# Actions of locally compact (quantum) groups on ternary rings of operators, their crossed products, and generalized Poisson boundaries

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**Abstract** Actions of locally compact groups and quantum groups on W*-ternary rings of operators are discussed, and related crossed products are introduced. The results generalize those for von Neumann algebraic actions with proofs based mostly on passing to the linking von Neumann algebra. They are motivated by the study of fixed-point spaces for convolution operators generated by contractive, not necessarily positive measures, both in the classical and in the quantum context.

The notion of a Poisson boundary, related to spaces of harmonic functions in the context of probability theory, that is, the fixed points of convolution operators associated to probability measures, has played an important role in the theory of random walks and various aspects of potential theory for more than 40 years. Since the groundbreaking work of Izumi [9], the concept (originally introduced for random walks on $\mathbb{Z}$, but later studied for any locally compact group) has also been extended to quantum groups. For the history of further developments we refer to article [12], where the abstract structure of noncommutative Poisson boundaries is studied in detail and connected to the crossed products of von Neumann algebras. The quantum extension shows very clearly that from the modern point of view the construction of the Poisson boundary is a special instance of the Choi–Effros product from the theory of operator algebras granting a von Neumann algebra structure to a space of fixed points of a given unital completely positive map on a von Neumann algebra; this is indeed one of the key observations which led to the current work.

When one considers convolution operators, be it in the classical or the quantum framework, it is natural to analyze not only those associated to probability measures, but also those arising from general signed measures. In general, the problem of the study of the fixed points then becomes far more complicated,

*Kyoto Journal of Mathematics*, Vol. 57, No. 3 (2017), 667–691
First published online 3, May 2017.
DOI 10.1215/21562261-2017-0011, 
Received December 24, 2015. Revised May 16, 2016. Accepted May 17, 2016.
*2010 Mathematics Subject Classification*: Primary 46L65; Secondary 43A05, 47L05.
Skalski's work supported in part by the National Centre of Science grant 2014/14/E/ST1/00525.

and even the characterization of idempotent signed measures (i.e., those whose convolution operators are idempotent maps) remains unknown for nonabelian groups (see [18] for a longer discussion). The questions become more tractable if one focuses on contractive signed measures. In the classical and the dual to classical context, related issues are studied, for example, in the monograph [2]; in the quantum framework the idempotent problem was solved in [18]. In both of these works, one can see the natural emergence of the W$^*$-TRO structures (*W$^*$-ternary rings of operators*). This was a motivation for a systematic study of the spaces of fixed points of completely contractive maps, which we conduct in this article.

It turns out that, whenever $\mathsf{M}$ is a von Neumann algebra and $P : \mathsf{M} \to \mathsf{M}$ is a completely contractive normal map, a Choi–Effros type construction, exploiting the algebraic properties of $P$ established in [23], equips the space $\operatorname{Fix} P$ with a unique structure of a W$^*$-TRO, which we may call a *generalized noncommutative Poisson boundary* related to $P$. To understand the structure of the resulting TRO for a contractive convolution operator, one needs to develop also the notion of (quantum) group actions on W$^*$-TROs and analyze appropriate crossed products. (In the classical context one can find related work in [8].) Several theorems follow here relatively easily from their von Neumann algebraic counterparts, as each W$^*$-TRO is a corner in its linking von Neumann algebra; some others require special care, as we mention below.

Once a satisfactory theory of crossed products is developed, it is natural to expect a generalization of the main result of [12], which would show that the generalized noncommutative Poisson boundary for an "extended" contractive convolution operator $\Theta_\mu$ is isomorphic to the crossed product of the analogous boundary for the "standard" convolution operator $R_\mu$ by a natural action of the underlying group. This is indeed what we prove here, but only for classical locally compact groups. (The positive case considered in [12] yielded the result for general locally compact quantum groups.) Here we see an example where the passage from the von Neumann algebra framework to the TRO case is highly nontrivial—very roughly speaking the reason is that the Choi–Effros-type construction connects a given concrete initial data (the pair $(\mathsf{M}, P)$) with an *abstract* W$^*$-TRO $\mathsf{X}$—and the linking von Neumann algebra of $\mathsf{X}$ arises naturally only once we fix a concrete realization of the latter, which need not be related in any explicit way to the original data.

The plan of the article is as follows: in the first section we recall basic facts regarding the ternary rings of operators. We also prove a few technical lemmas and connect the fixed points of the completely contractive maps to the TROs, introducing the corresponding version of the Choi–Effros product and discussing its basic properties. In Section 2 we develop the notion of actions of locally compact groups on TROs and construct respective crossed products, carefully developing various points of view on this concept. The next section extends the construction to the case of locally compact quantum groups and deals with certain technicalities, describing the way in which one can induce actions on TROs arising as fixed-point spaces. These are applied in Section 4 to the discussion of the TROs arising from fixed-point spaces of contractive convolution operators on locally compact (quantum) groups.

The angled brackets will denote the closed linear span. Hilbert space scalar products will be linear on the right. For a locally compact group $G$ we write $L^2(G)$ for the $L^2$-space with respect to the *left* invariant Haar measure; the group von Neumann algebra $\mathrm{VN}(G)$, will be the von Neumann algebra generated by the *left* regular representation.

## 1. W*-ternary rings of operators and fixed points of completely contractive maps

Recall that a (concrete) *TRO*, that is, a *ternary ring of operators*, $\mathsf{X}$ is a closed subspace of $B(\mathsf{H};\mathsf{K})$, where $\mathsf{H}$ and $\mathsf{K}$ are some Hilbert spaces, which is closed under the ternary product: $(a,b,c)\mapsto ab^*c$. TROs possess natural operator space structure and in fact can also be characterized abstractly, as operator spaces with a ternary product satisfying certain properties (see [17]). To each TRO $\mathsf{X}$ one can associate a C*-algebra $\mathsf{A}_\mathsf{X}\subset B(\mathsf{K}\oplus\mathsf{H})$, called the *linking algebra* of $\mathsf{X}$. It is explicitly defined as

$$\mathsf{A}_\mathsf{X} := \begin{pmatrix} \langle \mathsf{X}\mathsf{X}^*\rangle & \mathsf{X} \\ \mathsf{X}^* & \langle \mathsf{X}^*\mathsf{X}\rangle \end{pmatrix} \subset B(\mathsf{K}\oplus\mathsf{H}).$$

We always view $\mathsf{X}$, $\mathsf{X}^*$ and the C*-algebras $\langle \mathsf{X}\mathsf{X}^*\rangle$, $\langle \mathsf{X}^*\mathsf{X}\rangle$ as subspaces of $\mathsf{A}_\mathsf{X}$. If $\mathsf{X}$ and $\mathsf{Y}$ are TROs, then a linear map $\alpha:\mathsf{X}\to\mathsf{Y}$ is said to be a *TRO morphism* if it preserves the ternary product. A TRO morphism admits a unique extension to a *-homomorphism $\gamma:\mathsf{A}_\mathsf{X}\to\mathsf{A}_\mathsf{Y}$; this was proved by M. Hamana [7] (see also [8] and [24]), and we will call this map the *Hamana extension* of $\alpha$. Note that $\gamma$ is defined in a natural way, so for example if $x\in\mathsf{X}$, then $\gamma(xx^*)=\alpha(x)\alpha(x)^*$. Further we call a TRO morphism $\alpha:\mathsf{X}\to\mathsf{Y}$ *nondegenerate* if the linear spans of $\alpha(\mathsf{X})\mathsf{Y}^*\mathsf{Y}$ and $\alpha(\mathsf{X})^*\mathsf{Y}\mathsf{Y}^*$ are norm dense, respectively, in $\mathsf{Y}$ and $\mathsf{Y}^*$; in other words the space $\alpha(\mathsf{X})$, which is a sub-TRO of $\mathsf{Y}$ by [7], is a *nondegenerate sub-TRO* of $\mathsf{Y}$, as defined for example in [19]. Then by using the Hamana extensions one can easily show that $\alpha$ is nondegenerate if and only if its Hamana extension $\gamma:\mathsf{A}_\mathsf{X}\to\mathsf{A}_\mathsf{Y}$ is nondegenerate (as a *-homomorphism between C*-algebras; see [19, Proposition 1.1] for this result phrased in terms of sub-TROs).

We say that $\mathsf{X}$ is a *W*-TRO* if it is weak*-closed in $B(\mathsf{H};\mathsf{K})$. We will usually assume that the TROs we study are nondegenerately represented, that is, $\langle\mathsf{X}\mathsf{H}\rangle=\mathsf{K}$, $\langle\mathsf{X}^*\mathsf{K}\rangle=\mathsf{H}$. The *linking von Neumann algebra* associated to $\mathsf{X}$, equal to $\mathsf{A}_\mathsf{X}''$, will be denoted by $\mathsf{R}_\mathsf{X}$, so that

$$\mathsf{R}_\mathsf{X} := \begin{pmatrix} \langle \mathsf{X}\mathsf{X}^*\rangle'' & \mathsf{X} \\ \mathsf{X}^* & \langle \mathsf{X}^*\mathsf{X}\rangle'' \end{pmatrix} \subset B(\mathsf{K}\oplus\mathsf{H}).$$

For a TRO morphism between W*-TROs nondegeneracy will mean that the linear spans of the spaces introduced in the paragraph above are weak*-dense in the respective TROs. The predual of a W*-TRO $\mathsf{X}$ will be denoted by $\mathsf{X}_*$; it is not difficult to see that $\mathsf{X}_*=\{\omega|_\mathsf{X}:\omega\in(\mathsf{R}_\mathsf{X})_*\}$.

