Homological algebra modulo exact zero-divisors

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Abstract We study the homological behavior of modules over local rings modulo exact zero-divisors. We obtain new results which are in some sense "opposite" to those known for modules over local rings modulo regular elements.

1. Introduction

Given a local (meaning also commutative and Noetherian) ring S and an ideal I, one may ask whether the homological behavior of modules over S/I is related to that over S. In general this is hopeless; one needs to restrict the ideal I. When I is generated by a regular sequence, there is a well-developed and powerful theory relating the homological properties of modules over these two rings (see for instance [Av2], [AvBu], [Be], [Dao1], [Eis], [Gu1], [Jo1], and [Jo2]). For example, suppose that the ideal I is generated by a single regular element $x \in S$, denote the factor ring $S/(x)$ by R, and let M and N be R-modules. Then a primary result is that

$$\operatorname{Tor}_1^R(M,N) = \operatorname{Tor}_2^R(M,N) = \cdots = \operatorname{Tor}_n^R(M,N) = 0,$$

for some $n \geq 2$, implies that

$$\operatorname{Tor}_2^S(M,N) = \cdots = \operatorname{Tor}_n^S(M,N) = 0.$$

In other words, the vanishing of homology over R implies the vanishing of homology over S. An analogous statement for cohomology also holds.

Another primary result compares the complexity of a finitely generated R-module with its complexity as an S-module (see Section 3 for the definition of complexity). Namely, there are inequalities (see [Av2, Remark 3.2(3)])

$$\operatorname{cx}_S(M) \leq \operatorname{cx}_R(M) \leq \operatorname{cx}_S(M) + 1.$$

In this paper we study the case where the element $x \in S$ is in some sense the "next best thing" to being a regular element. More precisely, we consider the case where the annihilator of x is a nonzero principal ideal whose annihilator is also principal (and therefore is the ideal (x)). In accord with [HeŞ], the element x is said to be an *exact zero-divisor* if it is nonzero, it belongs to the maximal

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ideal of S, and there exists another element $y \in S$ such that $\operatorname{ann}_S(x) = (y)$ and $\operatorname{ann}_S(y) = (x)$. In this case we say that (x, y) is a *pair of exact zero-divisors* of S. The ideal (x) is then an example of a *quasicomplete intersection ideal*, a notion introduced in [AHŞ]. In that same paper, results relating certain invariants of modules over S with those over $S/(x)$ are proved. We continue along similar, but more homological lines, and show that even if the element x is the next best thing to being regular, namely, an exact zero-divisor, then the homological relationships between $S/(x)$-modules over $S/(x)$ and over S change dramatically compared to the case where x is regular. Two of our main results in Section 2 concern the vanishing of (co)homology. In particular, the result for homology takes the following form.

THEOREM

Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\operatorname{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq n$, then $\operatorname{Tor}_i^S(M, N) \cong M \otimes_S N$ for $1 \leq i \leq n - 1$.

Compared with the vanishing result for the case where x is regular, the conclusion of the previous theorem is opposite in the sense that it is a nonvanishing result: the vanishing of homology over R implies the *nonvanishing* of homology over S (when the modules involved are nonzero and finitely generated). For cohomology, we obtain the following analogue.

THEOREM

Let $R = S/(x)$ where S is a local ring and (x, y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules such that $yN = 0$. If there exists an integer $n \geq 2$ such that $\operatorname{Ext}_R^i(M, N) = 0$ for $1 \leq i \leq n$, then $\operatorname{Ext}_S^i(M, N) \cong \operatorname{Hom}_S(M, N)$ for $1 \leq i \leq n - 1$.

We also compare the complexities of finitely generated modules over R and over S. Similar to our previous results, we show that such a comparison is quite different from the case where x is regular. The following theorem is the main result of Section 3.

THEOREM

Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S. If M is a finitely generated R-module, then for any n there are inequalities

$$\beta_n^R(M) - \sum_{i=0}^{n-2} \beta_i^R(M) \leq \beta_n^S(M) \leq \sum_{i=0}^{n} \beta_i^R(M)$$

of Betti numbers. In particular, the inequality $\operatorname{cx}_S(M) \leq \operatorname{cx}_R(M) + 1$ holds.

In the final section, Section 4, we discuss canonical endomorphisms of complexes of finitely generated free R-modules, and canonical elements of $\operatorname{Ext}^2_R(M,M)$, for finitely generated R-modules M, in the case where $R = S/(x)$ and (x,x) is a pair of exact zero-divisors of S. The main result, Theorem 4.2, equates the ability to lift a finitely generated R-module M from R to S to the triviality of the canonical element in $\operatorname{Ext}^2_R(M,M)$. This generalizes classical results (see, e.g., [ADS]) on lifting modules from $T/(x)$ to $T/(x^2)$ in the case where x is a nonzero-divisor of the local ring T (cf. Example 4.3 below).

2. Vanishing results

In this section we prove our vanishing results, starting with the homology version. We fix a local ring S and a pair of exact zero-divisors (x,y), and denote the local ring $S/(x)$ by R. It should be mentioned that the modules we consider in this section are not necessarily assumed to be finitely generated.

Since a (deleted) free resolution of R over S has the form

$$\cdots \to S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \to 0$$

one has for any R-module N the following:

$$(\dagger) \qquad \operatorname{Tor}^S_q(R,N) \cong \begin{cases} N & \text{for } q = 0, \\ N/yN & \text{for } q > 0 \text{ odd}, \\ \operatorname{ann}_N(y) & \text{for } q > 0 \text{ even}, \end{cases}$$

and

$$(\ddagger) \qquad \operatorname{Ext}_S^q(R,N) \cong \begin{cases} N & \text{for } q = 0, \\ \operatorname{ann}_N(y) & \text{for } q > 0 \text{ odd}, \\ N/yN & \text{for } q > 0 \text{ even}. \end{cases}$$

Our main theorem on the vanishing of homology is the following.

