\bigcup_t \mathfrak{S}_t)$, the least σ-algebra containing all $\mathfrak{S}_t$, with the property that $\nu(\bigcap_1^\infty \tilde{B}_{t_i}) = \prod_1^\infty \nu_{t_i}(\tilde{B}_{t_i})$ for each σ-constituent $\bigcap_1^\infty \tilde{B}_{t_i}$ of $(\bigcup_t \mathfrak{S}_t)$, i.e., whenever $\tilde{B}_{t_i} \in \mathfrak{S}_{t_i}$ for all i, and $t_i \neq t_j$ for $i \neq j$. (See Section 2α). Consequently, the $\{\mathfrak{S}_t\}$ are stochastically independent with respect to the ν_t, i.e., satisfy (C_1).

It is also known that $\times_t B_t$ is empty if and only if one of the B_t is empty. But the σ-constituent $\bigcap_1^\infty \tilde{B}_{t_i} = \times_t A_t$ where

$$A_t = \begin{cases} Y_t & \text{if} \quad t \in \hat{\mathfrak{H}} - \{t_1, t_2, \cdots\} \\ B_{t_i} & \text{if} \quad t = t_i, \quad i = 1, 2, \cdots. \end{cases}$$

Consequently, a σ-constituent $\bigcap_1^\infty \tilde{B}_{t_i}$ is empty if and only if one of the $\tilde{B}_{t_i}$ is empty. Therefore, the $\{\mathfrak{S}_t\}$ satisfy conditions (C_0), (C_2), and (C_3).

Example 2. Let I be the unit interval. For each n, let I_n be the set of all points of I whose dyadic expansions contain 1 as the n^{th} digit. (If a point has two different expansions, only the terminating one will be considered.) For each n, the least σ-algebra $\mathcal{S}(\{I_n\}) = \{I, \emptyset, I_n' = I_n, I_n^0 = I - I_n\}$. (See Section 2α.) Define $\mu_n(I_n') = \mu_n(I_n^0) = \frac{1}{2}$, $\mu(I) = 1$, and $\mu_n(\emptyset) = 0$ for all n. Each μ_n is, therefore, a probability σ-measure, and each $(I, \mathcal{S}(\{I_n\}), \mu_n)$ is a probability σ-measure space.

Now, for any arbitrary sequence $\{j_1, j_2, j_3, \cdots\}$ of 0's and 1's, $x = \sum_1^\infty j_n/2^n \in \bigcap_1^\infty I_n^{j_n}$. Consequently, each $\bigcap_1^\infty I_n^{j_n}$ is nonempty, and conditions (C_0), (C_2), and (C_3) are satisfied.

Further, it is known (e.g. [3], p. 159) that the Lebesgue measure is the stochastic extension of the $\{\mu_n\}$. Hence, (C_1) is satisfied.

Example 3. Let I and I_n be the same as in Example 2; let C^* be the set of all points of I whose dyadic expansions contain an infinity of 0's; $I_n^* = I_n \cap C^*$ and $\lambda_n(A_n) = 1$ or 0 according as the point $0 \in A_n$ or not, for all $A_n \in \mathcal{S}(\{I_n^*\}) = \{C^*, \emptyset, I_n^*, (I_n^*)^0\}$ and all n. Each λ_n is, then, a probability σ-measure.

Since $\bigcap_1^\infty I_n^* = C^* \cap \bigcap_1^\infty I_n = \emptyset$, the $\{\mathcal{S}(\{I_n^*\})\}$ do not satisfy (C_0). On the other hand, since the sets of positive measure are exactly those sets which contain 0, any intersection of sets of positive measure is nonempty. Therefore, (C_2) and (C_3) are satisfied.

Further, if $\lambda(A) = 1$ or 0 according as the point $0 \in A$ or not for all $A \in \mathcal{S}(\bigcup_1^\infty \mathcal{S}(\{I_n^*\}))$, then λ is a probability σ-measure, which is the stochastic extension of the $\{\lambda_n\}$. (C_1), then, is satisfied.

