SOME SUFFICIENT CONDITIONS FOR A GROUP TO BE NILPOTENT

In commemoration of G. A. Miller

BY
P. HALL

1. Let

$$G = G_0 > G_1 > G_2 > \cdots > G_m = 1 \tag{1}$$

be a chain of subgroups of the group G. Following Kaloujnine [1], we define the *stability group* of the chain (1) to be the group A of all automorphisms α of G such that

$$(G_i\, x)^\alpha = G_i\, x \tag{2}$$

holds for all $x \in G_{i-1}$ and for each $i = 1, 2, \cdots, m$.

If the subgroups G_i are all normal in G, then it is easy to show that A is nilpotent and of class at most $m - 1$. But without some such assumption of normality, the nature of the group A is not so clear. In [1], however, Kaloujnine proved that A is always at least a soluble group, and the length d of its derived series cannot exceed $m - 1$. He remarks of this result that it is "wahrscheinlich nicht endgültig." In fact, we shall find that A is still nilpotent even in the general case. This is stated in

THEOREM 1. *The stability group A of any subgroup-chain* (1) *of length m is nilpotent and of class at most $\frac{1}{2}m(m - 1)$.*

It was shown in [3] that a nilpotent group A of derived length d must be of class at least 2^{d-1}. Thus Theorem 1 yields the bound

$$d \leqq [\log_2 m(m - 1)] \tag{3}$$

for the derived length of the stability group A. This bound never exceeds $m - 1$ and is smaller than $m - 1$ for $m > 5$. Indeed, it is of a smaller order of magnitude as $m \to \infty$. Hence Kaloujnine's theorem follows from (3).

For the class of A we have the bounds $m - 1$ and $\frac{1}{2}m(m - 1)$ which apply in the normal case and the general case, respectively. These bounds first differ when $m = 3$. That the difference is significant we show by constructing a group with a subgroup-chain of length 3 for which the stability group is of class 3. It will also be proved that the subgroup of G generated by all the commutators $x^{-1}x^\alpha$ with $x \in G$ and $\alpha \in A$ is always locally nilpotent. This commutator subgroup is known to be always nilpotent in the normal case: cf. [1], Satz 4. We show by an example that this need not be so in the general case.

Received January 5, 1959.

Let M be a nilpotent normal subgroup of G. If G/M is nilpotent, G is soluble but need not be nilpotent. However, if G/M' is nilpotent, where M' is the derived group of M, then as we shall show G is also nilpotent. These results will be used to derive sufficient conditions for the join $\{H, K\}$ of two subgroups of a group to be nilpotent.

2. If we replace G in (1) by its regular representation $\bar{G}$, the condition (2) on the elements α of A becomes

$$(4) \qquad [\bar{G}_{i-1}, A] \leqq \bar{G}_i \qquad (i = 1, 2, \cdots, m),$$

where $\bar{G}_i$ is the subgroup of $\bar{G}$ corresponding to G_i. The notation here is the standard one: if H and K are subgroups, then $[H, K]$ is the subgroup generated by all the commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in H$ and $y \in K$. Since $[y, x] = [x, y]^{-1}$, we have $[K, H] = [H, K]$. As usual we also write x^y for $y^{-1}xy$.

However, when repeated commutations are needed, this notation is inconvenient. Instead, we shall use a bracketless notation and write

$$(5) \qquad [H, K] = \gamma HK,$$

the symbol γ standing for the operation of commutation. For example,

$$[[H, K], L] = \gamma^2 HKL; \qquad [\cdots[[H, \underbrace{K], K], \cdots, K}_{n}] = \gamma^n HK^n;$$

the lower central series of a group G is

$$(6) \qquad G, \quad \gamma G^2 = G', \quad \gamma^2 G^3, \quad \cdots, \quad \gamma^{n-1}G^n, \quad \cdots;$$

and so on. G is nilpotent of class less than n if and only if $\gamma^{n-1}G^n = 1$. Since the operation of commutation is commutative, though not associative, we have $\gamma XY = \gamma YX$ and $\gamma^2 HKL = \gamma^2 KHL = \gamma L\gamma HK = \gamma L\gamma KH$ for any three subgroups H, K, and L. On the other hand the three subgroups $\gamma^2 HKL$, $\gamma^2 KLH$, and $\gamma^2 LHK$ are usually distinct. It will be understood, of course, that $\gamma^2 G^3$, for example, is an abbreviation for $\gamma\gamma GGG$. In spite of the absence of brackets, this symbolism is unambiguous.

The relations (4) now take the form $\gamma\bar{G}_{i-1}A \leqq \bar{G}_i$; and since $\bar{G}_0 = \bar{G}$, $\bar{G}_m = 1$, they imply that $\gamma^m \bar{G}A^m = 1$. Theorem 1 therefore follows from

THEOREM 2. *Let H and K be subgroups of a group, and suppose that $\gamma^m HK^m = 1$. Then $\gamma^{n+1}K^{n+1}H = 1$, where $n = \frac{1}{2}m(m-1)$.*

For if we take $H = \bar{G}$ and $K = A$ in this theorem, we obtain $\gamma B\bar{G} = 1$, where $B = \gamma^n A^{n+1}$. But this means that every element $\beta \in B$ commutes with every element of $\bar{G}$ and therefore also with every element of G. Hence B consists only of the identical automorphism of G, and the stability group A is nilpotent of class at most n. However, Theorem 2 is more general than Theorem 1 only in appearance.

3. Before proving Theorem 2, we must recall a few well known facts about commutators.

Let x, y, and z be elements of a group. Then we have the identities

$$[xy, z] = [x, z]^y[y, z]; \tag{7}$$

$$[x, yz] = [x, z][x, y]^z; \tag{8}$$

and

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1. \tag{9}$$

In (9), we have used the convention $[u, v, w] = [[u, v], w]$. Of these formulae, (7) and (8) are immediate. To obtain (9), let $a = xzx^{-1}yx$, and let b and c be derived from a by cyclic permutation of x, y, and z. Then

$$[x, y^{-1}, z]^y = a^{-1}b,$$

so that (9) becomes $a^{-1}bb^{-1}cc^{-1}a = 1$.