THEOREM 2.1

Let $R = S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules. If there exists an integer $n \geq 2$ such that $\operatorname{Tor}^R_i(M,N) = \operatorname{Tor}^R_i(M,N/yN) = \operatorname{Tor}^R_i(M,\operatorname{ann}_N(y)) = 0$ for $1 \leq i \leq n$, then

$$\operatorname{Tor}^S_i(M,N) \cong \begin{cases} M \otimes_S N & \textit{for } i = 0, \\ M \otimes_S N/yN & \textit{for } 0 < i < n \textit{ and } i \textit{ odd}, \\ M \otimes_S \operatorname{ann}_N(y) & \textit{for } 0 < i < n \textit{ and } i \textit{ even}. \end{cases}$$

Proof

Consider the first quadrant change of rings spectral sequence (see [Rot, Theorem 10.73])

$$\operatorname{Tor}^R_p\big(M, \operatorname{Tor}^S_q(R,N)\big) \underset{p}{\Longrightarrow} \operatorname{Tor}^S_{p+q}(M,N).$$

From (†), the term $E^2_{p,q}$ is given by

$$E^2_{p,q} \cong \begin{cases} \operatorname{Tor}^R_p(M,N) & \text{for } q=0, \\ \operatorname{Tor}^R_p(M,N/yN) & \text{for } q>0 \text{ odd}, \\ \operatorname{Tor}^R_p(M,\operatorname{ann}_N(y)) & \text{for } q>0 \text{ even}. \end{cases}$$

The vanishing assumptions imply that columns 1 through n of the E^2-page of this spectral sequence vanish, that is, $E^2_{p,q}=0$ for all $q \in \mathbb{Z}$ and $1 \le p \le n$. Fixing such p and q, we see that $E^\infty_{p,q}$ also vanishes since this term is a subquotient of $E^2_{p,q}$. Letting H_i denote $\operatorname{Tor}^S_i(M,N)$ for all i, we have a filtration $\{\Phi^j H_i\}$ of H_i satisfying

$$0=\Phi^{-1}H_i \subseteq \Phi^0 H_i \subseteq \cdots \subseteq \Phi^{i-1}H_i \subseteq \Phi^i H_i = H_i,$$

with $E^\infty_{j,i-j} \cong \Phi^j H_i/\Phi^{j-1}H_i$ for all i and j. Thus the vanishing of $E^\infty_{p,q}$ implies that $\Phi^p H_{p+q} = \Phi^{p-1}H_{p+q}$, that is, $\Phi^p H_q = \Phi^{p-1}H_q$ for all $q \in \mathbb{Z}$ and $1 \le p \le n$.

Now consider the zeroth column of the E^2-page. For a positive q, the $E^2_{0,q}$-term is isomorphic to $M \otimes_R N/yN$ when q is odd, and isomorphic to $M \otimes_R \operatorname{ann}_N(y)$ when q is even. Since $E^2_{p,q}=0$ for all $q \in \mathbb{Z}$ and $1 \le p \le n$, there is an isomorphism $E^\infty_{0,q} \cong E^2_{0,q}$ for $q \le n-1$, giving

$$E^\infty_{0,q} \cong \begin{cases} M \otimes_S N & \text{for } q=0, \\ M \otimes_R N/yN & \text{for } 0<q<n \text{ and } q \text{ odd}, \\ M \otimes_R \operatorname{ann}_N(y) & \text{for } 0<q<n \text{ and } q \text{ even}. \end{cases}$$

But it follows from above that the equalities

$$E^\infty_{0,q} \cong \Phi^0 H_q = \Phi^1 H_q = \cdots = \Phi^q H_q$$

hold when $q<n$. Therefore, since $\Phi^q H_q = \operatorname{Tor}^S_q(M,N)$, we are done. □

As an immediate corollary we obtain the result, for the vanishing of homology, stated in the introduction.

COROLLARY 2.2
Let $R=S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules such that $yN=0$. If there exists an integer $n \ge 2$ such that $\operatorname{Tor}^R_i(M,N)=0$ *for $1 \le i \le n$, then* $\operatorname{Tor}^S_i(M,N) \cong M \otimes_S N$ *for $0 \le i \le n-1$. Consequently, if M and N are nonzero and finitely generated, then* $\operatorname{Tor}^S_i(M,N) \ne 0$ *for $0 \le i \le n-1$.*

Thus when the modules involved are finitely generated and nonzero, the corollary shows that the vanishing of homology over R implies the *nonvanishing* of homology over S. This is in stark contrast to the case when x is a regular element.

In certain cases we can show that the Tor's over S cannot vanish irrespective of the vanishing of the Tor's over R.

PROPOSITION 2.3

Let $R=S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors, both of which are minimal *generators of the maximal ideal of S. Furthermore, let M and N be nonzero finitely generated R-modules such that $yN=0$. Then* $\operatorname{Tor}_i^S(M,N)\neq 0$ *for all $i\geq 0$.*

Proof

Consider a minimal free resolution of M over S:

$$F:\cdots\to F_2\xrightarrow{\partial_2}F_1\xrightarrow{\partial_1}F_0\to 0.$$

Letting m denote a minimal generator of M, we can define the homomorphism $f:S/(x)\to M$ sending $\bar{1}$ to m. Because x and y are minimal generators of the maximal ideal of S, we can lift this homomorphism to a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & S \arrow[r, "y"] \arrow[d, "f_2"] & S \arrow[r, "x"] \arrow[d, "f_1"] & S \arrow[r] \arrow[d, "f_0"] & S/(x) \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

in such a way that each f_i is a split injection. Tensoring the entire diagram with N we get the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & N \arrow[r, "0"] \arrow[d, "f_2\otimes N"] & N \arrow[r, "0"] \arrow[d, "f_1\otimes N"] & N \arrow[r, "="] \arrow[d, "f_0\otimes N"] & N \arrow[r] \arrow[d, "f\otimes N"] & 0 \\
\cdots \arrow[r] & F_2\otimes_S N \arrow[r, "\partial_2\otimes N"] & F_1\otimes_S N \arrow[r, "\partial_1\otimes N"] & F_0\otimes_S N \arrow[r] & M\otimes_S N \arrow[r] & 0
\end{tikzcd}$$

Now let n be a minimal generator of N. Then $f_i(1)\otimes_S n$ is a minimal generator of $F_i\otimes_S N$ and, by commutativity, is in $\ker(\partial_i\otimes_S N)$ for all $i\geq 1$. This element is not a boundary, however, since $\partial_{i+1}\subseteq \mathfrak{m}F_i$, and no element in the image of $\partial_{i+1}\otimes_S N$ is a minimal generator of $F_i\otimes_S N$. It follows that $\operatorname{Tor}_i^S(M,N)\neq 0$ for all $i\geq 0$. $\square$

We next state the cohomological versions of Theorem 2.1 and Corollary 2.2; they are proved dually.