There is also an abstract characterization of TROs and W*-TROs due to Zettl [24], which we now recall. An *abstract TRO* is a Banach space $\mathsf{X}$ equipped with a ternary operation

$$\{\cdot,\cdot,\cdot\} : \mathsf{X}\times\mathsf{X}\times\mathsf{X}\to\mathsf{X}$$

such that the following conditions hold ($x,y,z,u,v\in\mathsf{X}$):

(1) the operation is linear in the first and the third variables and conjugate linear in the second;
(2) $\{\{x,y,z\},u,v\}=\{x,\{u,z,y\},v\}=\{x,y,\{z,u,v\}\}$;
(3) $\|\{x,y,z\}\|\leq\|x\|\|y\|\|z\|$;
(4) $\|\{x,x,x\}\|=\|x\|^3$.

An *abstract W*-TRO* is an abstract TRO that is a dual Banach space. Zettl [24] proved that these abstractly defined objects have concrete representations as TROs and W*-TROs, respectively.

The next result is a W*-version of the fact due to Hamana regarding images of TROs, observed in [1, Section 8.5.18].

LEMMA 1.1
*If $\mathsf{X}$ and $\mathsf{Y}$ are W*-TROs and $\alpha:\mathsf{X}\to\mathsf{Y}$ is a normal TRO morphism, then $\alpha(\mathsf{X})$ is a W*-TRO.*

Proposition 3.1 of [19] shows that a TRO morphism $\alpha:\mathsf{X}\to\mathsf{Y}$ is nondegenerate if and only if the W*-TRO $\alpha(\mathsf{X})$ is nondegenerately represented. Lemma 1.1 can be used to note that Hamana extensions can be considered also in the W*-category and, moreover, have the expected properties.

PROPOSITION 1.2
*Let $\mathsf{X}$ and $\mathsf{Y}$ be W*-TROs, and let $\alpha:\mathsf{X}\to\mathsf{Y}$ be a normal TRO morphism. Then there exists a unique normal *-homomorphism $\beta:\mathsf{R}_\mathsf{X}\to\mathsf{R}_\mathsf{Y}$ such that*

$$\beta\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \alpha(x)\\ 0 & 0\end{pmatrix},\quad x\in\mathsf{X}.$$

*Moreover, $\beta|_{\mathsf{A}_\mathsf{X}}$ is the Hamana extension discussed above, the extension construction preserves the composition, and moreover,*

(i) *$\alpha$ is injective if and only if $\beta$ is injective;*
(ii) *$\alpha$ is nondegenerate if and only if $\beta$ is unital.*

*Proof*
There are at least two ways to see the first statement. (The rest is relatively easy.) In the first step one observes that we can assume that $\alpha$ is surjective, using Lemma 1.1. (Indeed, if $\mathsf{Z}=\alpha(\mathsf{X})$, then $\mathsf{R}_\mathsf{Z}$ is a von Neumann subalgebra of $\mathsf{R}_\mathsf{Y}$.)

Now we can either proceed directly, as in [19], using nondegeneracy, or first pass to the situation where $\alpha$ is isometric, quotienting out its kernel (this leads to another W*-TRO, as follows from [7]) and then by using the proof of [21, Corollary 3.4]. The reason we cannot use this corollary directly is that we need to verify that the map obtained there (or, rather, via [21, Theorem 2.1]) coincides with the weak*-continuous extension of the Hamana extension. This, however, can be checked directly, following the arguments in [21, Corollary 3.4 and Lemma 2.5].

The injectivity of $\alpha$ (resp., $\beta$) is equivalent to $\alpha$ (resp., $\beta$) being isometric. Thus, [21] implies that the injectivity of $\alpha$ is equivalent to that of $\beta$.

For a W*-TRO $\mathsf{X} \subset B(\mathsf{H};\mathsf{K})$ it is elementary to check that $\mathsf{X}$ is nondegenerately represented if and only if $\mathsf{R}_\mathsf{X}$ contains the unit of $B(\mathsf{K} \oplus \mathsf{H})$. This together with the comments before the proposition implies the last statement. □

Note that in the situation above, by the boundedness and normality of the maps in question, we have the following consequence of the algebraic form of Hamana extensions (in which we view both $\mathsf{X}$ and $\langle \mathsf{X}\mathsf{X}^* \rangle''$ as subspaces of $\mathsf{R}_\mathsf{X}$):

$$\beta(z)\alpha(x) = \alpha(zx), \quad x \in \mathsf{X},\ z \in \langle \mathsf{X}\mathsf{X}^* \rangle''. \tag{1.1}$$

If $\mathsf{X}$ happens to be a von Neumann algebra, then $\mathsf{R}_\mathsf{X} \cong M_2(\mathsf{X})$; if $\mathsf{Y}$ is another von Neumann algebra and we assume that $\alpha : \mathsf{X} \to \mathsf{Y}$ is a *-homomorphism, then $\beta$ is the usual matrix lifting of $\alpha$. Finally we note an easy observation which will be useful later.

COROLLARY 1.3

*Let $\mathsf{X}$ and $\mathsf{Y}$ be W*-TROs, and let $\beta : \mathsf{R}_\mathsf{X} \to \mathsf{R}_\mathsf{Y}$ be a normal *-homomorphism. Then $\beta$ is the Hamana extension of a normal TRO morphism between $\mathsf{X}$ and $\mathsf{Y}$ if and only if $\beta(\mathsf{X}) \subset \mathsf{Y}$. If $\mathsf{X}$ and $\mathsf{Y}$ are, respectively, nondegenerately represented in $B(\mathsf{H}_1;\mathsf{K}_1)$ and in $B(\mathsf{H}_2;\mathsf{K}_2)$, then the conditions above are equivalent to the equality*

$$P_{\mathsf{K}_2}\beta(P_{\mathsf{K}_1} x P_{\mathsf{H}_1})P_{\mathsf{H}_2} = \beta(P_{\mathsf{K}_1} x P_{\mathsf{H}_1})$$

*being valid for all $x \in \mathsf{R}_\mathsf{X}$.*

*Proof*

A simple calculation shows that if $\beta(\mathsf{X}) \subset \mathsf{Y}$, then $\beta|_\mathsf{X}$ is a TRO morphism. Then the equivalence follows from the uniqueness of Hamana extensions, and the second statement is an easy consequence of the definitions of $\mathsf{R}_\mathsf{X}$ and $\mathsf{R}_\mathsf{Y}$. □

Given two W*-TROs $\mathsf{X} \subset B(\mathsf{H}_1;\mathsf{K}_1)$ and $\mathsf{Y} \subset B(\mathsf{H}_2;\mathsf{K}_2)$, we can naturally consider the W*-TRO $\mathsf{X} \overline{\otimes} \mathsf{Y}$ defined as the weak* closure of the algebraic tensor product $\mathsf{X} \odot \mathsf{Y} \subset B(\mathsf{H}_1 \otimes \mathsf{H}_2; \mathsf{K}_1 \otimes \mathsf{K}_2)$. The fact that it is closed under the ternary product can be easily checked. Note that if $\mathsf{M}$ is a von Neumann algebra, then we have a natural identification of $\mathsf{R}_{\mathsf{X}\overline{\otimes}\mathsf{M}}$ with $\mathsf{R}_\mathsf{X} \overline{\otimes} \mathsf{M}$; this will be of use later.

Similarly note for future use that if $\mathsf{Z} \subset B(\mathsf{H})$ is a weak*-closed subalgebra and $P \in \mathsf{Z}$ is a projection, then $P\mathsf{Z}$ is weak*-closed. This implies that if, say,

$Q \in \mathsf{Z}$ is another projection and $\mathsf{W} \subset B(\mathsf{K})$ is a weak*-closed subalgebra, then

$$PZQ \mathbin{\overline{\otimes}} \mathsf{W} = (P \otimes I_{\mathsf{K}})(\mathsf{Z} \mathbin{\overline{\otimes}} \mathsf{W})(Q \otimes I_{\mathsf{K}}).$$

As usual, $\mathsf{Z} \mathbin{\overline{\otimes}} \mathsf{W}$ denotes the weak* closure of the algebraic tensor product $\mathsf{Z} \odot \mathsf{W}$ inside $B(\mathsf{H}) \mathbin{\overline{\otimes}} B(\mathsf{K})$.

Finally recall (see, e.g., [4, Chapter 7]) that if $\mathsf{X}$ and $\mathsf{Y}$ are dual operator spaces, then their *Fubini tensor product* $\mathsf{X} \otimes_F \mathsf{Y}$ is defined abstractly as the operator space dual of $\mathsf{X}_* \widehat{\otimes} \mathsf{Y}_*$; if $\mathsf{X}$ and $\mathsf{Y}$ are weak*-closed subspaces of, say, $B(\mathsf{H})$ and $B(\mathsf{K})$, then $\mathsf{X} \otimes_F \mathsf{Y}$ can be realized as

$$\big\{u \in B(\mathsf{H}) \mathbin{\overline{\otimes}} B(\mathsf{K}) : (\omega \otimes \mathrm{id})u \in \mathsf{Y} \text{ and } (\mathrm{id} \otimes \sigma)u \in \mathsf{X} \\ \text{for every } \omega \in B(\mathsf{H})_*, \sigma \in B(\mathsf{K})_*\big\}.$$

Clearly, $\mathsf{X} \mathbin{\overline{\otimes}} \mathsf{Y}$ is contained in $\mathsf{X} \otimes_F \mathsf{Y}$.