Let H and K be subgroups of a group G, and let $M = \gamma HK$. If, in (7), we take x and y in H, and z in K, we obtain $M^y \leqq M$. Hence H normalizes M. Since $M = \gamma KH$, K also normalizes M. Therefore

$$\gamma HK \lhd \{H, K\}. \tag{10}$$

Here, following Wielandt [4], we use $M \lhd J$ to mean that M is a normal subgroup of the group J.

Now let L be a third subgroup of G. In (9), choose $x \in H$, $y \in K$, and $z \in L$; and write

$$U = \gamma^2 KLH, \qquad V = \gamma^2 LHK, \qquad W = \gamma^2 HKL. \tag{11}$$

The three factors on the left of (9) then belong, respectively, to W^y, U^z, and V^x. Let N be a normal subgroup of G containing both U and V. Then $U^z \leqq N$, $V^x \leqq N$; and so, since N is normal, (9) gives $[x, y^{-1}, z] \in N$. As x runs through H, and y through K, the commutators $[x, y^{-1}]$ generate γHK. Consequently, every element z of L commutes modulo N with every element of γHK. In other words, $W = \gamma^2 HKL$ is also contained in N. This gives

LEMMA 1. *Let H, K, and L be subgroups of a group G. Then any normal subgroup of G which contains two of the three subgroups* (11) *contains also the third.*

COROLLARY. *If the subgroups* (11) *are themselves normal in G, then each of them is contained in the product of the other two.*

This result was first proved in [3] for the special case in which H, K, and L are all normal in G, and this case is sufficient for many applications. However, we shall need the general case, as stated in Lemma 1. This is due to Kaloujnine [2]; cf. the essentially equivalent Fundamentalhilfssatz of [1], p. 165.

LEMMA 2. *Suppose that $L \leqq J = \{H, K\}$ and that $\gamma^2 HKL = 1$. Then $\gamma^2 LHK = \gamma^2 LKH$, and this group is normal in J.*

For, let C be the centralizer of γHK in J. Then $C \lhd J$, by (10); and $L \leqq C$, by hypothesis. Hence $\gamma LH \leqq C$. Let $x \in H$, $y \in K$, and $t \in \gamma LH$. Then $t \in C$ and $[x, y^{-1}] \in \gamma HK$; and so t commutes with $[x, y^{-1}]$. But $y^x = [x, y^{-1}]y$, and (8) gives $[t, y^x] = [t, y]$. Hence $\gamma^2 LHK^x = \gamma^2 LHK$. Since $x \in H$, we also have $\gamma LH = (\gamma LH)^x$, by (10). Therefore $(\gamma^2 LHK)^x = \gamma(\gamma LH)^x K^x = \gamma^2 LHK^x = \gamma^2 LHK$. Thus H normalizes $\gamma^2 LHK$. By (10), K also normalizes $\gamma^2 LHK$. Hence $\gamma^2 LHK$ is normal in J. Since $\gamma^2 KHL = \gamma^2 HKL = 1$, we find similarly by interchanging H and K that $\gamma^2 LKH$ is normal in J. By noting that $\gamma^2 LKH = \gamma^2 KLH$, Lemma 2 now follows from the corollary to Lemma 1.

We now deduce Theorem 2 by induction on m. When $m = 1$, $n = 0$, and the result is immediate. Let $m > 1$, and write $H_1 = \gamma HK$. Then $\gamma^{m-1} H_1 K^{m-1} = \gamma^m HK^m = 1$, by hypothesis. Therefore we may suppose inductively that $\gamma^{l+1} K^{l+1} H_1 = 1$, where $l = n - m + 1$ and $n = \frac{1}{2}m(m-1)$. Hence $K_r = \gamma^{r-1} K^r$ centralizes H_1 for all $r > l$. Lemma 2 now gives $\gamma^2 K_r HK = \gamma^2 K_r KH = \gamma K_{r+1} H$ for $r > l$; and so

$$\gamma^m K_{l+1} HK^{m-1} = \gamma^{m-1} K_{l+2} HK^{m-2} = \cdots = \gamma^2 K_n HK = \gamma K_{n+1} H.$$

Since $K_{l+1} \leqq K$, we have $\gamma K_{l+1} H \leqq \gamma KH = \gamma HK$; and so

$$\gamma K_{n+1} H = \gamma^m K_{l+1} HK^{m-1} \leqq \gamma^m HK^m.$$

But $\gamma^m HK^m = 1$, and $\gamma K_{n+1} H = \gamma^{n+1} K^{n+1} H$, so Theorem 2 follows.

4. We consider next, though without solving it, the problem of the least upper bound $c(m)$ for the class of the stability group A in terms of the length m of the chain (1).

Let H and K be subgroups of a group G, and let

$$H = H_0 \geqq H_1 \geqq H_2 \geqq \cdots \tag{12}$$

be a series of normal subgroups of H such that

$$\gamma KH_i \leqq H_{i+1} \qquad (i = 0, 1, 2, \cdots). \tag{13}$$

Then we have

$$\gamma^j K^j H_i \leqq H_{i+j} \tag{14}$$

for all $i \geqq 0$ and $j \geqq 1$. For, by (12) and (13), K normalizes each H_i; and since $H_i \lhd H$, by hypothesis, we have $H_i \lhd \{H, K\}$. For $j = 1$, (14) reduces to (13). Assume (14) for a given $j \geqq 1$ and all i, and apply Lemma 1 with H_i, K, and $K_j = \gamma^{j-1} K^j$ for H, K, and L. By (13) and (14), $\gamma^2 H_i KK_j \leqq \gamma H_{i+1} K_j = \gamma K_j H_{i+1} \leqq H_{i+j+1}$; and $\gamma^2 K_j H_i K \leqq \gamma H_{i+j} K \leqq H_{i+j+1}$. Since H_{i+j+1} is normal in $\{H, K\}$, Lemma 1 gives $\gamma^2 KK_j H_i \leqq H_{i+j+1}$, or $\gamma^{j+1} K^{j+1} H_i \leqq H_{i+j+1}$. Thus (14) holds generally by induction on j.