THEOREM 2.4

Let $R=S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules. If there exists an integer $n\geq 2$ such that $\operatorname{Ext}_R^i(M,N)=\operatorname{Ext}_R^i(M,N/yN)=\operatorname{Ext}_R^i(M,\operatorname{ann}_N(y))=0$ for $1\leq i\leq n$, then

$$\operatorname{Ext}_S^i(M,N)\cong\begin{cases}\operatorname{Hom}_S(M,N) & \text{for } i=0,\\ \operatorname{Hom}_S(M,\operatorname{ann}_N(y)) & \text{for } 0<i<n \text{ and } i \text{ odd},\\ \operatorname{Hom}_S(M,N/yN) & \text{for } 0<i<n \text{ and } i \text{ even}.\end{cases}$$

COROLLARY 2.5

Let $R = S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be R-modules with $yN = 0$. If there exists an integer $n \geq 2$ such that $\operatorname{Ext}^i_R(M,N) = 0$ for $1 \leq i \leq n$, then $\operatorname{Ext}^i_S(M,N) \cong \operatorname{Hom}_S(M,N)$ for $0 \leq i < n$. Consequently, if $N = M \neq 0$, then $\operatorname{Ext}^i_S(M,M) \neq 0$ for $0 < i < n$.

3. Complexity

As in the previous section, we fix a local ring S and a pair of exact zero-divisors (x,y), and denote the local ring $S/(x)$ by R. In this section all modules are assumed to be finitely generated. Our aim is to compare free resolutions of modules over R with those over S and determine relationships involving complexities.

Given a local ring A and an A-module M, there exists a (deleted) free resolution of M

$$\cdots \to F_2 \to F_1 \to F_0 \to 0,$$

which is minimal, that is, it appears as a direct summand of every free resolution of M. The cokernel of the map $F_{n+1} \to F_n$ is the nth *syzygy* module of M and is denoted by $\Omega^n_A(M)$. Minimal free resolutions are unique up to isomorphism and hence the syzygies are uniquely determined up to isomorphism. Moreover, for every nonnegative integer n, the nth *Betti number* $\beta^A_n(M) \overset{\text{def}}{=} \operatorname{rank} F_n$ is a well-defined invariant of M. It is well known that $\dim_k \operatorname{Ext}^n_A(M,k) = \beta^A_n(M) = \dim_k \operatorname{Tor}^A_n(M,k)$ for every integer n where k is the residue field of A. It is also clear that the projective dimension of M is finite if and only if the Betti numbers of M eventually vanish. Thus the asymptotic behavior of the Betti sequence $\beta^A_0(M), \beta^A_1(M), \beta^A_2(M), \ldots$ determines an important homological property of M. Following ideas from modular representation theory (see [Alp]), an invariant measuring how "fast" the Betti sequence grows was introduced by Avramov [Av2] (see also [Av1]). The *complexity* of M, denoted by $\operatorname{cx}_A(M)$, is defined as

$$\operatorname{cx}_A(M) \overset{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta^A_n(M) \leq a n^{t-1} \text{ for all } n\},$$

and measures the polynomial rate of growth of the Betti sequence of M. It follows from the definition that M has finite projective dimension if and only if $\operatorname{cx}_A(M) = 0$, whereas $\operatorname{cx}_A(M) = 1$ if and only if the Betti sequence of M is bounded. For an arbitrary local ring, the complexity of a module is not necessarily finite (see [Av3, Example 4.2.2]). In fact, by [Gu2, Theorem 2.3], the finiteness of the complexity for all finitely generated A-modules is equivalent to A being a complete intersection.

We now return to our previous setting of exact zero-divisors. We first remark that every nonzero R-module has infinite projective dimension over S, that is, every such module has positive complexity over S. Indeed, (†) from the second paragraph of Section 2 shows that if $\operatorname{Tor}^S_i(R,M) = 0$ for all $i \gg 0$, then $M/yM = 0$. Thus $M = 0$ by Nakayama's lemma.

Over a local ring A, the complexity of a module equals the complexity of any of its syzygies: their minimal free resolutions are the same except at the beginning. Moreover, given a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of A-modules, the inequality

$$\operatorname{cx}_A(M_u) \le \max\{\operatorname{cx}_A(M_v), \operatorname{cx}_A(M_w)\} \tag{$*$}$$

holds for $\{u, v, w\} = \{1, 2, 3\}$. This follows simply by comparing the k-vector space dimensions of the Tor modules in the long exact sequence

$$\cdots \to \operatorname{Tor}_n^A(M_1, k) \to \operatorname{Tor}_n^A(M_2, k) \to \operatorname{Tor}_n^A(M_3, k) \to \operatorname{Tor}_{n-1}^A(M_1, k) \to \cdots,$$

where k is the residue field of A.

In the next proposition we use the inequality ($*$) and prove that if M is an R-module with $\operatorname{cx}_S(M) \ne 1$, then $\operatorname{cx}_S(M) = \operatorname{cx}_S(\Omega_R^n(M))$ for all n. Here the assumption $\operatorname{cx}_S(M) \ne 1$ is necessary: the S-module R has a minimal free resolution

$$\cdots \to S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \to 0$$

and hence has complexity one over S. However its syzygies $\Omega_R^n(R)$ are all zero for $n > 0$.

PROPOSITION 3.1

Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S. Then, for every finitely generated R-module M with $\operatorname{cx}_S(M) \ne 1$, the equality $\operatorname{cx}_S(M) = \operatorname{cx}_S(\Omega_R^n(M))$ holds for all n.