LEMMA 1.4

*Let $\mathsf{X}$ and $\mathsf{Y}$ be dual operator spaces that are weak* completely contractively complemented in von Neumann algebras. (Note that, in particular, W*-TROs satisfy these assumptions.) Then the natural weak*-continuous completely isometric embedding $\mathsf{X} \mathbin{\overline{\otimes}} \mathsf{Y} \hookrightarrow \mathsf{X} \otimes_F \mathsf{Y}$ is in fact an isomorphism.*

*Proof*

By the assumptions, there are von Neumann algebras $R_{\mathsf{X}}$ and $R_{\mathsf{Y}}$ containing $\mathsf{X}$ and $\mathsf{Y}$, respectively, and normal completely contractive projections $P_{\mathsf{X}} : R_{\mathsf{X}} \to \mathsf{X}$ and $P_{\mathsf{Y}} : R_{\mathsf{Y}} \to \mathsf{Y}$. The algebraic tensor product $P_{\mathsf{X}} \odot P_{\mathsf{Y}}$ extends uniquely to a normal map $P_{\mathsf{X}} \otimes P_{\mathsf{Y}}$ from $R_{\mathsf{X}} \mathbin{\overline{\otimes}} R_{\mathsf{Y}} = R_{\mathsf{X}} \otimes_F R_{\mathsf{Y}}$ to $\mathsf{X} \otimes_F \mathsf{Y}$ (see [4, Chapter 7] and [5, Proposition 4.3]). As $\mathsf{X} \otimes_F \mathsf{Y} \subset R_{\mathsf{X}} \mathbin{\overline{\otimes}} R_{\mathsf{Y}}$, the uniqueness of extensions implies that $P_{\mathsf{X}} \otimes P_{\mathsf{Y}}$ is the identity map when restricted to $\mathsf{X} \otimes_F \mathsf{Y}$. Let $u \in \mathsf{X} \otimes_F \mathsf{Y}$, and let $(u_i)_{i \in \mathcal{I}}$ be a net in the algebraic tensor product $R_{\mathsf{X}} \odot R_{\mathsf{Y}}$ that converges to $u$ in the weak* topology. Then

$$u = (P_{\mathsf{X}} \otimes P_{\mathsf{Y}})(u) = \mathrm{w}^*\text{-}\lim_{i \in \mathcal{I}} (P_{\mathsf{X}} \odot P_{\mathsf{Y}})(u_i) \in \mathsf{X} \mathbin{\overline{\otimes}} \mathsf{Y}. \qquad \square$$

Consider a TRO morphism $\alpha : \mathsf{X} \to \mathsf{Y}$. It follows from [7] that $\alpha$ is completely contractive. Moreover, [8, Proposition 1.1] implies that if $\mathsf{Z}$ is another W*-TRO, then the map $\mathrm{id}_{\mathsf{Z}} \otimes \alpha$ extends uniquely to a complete contraction from $\mathsf{Z} \otimes_F \mathsf{X}$ to $\mathsf{Z} \otimes_F \mathsf{Y}$—this does not require that the original map be normal. If $\alpha$ is in addition normal, then the resulting extension is also normal, as follows for example from the identification of the predual of the Fubini tensor product as the projective tensor product of the preduals of the individual factors. So when $\alpha$ is normal, we can view $\mathrm{id}_{\mathsf{Z}} \otimes \alpha$ as a normal TRO morphism from $\mathsf{Z} \mathbin{\overline{\otimes}} \mathsf{X}$ to $\mathsf{Z} \mathbin{\overline{\otimes}} \mathsf{Y}$. If we want to stress that we are working with a not necessarily normal extension, we will write $\mathrm{id}_{\mathsf{Z}} \otimes_F \alpha$.

The celebrated Choi–Effros construction equips a fixed-point space of a completely positive map with a C*-algebra structure. Below we present an analogous result for completely contractive maps and TRO structures.

The first proposition is essentially a theorem of Youngson [23] (see also [1, Theorem 4.4.9]).

PROPOSITION 1.5

*Let* $\mathsf{A}$ *be a* $C^*$*-algebra, and let* $P : \mathsf{A} \to \mathsf{A}$ *be a completely contractive projection. Then* $\widetilde{\mathsf{X}} := P(\mathsf{A})$ *possesses a TRO structure, with the product given by the formula*

$$\{a, b, c\} := P(ab^*c), \quad a, b, c \in \widetilde{\mathsf{X}}.$$

*Denote the resulting TRO by* $\mathsf{X}$*. Then the identity map* $\iota : \widetilde{\mathsf{X}} \to \mathsf{X}$ *is a completely isometric isomorphism (where* $\widetilde{\mathsf{X}}$ *inherits the operator space structure from* $\mathsf{A}$ *and* $\mathsf{X}$ *is an operator space as a TRO). If* $\mathsf{A}$ *is a von Neumann algebra and* $\widetilde{\mathsf{X}}$ *happens to be weak*$^*$*-closed, then* $\mathsf{X}$ *is a* $W^*$*-TRO and* $\iota : \widetilde{\mathsf{X}} \to \mathsf{X}$ *is in addition a homeomorphism for weak*$^*$ *topologies.*

*Proof*

The fact that the displayed formula defines a TRO structure (with the norm induced from $\mathsf{A}$) is [23, p. 508, Theorem]—it follows also from the abstract description due to Zettl mentioned earlier. The map $\iota$ is thus an isometry. Applying the same construction to $P^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{A})$ gives a TRO based on $M_n(\widetilde{\mathsf{X}})$, and this TRO is naturally isomorphic to $M_n(\mathsf{X})$. Proposition 2.1 of [7] implies that this isomorphism is an isometry. Thus, $\iota$ is in fact a complete isometry. The second part follows from the uniqueness of a predual of a W$^*$-TRO (see [3, Proposition 2.4]). □

THEOREM 1.6

*Let* $\mathsf{M}$ *be a von Neumann algebra, and let* $P : \mathsf{M} \to \mathsf{M}$ *be a completely contractive normal map. Consider the space* $\operatorname{Fix} P = \{x \in \mathsf{M} : Px = x\}$*. Then* $\operatorname{Fix} P$ *is a weak*$^*$*-closed subspace of* $\mathsf{M}$ *and so, in particular, is a dual operator space. It possesses a unique ternary product, which makes it a* $W^*$*-TRO. It is explicitly given by the formula*

$$\{a, b, c\} := \widetilde{P}_\beta(ab^*c), \quad a, b, c \in \operatorname{Fix} P,$$

*where* $\beta$ *is a fixed free ultrafilter,*

$$\widetilde{P}_\beta(x) = \beta - \lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x), \quad x \in \mathsf{M},$$

*and the limit is understood in a weak*$^*$ *topology.*

*Proof*

To show the existence of the ternary product described above, it suffices to verify that $\widetilde{P}_\beta : \mathsf{M} \to \mathsf{M}$ is a completely contractive projection onto $\operatorname{Fix} P$. The fact that $\widetilde{P}_\beta$ is a projection onto $\operatorname{Fix} P$ follows by standard Cesàro limit arguments; the (complete) contractivity of $P_\beta$ follows from the analogous property of $P$ and the fact that a weak$^*$ limit of contractions is a contraction. The uniqueness of the ternary product follows once again from [7, Proposition 2.1]. □

REMARK 1.7

Let us stress that Proposition 1.5 implies, in particular, that the W$^*$-TRO structure of $\mathrm{Fix}\,P$ does not depend on the choice of the ultrafilter in the above proof (although the map $\widetilde{P}_\beta$ may well do so).

The following result is an abstract extension of [2, Proposition 3.3.1]; the proof is essentially the same.

PROPOSITION 1.8

*Suppose that the assumptions of Theorem 1.6 hold. If there exists a normal (i.e., weak$^*$–weak$^*$-continuous) projection $Q : \mathsf{M} \to \mathrm{Fix}\,P$ such that $Q \circ P = P \circ Q$, then $\widetilde{P}_\beta = Q$ for any free ultrafilter $\beta$.*

*Proof*

Recall that a normal projection is necessarily bounded. Thus, we have for each $x \in \mathsf{M}$ (and a free ultrafilter $\beta$)

$$\begin{aligned}\widetilde{P}_\beta(x) = Q\widetilde{P}_\beta(x) &= Q\Big(\beta\text{–}\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=0}^{n-1} P^k(x)\Big)\\ &= \beta\text{–}\lim_{n\in\mathbb{N}} Q\Big(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x)\Big)\\ &= \beta\text{–}\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=0}^{n-1} P^k(Qx) = Qx.\end{aligned}$$

□

## 2. Actions of locally compact groups on W$^*$-TROs and resulting crossed products

In this section we discuss actions of locally compact groups on W$^*$-TROs and the associated crossed products. A correspondent study in the operator space context was undertaken in [8]; we will comment on some specific analogies at the end of this section.

DEFINITION 2.1

Let $G$ be a locally compact group, and let $\mathsf{X}$ be a W$^*$-TRO. Denote by $\mathrm{Aut}(\mathsf{X})$ the set of all normal automorphisms of $\mathsf{X}$, that is, normal bijective TRO morphisms from $\mathsf{X}$ onto itself. A (continuous) *action* of $G$ on $\mathsf{X}$ is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ such that for each $x \in \mathsf{X}$ the map $\alpha^x : G \to \mathsf{X}$ defined by

$$\alpha^x(s) = \big(\alpha(s)\big)(x), \quad s \in G,$$

is weak$^*$-continuous. We shall write $\alpha_s = \alpha(s)$ for $s \in G$.

The continuity condition above has several equivalent formulations which can be deduced from [22, Sections 13.4 and 13.5]. We record one of them in the following proposition.

PROPOSITION 2.2

*Let $G$ be a locally compact group, and let $\mathsf{X}$ be a W*-TRO. A homomorphism $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ is a continuous action of $G$ on $\mathsf{X}$ if and only if the map $G \times \mathsf{X}_* \ni (s,\varphi) \mapsto \varphi \circ \alpha_s \in \mathsf{X}_*$ is norm-continuous.*

We are ready to connect the action of $G$ on $\mathsf{X}$ with the action on $\mathsf{R}_\mathsf{X}$.