In particular, if $H_m = 1$, we obtain $\gamma^m K^m H = 1$. This gives the result already mentioned which is stated in

LEMMA 3. *The stability group of a chain of normal subgroups of length m is nilpotent of class at most $m - 1$.*

It is also well known that this bound $m - 1$ is best possible. For example, let G be an elementary Abelian p-group of order p^m with a basis $a_1, a_2, \cdots, a_m$, and let $G_i = \{a_{i+1}, a_{i+2}, \cdots, a_m\}$. The stability group A of this chain (G_i) contains the elements τ_i $(i = 1, 2, \cdots, m - 1)$, where

$$a_i^{\tau_i} = a_i a_{i+1}, \qquad a_j^{\tau_i} = a_j \quad (j \neq i);$$

and $[\tau_1, \tau_2, \cdots, \tau_{m-1}]$ maps a_1 into $a_1 a_m$; so that the class of A is at least $m - 1$, and therefore equal to $m - 1$ by Lemma 3. In fact,

$$A = \{\tau_1, \tau_2, \cdots, \tau_{m-1}\}$$

and is a Sylow p-subgroup of the group of automorphisms of G. Also it is easy to see that $A \cong T_m(p)$, the group of all unitriangular $m \times m$ matrices with coefficients in the prime field of p elements.

This example, in conjunction with Theorem 2, yields

$$m - 1 \leqq c(m) \leqq \tfrac{1}{2}m(m - 1), \tag{15}$$

so that $c(1) = 0$ and $c(2) = 1$. We shall now show by another example that $c(3) = 3$, so that there is a genuine difference between the normal case and the general one.

First, it will be convenient to have a suitable system of generators for a commutator group.

LEMMA 4. *Let $H = \{X\}$ and $K = \{Y\}$ be subgroups of a group J; and let T be the set of all elements of J of the form $[x, y]^{vu}$, with $x \in X$, $y \in Y$, $u \in H$, and $v \in K$. Then $\gamma HK = \{T\}$.*

Since $[x, y] \in \gamma HK$, which is normal in $\{H, K\}$ by (10), we have $\{T\} \leqq \gamma HK$. Thus we need only show that every commutator $[a, b]$ with $a \in H$ and $b \in K$ is expressible in terms of elements of T. This may be done by using a "collecting process" represented by the formula

$$a_1 b_1 a_2 b_2 \cdots a_n b_n = a_1 a_2 \cdots a_n b_1' b_2' \cdots b_n', \tag{16}$$

where $b_i' = b_i^{a_{i+1}a_{i+2}\cdots a_n}$. Here, the a_i and b_j are arbitrary elements of any group. Since $H = \{X\}$, every element a of H is expressible in the form $a = x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}$, where each $x_i \in X$ and the r_i are integers. Then

$$b^{-1}ab = \textstyle\prod_{i=1}^{m}(x_i [x_i, b])^{r_i}.$$

Applying (16), we obtain

$$[a, b] = t_1 t_2 \cdots t_r, \tag{17}$$

where $r = |r_1| + |r_2| + \cdots + |r_m|$, and each t_i is either the transform by an element of H of some $[x_i, b]$, or else the inverse of such a transform. If $b \in K$, we have $b = y_1^{s_1} y_2^{s_2} \cdots y_l^{s_l}$, where each $y_j \in Y$ and the s_j are integers. Then

$$x_i^{-1} b x_i = \prod_{j=1}^{l} (y_j [y_j, x_i])^{s_j}.$$

Applying (16) again, and noting that $[x_i, b] = [b, x_i]^{-1}$ and $[x_i, y_j] = [y_j, x_i]^{-1}$, we obtain $[x_i, b] = z_1 z_2 \cdots z_s$, where $s = |s_1| + |s_2| + \cdots + |s_l|$, and each z_j is either the transform by an element of K of some $[x_i, y_j]$, or else the inverse of such a transform. Substituting for the $[x_i, b]$ in (17), we find the required expression for $[a, b]$ in terms of elements of T.

5. We now prove

THEOREM 3. *There exists a nilpotent group G of class 2 with a subgroup-chain of length 3 whose stability group is of class 3.*

We define G to be the group generated by the elements

(18) $$x, \quad x_1, \quad x_2, \quad x_{12}, \quad x_{21}$$

subject only to the following defining relations:

(19) All commutators of weight 3 in the generators (18) are equal to 1;

(20) $$[x_{12}, x_{21}] = 1;$$

(21) $$[x_1, x_{12}] = [x_1, x_{21}] = c_1; \qquad [x_2, x_{12}] = [x_2, x_{21}] = c_2;$$

and

(22) $$c_1^2 = c_2^2 = 1.$$

The relations (19) by themselves would define the free nilpotent group F of class 2 with (18) as a system of free generators. F' is then a free Abelian group with the ten commutators $[x, x_1]$, $[x, x_2]$, $\cdots$ $[x_{12}, x_{21}]$ as a system of free generators. F' is also the centre of F. Consequently, in the group G obtained by imposing the additional relations (20), (21), and (22), the elements c_1 and c_2 are actually of order 2. In particular,

(23) $$c_1 \neq 1.$$

The defining relations of G are easily seen to be invariant under the two transformations which map (18) into

(24) $$xx_1, \quad x_1, \quad x_2 x_{21}, \quad x_{12}, \quad x_{21}$$

and

(25) $$xx_2, \quad x_1 x_{12}, \quad x_2, \quad x_{12}, \quad x_{21},$$

respectively. Hence these transformations define two endomorphisms η_1 and

η_2 of G. Clearly, $G^{\eta_1} = G^{\eta_2} = G$. In fact, η_1 and η_2 are automorphisms of G; for η_1 has an inverse which maps (18) into

$$xx_1^{-1}, \quad x_1, \quad x_2 x_{21}^{-1}, \quad x_{12}, \quad x_{21}, \tag{26}$$

and similarly for η_2.