Proof

If $\operatorname{cx}_S(M) = 0$, then $M = 0$ (see the third paragraph of Section 3). Thus the result is trivial in this case. Next suppose that $\operatorname{cx}_S(M) > 1$. Consider the short exact sequence

$$0 \to \Omega_R^1(M) \to F \to M \to 0,$$

where F is a free R-module. Since the S-module R has complexity one so does F. Hence the result follows from the inequality ($*$) and the short exact sequence considered above. □

Next we will compare the Betti numbers and complexities of modules over R with those over S. For that we first set some notations that generalize the notion of the Betti number and the complexity of a module.

Let $(A, \mathfrak{m})$ be a local ring with residue field k, and let M and N be A-modules with the property that $M \otimes_A N$ has finite length. Then, for every nonnegative integer n, the length of $\operatorname{Tor}_n^A(M, N)$ is finite. We define this length to be the nth Betti number $\beta_n^A(M, N)$ of the pair (M, N), that is, $\beta_n^A(M, N) \stackrel{\text{def}}{=} \ell(\operatorname{Tor}_n^A(M, N))$. The *length complexity of the pair* (M, N), denoted by $\ell \operatorname{cx}_A(M, N)$, is then defined as (see [Dao2, the discussion preceding Definition 2.1])

$$\ell\,\mathrm{cx}_A(M,N) \stackrel{\mathrm{def}}{=} \inf\bigl\{t \in \mathbb{N}\cup\{0\} \bigm| \exists a \in \mathbb{R} \text{ such that } \beta_n^A(M,N) \le a n^{t-1} \text{ for all } n\bigr\}.$$

Although, letting $N=k$, we recover the Betti number and the ordinary complexity of M, that is,

$$\beta_n^A(M) = \beta_n^A(M,k) \qquad \text{and} \qquad \mathrm{cx}_A(M) = \ell\,\mathrm{cx}_A(M,k),$$

our definition for $\ell\,\mathrm{cx}_A(M,N)$ of the pair (M,N) is different than the one originally given by Avramov and Buchweitz [AvBu, p. 286, para 2], where the minimal number of generators of the cohomology modules $\mathrm{Ext}_A^n(M,N)$ is used. In general there is no comparison between these two definitions of Betti numbers of the pair (M,N) (see also [Dao2, Theorem 5.4]).

THEOREM 3.2

Let $R=S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be finitely generated R-modules such that $yN=0$ and $M\otimes_R N$ has finite length. Then, for all n,

$$\beta_n^R(M,N) - \sum_{i=0}^{n-2}\beta_i^R(M,N) \le \beta_n^S(M,N) \le \sum_{i=0}^{n}\beta_i^R(M,N). \tag{3.2}$$

REMARK

We have used the convention that negative Betti numbers are zero.

Proof

As in the proof of Theorem 2.1, we consider the first quadrant change of rings spectral sequence:

$$\mathrm{Tor}_p^R\bigl(M, \mathrm{Tor}_q^S(R,N)\bigr) \underset{p}{\Longrightarrow} \mathrm{Tor}_{p+q}^S(M,N).$$

Since $yN=0$, we see from (†) that the E^2-page entries are given by $E^2_{p,q} = \mathrm{Tor}_p^R(M,N)$.

We first prove the left-hand inequality of (3.2). Fix an integer n, and consider the short exact sequence

$$0 \to \Phi^{n-1}H_n \to \mathrm{Tor}_n^S(M,N) \to E^\infty_{n,0} \to 0,$$

where $\Phi^i H_n$ is the filtration of H_n from the proof of Theorem 2.1. Since $E^\infty_{n,0} = \mathrm{Ker}\, d^n_{n,0}$, we obtain the inequality $\ell(\mathrm{Tor}_n^S(M,N)) \ge \ell(\mathrm{Ker}\, d^n_{n,0})$. Now for all $2 \le p \le n$, there is an exact sequence

$$0 \to \mathrm{Ker}\, d^p_{n,0} \to \mathrm{Ker}\, d^{p-1}_{n,0} \to \mathrm{Im}\, d^p_{n,0} \to 0,$$

which implies that

$$\begin{aligned}\ell(\mathrm{Ker}\, d^n_{n,0}) &= \ell(\mathrm{Ker}\, d^{n-1}_{n,0}) - \ell(\mathrm{Im}\, d^n_{n,0})\\ &= \ell(\mathrm{Ker}\, d^{n-2}_{n,0}) - \bigl(\ell(\mathrm{Im}\, d^{n-1}_{n,0}) + \ell(\mathrm{Im}\, d^n_{n,0})\bigr)\end{aligned}$$

$$\vdots$$

$$= \ell(\operatorname{Ker} d^1_{n,0}) - \sum_{i=2}^{n} \ell(\operatorname{Im} d^i_{n,0}).$$

For $2 \le i \le n$, the image of $d^i_{n,0}$ is a submodule of $E^i_{n-i,i-1}$, and the latter is a subquotient of $E^2_{n-i,i-1}$. Then since $E^2_{n-i,i-1} = \operatorname{Tor}^R_{n-i}(M,N)$, there is an inequality $\ell(\operatorname{Im} d^i_{n,0}) \le \ell(\operatorname{Tor}^R_{n-i}(M,N))$. Moreover, the module $E^2_{n,0}$ is a subquotient of $\operatorname{Ker} d^1_{n,0}$. Thus, since $E^2_{n,0} = \operatorname{Tor}^R_n(M,N)$, we have that $\ell(\operatorname{Ker} d^1_{n,0}) \ge \ell(\operatorname{Tor}^R_n(M,N))$. This gives

$$\begin{aligned}
\ell\big(\operatorname{Tor}^S_n(M,N)\big) &\ge \ell(\operatorname{Ker} d^n_{n,0}) \\
&= \ell(\operatorname{Ker} d^1_{n,0}) - \sum_{i=2}^{n} \ell(\operatorname{Im} d^i_{n,0}) \\
&\ge \ell\big(\operatorname{Tor}^R_n(M,N)\big) - \sum_{i=0}^{n-2} \ell\big(\operatorname{Tor}^R_i(M,N)\big),
\end{aligned}$$

proving the left-hand inequality.