Example 4. Let I, I_n, I_n^*, C^*, $\mathcal{S}(\{I_n\})$, $\mathcal{S}(\{I_n^*\})$, $\{\mu_n\}$, etc. be the same as in Examples 2 and 3. Let $\nu_n(I_n^*) = \nu_n((I_n^*)^0) = \frac{1}{2}$, $\nu_n(C^*) = 1$, and $\nu_n(\emptyset) = 0$ for all n. ν_n is, then, a probability σ-measure on $\mathcal{S}(\{I_n^*\})$ for all n.

Since $\bigcap_1^\infty I_n^* = C^* \cap \bigcap_1^\infty I_n = \emptyset$, the least σ-algebras $\{\mathcal{S}(\{I_n^*\})\}$ do not satisfy (C_0), and do not satisfy (C_2) with respect to the $\{\nu_n\}$.

However, since the $\{\nu_n\}$ and $\{\mu_n\}$ coincide with Lebesgue measure on their respective domains of definition; since each set of $\bigcup_1^\infty \mathcal{S}(\{I_n^*\})$ differs from a set of $\bigcup_1^\infty \mathcal{S}(\{I_n\})$ by a set of Lebesgue measure zero; and, since the $\{\mathcal{S}(\{I_n\})\}$ satisfy (C_1) and (C_3) with respect to the $\{\mu_n\}$, it can be easily demonstrated that the $\{\mathcal{S}(\{I_n^*\})\}$ do satisfy (C_1) and (C_3) with respect to the $\{\nu_n\}$.

Example 5. Let I, I_n, μ_n, etc., be the same as in the preceding examples; R, the set of rational numbers in the unit interval; $R_n = I_n \cap R$ and $\rho_n(R \cap A_n) = \mu_n(A_n)$ for all $A_n \in \mathcal{S}(\{I_n\})$ and all n.

Then $\mathcal{S}(\{R_n\}) = \{R, \emptyset, R_n', R_n^0\}$ and ρ_n is a probability σ-measure on $\mathcal{S}(\{R_n\})$ for each n.

Now if $\{j_1, j_2, \cdots\}$ is any sequence of 0's and 1's such that $x = \sum_1^\infty j_n/2^n$ is an irrational number, then $\bigcap_1^\infty R_n^{j_n} = R \cap \bigcap_1^\infty I_n^{j_n} = \emptyset$. Consequently, the $\mathcal{S}(\{R_n\})$ do not satisfy (C_0), and do not satisfy (C_2) with respect to the $\{\rho_n\}$.

Further, if ρ were a stochastic extension of the $\{\rho_n\}$, then ρ must assign measure $\prod_1^\infty \rho_n(R_n^{j_n}) = 0$ to each rational number $\{\sum_1^\infty j_n/2^n\} = \bigcap_1^\infty R_n^{j_n}$; and measure 1 to R which is a countable union of rational numbers. There, of course, can exist no such probability σ-measure, and hence, condition (C_1) is not satisfied.

On the other hand, one finds that $\prod_1^\infty \rho_n(B_n) > 0$ if and only if all but a finite number of the $B_n = R$; that any finite intersection of nonempty elements of $\bigcup_1^\infty \mathcal{S}(\{R_n\})$ contains at least one set of the form $\bigcap_1^m R_n^{j_n}$; and that each $\bigcap_1^m R_n^{j_n}$ contains the rational number $x = \sum_1^{m-1} j_n/2^n + 1/2^{m+1}$

and is, hence, nonempty. From these considerations one can conclude that the $\{\mathcal{S}(\{R_n\})\}$ satisfy (C_3) with respect to the $\{\rho_n\}$.

2. Terminology, notation, and preliminaries

In the product representation of Section 3 certain known results concerning algebras, measures and homomorphisms will be used repeatedly. These results and pertinent definitions are presented in this section.

(α) *Algebras and semialgebras.* A space, Y, is a nonvoid collection of objects. $\emptyset$ will denote the empty subset of each space. A collection, $\mathcal{S}$, of subsets of Y is called a σ-algebra if $\mathcal{S}$ is closed under countable unions and complementation. A *semialgebra*, $\mathcal{H}$, of subsets of Y is a class containing Y, closed under finite intersections, and satisfying the chain condition (ω): whenever M and N are elements of $\mathcal{H}$ such that M contains N, there exists a finite subclass $\{N_0, N_1, \cdots, N_n\}$ of $\mathcal{H}$ with the property that $N = N_0 \subset N_1 \subset \cdots \subset N_n = M$ and $F_i = N_i - N_{i-1} \in \mathcal{H}$ for $i = 1, 2, \cdots, n$.