THEOREM 2.3

*Let $\alpha$ be an action of a locally compact group $G$ on a W*-TRO $\mathsf{X}$. Then it possesses a unique extension to an action of $G$ on $\mathsf{R}_\mathsf{X}$.*

*Proof*

First fix $g \in G$, and extend $\alpha_g \in \mathrm{Aut}(\mathsf{X})$ to a normal automorphism $\beta_g \in \mathrm{Aut}(\mathsf{R}_\mathsf{X})$ via Proposition 1.2. The uniqueness of the extensions implies that the resulting family $\{\beta_g : g \in G\}$ defines a homomorphism $\beta : G \to \mathrm{Aut}(\mathsf{R}_\mathsf{X})$. It remains to check that it satisfies the continuity requirement. We do it separately for each corner of the map $\beta$, presenting the argument only for the upper-left corner.

Take $z \in \langle \mathsf{X}\mathsf{X}^* \rangle''$, and consider the map $\beta^z : G \to \mathsf{R}_\mathsf{X}$. We need to show that it is weak*-continuous. As all the maps in question are contractive and we may assume that $\mathsf{X}$ is nondegenerately represented in $B(\mathsf{H};\mathsf{K})$, it suffices to check that, for all $\xi \in \mathsf{K}$, $x \in \mathsf{X}$, $\eta \in \mathsf{H}$, and a net of elements $(s_i)_{i\in\mathcal{I}}$ of $G$ converging to $e \in G$, we have

$$\big\langle \xi, \beta_{s_i}(z)x\eta \big\rangle \xrightarrow{i\in\mathcal{I}} \langle \xi, zx\eta \rangle.$$

Note that, by (1.1),

$$\begin{aligned}\big\langle \xi, \beta_{s_i}(z)x\eta \big\rangle &= \big\langle \xi, \beta_{s_i}(z)\alpha_{s_i}\big(\alpha_{s_i^{-1}}(x)\big)\eta \big\rangle \\ &= \big\langle \xi, \alpha_{s_i}\big(z\alpha_{s_i^{-1}}(x)\big)\eta \big\rangle \\ &= \omega_{\xi,\eta} \circ \alpha_{s_i}\big(z\alpha_{s_i^{-1}}(x)\big),\end{aligned}$$

so putting $\omega := \omega_{\xi,\eta}$ we obtain

$$\big\langle \xi, \beta_{s_i}(z)x\eta \big\rangle - \langle \xi, zx\eta \rangle = (\omega \circ \alpha_{s_i} - \omega)\big(z\alpha_{s_i^{-1}}(x)\big) + \omega\big(z\alpha_{s_i^{-1}}(x) - zx\big).$$

Applying Proposition 2.2, we see that the upper-left corner of $\beta^z$ is weak*-continuous. The remaining parts of the proof follow analogously. $\square$

For an action $\alpha$ of $G$ on a W*-TRO $\mathsf{X}$ we define the fixed-point space $\mathrm{Fix}\,\alpha$ as

$$\mathrm{Fix}\,\alpha = \big\{x \in \mathsf{X} : \forall_{g\in G}\, \alpha_g(x) = x\big\}.$$

It is clear that $\mathrm{Fix}\,\alpha$ is a W*-sub-TRO of $\mathsf{X}$.

COROLLARY 2.4

*Assume that* $\mathsf{X}$ *is a* $W^*$*-TRO that is nondegenerately represented in some* $B(\mathsf{H};\mathsf{K})$, $\alpha$ *is an action of* $G$ *on* $\mathsf{X}$, *and* $\beta$ *is an action of* $G$ *on* $\mathsf{R}_{\mathsf{X}}$ *introduced in Theorem 2.3. Then* $\mathrm{Fix}\,\alpha = P_{\mathsf{K}}(\mathrm{Fix}\,\beta)P_{\mathsf{H}}$.

*Proof*

Let $x \in \mathsf{X}$, $g \in G$. If $\alpha_g(x) = x$, then we also have $\beta_g(x) = x$, and naturally $P_{\mathsf{K}} x P_{\mathsf{H}} = x$, which proves the inclusion $\subset$ in the desired equality. On the other hand if $x = P_{\mathsf{K}} z P_{\mathsf{H}}$ for some $z \in \mathrm{Fix}\,\beta$, then

$$\alpha_g(x) = \beta_g(x) = \beta_g(P_{\mathsf{K}} z P_{\mathsf{H}}) = \beta_g(P_{\mathsf{K}})\beta_g(z)\beta_g(P_{\mathsf{H}}) = P_{\mathsf{K}} z P_{\mathsf{H}} = x,$$

where we used the fact that $\beta_g$ is a homomorphism and that (by construction) it preserves the projections $P_{\mathsf{K}}$ and $P_{\mathsf{H}}$. □

We now discuss the connection of the pointwise actions defined above with their integrated incarnations. The interaction between the two plays a crucial role in [8]—the situation studied there is, however, subtler, as the W*-context (as opposed to the C*-problems studied by Hamana) and the presence of linking von Neumann algebras lead to certain simplifications. Recall that if $G$ is a locally compact group, then $L^\infty(G)$ admits a natural coproduct (see also Section 3) $\Delta : L^\infty(G) \to L^\infty(G) \mathbin{\overline{\otimes}} L^\infty(G)$, defined via the isomorphism $L^\infty(G) \mathbin{\overline{\otimes}} L^\infty(G) \cong L^\infty(G \times G)$ and the formula

$$\Delta(f)(g,h) = f(gh), \quad f \in L^\infty(G),\ g,h \in G.$$

THEOREM 2.5

*Suppose that* $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ *is an action of a locally compact group* $G$ *on a* $W^*$*-TRO* $\mathsf{X}$. *Then there exists a unique map* $\pi_\alpha : \mathsf{X} \to L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{X}$ *such that for each* $f \in L^1(G)$, $\phi \in \mathsf{X}_*$, *and* $x \in \mathsf{X}$ *we have*

$$(2.1) \qquad (f \otimes \phi)\big(\pi_\alpha(x)\big) = \int_G f(g)\phi\big(\alpha_{g^{-1}}(x)\big)\,dg.$$

*Moreover, if we write* $\gamma := \pi_\alpha$, *then* $\gamma$ *is an injective, normal, nondegenerate TRO morphism such that*

$$(2.2) \qquad (\Delta \otimes \mathrm{id}_{\mathsf{X}}) \circ \gamma = (\mathrm{id}_{L^\infty(G)} \otimes \gamma) \circ \gamma.$$

*Conversely, if* $\gamma : \mathsf{X} \to L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{X}$ *is an injective, normal, nondegenerate TRO morphism satisfying (2.2), then there exists a unique action* $\alpha$ *of* $G$ *on* $\mathsf{X}$ *such that* $\gamma = \pi_\alpha$.

*Proof*

We may assume that $\mathsf{X}$ is nondegenerately represented in $B(\mathsf{H};\mathsf{K})$. Assume that we are given an action $\alpha : G \to \mathrm{Aut}(\mathsf{X})$, and extend it pointwise, using Theorem 2.3, to a continuous action $\beta : G \to \mathrm{Aut}(\mathsf{R}_{\mathsf{X}})$. The discussion in [22, Section 18.6] implies that there exists an injective, normal, unital *-homomorphism

$\pi_\beta : \mathsf{R}_\mathsf{X} \to L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{R}_\mathsf{X}$ such that for each $f \in L^1(G)$, $\phi \in (\mathsf{R}_\mathsf{X})_*$, and $z \in \mathsf{R}_\mathsf{X}$ we have

$$(f \otimes \phi)\big(\pi_\beta(z)\big) = \int_G f(g)\phi\big(\beta_{g^{-1}}(z)\big)\,dg \tag{2.3}$$

and $(\Delta \otimes \mathrm{id}) \circ \pi_\beta = (\mathrm{id} \otimes \pi_\beta) \circ \pi_\beta$. (Note that our formulas are formally different from Strătilă's: the difference is, however, only in the fact that we choose to work with maps taking values in $L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{R}_\mathsf{X}$, and not in $\mathsf{R}_\mathsf{X} \mathbin{\overline{\otimes}} L^\infty(G)$, which allows us to work with the standard coproduct of $L^\infty(G)$, and not with the opposite one as in [22].) Consider the map $\pi_\beta|_\mathsf{X}$. We want to show that it takes values in $\mathsf{Y} := L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{X}$. Considerations before Lemma 1.4 imply that the latter space is a W*-TRO equal to $(I_{L^2(G)} \otimes P_\mathsf{K})(L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{R}_\mathsf{X})(I_{L^2(G)} \otimes P_\mathsf{H})$; moreover, $\mathsf{R}_\mathsf{Y} = L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{R}_\mathsf{X}$. Then let $x \in \mathsf{X}$. It suffices to show that for any $f \in L^1(G)$ and $\phi \in (\mathsf{R}_\mathsf{X})_*$ we have

$$(f \otimes \phi)\big(\pi_\beta(x)\big) = (f \otimes \phi)\big((I_{L^2(G)} \otimes P_\mathsf{K})\pi_\beta(x)(I_{L^2(G)} \otimes P_\mathsf{H})\big).$$

This, however, follows immediately from (2.3) once we note that $\beta_g(x) = \alpha_g(x)$ for all $g \in G$ and that $y \mapsto \phi(P_\mathsf{K} y P_\mathsf{H})$ is a normal functional on $\mathsf{R}_\mathsf{X}$. Thus, we showed that $\gamma := \pi_\beta|_\mathsf{X}$ maps $\mathsf{X}$ into $\mathsf{Y}$. It is an injective, normal TRO morphism satisfying (2.2) (as a restriction of an injective, normal *-homomorphism satisfying (2.2)). By the uniqueness of Hamana extensions and the above identification of $\mathsf{R}_\mathsf{Y}$, we deduce that $\pi_\beta$ is the Hamana extension of $\gamma$, so that the nondegeneracy of $\gamma$ follows from the unitality of $\pi_\beta$ via Proposition 1.2.