Let $K = \{\eta_1, \eta_2\}$. We shall suppose G identified in the natural way with its regular representation, so that G and K can be considered as subgroups of the holomorph of G. Let $J = \{G, K\}$, so that $\gamma GK \lhd J$ by (10); and γGK contains the group

$$G_1 = \{G', x_1, x_2, x_{12}, x_{21}\}, \tag{27}$$

since $[x, \eta_i] = x_i$, $x_{12} = [x_1, \eta_2]$ and $x_{21} = [x_2, \eta_1]$. But $G_1 \lhd J$, and J/G_1 is the direct product of $G_1 K/G_1$ and G/G_1. Hence

$$G_1 = \gamma GK. \tag{28}$$

To calculate $G_2 = \gamma G_1 K = \gamma^2 GK^2$, we use Lemma 4. By (27), G_1 is generated by the set X consisting of x_1, x_2, x_{12}, x_{21} together with the commutators of these four elements with x; while K is generated by η_1 and η_2. Let ξ be one of x_1, x_2, x_{12}, x_{21}. Since G is of class 2, we have $[x, \xi]^{\eta_i} = [xx_i, \xi^{\eta_i}] = [x, \xi^{\eta_i}][x_i, \xi^{\eta_i}]$. If $\xi \neq x_2$, then $\xi^{\eta_1} = \xi$, and so $[x, \xi, \eta_1] = [x_1, \xi]$. But

$$[x, x_2, \eta_1] = [x, x_{21}][x_1, x_2 x_{21}] = [x, x_{21}][x_1, x_2]c_1.$$

Similarly, if $\xi \neq x_1$, then $\xi^{\eta_2} = \xi$ and $[x, \xi, \eta_2] = [x_2, \xi]$; while

$$[x, x_1, \eta_2] = [x, x_{12}][x_2, x_1]c_2.$$

It now follows from Lemma 4 that G_2 is generated by the transforms of the six elements

$$x_{12}, \quad x_{21}, \quad c_1, \quad c_2, \quad t_1, \quad t_2 \tag{29}$$

by the elements of the product KG_1, where

$$t_1 = [x, x_{21}][x_1, x_2], \qquad t_2 = [x, x_{12}][x_2, x_1].$$

But the defining relations of G show that the elements (29) all commute with η_1 and η_2 and hence with every element of K. Since the c's and t's belong to the centre of G and $x_{12}^{x_i} = x_{12} c_i$, $x_{21}^{x_i} = x_{21} c_i$, it follows that G_2 is generated by (29) and

$$\gamma G_2 K = \gamma^3 GK^3 = 1.$$

Hence K is contained in the stability group of the chain of subgroups (G_i) of G, where $G_0 = G$ and $G_3 = 1$.

Using (24), (25), and (26) to transform x successively by $\eta_1^{-1}, \eta_2^{-1}, \eta_1, \eta_2$, we obtain the sequence

$$x; \quad xx_1^{-1}; \quad xx_2^{-1}x_{12}x_1^{-1}; \quad xx_1 x_{21}^{-1} x_2^{-1} x_{12} x_1^{-1}; \quad xx_2 x_1 x_{12} x_{21}^{-1} x_2^{-1} x_1^{-1}.$$

Using (19)–(22), this gives

(30) $$[x,[\eta_1, \eta_2]] = x_{12}\, x_{21}^{-1}[x_2, x_1].$$

Similarly,

(31) $$[x,[\eta_2, \eta_1]] = x_{21}\, x_{12}^{-1}[x_1, x_2].$$

Since $G_2 \lhd G_1$, Lemma 3 applied to the chain G_1, G_2, 1 shows that $[\eta_1, \eta_2]$ commutes with every element of G_1. Using (24), (26), (30), and (31) to transform x successively by $[\eta_2, \eta_1]$, η_1^{-1}, $[\eta_1, \eta_2]$, and η_1, we find the sequence

$$x; \quad xx_{21}\, x_{12}^{-1}[x_1, x_2]; \quad xx_1^{-1}x_{21}\, x_{12}^{-1}[x_1, x_2][x_1, x_{21}^{-1}];$$

$$xx_{12}\, x_{21}^{-1}[x_2, x_1]x_1^{-1}x_{21}\, x_{12}^{-1}[x_1, x_2]c_1 = xx_1^{-1}c_1; \quad xc_1.$$

In this calculation, we have used the defining relations of G, from which it follows that $[x_1, x_{21}^{-1}] = c_1$ and that x_1^{-1} commutes with $x_{12}\, x_{21}^{-1}$. The final term xc_1 shows that $[x,[\eta_1, \eta_2, \eta_1]] = c_1$; and so, by (23), K is of class at least 3. By Theorem 1, the class of K must be exactly 3. Thus Theorem 3 is proved.

The group G used in this example is infinite. If we impose additional defining relations to make G, for example, a group of exponent 4, the above calculations are unaffected, but G becomes a finite 2-group of order 2^{22}. No doubt smaller groups could also be found with the relevant property.

6. We return now to the case of general m as in Theorem 2. Let H and K be subgroups of any group G, and let $J = \{H, K\}$. We write

(32) $$H_r = \gamma^r HK^r \qquad (r = 1, 2, 3, \cdots),$$

and

(33) $$K_r = KH_r.$$

Since K normalizes H_r, the product K_r is a subgroup of J. In particular, K_1 is the normal closure of K in J; and

(34) $$\bar{H} = HH_1$$

is the normal closure of H in J. It is clear that

(35) $$H_1 = \gamma HK = \gamma \bar{H}K.$$

In considering the consequences of the relation $H_m = 1$, there would therefore be no loss of generality in replacing H by $\bar{H}$; this would be equivalent to assuming H normal in J, or $H \geqq H_1$.