To understand the structure of σ-algebras and semialgebras it is expedient to introduce two entities, the constituent and σ-constituent, first used by Marczewski [11].

If $\mathcal{K}$ is an arbitrary class of subsets of some space Y, then each set which can be represented in the form $\bigcap_1^n A_i^{j_i}$ where $A_i \in \mathcal{K}$, $A_i^0 = Y - A_i$, $A_i' = A_i$, and $j_i = 0$ or 1 for $i = 1, 2, \cdots, n$, is called a *constituent* of $\mathcal{K}$; a set which can be represented in the form $\bigcap_1^\infty A_i^{j_i}$ is called a *σ-constituent* of $\mathcal{K}$; and the $A_i^{j_i}$ are called *sides* of the [σ-] constituent.

It is easily proved (e.g. [4], p. 56) that

(i) the class $\mathcal{H}(\mathcal{K})$ of all constituents of $\mathcal{K}$ is a semialgebra.

Further, it is known (e.g. [6], pp. 485–486) that

(ii) there exists a *least σ-algebra* $\mathcal{S}(\mathcal{K})$ containing $\mathcal{K}$; that each element of $\mathcal{S}(\mathcal{K})$ is a union of σ-constituents of $\mathcal{K}$; and $\mathcal{S}(\mathcal{K}) = \mathcal{S}(\mathcal{H}(\mathcal{K}))$.

Since Section 3 will deal with families of σ-algebras, it is worthwhile to put some of the sets and classes above in a form more adapted to work with families of σ-algebras.

If $\{\mathfrak{M}_t\}$ $(t \in \mathfrak{H})$ is a family of σ-algebras of some space, Y, then because each σ-algebra is closed under complementation and countable unions, it is possible to represent each constitutent of $(\bigcup \mathfrak{M}_t)$ in the form $\bigcap_1^n A_{t_i}$ and each σ-constituent in the form $\bigcap_1^\infty A_{t_i}$, where $A_{t_i} \in \mathfrak{M}_{t_i}$ and $t_i \in \mathfrak{H}$, for all i, and $t_i \neq t_j$ for $i \neq j$.

(Throughout the sequel it will be understood that any set represented in one of the above forms, i.e., $\bigcap_1^n A_{t_i}$ or $\bigcap_1^\infty A_{t_i}$, is a constituent or σ-constituent, respectively, of the designated union of families of σ-algebras. Further, unless something to the contrary is specified, it will be understood that $\mathfrak{H}$ is the index set for the indices t. Italic capitals will denote sets, and script capitals will denote classes of sets in the sequel.)

As one might ascertain from the statements of conditions (C_1) and (C_2), it will be necessary to employ some special properties of measures in the investigation of these conditions. For that reason these measure properties will be discussed next.

(β) *Measures.*

A *probability σ-measure* is a real-valued, nonnegative, countably additive set function defined on a semialgebra of subsets of some space, with the added properties that the measures of the whole space and the empty set, $\emptyset$, are respectively, 1 and 0. A triplet $(Y, \mathcal{S}, \mu)$ is called a *probability σ-measure space*, whenever $\mathcal{S}$ is a σ-algebra of subsets of Y and μ is a probability σ-measure on $\mathcal{S}$.

The measure theorem which is most important to the product representation is essentially due to von Neumann ([1], p. 94). It is as follows:

(iii) If μ is a probability σ-measure on a semialgebra $\mathcal{H}$, then there exists a unique probability σ-measure $\bar{\mu}$ on $\mathcal{S}(\mathcal{H})$ such that $\bar{\mu}$ coincides with μ on $\mathcal{H}$.

This unique extension theorem allows one to work with the semialgebras and still obtain measure theorems applicable to the generated σ-algebras.

Since measures, constituents, etc., have been defined, it is now possible to restate the independence conditions in terms of these.