Assume now that $\gamma : \mathsf{X} \to L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{X}$ is an injective, normal, nondegenerate TRO morphism satisfying the action equation (2.2). Again write $\mathsf{Y} = L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{X}$, and let $\pi : \mathsf{R}_\mathsf{X} \to \mathsf{R}_\mathsf{Y} = L^\infty(G) \mathbin{\overline{\otimes}} \mathsf{R}_\mathsf{X}$ denote the Hamana extension of $\gamma$. Proposition 1.2 implies that $\pi$ is a unital, injective, normal *-homomorphism. The normality of $\pi$ and $\Delta$ implies that it suffices to check the validity of the action equation with $\gamma$ replaced by $\pi$ on a weak*-dense subset; this follows in turn from computations of the type $(x, z \in \mathsf{X})$:

$$\begin{aligned}
(\Delta \otimes \mathrm{id}_\mathsf{X})\big(\pi(xz^*)\big) &= (\Delta \otimes \mathrm{id}_\mathsf{X})\big(\gamma(x)\gamma(z)^*\big) \\
&= (\Delta \otimes \mathrm{id}_\mathsf{X})\big(\gamma(x)\big)(\Delta \otimes \mathrm{id}_\mathsf{X})\big(\gamma(z)\big)^* \\
&= (\mathrm{id}_{L^\infty(G)} \otimes \gamma)\big(\gamma(x)\big)(\mathrm{id}_{L^\infty(G)} \otimes \gamma)\big(\gamma(z)\big)^* \\
&= (\mathrm{id}_{L^\infty(G)} \otimes \pi)\big(\gamma(x)\gamma(z)^*\big) \\
&= (\mathrm{id}_{L^\infty(G)} \otimes \pi)\big(\pi(xz^*)\big).
\end{aligned}$$

It follows from [22, Section 18.6] (or rather its left version) that there exists an action $\beta : G \to \mathrm{Aut}(\mathsf{R}_\mathsf{X})$ such that $\pi = \pi_\beta$, where $\pi_\beta$ is defined via formula (2.3). It remains to show that for each $g \in G$ the map $\alpha_g := \beta_g|_\mathsf{X}$ takes values in $\mathsf{X}$, as then it will be easy to check that the family $(\alpha_g)_{g\in G}$ defines an action of $G$ on $\mathsf{X}$ and that $\gamma$ arises from this action via the formulas given in the theorem. Then fix $x \in \mathsf{X}$, and define $z_g = \beta_{g^{-1}}(x)$ for each $g \in G$. Then $z : G \to \mathsf{R}_\mathsf{X}$ is a weak*-continuous function, and we know that for all $f \in L^1(G)$ and all $\phi \in (\mathsf{R}_\mathsf{X})_*$

such that $\phi(\mathsf{X}) = \{0\}$ we have

$$\int_G f(g)\phi(z_g)\,dg = 0.$$

But then we deduce immediately that $\phi(z_g)$ is 0 for almost all $g \in G$, and as it is a continuous function it must actually be 0 everywhere. This in turn means that $z_g \in \mathsf{X}$ for all $g \in G$, which ends the proof. □

The next proposition is also familiar from the von Neumann algebraic context.

PROPOSITION 2.6

*Suppose that $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ is an action of a locally compact group $G$ on a $W^*$-TRO $\mathsf{X}$, and assume that $\mathsf{X}$ is nondegenerately represented in $B(\mathsf{H};\mathsf{K})$. Then the map $\pi_\alpha$ introduced in Theorem 2.5 may be viewed as a faithful representation of $\mathsf{X}$ in $B(L^2(G;\mathsf{H}); L^2(G;\mathsf{K}))$, and we have for all $x \in \mathsf{X}$ and $\zeta \in L^2(G;\mathsf{H})$*

$$\big(\pi_\alpha(x)(\zeta)\big)(g) = \alpha_{g^{-1}}(x)\zeta(g) \quad \text{for almost every } g \in G.$$

*Proof*

The fact that $\pi_\alpha$ can be viewed as a faithful representation of $\mathsf{X}$ in $B(L^2(G;\mathsf{H}); L^2(G;\mathsf{K}))$ follows from Theorem 2.5.

It remains to prove the displayed formula. We identify $L^2(G;\mathsf{H})$ with $L^2(G) \otimes \mathsf{H}$ and $L^2(G;\mathsf{K})$ with $L^2(G) \otimes \mathsf{K}$, and let $\xi \in \mathsf{H}$, $\eta \in \mathsf{K}$, and $f, h \in L^2(G)$. Then

$$\langle h \otimes \eta, \pi_\alpha(x)(f \otimes \xi)\rangle = \int \big\langle h(g)\eta, \big(\pi_\alpha(x)(f \otimes \xi)\big)(g)\big\rangle\,dg.$$

By Theorem 2.5, the left-hand side of the previous identity is equal to

$$\int f(g)\overline{h(g)}\langle \eta, \alpha_{g^{-1}}(x)\xi\rangle\,dg = \int \big\langle h(g)\eta, f(g)\alpha_{g^{-1}}(x)\xi\big\rangle\,dg.$$

Then the displayed formula follows by density. □

In the next lemma we show how implemented actions of $G$ on W*-TROs look.

LEMMA 2.7

*Assume that $\mathsf{X}$ is a concrete $W^*$-TRO in $B(\mathsf{H};\mathsf{K})$, that $\sigma : G \to B(\mathsf{H})$, $\tau : G \to B(\mathsf{K})$ are so-continuous representations of $G$, and that for each $g \in G$ and $x \in \mathsf{X}$ the operator $\tau_g x \sigma_g^*$ belongs to $\mathsf{X}$. Then the map $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ defined by*

$$\alpha_g(x) = \tau_g x \sigma_g^*, \quad g \in G,\ x \in \mathsf{X},$$

*is an action of $G$ on $\mathsf{X}$.*

*Proof*

We can conduct straightforward checks: we first observe that $\alpha_g$ is indeed a normal TRO automorphism of $\mathsf{X}$ and then verify that $\alpha$ is a homomorphism and that the continuity conditions are satisfied. □

In fact, all actions of groups on TROs can be put in this form, at the cost of extending the TRO in question. This is analogous to the crossed product construction for the actions of groups on von Neumann subalgebras.

LEMMA 2.8

*Let $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ be an action of a locally compact group $G$ on a $W^*$-TRO $\mathsf{X}$. Assume that $\mathsf{X}$ is concretely represented as a $W^*$-sub-TRO of $B(\mathsf{H};\mathsf{K})$. Let $\pi := \pi_\alpha : \mathsf{X} \to B(L^2(G) \otimes \mathsf{H}; L^2(G) \otimes \mathsf{K})$ be the representation of $\mathsf{X}$ introduced in Proposition 2.6, and let $\tau = \lambda \otimes I_{\mathsf{K}}$, $\sigma = \lambda \otimes I_{\mathsf{H}}$ denote the respective amplifications of the left regular representation of $G$. Then the space*

$$\text{w}^*\text{-cl}\,\mathrm{Lin}\big\{\big(\mathrm{VN}(G) \otimes I_{\mathsf{K}}\big)\pi(\mathsf{X})\big\} \tag{2.4}$$

*is equal to*

$$\begin{aligned}\text{w}^*\text{-cl}\,\mathrm{Lin}\big\{\tau_g\pi(x) : g \in G, x \in \mathsf{X}\big\} &= \text{w}^*\text{-cl}\,\mathrm{Lin}\big\{\tau_g\pi(x)\sigma_{g'} : g, g' \in G, x \in \mathsf{X}\big\} \\ &= \text{w}^*\text{-cl}\,\mathrm{Lin}\big\{\pi(x)\sigma_g : g \in G, x \in \mathsf{X}\big\}\end{aligned}$$

*and is a $W^*$-TRO. Moreover, we have the following formula:*

$$\pi\big(\alpha_g(x)\big) = \tau_g\pi(x)\sigma_g^*, \quad g \in G, \ x \in \mathsf{X}. \tag{2.5}$$

*Proof*

It suffices to prove the formula (2.5). The rest is based on easy checks.

For $x \in \mathsf{X}$, $\xi \in L^2(G;\mathsf{H})$, and almost every $g, h \in G$, we have

$$\begin{aligned}\big(\tau_g\pi(x)\sigma_g^*\xi\big)(h) &= \big(\pi(x)\sigma_{g^{-1}}\xi\big)(g^{-1}h) = \alpha_{h^{-1}g}(x)\big((\sigma_{g^{-1}}\xi)(g^{-1}h)\big) \\ &= \alpha_{h^{-1}}\big(\alpha_g(x)\big)\big(\xi(h)\big) = \pi\big(\alpha_g(x)\xi\big)(h),\end{aligned}$$

as claimed. □

DEFINITION 2.9

Let $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ be an action of a locally compact group $G$ on a W*-TRO $\mathsf{X}$. The W*-TRO described by formula (2.4) above is called the *crossed product* of $\mathsf{X}$ by $\alpha$ and is denoted $G \ltimes_\alpha \mathsf{X}$.