According to Theorem 2, the relation $H_m = 1$ implies $\gamma^{n+1}K^{n+1}H = 1$ with $n = \frac{1}{2}m(m-1)$. But (35) shows that $H_m = 1$ is equivalent to $\gamma^m \bar{H}K^m = 1$; therefore it implies $\gamma^{n+1}K^{n+1}\bar{H} = 1$, which represents a limitation on the class of the group of automorphisms induced by K in $\bar{H}$. In this way, we recover Theorem 1 from Theorem 2.

We shall now show that $H_m = 1$ implies that the group H_1 is locally nilpotent. More precisely, the result is

THEOREM 4. *Let H and K be subgroups of a group G such that $\gamma^m HK^m = 1$. Let $H_1 = \gamma HK$, and let $\bar{H} = HH_1$ and $K_1 = KH_1$ be the normal closures of H and K, respectively, in $J = \{H, K\}$. Let C be the centralizer of $\bar{H}$ in K_1. Then the groups K_1/C and H_1 are locally nilpotent.*

It should be noted that $C \lhd J$, since both $\bar{H}$ and K_1 are normal in J. It will be sufficient to consider the group K_1/C. For if this is locally nilpotent, so is its subgroup $CH_1/C \cong H_1/C \cap H_1$. Since $C \cap H_1$ is contained in the centre of H_1, it then follows that H_1 itself is locally nilpotent.

To prove Theorem 4, we need an important result due to Hirsch [5]. This is stated in

THEOREM 5. *In any group G, the join $\lambda(G)$ of all normal locally nilpotent subgroups of G is itself locally nilpotent.*

Obviously, $\lambda(G)$ is a characteristic subgroup of G.

COROLLARY. *If K is a subnormal subgroup of G, then $\lambda(K) = \lambda(G) \cap K$.*

For, to say that K is subnormal in G (*nachinvariant* in the sense of Wielandt [4]) means that there is a finite chain of subgroups

$$K = K_m \lhd K_{m-1} \lhd \cdots \lhd K_1 \lhd K_0 = G$$

stretching from K to G, each member of the chain being normal in the next. Since $L_r = \lambda(K_r)$ is characteristic in K_r, it is normal in K_{r-1}. Since L_r is locally nilpotent, by Hirsch's theorem, it follows that $L_r \leqq L_{r-1}$. Hence $\lambda(K) = L_m \leqq M = \lambda(G) \cap K = L_0 \cap K$. But $\lambda(G) \lhd G$ and so $M \lhd K$. As a subgroup of $\lambda(G)$, M is also locally nilpotent. Hence $M \leqq \lambda(K)$. Combining, we find $M = \lambda(K)$ as required.

In the proof of Theorem 4, we shall use the notations (32) and (33), so that $H_m = 1$ and $K_m = K$. By (10), K normalizes H_r, and so $H_{r+1} \leqq H_r$; and then $H_{r+1} \lhd H_r$, again by (10). Hence

$$1 = H_m \lhd H_{m-1} \lhd \cdots \lhd H_1 \lhd J. \tag{36}$$

We shall show that

$$K = K_m \lhd K_{m-1} \lhd \cdots \lhd K_1 \lhd J. \tag{37}$$

For (36) shows that $K_{r+1} \leqq K_r$. Let y and y_1 be elements of K, let $x \in H_r$ and $x_1 \in H_{r+1}$. Then $y_2 = y_1^y \in K$, and $x_2 = x_1^{yx} \in H_{r+1}$, since K and H_r both normalize H_{r+1}. Hence $(y_1 x_1)^{yx} = y_2^x x_2 = y_2 [y_2, x] x_2$. Since therefore $[y_2, x] \in \gamma KH_r = H_{r+1}$, we have $(y_1 x_1)^{yx} \in KH_{r+1} = K_{r+1}$. But yx and $y_1 x_1$ are arbitrary elements of K_r and K_{r+1}, respectively. Thus $K_{r+1} \lhd K_r$ for $r = 1, 2, \cdots, m - 1$, and (37) is proved.

As already noted, $H_m = 1$ implies $\gamma^{n+1} K^{n+1} \bar{H} = 1$, so that $\gamma^n K^{n+1}$ is contained in C. Hence $CK/C \cong K/C \cap K$ is nilpotent. It follows from (37) that CK/C is subnormal in J/C. By the corollary to Theorem 5, we now have $CK/C = \lambda(CK/C) \leqq \lambda(J/C) = L/C$, say. Then $K \leqq L \lhd J$, and consequently L contains the normal closure K_1 of K in J. Since L/C is locally nilpotent, it follows that K_1/C is locally nilpotent. This concludes the proof of Theorem 4.

7. It follows from Theorem 4 that, if $\gamma^m HK^m = 1$, then γHK is nilpotent provided that it is finitely generated. This will certainly be the case if $\{H, K\}$ is finite. We shall now show by examples that γHK need not be nilpotent in general; and that even when $\{H, K\}$ is finite, and so γHK is nilpotent, there is no bound for the class of γHK in terms of m provided that $m > 2$. This is stated in

THEOREM 6. *There exists a group $\{H, K\}$ such that $\gamma^3 HK^3 = 1$ and γHK is not nilpotent. Given any integer n, there exists a finite group $\{H, K\}$ such that $\gamma^3 HK^3 = 1$ and γHK is nilpotent of class at least n. On the other hand, $\gamma^2 HK^2 = 1$ implies that γHK is Abelian.*

The last remark was already noted by Kaloujnine in [1]. If $\gamma^2 HK^2 = 1$, then K centralizes $H_1 = \gamma HK$. Since $H_1 \lhd J = \{H, K\}$, it follows that the normal closure K_1 of K in J also centralizes H_1. But $K_1 = KH_1$, so that in this case H_1 is contained in the centre of K_1.