(γ) *Independence conditions.* If $\{Y, \mathfrak{M}_t, \mu_t\}$ is an arbitrary at-least-countable family of probability σ-measure spaces, the new statements are as follows:

(C_1) (Stochastic Independence) There exists a probability σ-measure μ on $\mathcal{S}(\bigcup \mathfrak{M}_t)$ such that $\mu(\bigcap_1^n A_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i})$ for each constituent $\bigcap_1^n A_{t_i}$ of $(\bigcup \mathfrak{M}_t)$.

(C_2) (Almost σ-independence) Each σ-constituent of $(\bigcup \mathfrak{M}_t)$ with all sides of positive measure is nonempty.

The next step is to phrase these conditions in terms of properties of the product space which will be constructed. However, before the construction can be effected, two types of transformations need to be introduced.

(δ) *Homomorphisms.* A *σ-homomorphism* of a class $\mathcal{K}$ of subsets of Z onto a class $\mathcal{L}$ of subsets of Y is defined to be a transformation, φ, of $\mathcal{K}$ onto $\mathcal{L}$ such that φ can be extended to a countably multiplicative complementative transformation, $\bar{\varphi}$, of $\mathcal{S}(\mathcal{K})$ onto $\mathcal{S}(\mathcal{L})$.

It is easily proved (e.g. [6], p. 487) that

(iv) each σ-homomorphism φ, of a σ-algebra $\mathcal{S}(\mathcal{K})$ of subsets of Z onto a σ-algebra $\mathcal{S}(\mathcal{L})$ of subsets of Y, is countably additive and subtractive; and, further, $\varphi(Z) = Y$ and $\varphi(\emptyset) = \emptyset$. Also, $\varphi(\mathcal{S}(\mathfrak{M})) = \mathcal{S}(\varphi(\mathfrak{M}))$ for all subclasses $\mathfrak{M}$ of $\mathcal{K}$.

Besides this it has essentially been shown by Sikorski [17], [20] and the author [6] that

(v) a transformation φ of $\mathcal{K}$ onto $\mathcal{L}$ is a σ-homomorphism of $\mathcal{K}$ onto $\mathcal{L}$ if and only if, for every σ-constituent $\bigcap_1^\infty A_i^{j_i}$ of $\mathcal{K}$, $\bigcap_1^\infty [\varphi(A_i)]^{j_i} = \emptyset$ whenever $\bigcap_1^\infty A_i^{j_i} = \emptyset$.

This result can be extended to the case of a family of σ-homomorphisms.

When it is extended, it takes on the following form which will be employed in Section 3.

(vi) If φ_t is a σ-homomorphism of σ-algebra $\mathfrak{M}_t$ of subsets of Z onto σ-algebra $\mathcal{M}_t$ of subsets of Y for all $t \in \mathfrak{H}$, then the $\{\varphi_t\}$ have a common extension φ which is a σ-homomorphism of $\mathcal{S}(\bigcup \mathfrak{M}_t)$ onto $\mathcal{S}(\bigcup \mathcal{M}_t)$ if and only if, for each σ-constituent $\bigcap_1^\infty A_{t_i}$ of $(\bigcup \mathcal{M}_t)$, $\bigcap_1^\infty \varphi_{t_i}(\tilde{A}_{t_i}) = \emptyset$ whenever $\bigcap_1^\infty \tilde{A}_{t_i} = \emptyset$. The extension φ is unique in either case.

(iv)–(vi) concern only the preservation of the algebraic properties. In view of the measure aspect of the problem, one must consider homomorphisms which preserve measure.

(ε) *Measure homomorphisms.* A transformation φ is a measure σ-homomorphism of $(Z, \mathfrak{S}, \nu)$ onto $(Y, \mathcal{S}, \mu)$ if φ is a σ-homomorphism of $\mathfrak{S}$ onto $\mathcal{S}$ and if $\nu(\tilde{A}) = \mu(\varphi(\tilde{A}))$ for all $\tilde{A} \in \mathfrak{S}$.

In view of (iii), φ will be a measure σ-homomorphism if it preserves measure on an appropriate semialgebra, i.e.,

(vii) a σ-homomorphism φ of $\mathfrak{S}$ onto $\mathcal{S}$ is a measure σ-homomorphism of $(Z, \mathfrak{S}, \mu)$ onto $(Y, \mathcal{S}, \mu)$ if $\nu(\tilde{A}) = \mu(\varphi(\tilde{A}))$ for all $\tilde{A} \in \mathfrak{H}$, where $\mathfrak{H}$ is a semialgebra which generates $\mathfrak{S}$, i.e., $\mathcal{S}(\mathfrak{H}) = \mathfrak{S}$.