For the right-hand inequality, we fix an integer n and consider the short exact sequence

$$0 \to \Phi^{p-1} H_n \to \Phi^p H_n \to E^\infty_{p,n-p} \to 0$$

for $0 \le p \le n$. Counting the lengths, we obtain the equalities

$$\begin{aligned}
\ell(\Phi^n H_n) &= \ell(\Phi^{n-1} H_n) + \ell(E^\infty_{n,0}) \\
&= \ell(\Phi^{n-2} H_n) + \ell(E^\infty_{n-1,1}) + \ell(E^\infty_{n,0}) \\
&\ \ \vdots \\
&= \sum_{i=0}^{n} \ell(E^\infty_{i,n-i}).
\end{aligned}$$

Each $E^\infty_{i,n-i}$ is a subquotient of $E^2_{i,n-i}$, and so since $E^2_{i,n-i} = \operatorname{Tor}^R_i(M,N)$, we obtain the inequality $\ell(E^\infty_{i,n-i}) \le \ell(\operatorname{Tor}^R_i(M,N))$. Then since $\Phi^n H_n = \operatorname{Tor}^S_n(M,N)$, we obtain

$$\begin{aligned}
\ell\big(\operatorname{Tor}^S_n(M,N)\big) &= \ell(\Phi^n H_n) \\
&= \sum_{i=0}^{n} \ell(E^\infty_{i,n-i}) \\
&\le \sum_{i=0}^{n} \ell\big(\operatorname{Tor}^R_i(M,N)\big),
\end{aligned}$$

proving the right-hand inequality. $\square$

As a consequence, using the right-hand side of the inequality (3.2), we obtain an upper bound for $\ell\,\mathrm{cx}_S(M,N)$ in terms of the complexity of (M,N) over R.

COROLLARY 3.3

Let $R=S/(x)$ where S is a local ring and (x,y) is a pair of exact zero-divisors in S. Furthermore, let M and N be finitely generated R-modules such that $yN=0$ and $M\otimes_R N$ has finite length. Then $\ell\,\mathrm{cx}_S(M,N)\leq \ell\,\mathrm{cx}_R(M,N)+1$.

Proof

If $\ell\,\mathrm{cx}_R(M,N)=\infty$, then there is nothing to prove. So suppose that $\ell\,\mathrm{cx}_R(M,N)=c<\infty$. Then, by the definition, there exists a real number a such that $\beta_n^R(M,N)\leq an^{c-1}$ for all n. By Theorem 3.2, the inequality

$$\beta_n^S(M,N)\leq \sum_{i=0}^{n}\beta_i^R(M,N)\leq \sum_{i=0}^{n} ai^{c-1}\leq (n+1)an^{c-1}$$

holds for all n. Therefore there is a real number b such that $\beta_n^S(M,N)\leq bn^c$ for all n. This shows that $\ell\,\mathrm{cx}_S(M,N)\leq c+1$. □

We are unaware of an example of a pair of R-modules for which equality holds on the left-hand side of (3.2). On the other hand, equality may occur on the right-hand side. Indeed, when the exact zero-divisors x and y are minimal generators of the maximal ideal of S, Henriques and Şega [HeŞ, Theorem 1.7] proved that the equality

$$\sum_{n=0}^{\infty}\beta_n^S(M)t^n=\frac{1}{1-t}\sum_{n=0}^{\infty}\beta_n^R(M)t^n$$

of Poincaré series holds for every finitely generated R-module M. This gives that

$$\beta_n^S(M)=\sum_{i=0}^{n}\beta_i^R(M).$$

However, when x and y are arbitrary, the equality of the Poincaré series stated above may fail.

EXAMPLE 3.4

Let $S=k[[x]]/(x^3)$ where k is a field. Then x^2 is an exact zero divisor in S. Set $R=S/(x^2)\cong k[[x]]/(x^2)$. It can be seen that

$$\sum_{n=0}^{\infty}\beta_n^S(k)t^n=\frac{1}{1-t}=\sum_{n=0}^{\infty}\beta_n^R(k)t^n.$$

This example also shows that the inequality of Corollary 3.3 can be strict.

We now give an example illustrating the fact that the left-hand inequality of (3.2) does give useful lower bounds in some cases.

EXAMPLE 3.5

Let $R = k[x_1, \dots, x_e]/(x_1, \dots, x_e)^2$, and let M be a finitely generated R-module. Then $\Omega^1_R(M)$ is a finite-dimensional vector space over k of dimension $\beta^R_1(M)$. It is easy to see that the Betti numbers of k are $\beta^R_n(k) = e^n$. It follows that $\beta^R_n(M) = \beta^R_1(M)e^{n-1}$ for all $n \geq 1$. From the left-hand inequality of (3.2) we have

$$\begin{aligned}
\beta^S_n(M) \geq \beta^R_n(M) - \sum_{i=0}^{n-2} \beta^R_i(M) & \\
&= \beta^R_1(M)e^{n-1} - \Big(\sum_{i=1}^{n-2} \beta^R_1(M)e^{i-1}\Big) - \beta^R_0(M) \\
&= \beta^R_1(M)\Big(e^{n-1} - \frac{e^{n-2}-1}{e-1}\Big) - \beta^R_0(M) \\
&= \beta^R_1(M)\Big(\frac{e^n - e^{n-1} - e^{n-2} + 1}{e-1}\Big) - \beta^R_0(M) \\
&\geq \frac{\beta^R_1(M)}{2} e^{n-1} - \beta^R_0(M)
\end{aligned}$$

for $e \geq 2$ and for any ring S such that there exists an exact zero-divisor x with $R \cong S/(x)$. Note that the last inequality follows since for $e \geq 2$ we have $e^2 - e - 2 \geq 0$. Then $e^{n-2}(e^2 - e - 2) \geq 0$, which implies that $e^n - e^{n-1} - 2e^{n-2} + 2 \geq 0$. Thus $2(e^n - e^{n-1} - e^{n-2} + 1) \geq e^{n-1}(e-1)$, and the desired inequality follows. In particular, R-modules must have exponential growth over S as well. As a specific example, let $S = k[x,y,z]/(x^2, y^2, z^2, yz)$. Then x is an exact zero-divisor in S, and $R = S/(x) \cong k[y,z]/(y,z)^2$ has the form above.