Let V be a vector space over the prime field of p elements, and let (v_n), $n = 0, \pm 1, \pm 2, \cdots$, be a basis of V. We take H to be $\{\xi\}$, where ξ is the linear transformation of V defined by

$$v_n \xi = v_{n+1} \tag{38}$$

for all n. We take K to be $\{\eta\}$, where η is defined by

$$v_0 \eta = v_0 + v_1 ; \qquad v_n \eta = v_n \quad \text{if} \quad n \neq 0. \tag{39}$$

The normal closure K_1 of K in $J = \{\xi, \eta\}$ is then the group generated by the conjugates $\eta_n = \xi^{-n}\eta\xi^n$ of η $(n = 0, \pm 1, \pm 2, \cdots)$. Hence $v_k \eta_k = v_k + v_{k+1}$ and $v_j \eta_k = v_j$ if $j \neq k$. If V_k is the subspace of V spanned by the vectors $v_k, v_{k+1}, v_{k+2}, \cdots$, then K_1 leaves each V_k invariant.

Let A be the group of all elements of K_1 which transform identically both V_1 and V/V_1. Then $K \leqq A$ by (39), and $A \lhd K_1$. Obviously A is Abelian. But $H_1 = \gamma HK \leqq K_1$, and so $H_2 = \gamma H_1 K \leqq \gamma K_1 K \leqq A$. Since A is Abelian and contains K, we have $\gamma^3 HK^3 = \gamma H_2 K = 1$. Now H_1 contains the elements $[\eta, \xi^n] = \eta^{-1}\eta_n = \zeta_n$, say. Since

$$v_1 [\zeta_1, \zeta_2, \cdots, \zeta_n] = v_1 + v_{n+1},$$

and n is arbitrary, H_1 cannot be nilpotent. This proves the first statement of Theorem 6.

To prove the second part of the theorem, we must proceed rather differently.

Let L_n be the group generated by n elements $x_1, x_2, \cdots, x_n$ subject only to the defining relations which express that

(i) $x^p = 1$ for all $x \in L_n$, where p is a given prime $> n$; and

(ii) x_i commutes with all its conjugates in L_n for each $i = 1, 2, \cdots, n$.

We show first that L_n is nilpotent of class n. We recall a well known theorem of Fitting [6]:

LEMMA 5. *Let X_1 and X_2 be normal subgroups of a group G. If X_1 is nilpotent of class c_1 and X_2 is nilpotent of class c_2, then $X_1 X_2$ is nilpotent of class at most $c_1 + c_2$.*

Now let X_i be the subgroup generated by the conjugates of x_i in L_n. Then $X_i \lhd L_n$ for each i, and $L_n = X_1 X_2 \cdots X_n$. By (ii), the X_i are all Abelian. Hence L_n is nilpotent of class at most n, by Lemma 5. To show that the class of L_n is actually equal to n, we compare L_n with the group $Y_n = \{\eta_1, \eta_2, \cdots, \eta_n\}$ of linear transformations of V, where $\eta_i = \xi^{-i}\eta\xi^i$ and ξ, η are defined by (38) and (39). It is easy to verify that each η_i commutes with all its conjugates in Y_n and so, as for L_n, Y_n is nilpotent of class at most n. Also $\eta_i^p = 1$ for each i. Since $n < p$, by hypothesis, it follows from the theory of regular p-groups developed in [3] (the relevant theorems are 4.13, p. 73 and 4.26, p. 76) that $y^p = 1$ for all $y \in Y_n$. Hence the mapping $x_i \rightarrow \eta_i$ $(i = 1, 2, \cdots, n)$ defines a homomorphism of L_n onto Y_n. But $[\eta_1, \eta_2, \cdots, \eta_n]$ maps v_1 onto $v_1 + v_{n+1}$ and so Y_n is of class at least n. Consequently, the class of L_n cannot be less than n.

Now L_n has an automorphism permuting the generators $x_1, x_2, \cdots, x_n$ cyclically. Hence L_n may be embedded in a group $G = \{L_n, t\}$ such that $x_i^t = x_{i+1}$ for $i < n$ and $x_n^t = x_1$. Define $H = \{t\}$ and $K = \{x_1\}$. Then $H_1 = \gamma HK \leqq L_n$, since $K \leqq L_n \lhd G$. And $H_2 = \gamma H_1 K \leqq X_1$, since $K \leqq X_1 \lhd L_n$. So $\gamma^3 HK^3 = \gamma H_2 K = 1$, since X_1 is Abelian. But H_1 contains the elements $[x_1, t^{i-1}] = x_1^{-1}x_i$ $(i = 2, 3, \cdots, n)$. Hence $X_1 H_1 = L_n$. But $L_n / X_1 \cong L_{n-1}$, which is a group of class $n - 1$. Since $H_1 / X_1 \cap H_1 \cong L_n / X_1$, it follows that H_1 is of class at least $n - 1$. This concludes the proof of Theorem 6.

For the sake of completeness we note the very simple result which is related to Theorem 4 in much the same way as Lemma 3 is related to Theorem 1. This is

LEMMA 6. *Let K and L be subgroups of any group, and suppose that there exists a chain of subgroups $L = L_0 \geqq L_1 \geqq \cdots \geqq L_m = 1$, all normal in L and such that $\gamma L_{i-1} K \leqq L_i$ for each $i = 1, 2, \cdots, m$. Let $M = \gamma LK$, and let $\bar{K} = KM$ be the normal closure of K in $J = \{K, L\}$. Let C be the centralizer of L in $\bar{K}$. Then the groups $\bar{K}/C$ and M are nilpotent of class at most $m - 1$.*

For in this case the groups L_i are all normal in J; and therefore $\gamma L_{i-1} K \leqq L_i$ implies $\gamma L_{i-1} \bar{K} \leqq L_i$ for $1 \leqq i \leqq m$. It now follows from Lemma 3 that $\gamma^j \bar{K}^j L_i \leqq L_{i+j}$ for $i + j \leqq m$. Hence $\gamma^{m-1} \bar{K}^m$ centralizes L, so that $\bar{K}/C$ is nilpotent of class less than m. Also $\gamma^{m-2}\bar{K}^{m-1}$ centralizes L_1. But $M \leqq \bar{K} \cap L_1$. Hence $\gamma^{m-2}M^{m-1}$ is contained in the centre of M, so that M also is nilpotent of class less than m. Cf. Satz 4 of Kaloujnine [1].