With these definitions and theorems it is now possible to effect the desired product representation.

3. Product representation

Let $\{(Y, \mathcal{M}_t, \mu_t)\}$ $(t \in \hat{\mathfrak{H}})$ be any at-least-countable family of probability σ-measure spaces with the same space Y. Form the product space, i.e., let $Z = \times_t Y$. Z can be considered the union of all $\hat{\mathfrak{H}}$-sequences of points of Y.

The *cylinder sets* of Z are those product sets of the form $\tilde{B}_{t_0} = \times_t A_t$ where

$$A_t = \begin{cases} Y & \text{for } t \neq t_0 \\ B_{t_0} \in \mathcal{M}_{t_0} & \text{for } t = t_0 \end{cases}.$$

If, now, for each $t \in \hat{\mathfrak{H}}$, $\mathfrak{M}_t = \{\tilde{B}_t \mid B_t \in \mathcal{M}_t\}$; $\nu_t(\tilde{B}_t) = \mu_t(B_t)$ and $\varphi_t(\tilde{B}_t) = B_t$ for all $B_t \in \mathcal{M}_t$, then one can conclude that Lemma 1 below is valid.

LEMMA 1. *$\mathfrak{M}_t$ is a σ-algebra of subsets of Z; ν_t is a probability σ-measure on $\mathfrak{M}_t$; φ_t is a measure σ-homomorphism of $(Z, \mathfrak{M}_t, \nu_t)$ onto $(Y, \mathcal{M}_t, \mu_t)$ for all $t \in \hat{\mathfrak{H}}$.*

The $\{\mathfrak{M}_t\}$ are called the *cylinder σ-algebras*; and $\mathfrak{S} = \mathcal{S}(\bigcup \mathfrak{M}_t)$ is the product σ-algebra.

It is well-known (e.g. Example 1; and [2], pp. 90–92) that there exists a product probability σ-measure ν on $\mathfrak{S} = \mathcal{S}(\bigcup \mathfrak{M}_t)$; and that the σ-constituents with nonempty sides are nonempty. Consequently,

LEMMA 2. *The $\{\mathfrak{M}_t\}$ are σ-independent; and are almost σ-independent, stochastically independent, and quasi-σ-independent with respect to the $\{\nu_t\}$; i.e., they satisfy conditions* (C_0)–(C_3).

Since the $\{\mathfrak{M}_t\}$ are σ-independent σ-algebras, the $\{\varphi_t\}$ satisfy the hypotheses of (vi), and the following lemma holds.

LEMMA 3. *There exists a unique σ-homomorphism φ of $\mathfrak{S} = \mathcal{S}(\bigcup \mathfrak{M}_t)$ onto $\mathcal{S} = \mathcal{S}(\bigcup \mathcal{M}_t)$ with the property $\varphi(\bigcap_1^\infty \tilde{A}_{t_i}) = \bigcap_1^\infty \varphi_{t_i}(\tilde{A}_{t_i})$ for each σ-constituent $\bigcap_1^\infty \tilde{A}_{t_i}$ of $(\bigcup \mathfrak{M}_t)$.*

Since it is known (Lemma 2) that the $\{\mathfrak{M}_t\}$ are stochastically independent with respect to the $\{\nu_t\}$, one is led to suspect that the same situation will hold for the $\{\mathcal{M}_t\}$ and the $\{\mu_t\}$, if φ is a measure σ-homomorphism of $(Z, \mathfrak{S}, \nu)$ onto $(Y, \mathcal{S}, \mu)$ for some measure μ. The necessity and sufficiency of this condition are established below.