When $N = k$, the assumptions that $M \otimes N$ has finite length and $yN = 0$ hold automatically. Therefore, in this situation, Theorem 3.2 and Corollary 3.3 can be summarized as follows.

COROLLARY 3.6

Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S. Then, for every finitely generated R-module M, the inequalities

$$\beta^R_n(M) - \sum_{i=0}^{n-2} \beta^R_i(M) \leq \beta^S_n(M) \leq \sum_{i=0}^{n} \beta^R_i(M)$$

hold for all n. Consequently $\mathrm{cx}_S(M) \leq \mathrm{cx}_R(M) + 1$ holds.

REMARK

It follows from [AHŞ, Remark 4.4] that R is a complete intersection if and only if S is a complete intersection. The complexity inequality obtained in Corollary 3.6 gives a different proof for the 'only if' direction of this result: if $\mathrm{cx}_R(k) < \infty$, where k is the residue field of R, then it follows from Corollary 3.6 that $\mathrm{cx}_S(k) < \infty$ and hence, by [Gu2, Corollary 2.5], S is a complete intersection.

Another observation related to the result stated above concerns commutative local Cohen–Macaulay Golod rings (see [Av3, Section 5.2]). Assume that S is such a ring. Since a finitely generated module has infinite complexity over S in the case in which it has infinite projective dimension over S and $\operatorname{codepth}(S) \geq 2$ (see [Av3, Theorem 5.3.3(2)]), we conclude that $\operatorname{codepth}(S) \leq 1$. (Recall that $\operatorname{cx}_S(R) = 1$.) Moreover, as x is not regular, $\operatorname{codepth}(S) = 1$. This implies that S is a hypersurface and hence R is a complete intersection.

As discussed in the introduction, when x is regular the complexity inequality is quite different than the one obtained in Corollary 3.6. More precisely, in that case the inequalities $\operatorname{cx}_S(M) \leq \operatorname{cx}_R(M) \leq \operatorname{cx}_S(M) + 1$ hold. In particular, the complexity of M over R is finite if and only if it is finite over S. However, in our situation, when x is an exact zero-divisor, we are unable to deduce any further inequalities, such as $\operatorname{cx}_R(M) \leq \operatorname{cx}_S(M)$, from Theorem 3.2. In fact we do not know whether there exists an R-module M with $\operatorname{cx}_S(M) < \infty$ and $\operatorname{cx}_R(M) = \infty$. We record this in the next question.

QUESTION
Let $R = S/(x)$ where S is a local ring and x is an exact zero-divisor in S. Is $\operatorname{cx}_R(M) \leq \operatorname{cx}_S(M)$ for all finitely generated R-modules M?

4. Canonical elements of $\operatorname{Ext}^2_R(M, M)$ and lifting

In this section we restrict our attention to the case where (x, x) is a pair of exact zero-divisors in the local ring S, and $R = S/(x)$. We discuss natural chain endomorphisms of complexes over R, following the construction in [Eis, Section 1], and show that whether or not they are null-homotopic dictates the liftability of R-modules to S. These results generalize classical results (see, e.g., [ADS]) for lifting modules modulo a regular element to modulo the square of the regular element.

4.1. Canonical endomorphisms of complexes

Let

(1) $$F : \cdots \to F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots$$

be a complex of finitely generated free R-modules. We let

(2) $$\widetilde{F} : \cdots \to \widetilde{F}_{i+1} \xrightarrow{\widetilde{\partial}_{i+1}} \widetilde{F}_i \xrightarrow{\widetilde{\partial}_i} \widetilde{F}_{i-1} \to \cdots$$

denote a preimage over S of the complex F, that is, a sequence of homomorphisms $\widetilde{\partial}_i : \widetilde{F}_i \to \widetilde{F}_{i-1}$ of free S-modules such that F and $\widetilde{F} \otimes_S R$ are isomorphic R-complexes. From the fact that $\widetilde{\partial}_{i-1}\widetilde{\partial}_i(\widetilde{F}_i) \subseteq x\widetilde{F}_{i-2}$ for all i, we can write that

(3) $$\widetilde{\partial}_{i-1}\widetilde{\partial}_i = x\widetilde{s}_i$$

for some homomorphism $\widetilde{s}_i : \widetilde{F}_i \to \widetilde{F}_{i-2}$. Now we define the homomorphisms $s_i : F_i \to F_{i-2}$ by

$$s_i = \widetilde{s}_i \otimes_S R \tag{4}$$

for all i.

There are several properties of the s_i's which we should like to mention. See [Eis, Section 1] for the proofs. (Note that in our case $(x)/(x)^2 = (x) \cong S/(x)$ is a free $S/(x)$-module.)

(a) The definition of s_i is independent of the factorization in (3).
(b) The family $s = \{s_i\}$ is a chain endomorphism of F of degree -2.
(c) Let

$$G : \cdots \to G_{i+1} \xrightarrow{\delta_{i+1}} G_i \xrightarrow{\delta_i} G_{i-1} \to \cdots$$

be another complex of finitely generated free R-modules, and assume that there exists a chain map $f : F \to G$. Let $t = \{t_i = \widetilde{t}_i \otimes_S R : G_i \to G_{i-2}\}$ be the chain map defined by the factorizations $\widetilde{\delta}_{i-1}\widetilde{\delta}_i = x\widetilde{t}_i$ for all i, where $\widetilde{G}$ is a preimage over S of G. Then the chain maps fs and tf are homotopic.

(d) From (c) it follows that the definition of the s_i's is independent, up to homotopy, of the preimage $\widetilde{F}$ of F chosen in (2).