PROPOSITION 2.10

*Let $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ be an action of a locally compact group $G$ on a $W^*$-TRO $\mathsf{X}$, and let $\beta : G \to \mathrm{Aut}(\mathsf{R}_{\mathsf{X}})$ be an action of $G$ on $\mathsf{R}_{\mathsf{X}}$ introduced in Theorem 2.3. Then the crossed product $G \ltimes_\alpha \mathsf{X}$ is the corner in the crossed product $G \ltimes_\beta \mathsf{R}_{\mathsf{X}}$: if we start from $\mathsf{X}$ represented nondegenerately in $B(\mathsf{H};\mathsf{K})$, we obtain*

$$(I_{L^2(G)} \otimes P_{\mathsf{K}})(G \ltimes_\beta \mathsf{R}_{\mathsf{X}})(I_{L^2(G)} \otimes P_{\mathsf{H}}) = G \ltimes_\alpha \mathsf{X}.$$

*Proof*

This is an immediate consequence of the fact that the space

$$(I_{L^2(G)} \otimes P_{\mathsf{K}})\big(\big(\mathrm{VN}(G) \otimes I_{\mathsf{K}\oplus\mathsf{H}}\big)\pi_\beta(\mathsf{R}_{\mathsf{X}})\big)(I_{L^2(G)} \otimes P_{\mathsf{H}})$$

coincides with $(\mathrm{VN}(G) \otimes I_{\mathsf{K}})\pi_\alpha(\mathsf{X})$, which was established in the proof of Theorem 2.5 and through the weak* density of respective spaces. □

The following corollary is now easy to observe, once again by using the von Neumann algebra result and referring to the properties of Hamana extensions.

COROLLARY 2.11
*The crossed product $G \ltimes_\alpha \mathsf{X}$ does not depend on the choice of the original faithful nondegenerate representation of $\mathsf{X}$.*

The final result in this section explains the connection between the definition of the crossed product introduced above and that of Hamana [8]. Before we formulate it we need to introduce another action: if $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ is an action, then $\mathrm{Ad}_\rho \otimes \alpha$ is an action of $G$ on the W*-TRO $B(L^2(G)) \overline{\otimes} \mathsf{X}$ given by the formula

$$(\mathrm{Ad}_\rho \otimes \alpha)_g(z) = (\mathrm{Ad}_{\rho_g} \otimes \alpha_g)(z), \quad z \in B\big(L^2(G)\big) \overline{\otimes} \mathsf{X}, \tag{2.6}$$

where $\rho : G \to B(L^2(G))$ is the right regular representation and we take the convention that $\mathrm{Ad}_{\rho_g}(z) = \rho_g z \rho_g^*$ for $z \in B(L^2(G))$. Note for further use the following fact: if we write $\delta := \mathrm{Ad}_\rho \otimes \alpha$, then the corresponding map $\pi_\delta : B(L^2(G)) \overline{\otimes} \mathsf{X} \to L^\infty(G) \overline{\otimes} B(L^2(G)) \overline{\otimes} \mathsf{X}$ is given explicitly by the formula

$$\begin{aligned} \pi_\delta(z) &= \chi_{12}\big((V^* \otimes \mathrm{id}_{\mathsf{X}})(\mathrm{id}_{B(L^2(G))} \otimes \pi_\alpha)(z)(V \otimes \mathrm{id}_{\mathsf{X}})\big), \\ & z \in B\big(L^2(G)\big) \overline{\otimes} \mathsf{X}, \end{aligned} \tag{2.7}$$

where $V \in B(L^2(G) \otimes L^2(G))$ is the *right multiplicative unitary of* $G$, defined by the formula

$$(Vf)(g,h) = \delta(h)^{1/2} f(gh,h), \quad f \in L^2(G),\ g,h \in G, \tag{2.8}$$

where $\delta$ is the modular function of $G$, and $\chi_{12}$ flips the first two legs of the tensor product.

PROPOSITION 2.12
*If $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ is an action of $G$ on a W*-TRO $\mathsf{X}$, then the following equality holds:*

$$G \ltimes_\alpha \mathsf{X} = \mathrm{Fix}(\mathrm{Ad}_\rho \otimes \alpha).$$

*Proof*
Assume that $\mathsf{X}$ is nondegenerately represented in $B(\mathsf{H};\mathsf{K})$, and consider the extension of $\alpha$ to an action $\beta$ of $G$ on the von Neumann algebra $\mathsf{R}_{\mathsf{X}}$ given by Lemma 2.3. The uniqueness of Hamana extensions shows that the action $\mathrm{Ad}_\rho \otimes \beta$ of $G$ on $B(L^2(G)) \overline{\otimes} \mathsf{R}_{\mathsf{X}}$ is the canonical extension of $\mathrm{Ad}_\rho \otimes \alpha$, the action of $G$ on $B(L^2(G)) \overline{\otimes} \mathsf{X}$. The left version of [22, Corollary 19.13], attributed there to M. Takesaki and T. Digernes, shows that $G \ltimes_\beta \mathsf{R}_{\mathsf{X}} = \mathrm{Fix}(\mathrm{Ad}_\rho \otimes \beta)$. In view of

Proposition 2.10 it remains to show that

$$\mathrm{Fix}(\mathrm{Ad}_\rho \otimes \alpha) = (I_{L^2(G)} \otimes P_{\mathsf{K}})\,\mathrm{Fix}(\mathrm{Ad}_\rho \otimes \beta)(I_{L^2(G)} \otimes P_{\mathsf{H}}).$$

This, however, follows from Corollary 2.4 in view of the comments above. □

REMARK 2.13

Hamana [8] defines the crossed product for an action of a group on an operator space directly via the fixed-point formula of the type above. Note, however, that his definition does not coincide explicitly with ours, as he follows the approach of [16], where everything is formulated in terms of the *right* invariant Haar measure of $G$ (so that the crossed product contains the amplification of the *right* group von Neumann algebra).

It should be clear from the above discussions that it is also possible to develop the TRO crossed product construction in the C*-setting, we will, however, not need it in the rest of this article.

## 3. Actions of locally compact quantum groups on W*-TROs and resulting crossed products

In this section we discuss actions of locally compact quantum groups on W*-TROs and define associated crossed products.

We follow the von Neumann algebraic approach to locally compact quantum groups due to Kustermans and Vaes [15] (see also [11] and [12] for more background). A *locally compact quantum group* $\mathbb{G}$, effectively a virtual object, is studied via the von Neumann algebra $L^\infty(\mathbb{G})$, playing the role of the algebra of essentially bounded measurable functions on $\mathbb{G}$, equipped with a *coproduct* $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$, which is a unital normal coassociative *-homomorphism. A locally compact quantum group $\mathbb{G}$ is by definition assumed to admit a left Haar weight $\phi$ and a right Haar weight $\psi$; these are faithful, normal, semifinite weights on $L^\infty(\mathbb{G})$ satisfying suitable invariance conditions. The Gelfand–Naimark–Segal representation space for the left Haar weight will be denoted by $L^2(\mathbb{G})$. All the information about $\mathbb{G}$ is contained in the *right multiplicative unitary* $V \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$; it is a unitary operator such that we have

$$\Delta(x) = V(x \otimes I_{L^2(\mathbb{G})})V^*, \quad x \in L^\infty(\mathbb{G}).$$

This fact enables us to define a natural extension of the coproduct, the map $\widetilde{\Delta} : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ given by the same formula

$$\widetilde{\Delta}(y) = V(y \otimes I_{L^2(\mathbb{G})})V^*, \quad y \in B\big(L^2(\mathbb{G})\big). \tag{3.1}$$

In fact, $\widetilde{\Delta}$ takes values in $B(L^2(\mathbb{G})) \overline{\otimes} L^\infty(\mathbb{G})$ as $V \in L^\infty(\widehat{\mathbb{G}})' \overline{\otimes} L^\infty(\mathbb{G})$, where $\widehat{\mathbb{G}}$ is the *dual* locally compact quantum group of $\mathbb{G}$. (The algebra $L^\infty(\widehat{\mathbb{G}})$ acts naturally on $L^2(\mathbb{G})$.) If $\mathbb{G} = G$ happens to be a locally compact group, then $L^\infty(\widehat{\mathbb{G}}) = \mathrm{VN}(G)$. Finally note that by analogy with the classical situation we denote the predual of $L^\infty(\mathbb{G})$ by $L^1(\mathbb{G})$.

Recall the standard definition of an action of $\mathbb{G}$ on a von Neumann algebra $\mathsf{M}$. A (continuous, left) action of $\mathbb{G}$ on $\mathsf{M}$ is an injective, normal, unital $^*$-homomorphism $\beta : \mathsf{M} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{M}$ satisfying the action equation

$$(\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \beta) \circ \beta = (\Delta_{\mathbb{G}} \otimes \mathrm{id}_{\mathsf{M}}) \circ \beta.$$

Replacing a von Neumann algebra with a W$^*$-TRO yields no extra complications.

DEFINITION 3.1

Let $\mathsf{X}$ be a W$^*$-TRO, and let $\mathbb{G}$ be a locally compact quantum group. An *action* of $\mathbb{G}$ on $\mathsf{X}$ is an injective, normal, nondegenerate TRO morphism $\alpha : \mathsf{X} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$ such that

$$(\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \alpha) \circ \alpha = (\Delta_{\mathbb{G}} \otimes \mathrm{id}_{\mathsf{M}}) \circ \alpha.$$

Theorem 2.5 implies that if $\mathbb{G} = G$ happens to be a classical locally compact group, then the definition above agrees with Definition 2.1.