THEOREM 1. (C_1) *holds if and only if* (a) *there exists a probability σ-measure μ on $\mathcal{S}$ such that φ is a measure σ-homomorphism of $(Z, \mathfrak{S}, \nu)$ onto $(Y, \mathcal{S}, \mu)$.*

Proof. (a) $\rightarrow$ (C_1). Let μ be the probability σ-measure on $\mathcal{S}$ satisfying (a). If now, $\bigcap_1^n A_{t_i}$ is an arbitrary constituent of $(\bigcup \mathcal{M}_t)$, then

$$\varphi(\bigcap_1^n \tilde{A}_{t_i}) = \bigcap_1^n A_{t_i}$$

and $\mu(\bigcap_1^n A_{t_i}) = \mu(\varphi(\bigcap_1^n \tilde{A}_{t_i})) = \nu(\bigcap_1^n \tilde{A}_{t_i}) = \prod_1^n \nu_{t_i}(\tilde{A}_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i})$. Therefore, μ is the desired stochastic extension of the $\{\mu_t\}$ and (C_1) holds.

(C_1) $\rightarrow$ (a). If (C_1) holds, let μ be the stochastic extension of the $\{\mu_t\}$. μ is a probability σ-measure on $\mathcal{S}$. In view of (vii) it will be sufficient to prove that $\mu\varphi$ and ν coincide on a semialgebra which generates $\mathfrak{S}$. From (i), (ii), and the definition of $\mathfrak{S}$, one can conclude that the class $\mathfrak{H}$ of all constituents of $(\bigcup \mathfrak{M}_t)$ is such a semialgebra.

But for an arbitrary element $\bigcap_1^n \tilde{A}_{t_i}$ of $\mathfrak{H}$, $\mu(\varphi(\bigcap_1^n \tilde{A}_{t_i})) = \mu(\bigcap_1^n \varphi_{t_i}(\tilde{A}_{t_i})) = \mu(\bigcap_1^n A_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i}) = \prod_1^n \nu_{t_i}(\tilde{A}_{t_i}) = \nu(\bigcap_1^n \tilde{A}_{t_i})$. Consequently, (a) holds.

4. The main theorem

The main result can now be established in terms of φ. It will be shown that whenever (C_2) holds, it is possible to construct a probability σ-measure μ on the semialgebra $\mathcal{H}$ of all constituents of $(\bigcup \mathcal{M}_t)$ such that $\mu(\varphi(\tilde{A})) = \nu(\tilde{A})$ for all $\tilde{A} \in \mathfrak{H}$, the semialgebra of all constituents of $(\bigcup \mathfrak{M}_t)$; and that (C_1) then follows from (vii) and Theorem 1.

The following three lemmas will be needed.

LEMMA 4. *If the $\{\mathcal{M}_t\}$ satisfy (C_2) with respect to the $\{\mu_t\}$; and if $\bigcap_1^\infty A_{t_i}$ and $\bigcap_1^\infty B_{t_i}$ are two σ-constituents of $(\bigcup \mathcal{M}_t)$ such that $\bigcap_1^\infty A_{t_i} \subset \bigcap_1^\infty B_{t_i}$ and $\mu_{t_i}(A_{t_i}) > 0$ for all i, then $\mu_{t_i}(A_{t_i} \cap B_{t_i}^0) = 0$ for all i, i.e., $\mu_{t_i}(A_{t_i}) = \mu_{t_i}(A_{t_i} \cap B_{t_i})$ for all i.*

Proof. From the hypotheses it is seen that for an arbitrary integer k, $\bigcap_1^\infty (A_{t_i} \cap B_{t_k}^0) = \emptyset$ and, hence, at least one of $\{\mu_{t_i}(A_{t_i})\}$ $(i \neq k)$ or $\mu_{t_k}(A_{t_k} \cap B_{t_k}^0)$ is zero. But since the $\mu_{t_i}(A_{t_i})$ are nonzero, then

$$\mu_{t_k}(A_{t_k} \cap B_{t_k}^0) = 0.$$

Since k was chosen arbitrarily, the conclusion follows.

LEMMA 5. *If the* $\{\mathfrak{M}_t\}$ *satisfy* (C_2) *with respect to the* $\{\mu_t\}$ *and if, further,* $\bigcap_1^\infty A_{t_i}$ *and* $\bigcap_1^\infty B_{t_i}$ *are* σ*-constituents of* $(\bigcup \mathfrak{M}_t)$ *such that* $\bigcap_1^\infty A_{t_i} = \bigcap_1^\infty B_{t_i}$ *and* $\mu_{t_i}(A_{t_i}) > 0$ *for all* i, *then* $\mu_{t_i}(A_{t_i}) = \mu_{t_i}(B_{t_i})$ *for all* i *and, consequently,* $\prod_1^\infty \mu_{t_i}(A_{t_i}) = \prod_1^\infty \mu_{t_i}(B_{t_i})$.