4.2. The group $\mathrm{Ext}^n_A(M,M)$

Let A be an associative ring, and let M be an A-module. Suppose that F is a projective resolution of M. Then $\mathrm{H}^n(\mathrm{Hom}_A(F,F))$ is the group of homotopy equivalence classes of chain endomorphisms of F of degree n. For a chain endomorphism s of F of degree n, we let $[s]$ denote the class of s in $\mathrm{H}^n(\mathrm{Hom}_A(F,F))$. Let G be another projective resolution of M over A. Then the comparison maps $f : F \to G$ and $g : G \to F$ lifting the identity map on M are homotopically equivalent. That is, fg is homotopic to the identity map on G and gf is homotopic to the identity map on F. It follows that the map

$$\theta^F_G : \mathrm{H}^n\big(\mathrm{Hom}_A(F,F)\big) \to \mathrm{H}^n\big(\mathrm{Hom}_A(G,G)\big) \tag{5}$$

given by $[s] \mapsto [fsg]$ is an isomorphism, with inverse $\theta^G_F : [s] \mapsto [gsf]$. It is well known that this group is $\mathrm{Ext}^n_A(M,M)$ (see, e.g., [AV]).

4.3. Canonical elements of $\mathrm{Ext}^2_R(M,M)$

Returning to the situation where $R = S/(x)$ for the pair (x,x) of exact zero-divisors, let F be a free resolution of M over R, and let s be the endomorphism of F defined by (4). Thus we have the element $[s] \in \mathrm{H}^2(\mathrm{Hom}_R(F,F))$. That we call $[s]$ a canonical element of $\mathrm{Ext}^2_R(M,M)$ is reinforced by the following lemma.

LEMMA 4.1

Let $R = S/(x)$ where S is a local ring and (x,x) is a pair of exact zero-divisors in S. Suppose that F and G are free resolutions of a finitely generated module M over R, that s is the canonical endomorphism of F as defined in (4), and that t

is the canonical endomorphism of G as defined in (4). Then we have that

$$\theta_F^G\big([t]\big) = [s],$$

where θ_F^G is the isomorphism defined in (5).

Proof

First assume that F is a minimal free resolution of M. Then the comparison map $f : F \to G$ lifting the identity map on M can be chosen to be a split injection, with splitting $g : G \to F$ also lifting the identity map on M. In particular, we have $gf = \mathrm{id}_F$, the identity map on F.

Denote the differential on F by ∂, and denote that on G by δ. Let $(\widetilde{F}, \widetilde{\partial})$ be a preimage over S of (F, ∂), and let $(\widetilde{G}, \widetilde{\delta})$ be a preimage over S of (G, δ). We choose preimages $\widetilde{f}$ of f and $\widetilde{g}$ of g over S such that $\widetilde{g}\widetilde{f} = \mathrm{id}_{\widetilde{F}}$.

As $g_{i-1}\delta_i = \partial_i g_i$ for all i, there exists $u_i : \widetilde{G}_i \to \widetilde{F}_{i-1}$ such that $\widetilde{g}_{i-1}\widetilde{\delta}_i = \widetilde{\partial}_i \widetilde{g}_i + x u_i$ for all i. Similarly, there exists $v_i : \widetilde{F}_i \to \widetilde{G}_{i-1}$ such that $\widetilde{\delta}_i \widetilde{f}_i = \widetilde{f}_{i-1}\widetilde{\partial}_i + x v_i$ for all i. Thus we have

$$\begin{aligned} x(\widetilde{g}_{i-2}\widetilde{t}_i\widetilde{f}_i - \widetilde{s}_i) &= \widetilde{g}_{i-2}\widetilde{\delta}_{i-1}\widetilde{\delta}_i\widetilde{f}_i - \widetilde{\partial}_{i-1}\widetilde{\partial}_i \\ &= (\widetilde{\partial}_{i-1}\widetilde{g}_{i-1} + x u_{i-1})(\widetilde{f}_{i-1}\widetilde{\partial}_i + x v_i) - \widetilde{\partial}_{i-1}\widetilde{\partial}_i \\ &= x(\widetilde{\partial}_{i-1}\widetilde{g}_{i-1} v_i + u_{i-1}\widetilde{f}_{i-1}\widetilde{\partial}_i). \end{aligned}$$

It follows that $g_{i-2}t_i f_i - s_i = \partial_{i-1}(g_{i-1}\overline{v}_i) + (\overline{u}_{i-1}f_{i-1})\partial_i$ for all i, where $\overline{u}_i = u_i \otimes_S R$ and $\overline{v}_i = v_i \otimes_S R$. We will have shown that gtf is homotopic to s with homotopy $h_i = \overline{u}_i f_i$ once we know that $\overline{u}_i f_i = g_{i-1}\overline{v}_i$ for all i. But this is easy:

$$\begin{aligned} x(u_i\widetilde{f}_i - \widetilde{g}_{i-1}v_i) &= (\widetilde{g}_{i-1}\widetilde{\delta}_i - \widetilde{\partial}_i\widetilde{g}_i)\widetilde{f}_i - \widetilde{g}_{i-1}(\widetilde{\delta}_i\widetilde{f}_i - \widetilde{f}_{i-1}\widetilde{\partial}_i) \\ &= -\widetilde{\partial}_i\widetilde{g}_i\widetilde{f}_i + \widetilde{g}_{i-1}\widetilde{f}_{i-1}\widetilde{\partial}_i \\ &= 0, \end{aligned}$$

hence the claim follows.

Notice that we also have $\theta_G^F([s]) = [t]$ when F is minimal. Therefore, for two arbitrary free resolutions F and G of M, $\theta_F^G([t]) = [s]$ follows from composing $\theta_F^G = \theta_L^G\theta_F^L$ where L is a minimal free resolution of M. □

4.4. Lifting

Let B be an associative ring, let I be an ideal of B, and let $A = B/I$. Recall that a finitely generated A-module M is said to *lift* to B with *lifting* M' if there exists a finitely generated B-module M' such that $M \cong M' \otimes_B A$ and $\mathrm{Tor}_i^B(M', A) = 0$ for all $i \geq 1$. Similarly, a complex of finitely generated free A-modules

$$F : \cdots \to F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots$$

is said to *lift* to B with *lifting* $\widetilde{F}$ if there exists a preimage $\widetilde{F}$ of F

$$\widetilde{F} : \cdots \to \widetilde{F}_{i+1} \xrightarrow{\widetilde{\partial}_{i+1}} \widetilde{F}_i \xrightarrow{\widetilde{\partial}_i} \widetilde{F}_{i-1} \to \cdots$$

such that $\widetilde{\partial}_{i-1}\widetilde{\partial}_i = 0$ for all i. A close connection between these two notions of lifting will be explained in the next theorem. We also want to show that, when $R = S/(x)$ for (x, x) a pair of exact zero-divisors, the triviality of the canonical element $[s]$ determines whether the module M lifts to S.