8. We conclude with a criterion for nilpotency of a rather different kind from those considered above. This is based on

LEMMA 7. *Let K and L be subgroups of any group, and suppose that K normalizes L and that $\gamma^{m}LK^{m} \leqq L' = \gamma L^{2}$, the derived group of L. Then $\gamma^{rm}L^{r}K^{rm-r+1} \leqq \gamma^{r}L^{r+1}$ for $r = 1, 2, 3, \cdots$.*

The case $m = 1$ states that, if K centralizes the first factor group L/L' of the lower central series of L, then K centralizes all the factors $\gamma^{r-1}L^{r}/\gamma^{r}L^{r+1}$ of that series; this is well known.

To prove the lemma, we form the series $L = L_0 \geqq L_1 \geqq \cdots \geqq L_m = L'$ where $L_i = L' \cdot (\gamma^{i}LK^{i})$. Then each L_i is normal in L, and $\gamma L_{i-1} K \leqq L_i$ for each $i = 1, 2, \cdots, m$. As in the proof of Lemma 6, it follows that $\gamma L_{i-1} \bar{K} \leqq L_i$ for each i, where $\bar{K}$ is the normal closure of K in $J = \{K, L\}$. Hence $\gamma^{m}L\bar{K}^{m} \leqq L'$. Therefore there will be no loss of generality if we assume K, as well as L, to be normal in J. Thus Lemma 7 is really a theorem about normal subgroups.

For any normal subgroup X of J, we write $X_0 = X$ and $X_n = \gamma^{n}XK^{n}$ for $n > 0$. Assuming $K \lhd J$, X_n is then also normal in J. The corollary to Lemma 1 then gives

$$(40) \qquad (\gamma XL)_1 \leqq (\gamma X_0 L_1)(\gamma X_1 L_0).$$

By induction on n we deduce that

$$(41) \qquad (\gamma XL)_n \leqq \prod_{i=0}^{n} (\gamma X_i L_{n-i}).$$

For let $P = P_1 P_2 \cdots P_n$, where each P_i is normal in J. Then by (7), we have $\gamma PK = Q_1 Q_2 \cdots Q_n$, where $Q_i = \gamma P_i K$. Taking $P_i = \gamma X_{i-1} L_{n-i}$, we have $Q_i \leqq R_{i-1} R_i$ by Lemma 1, where $R_i = \gamma X_i L_{n-i}$. This gives the induction step from $n - 1$ to n in (41).

We prove Lemma 7 by induction on r, the case $r = 1$ being true by hypothesis. Assume that the result holds for a given $r \geqq 1$, and take $X = \gamma^{r-1}L^{r}$ and $n = (r + 1)m - r$ in (41). By the induction hypothesis, $X_{n-m+1} \leqq \gamma XL$. Of the factors on the right of (41), those for which $i > n - m$ have $X_i \leqq \gamma XL$; and since $L_{n-i} \leqq L$, each of these factors is contained in $\gamma^{2}XL^{2}$. In each of the remaining factors we have $n - i \geqq m$, so that $L_{n-i} \leqq L'$; and since $X_i \leqq X$, each of these factors is contained in $\gamma XL' = \gamma^{2}L^{2}X$. But $\gamma^{2}L^{2}X \leqq \gamma^{2}XL^{2}$ by Lemma 1. Hence $(\gamma XL)_n \leqq \gamma^{2}XL^{2}$, which is the result required for the next value of r. This concludes the proof of the lemma. An immediate corollary is

THEOREM 7. *Let L be a normal subgroup of a group K. If L is nilpotent of class c and K/L′ is nilpotent of class d, then K is nilpotent of class at most* $f(c) = \binom{c+1}{2}d - \binom{c}{2}$.

For $\gamma^d K^{d+1} \leqq L'$, and so $\gamma^d LK^d \leqq L'$. Using the notation $X_n = \gamma^n XK^n$, Lemma 7 gives $(\gamma^{r-1}L^r)_{rd-r+1} \leqq \gamma^r L^{r+1}$ for $r = 1, 2, 3, \cdots$. Since $L' = \gamma L^2$, we obtain $\gamma^{f(i)}K^{f(i)+1} \leqq \gamma^i L^{i+1}$ for $i = 1, 2, 3, \cdots$ by induction on i, owing to $f(i) - f(i-1) = id - i + 1$. But $\gamma^c L^{c+1} = 1$, and so $\gamma^{f(c)}K^{f(c)+1} = 1$, so that K is nilpotent of class at most $f(c)$.

COROLLARY 1. *If H is a normal subgroup of G such that* $H' = G'$, *then* $\gamma^{r-1}H^r = \gamma^{r-1}G^r$ *for all* $r > 1$. *If H and M are normal subgroups of G such that* γHM *and* H' *are both contained in* M', *then* $\gamma^{r-1}H^r$ *is contained in* $\gamma^{r-1}M^r$ *for all* $r > 1$.

For $N = \gamma^{r-1}H^r \triangleleft G$. Applying Theorem 7 with $d = 1$ to the groups $L = H/N$ and $K = G/N$ gives the first part of the corollary. If $J = HM$, then $J' = H'M' \cdot \gamma HM = M'$ and so $\gamma^{r-1}H^r \leqq \gamma^{r-1}J^r = \gamma^{r-1}M^r$ by the first part.

A discussion of the question whether the bound $f(c)$ in Theorem 7 is the best possible one for given c and d would probably be rather tedious, and we shall not attempt it here. Instead, we note the following criterion for the nilpotency of the join of two subgroups:

COROLLARY 2. *Let H and K be subgroups of any group, let* $J = \{H, K\}$ *and* $M = \gamma HK$. *If J/M and M are both nilpotent, and if there exist integers m and n such that* $\gamma^m MH^m \leqq M'$ *and* $\gamma^n MK^n \leqq M'$, *then J is also nilpotent.*

It is clear that these conditions are necessary if J is to be nilpotent. Note also that J/M is in any case a homomorphic image of the direct product of H and K, so that J/M will certainly be nilpotent if both H and K are nilpotent.