Proof. From Lemma 4 one can conclude that $\mu_{t_i}(A_{t_i} \cap B_{t_i}^0) = 0$ and, hence $\mu_{t_i}(A_{t_i}) = \mu_{t_i}(A_{t_i} \cap B_{t_i})$ for all i. But $A_{t_i} \cap B_{t_i} \subset B_{t_i}$ and, hence, $\mu_{t_i}(B_{t_i}) \geqq \mu_{t_i}(A_{t_i} \cap B_{t_i}) = \mu_{t_i}(A_{t_i}) > 0$ for all i. On applying Lemma 4 with the roles of the $\{B_{t_i}\}$ and $\{A_{t_i}\}$ interchanged, one finds that $\mu_{t_i}(B_{t_i} \cap A_{t_i}^0) = 0$ and, therefore, $\mu_{t_i}(A_{t_i} \cap B_{t_i}) = \mu_{t_i}(B_{t_i})$ for all i. The second part of the conclusion follows immediately from the first part.

LEMMA 6. *If the* $\{\mathfrak{M}_t\}$ *satisfy* (C_2) *with respect to the* $\{\mu_t\}$; *if* $\bigcap_1^\infty A_{t_i}$ *and* $\bigcap_1^\infty B_{t_i}$ *are* σ*-constituents of* $(\bigcup \mathfrak{M}_t)$ *such that* $\bigcap_1^\infty A_{t_i} = \bigcap_1^\infty B_{t_i}$; *and if* J *is a nonempty set of natural numbers such that* $\mu_{t_i}(A_{t_i}) = 0$ *for* $i \in J$ *and* $\mu_{t_i}(A_{t_i}) > 0$ *for* $i \notin J$, *then* $\prod_1^\infty \mu_{t_i}(A_{t_i}) = \prod_1^\infty \mu_{t_i}(B_{t_i})$.

Proof. For each $k \in J$, $\bigcap_1^\infty (B_{t_i} \cap A_{t_k}^0) = 0$. Therefore, at least one of $\{\mu_{t_i}(B_{t_i})\}$ $(i \notin J)$ or $\{\mu_{t_k}(B_{t_k} \cap A_{t_k}^0)\}$ $(k \in J)$ is zero. But for all $k \in J$, $\mu_{t_k}(B_{t_k} \cap A_{t_k}^0) = \mu_{t_k}(B_{t_k})$, since $\mu_{t_k}(A_{t_k}) = 0$. Therefore, at least one of the $\{\mu_{t_i}(B_{t_i})\}$ $(i = 1, 2, \cdots)$ is zero, and, hence $\prod_1^\infty \mu_{t_i}(A_{t_i}) = \prod_1^\infty \mu_{t_i}(B_{t_i}) = 0$.

It is now possible to define the desired stochastic extension and prove the theorem.

THEOREM 2. (C_2) $\rightarrow$ (C_1), *i.e., almost* σ*-independence implies stochastic independence.*

Proof. In view of Theorem 1, the definition of a measure σ-homomorphism, and (vii), it will be sufficient to prove that whenever (C_2) holds, there exists a probability measure μ on the semialgebra, $\mathfrak{K}$, of all constituents of $(\bigcup \mathfrak{M}_t)$ with the property that $\mu\varphi$ and ν coincide on the semialgebra $\mathfrak{H}$ of all constituents of $(\bigcup \mathfrak{M}_t)$.

For each element $\bigcap_1^m A_{t_i}$ of $\mathfrak{K}$ define the set function μ as follows: $\mu(\bigcap_1^m A_{t_i}) = \prod_1^m \mu_{t_i}(A_{t_i})$.