THEOREM 4.2

Let $R = S/(x)$ where S is a local ring and (x, x) is a pair of exact zero-divisors in S. Then for every finitely generated R-module M, the following are equivalent.

(a) *M lifts to S.*
(b) *The canonical element $[s]$ in $\operatorname{Ext}^2_R(M, M)$ is trivial.*
(c) *Every free resolution of M by finitely generated free R-modules lifts to S.*
(d) *Some free resolution of M by finitely generated free R-modules lifts to S.*

Proof

(a) $\Longrightarrow$ (b). Suppose that M' is a lifting of M to S. Let

$$\widetilde{F} : \cdots \to \widetilde{F}_2 \xrightarrow{\widetilde{\partial}_2} \widetilde{F}_1 \xrightarrow{\widetilde{\partial}_1} \widetilde{F}_0 \to 0$$

be a resolution of M' by finitely generated free S-modules. Since $\operatorname{Tor}^S_i(M', R) = 0$ for all $i > 0$, $F = \widetilde{F} \otimes_S R$ is a resolution of $M \cong M' \otimes_S R$ by finitely generated free R-modules. Computing the endomorphism s from the preimage $\widetilde{F}$ of F, which is exact, we see that s is actually the zero endomorphism, and is therefore certainly trivial in $\operatorname{Ext}^2_R(M, M)$.

(b) $\Longrightarrow$ (c). By Lemma 4.1 the canonical element of $\operatorname{Ext}^2_R(M, M)$ is trivial regardless of which resolution by finitely generated free R-modules F of M we choose to define it. Therefore let F be an arbitrary such resolution of M, and let s be the canonical chain endomorphism defined as in (4) of the section above on canonical endomorphisms of complexes. By assumption, s is homotopic to zero. Therefore there exists a homotopy $h = \{h_i\}$ with $h_i : F_i \to F_{i-1}$ such that $s_i = \partial_{i-1}h_i + h_{i-1}\partial_i$ for all i. Let $\widetilde{F}$ be an arbitrary preimage of F, with maps $\widetilde{\partial}$. Let $\widetilde{h}_i : \widetilde{F}_i \to \widetilde{F}_{i-1}$ be a preimage of h_i for all i. There exists $u_i : \widetilde{F}_i \to \widetilde{F}_{i-2}$ such that $\widetilde{s}_i = \widetilde{\partial}_{i-1}\widetilde{h}_i + \widetilde{h}_{i-1}\widetilde{\partial}_i + xu_i$ for all i. Now consider the preimage $F^\sharp$ of F where we take $F^\sharp_i = \widetilde{F}_i$ for all i, but we take the maps $\partial^\sharp_i = \widetilde{\partial}_i - x\widetilde{h}_i$ instead. We have

$$\begin{aligned}
\partial^\sharp_{i-1}\partial^\sharp_i &= (\widetilde{\partial}_{i-1} - x\widetilde{h}_{i-1})(\widetilde{\partial}_i - x\widetilde{h}_i) \\
&= \widetilde{\partial}_{i-1}\widetilde{\partial}_i - x(\widetilde{\partial}_{i-1}\widetilde{h}_i + \widetilde{h}_{i-1}\widetilde{\partial}_i) \\
&= x(\widetilde{s}_i - \widetilde{\partial}_{i-1}\widetilde{h}_i - \widetilde{h}_{i-1}\widetilde{\partial}_i) \\
&= 0.
\end{aligned}$$

Thus $F^\sharp$ is a lifting of F to S.

(c) $\Longrightarrow$ (d). This is trivial. To show that (d) $\Longrightarrow$ (a), assume that F is a free resolution of M by finitely generated free R-modules which lifts to the complex

$\widetilde{F}$ over S. We claim that $\mathrm{H}_i(\widetilde{F}) = 0$ for $i \neq 0$. Indeed, if $\widetilde{\partial}_i(a) = 0$ for some $a \in \widetilde{F}_i$, then $a = \widetilde{\partial}_{i+1}(b) + xc$ for some $b \in \widetilde{F}_{i+1}$ and $c \in \widetilde{F}_i$ by the exactness of F. Since $x\widetilde{\partial}_i(c) = 0$, we have that $\widetilde{\partial}_i(c) \in x\widetilde{F}_{i-1}$. Again by the exactness of F we have that $c = \widetilde{\partial}_{i+1}(d) + xe$ for some $d \in \widetilde{F}_{i+1}$ and $e \in \widetilde{F}_i$. Therefore $a = \widetilde{\partial}_{i+1}(b + xd)$. It follows that $\widetilde{F}$ is a resolution of $M' = \mathrm{H}_0(\widetilde{F})$ by finitely generated free S-modules, and thus M' is a lifting of M to S. □

We end with an example showing that there are local rings S admitting a pair of exact zero-divisors (x, x), but no local ring T with regular element $\widetilde{x}$ such that $S = T/(\widetilde{x}^2)$ and $x = \widetilde{x} + (\widetilde{x}^2)$. Therefore the notion of lifting modulo an exact zero-divisor is a more general notion than lifting from modulo a regular element to modulo the square of the regular element.

EXAMPLE 4.3
Let k be field, let $S = k[V, X, Y, Z]/I$ where I is the ideal

$$(V^2, Z^2, XY, VX + XZ, VY + YZ, VX + Y^2, VY - X^2),$$

and set $v = V + I$. Then (v, v) is a pair of exact zero-divisors. Moreover, it is shown in [AGP] that S does not have an embedded deformation. As a consequence there is no local ring T and nonzero-divisor $\widetilde{V}$ of T such that $S \cong T/(\widetilde{V}^2)$.

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