PROPOSITION 3.2

*Let $\mathsf{X}$ be a W$^*$-TRO, and let $\mathbb{G}$ be a locally compact quantum group acting on $\mathsf{X}$ via $\alpha : \mathsf{X} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$. Then the extension provided by Proposition 1.2 defines an action $\beta = \mathsf{R}_{\mathsf{X}} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{R}_{\mathsf{X}}$ of $\mathbb{G}$ on $\mathsf{R}_{\mathsf{X}}$.*

*Proof*

Similar to the proof of Theorem 2.5: effectively we use the fact that if we denote the W$^*$-TRO $L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$ by $\mathsf{Y}$, then we have $\mathsf{R}_{\mathsf{Y}} \cong L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{R}_{\mathsf{X}}$. □

If $\beta : \mathsf{M} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{M}$ is an action of $\mathbb{G}$ on a von Neumann algebra $\mathsf{M}$, then the crossed product $\mathbb{G} \ltimes_\beta \mathsf{M}$ is defined as the von Neumann algebra $((L^\infty(\widehat{\mathbb{G}}) \otimes I_{\mathsf{M}})\beta(\mathsf{M}))''$. Equivalently,

$$\mathbb{G} \ltimes_\beta \mathsf{M} = \text{w}^*\text{-cl}\,\mathrm{Lin}\big\{(y \otimes I_{\mathsf{M}})\beta(m) : y \in L^\infty(\widehat{\mathbb{G}}), m \in \mathsf{M}\big\}.$$

The last equality amounts to the fact that the weak$^*$ closure of $(L^\infty(\widehat{\mathbb{G}}) \otimes I_{\mathsf{M}})\beta(\mathsf{M})$ is a $^*$-subspace of $B(L^2(\mathbb{G})) \overline{\otimes} \mathsf{M}$. This is a well-known fact, formulated for example in [13, Proposition 2.3]: it can be shown using a simpler version of the C$^*$-algebraic calculations in [20] after [20, Definition 2.4].

DEFINITION 3.3

Let $\mathsf{X}$ be a W$^*$-TRO nondegenerately represented in $B(\mathsf{H};\mathsf{K})$, and let $\mathbb{G}$ be a locally compact quantum group acting on $\mathsf{X}$ via $\alpha : \mathsf{X} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$. The *W$^*$-TRO crossed product* $\mathbb{G} \ltimes_\alpha \mathsf{X}$ is defined as the weak$^*$ closure of the linear span of $(L^\infty(\widehat{\mathbb{G}}) \otimes I_{\mathsf{K}})\alpha(\mathsf{X})$.

We will soon note that again the crossed product has several other descriptions, but we first need to record the quantum version of Proposition 2.10.

PROPOSITION 3.4

*Let* $\mathsf{X}$ *be a* $W^*$*-TRO nondegenerately represented in* $B(\mathsf{H};\mathsf{K})$*, and let* $\mathbb{G}$ *be a locally compact quantum group acting on* $\mathsf{X}$ *via* $\alpha:\mathsf{X}\to L^\infty(\mathbb{G})\,\overline{\otimes}\,\mathsf{X}$*. Let* $\beta:\mathsf{R}_\mathsf{X}\to L^\infty(\mathbb{G})\,\overline{\otimes}\,\mathsf{R}_\mathsf{X}$ *be the action of* $\mathbb{G}$ *on* $\mathsf{R}_\mathsf{X}$ *provided by Proposition 3.2. Then*

$$(I_{L^2(G)}\otimes P_\mathsf{K})(\mathbb{G}\ltimes_\beta \mathsf{R}_\mathsf{X})(I_{L^2(G)}\otimes P_\mathsf{H})=\mathbb{G}\ltimes_\alpha \mathsf{X}.$$

*Proof*

This result follows exactly as in the case of Proposition 2.10. □

COROLLARY 3.5

*Under the assumptions of Definition 3.3, we have the following equalities:*

$$\begin{aligned}\mathbb{G}\ltimes_\alpha \mathsf{X}&=\mathrm{w}^*\text{-cl}\,\mathrm{Lin}\big\{\big(L^\infty(\widehat{\mathbb{G}})\otimes I_\mathsf{K}\big)\alpha(\mathsf{X})\big(L^\infty(\widehat{\mathbb{G}})\otimes I_\mathsf{H}\big)\big\}\\&=\mathrm{w}^*\text{-cl}\,\mathrm{Lin}\big\{\alpha(\mathsf{X})\big(L^\infty(\widehat{\mathbb{G}})\otimes I_\mathsf{H}\big)\big\}.\end{aligned}$$

*Moreover,* $\mathbb{G}\ltimes_\alpha \mathsf{X}$ *is the* $W^*$*-TRO generated in* $B(L^2(\mathbb{G})\otimes\mathsf{H};L^2(\mathbb{G})\otimes\mathsf{K})$ *by the set* $(L^\infty(\widehat{\mathbb{G}})\otimes I_\mathsf{K})\alpha(\mathsf{X})$*. It does not depend (up to an isomorphism) on the initial choice of a faithful nondegenerate representation of* $\mathsf{X}$*.*

*Proof*

This follows immediately from the analogous facts for the von Neumann crossed products and Proposition 3.4. □

If $\alpha:\mathsf{X}\to L^\infty(\mathbb{G})\,\overline{\otimes}\,\mathsf{X}$ is an action of $\mathbb{G}$ on a W*-TRO $\mathsf{X}$, then the fixed-point space of $\alpha$ is defined as

$$\mathrm{Fix}\,\alpha=\big\{x\in\mathsf{X}:\alpha(x)=I_{L^2(\mathbb{G})}\otimes x\big\}.$$

PROPOSITION 3.6

*Assume that* $\mathsf{X}$ *is a* $W^*$*-TRO nondegenerately represented in some* $B(\mathsf{H};\mathsf{K})$*,* $\alpha$ *is an action of* $\mathbb{G}$ *on* $\mathsf{X}$*, and* $\beta$ *is an action of* $\mathbb{G}$ *on* $\mathsf{R}_\mathsf{X}$ *as introduced in Proposition 3.2. Then* $\mathrm{Fix}\,\alpha=P_\mathsf{K}(\mathrm{Fix}\,\beta)P_\mathsf{H}$*.*

*Proof*

This result follows as in Corollary 2.4. □

Let $\alpha:\mathsf{X}\to L^\infty(\mathbb{G})\,\overline{\otimes}\,\mathsf{X}$ be an action of $\mathbb{G}$ on $\mathsf{X}$, and consider the map $\delta:B(L^2(\mathbb{G}))\,\overline{\otimes}\,\mathsf{X}\to L^\infty(\mathbb{G})\,\overline{\otimes}\,B(L^2(\mathbb{G}))\,\overline{\otimes}\,\mathsf{X}$:

$$\begin{aligned}&\delta(z)=\chi_{12}\big((V^*\otimes\mathrm{id}_\mathsf{X})(\mathrm{id}_{B(L^2(G))}\otimes\alpha)(z)(V\otimes\mathrm{id}_\mathsf{X})\big),\\&\qquad z\in B\big(L^2(\mathbb{G})\big)\,\overline{\otimes}\,\mathsf{X},\end{aligned}\tag{3.2}$$

where $V$ is the right multiplicative unitary of $\mathbb{G}$. (Compare this to the formula (2.7), remembering that for quantum groups we denote simply by $\alpha$ what used to be $\pi_\alpha$.)

The following result is a quantum version of Proposition 2.12, this time following from the von Neumann algebraic result due to Enock [6] (see also [12]).

THEOREM 3.7

*Let* $\mathsf{X}$ *be a* $W^*$*-TRO, and let* $\mathbb{G}$ *be a locally compact quantum group acting on* $\mathsf{X}$ *via* $\alpha : \mathsf{X} \to L^\infty(\mathbb{G}) \,\overline{\otimes}\, \mathsf{X}$*. The map* $\delta : B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{X} \to L^\infty(\mathbb{G}) \,\overline{\otimes}\, B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{X}$ *defined by (3.2) is an action of* $\mathbb{G}$ *on the* $W^*$*-TRO* $B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{X}$*. Moreover, we have the following equality:*

$$\mathbb{G} \ltimes_\alpha \mathsf{X} = \mathrm{Fix}\,\delta.$$

*Proof*

Let $\mathsf{X}$ be nondegenerately represented in $B(\mathsf{H};\mathsf{K})$, and let $\beta$ be the action of $\mathbb{G}$ on $\mathsf{R}_\mathsf{X}$ provided in Theorem 3.2. Then [12, Theorem 2.3], which is a simplified version of [6, Theorem 11.6], says that the map $\gamma : B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{R}_\mathsf{X} \to L^\infty(\mathbb{G}) \,\overline{\otimes}\, B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{R}_\mathsf{X}$,

$$\begin{aligned} \gamma(t) &= \chi_{12}\big((V^* \otimes \mathrm{id}_\mathsf{X})(\mathrm{id}_{B(L^2(G))} \otimes \beta)(t)(V \otimes \mathrm{id}_\mathsf{X})\big), \\ &\quad t \in B\big(L^2(\mathbb{G})\big) \,\overline{\otimes}\, \mathsf{R}_\mathsf{X}, \end{aligned} \tag{3.3}$$

is an action of $\mathbb{G}$ on $\mathsf{R}_\mathsf{X}$ and

$$\mathbb{G} \ltimes_\beta \mathsf{R}_\mathsf{X} = \mathrm{Fix}\,\gamma. \tag{3.4}$$

It is easy to verify that in fact for $z \in B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{X} \subset B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{R}_\mathsf{X}$ we have

$$\delta(z) = (I_{L^2(\mathbb{G})} \otimes I_{L^2(\mathbb{G})} \otimes P_\mathsf{K})\gamma(z)(I_{L^2(\mathbb{G})} \otimes I_{L^2(\mathbb{G})} \otimes P_\mathsf{H}).$$

This implies, via Corollary 1.3, that $\delta$ is a normal TRO morphism whose Hamana extension is $\gamma$. An explicit computation and another application of Proposition 1.2 show that $\delta$ is in fact an action of $\mathbb{G}$ on the W*-TRO $B(L^2(\mathbb{G})) \,\overline{\otimes}\, \mathsf{X}$, with $\gamma$ clearly being the extension of $\delta$ provided by Proposition 3.2. Then formula (3.4) and Propositions 3.4 and 3.6 end the proof. □

We finish this section by discussing certain connections between the TROs arising as fixed-point spaces of completely contractive maps, which were studied in Section 1, and (quantum) group actions.