Without loss of generality one may assume that any two different representations of the same element of $\mathfrak{K}$ are in the forms $\bigcap_1^m A_{t_i}$ and $\bigcap_1^n B_{t_i}$. In this case $(\bigcap_1^m A_{t_i} \cap \bigcap_{m+1}^\infty Y) = (\bigcap_1^n B_{t_i} \cap \bigcap_{n+1}^\infty Y)$, and Lemmas 5 and 6 guarantee that

$$\mu(\bigcap_1^m A_{t_i}) = \prod_1^m \mu_{t_i}(A_{t_i}) \cdot \prod_{m+1}^\infty 1 = \prod_1^m \mu_{t_i}(A_{t_i}) \cdot \prod_{m+1}^\infty \mu_{t_i}(Y)$$
$$= \prod_1^n \mu_{t_i}(B_{t_i}) \cdot \prod_{m+1}^\infty \mu_{t_i}(Y) = \prod_1^n \mu_{t_i}(B_{t_i}) = \mu(\bigcap_1^n B_{t_i})$$

where $\{t_{n+1}, t_{n+2}, \cdots\}$ is any collection of the t's of $\mathfrak{H}$ other than $\{t_1, t_2, \cdots, t_n\}$. Therefore, μ is well-defined on $\mathfrak{K}$.

Now, for any arbitrary element $\bigcap_1^n A_{t_i}$ of $\mathfrak{K}$, $\mu(\varphi(\bigcap_1^n \tilde{A}_{t_i})) = \mu(\bigcap_1^n A_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i}) = \prod_1^n \nu_{t_i}(\tilde{A}_{t_i}) = \nu(\bigcap_1^n \tilde{A}_{t_i})$. Therefore $\mu\varphi$ and ν coincide on $\mathfrak{H}$.

Further, since $\varphi(\mathfrak{H}) = \mathcal{H}$, and ν is nonnegative and real-valued on $\mathfrak{H}$, then μ is nonnegative and real-valued on $\mathcal{H}$. Also, $\mu(Y) = \mu(\varphi(Z)) = \nu(Z) = 1$, and $\mu(\emptyset) = \mu(\varphi(\emptyset)) = \nu(\emptyset) = 0$.

Therefore, in order to demonstrate that μ is a probability σ-measure it will be sufficient to show that μ is countably additive on $\mathcal{H}$. This demonstration will complete the proof of the theorem.

Let $\{A_i\}$ be an arbitrary countable collection of mutually disjoint elements of $\mathcal{H}$, whose union is also an element of $\mathcal{H}$. Let $\{\tilde{A}_i\}$ be a collection of elements of $\mathfrak{H}$ such that $\varphi(\tilde{A}_i) = A_i$ for all i. Then for each pair of natural numbers k and j such that $k \neq j$, $A_k \cap A_j = \emptyset$, and $\nu(\tilde{A}_k \cap \tilde{A}_j) = \mu\varphi(\tilde{A}_k \cap \tilde{A}_j) = \mu(A_k \cap A_j) = \mu(\emptyset) = 0$, and, hence $\nu(\bigcup_1^\infty \tilde{A}_i) = \sum_1^\infty \nu(\tilde{A}_i) = \sum_1^\infty \mu(\varphi(\tilde{A}_i)) = \sum_1^\infty \mu(A_i)$.

However, $\bigcup_1^\infty \tilde{A}_i$ is not necessarily an element of $\mathfrak{H}$, and, hence, $\mu\varphi$ is not necessarily defined on $\bigcup_1^\infty \tilde{A}_i$. But, $\bigcup_1^\infty \tilde{A}_i \in \mathfrak{S}$ and $\mu\varphi$ has a unique extension $\bar{\mu}\varphi$ which is a probability σ-measure on $\mathfrak{S}$ and which coincides with ν on $\mathfrak{S}$. Therefore $\mu(\bigcup_1^\infty A_i) = \mu(\varphi(\bigcup_1^\infty \tilde{A}_i)) = \bar{\mu}(\varphi(\bigcup_1^\infty \tilde{A}_i)) = \nu(\bigcup_1^\infty \tilde{A}_i) = \sum_1^\infty \nu(\tilde{A}_i) = \sum_1^\infty \mu(\varphi(\tilde{A}_i)) = \sum_1^\infty \mu(A_i)$ and μ is countably additive on $\mathcal{H}$. This completes the proof.

The solution to the original problem can be summarized as follows: No two of the conditions are equivalent; and $(C_0) \to (C_2) \to (C_1) \to (C_3)$.

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