Let $\bar{H} = HM$ be the normal closure of H in J. Suppose first that $M' = 1$ so that M is Abelian. If u and v are in M and x in H, we then have $[u, vx] = [u, x]$; and so $\gamma^m M\bar{H}^m = \gamma^m MH^m = 1$. Hence M is contained in the m^{th} term of the upper central series of $\bar{H}$. Since $\bar{H}/M$ is a subgroup of J/M, it is nilpotent. Hence $\bar{H}$, and similarly $\bar{K} = KM$, are both nilpotent. Thus $J = \bar{H}\bar{K}$ is the product of two normal nilpotent subgroups. So J is nilpotent by Lemma 5. In the general case where $M' > 1$, we conclude that J/M' is nilpotent. By hypothesis, M is nilpotent; and since $M \triangleleft J$ by (10), it follows from Theorem 7 that J is nilpotent in this case also.

The criterion of Corollary 2 may be compared with the following criterion which follows easily from Hirsch's Theorem 5.

THEOREM 8. *Let H and K be subgroups of any group such that* $\gamma^m HK^m = \gamma^n KH^n = 1$ *for suitable integers m and n. If H and K are both locally nilpo-*

tent, then so is $J = \{H, K\}$. *If* H *and* K *are both finitely generated nilpotent groups, then so is* J.

The second statement follows immediately from the first. As we saw in proving Theorem 4, the equation $\gamma^m HK^m = 1$ implies that K is subnormal in J, equation (37). Consequently, if K is locally nilpotent, we have $K = \lambda(K) \leqq \lambda(J)$ by the corollary to Hirsch's theorem. Hence the normal closure $\bar{K}$ of K in J is contained in $\lambda(J)$. Similarly, if $\gamma^n KH^n = 1$, and if H is locally nilpotent, the normal closure $\bar{H}$ of H in J is also contained in $\lambda(J)$. Hence $J = \bar{H}\bar{K} = \lambda(J)$ is locally nilpotent.

In Theorem 8, no use has been made of our main result, Theorem 2. According to this theorem, if C is the centralizer of $\bar{H}$ in K, then $\gamma^m HK^m = 1$ implies that K/C is nilpotent. Consequently, K will be itself nilpotent, if we assume in addition that $\gamma^r CK^r \leqq \gamma HK$ for some r. Similarly, if D is the centralizer of H in $\bar{K}$, then $\gamma^n KH^n = 1$ and $\gamma^s DH^s \leqq \gamma HK$ together imply that H is nilpotent. Thus we may state

LEMMA 8. *Let* $\bar{H}$ *and* $\bar{K}$ *be the normal closures of the subgroups* H *and* K *in* $J = \{H, K\}$; *let* C *and* D *be the centralizers of* $\bar{H}$ *in* K *and of* $\bar{K}$ *in* H, *respectively; and let* $M = \gamma HK$. *Then, if* $\gamma^m HK^m = \gamma^n KH^n = 1$, *and if* $\gamma^r CK^r$ *and* $\gamma^s DK^s$ *are both contained in* M *for suitable integers* m, n, r, *and* s, *it follows that* J/M *is nilpotent.*

We could infer at once that J itself is nilpotent, by Corollary 2 to Lemma 7, provided we knew that M was nilpotent. By Theorem 4, the equation $\gamma^m HK^m = 1$ by itself already implies that M is locally nilpotent. It seems reasonable to think that the equations $\gamma^m HK^m = \gamma^n KH^n = 1$ taken together should enable us to show that M is in fact nilpotent. But we have not been able to prove this. The doubt disappears when J is finite. Hence we may state the

COROLLARY. *If* J *is finite, the hypotheses of Lemma* 8 *are sufficient to ensure that* J *is nilpotent.*

Indeed, in this case, the condition $\gamma^n KH^n = 1$ may be weakened to $\gamma^n KH^n \leqq M'$. For the similar condition $\gamma^m HK^m = 1$ already ensures the nilpotency of M; and that being so, $\gamma^n KH^n = \gamma^{n-1} MH^{n-1} \leqq M'$ implies $\gamma^p KH^p = 1$ for some p, by Lemma 7.

On the other hand, the conditions $\gamma^r CK^r \leqq M$ and $\gamma^s DH^s \leqq M$ of Lemma 8 cannot be omitted. For example, if G is the icosahedral group, and if we form $J^* = J \times G$, $K^* = K \times G$ and $H^* = H$, then $J^* = \{H^*, K^*\}$ fulfils all the hypotheses of Lemma 8 except the one involving $C^* = C \times G$, the centralizer of $\overline{H^*} = \bar{H}$ in K^*. And obviously $M^* = \gamma H^* K^* = M$, so that $J^*/M^* \cong G \times J/M$ and is not nilpotent.

REFERENCES

1. L. KALOUJNINE, *Über gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen*, Berliner Mathematische Tagung, 1953, pp. 164–172.
2. ———, *Sur quelques propriétés des groupes d'automorphismes d'un groupe abstrait*, C. R. Acad. Sci. Paris, vol. 230 (1950), pp. 2067–2069 and vol. 231 (1950), pp. 400–402.
3. P. HALL, *A contribution to the theory of groups of prime-power order*, Proc. London Math. Soc. (2), vol. 36 (1933), pp. 29–95.
4. H. WIELANDT, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Zeit., vol. 45 (1939), pp. 209–244.
5. K. A. HIRSCH, *Über lokal-nilpotente Gruppen*, Math. Zeit., vol. 63 (1955), pp. 290–294.
6. H. FITTING, *Beiträge zur Theorie der Gruppen von endlicher Ordnung*, Jber. Deutsch. Math. Verein., vol. 48 (1938), pp. 77–141.

KING'S COLLEGE
CAMBRIDGE, ENGLAND