PROPOSITION 3.8

*Let* $\mathsf{M}$ *be a von Neumann algebra, and let* $\beta : \mathsf{M} \to L^\infty(\mathbb{G}) \,\overline{\otimes}\, \mathsf{M}$ *be an action of a locally compact quantum group on* $\mathsf{M}$*. Let* $\widetilde{\mathsf{X}}$ *be a weak*$^*$*-closed subspace of* $\mathsf{M}$*, and let* $P : \mathsf{M} \to \mathsf{M}$ *be a completely contractive idempotent map such that* $P(\mathsf{M}) = \widetilde{\mathsf{X}}$ *and*

$$\beta \circ P = (\mathrm{id}_{L^\infty(\mathbb{G})} \otimes_F P) \circ \beta. \tag{3.5}$$

*Then the formula (see Proposition 1.5)*

$$\alpha = (\mathrm{id}_{L^\infty(\mathbb{G})} \,\overline{\otimes}\, \iota) \circ \beta \circ \iota^{-1}$$

*defines a normal, injective TRO morphism* $\alpha : \mathsf{X} \to L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$ *satisfying the action equation*

$$(\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \alpha) \circ \alpha = (\Delta_{\mathbb{G}} \otimes \mathrm{id}_{\mathsf{M}}) \circ \alpha.$$

*Proof*
For $x, y, z \in \widetilde{\mathsf{X}}$, we have

$$\begin{aligned}\alpha\big(\{\iota(x), \iota(y), \iota(z)\}_{\mathsf{X}}\big) &= (\mathrm{id}_{L^\infty(\mathbb{G})} \overline{\otimes}\, \iota) \circ \beta\big(P(xy^*z)\big) \\ &= (\mathrm{id}_{L^\infty(\mathbb{G})} \overline{\otimes}\, \iota) \circ (\mathrm{id}_{L^\infty(\mathbb{G})} \otimes_F P)\big(\beta(x)\beta(y)^*\beta(z)\big) \\ &= \big\{\alpha\big(\iota(x)\big), \alpha\big(\iota(y)\big), \alpha\big(\iota(z)\big)\big\}_{L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}}.\end{aligned}$$

Hence, $\alpha$ is a TRO morphism, and it is normal, because $\iota$ is a weak$^*$-homeomorphism. It is also easy to check that $\alpha$ is injective and satisfies the coassociativity condition, using the corresponding properties of $\beta$. □

REMARK 3.9

We do not know whether the map constructed above is an action of $\mathbb{G}$ on $\mathsf{X}$, as it is not clear whether it is nondegenerate. Let us sketch a natural approach to proving nondegeneracy, so that it is clear where it breaks down. Using the notation of the last proposition we should show that the weak$^*$-closed linear span of elements of the form

$$\big\{\alpha\big(\iota(x)\big), a \otimes \iota(y), b \otimes \iota(z)\big\}_{L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}}, \quad x, y, z \in \widetilde{\mathsf{X}},\ a, b \in L^\infty(\mathbb{G}),$$

is equal to $L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{X}$. Writing $x = P(m)$ for $m \in \mathsf{M}$, we have

$$\begin{aligned}&\big\{\alpha\big(\iota(x)\big), a \otimes \iota(y), b \otimes \iota(z)\big\}_{L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}} \\ &\quad = \big\{(\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \iota P)\beta(m), a \otimes \iota(y), b \otimes \iota(z)\big\}_{L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}} \\ &\quad = (\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \iota P)\big(\beta(m)(a^*b \otimes y^*z)\big),\end{aligned}$$

where we used Youngson's identity

$$P\big(P(m_1)P(m_2)^*P(m_3)\big) = P\big(m_1 P(m_2)^* P(m_3)\big).$$

In other words, it suffices to show that $(\mathrm{id} \otimes P)(\beta(\mathsf{M})(L^\infty(\mathbb{G}) \overline{\otimes} \widetilde{\mathsf{X}}^*\widetilde{\mathsf{X}}))$ is linearly weak$^*$-dense in $L^\infty(\mathbb{G}) \overline{\otimes} \widetilde{\mathsf{X}}$. Now we know on one hand via [13, Proposition 2.9] that $\beta(\mathsf{M})(L^\infty(\mathbb{G}) \otimes 1)$ is linearly weak$^*$-dense in $L^\infty(\mathbb{G}) \overline{\otimes} \mathsf{M}$ and on the other hand that $P(\mathsf{M}\widetilde{\mathsf{X}}^*\widetilde{\mathsf{X}})$ is weak$^*$-dense in $\widetilde{\mathsf{X}}$ (essentially because $\mathsf{X}$ is a W$^*$-TRO). Combining these two facts brings us close to completing the proof, but $P$ is not assumed to be normal (and cannot be for applications; see, e.g., [2, Propositions 3.1.1 and 3.1.3], which show that, given a locally compact group $G$ and a norm one function $\sigma$ in the Fourier–Stieltjes algebra $B(G)$ such that the set $\sigma^{-1}(\{1\})$ is not open, the projection from $\mathrm{VN}(G)$ onto the set $I_\sigma^\perp \subset \mathrm{VN}(G)$ is never normal). Note that in the positive case (by which we mean the case where $P$ is a completely positive projection and $\mathsf{X}$ is a von Neumann algebra) the argument goes through simply by choosing $m = 1$.

The problem disappears in the case in which $\mathbb{G}$ is a classical group, as then we can use the pointwise picture of the actions.

THEOREM 3.10

*Let* $\mathsf{M}$ *be a von Neumann algebra, and let* $\beta : G \to \mathrm{Aut}(\mathsf{M})$ *be an action of a locally compact group* $G$ *on* $\mathsf{M}$*. Let* $\widetilde{\mathsf{X}}$ *be a weak*-closed subspace of* $\mathsf{M}$*, and let* $P : \mathsf{M} \to \mathsf{M}$ *be a completely contractive idempotent map such that* $P(\mathsf{M}) = \widetilde{\mathsf{X}}$ *and*

$$\beta_g \circ P = P \circ \beta_g, \quad g \in G. \tag{3.6}$$

*Then the formula (see Proposition 1.5)*

$$\alpha_g = \iota \circ \beta_g \circ \iota^{-1}, \quad g \in G,$$

*defines an action* $\alpha$ *of* $G$ *on the W*-TRO* $\mathsf{X}$*.*

*Proof*

The fact that each $\alpha_g$ $(g \in G)$ is a normal TRO morphism follows as in the last proposition; as we have for $g, h \in G$

$$\alpha_g \circ \alpha_h = \iota \circ \beta_g \circ \iota^{-1} \circ \iota \circ \beta_h \circ \iota^{-1} = \iota \circ \beta_g \circ \beta_h \circ \iota^{-1} = \iota \circ \beta_{gh} \circ \iota^{-1} = \alpha_{gh}$$

and $\alpha_e = \mathrm{id}_{\mathsf{X}}$, each $\alpha_g$ is in fact an automorphism, and $\alpha : G \to \mathrm{Aut}(\mathsf{X})$ is a homomorphism. Finally the continuity condition follows from that for $\beta$: if $(g_i)_{i\in\mathcal{I}}$ is a net of elements of $G$ converging to $g \in G$ and $x \in \mathsf{X}$, then, as $\iota$ is a homeomorphism with respect to weak* topologies, we have

$$\begin{aligned} \mathrm{w}^*\text{-}\lim_{i\in\mathcal{I}} \alpha^x(g_i) &= \mathrm{w}^*\text{-}\lim_{i\in\mathcal{I}} \alpha_{g_i}(x) = \mathrm{w}^*\text{-}\lim_{i\in\mathcal{I}} \iota\big(\beta_{g_i}\big(\iota^{-1}(x)\big)\big) \\ &= \iota\big(\mathrm{w}^*\text{-}\lim_{i\in\mathcal{I}} \beta_{g_i}\big(\iota^{-1}(x)\big)\big) \\ &= \iota\big(\beta_g\big(\iota^{-1}(x)\big)\big) = \alpha^x(g). \end{aligned}$$

□

Note that, although the projection $P$ features in one of the conditions in both Proposition 3.8 and Theorem 3.10, the actual maps constructed there depend only on its image.

## 4. Poisson boundaries associated with contractive functionals in $C_0^u(\mathbb{G})^*$

Let $\mathbb{G}$ be a locally compact quantum group, let $C_0^u(\mathbb{G})$ be the universal C*-algebra associated with $\mathbb{G}$, and let $\Delta_u$ be the coproduct on $C_0^u(\mathbb{G})$ (see [14]). The Banach space dual of $C_0^u(\mathbb{G})$ will be denoted $M^u(\mathbb{G})$ and called the *measure algebra of* $\mathbb{G}$. It is a Banach algebra with the product defined by

$$\mu \star \nu := (\mu \otimes \nu) \circ \Delta_u, \quad \mu, \nu \in M^u(\mathbb{G}).$$

Given $\mu \in M^u(\mathbb{G})$ the associated *right convolution* operator $R_\mu : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is defined by the formula

$$\big\langle R_\mu(x), \omega \big\rangle = \langle \omega \star \mu, x \rangle, \quad x \in L^\infty(\mathbb{G}),\ \omega \in L^1(\mathbb{G}).$$

This is well defined, as $L^1(\mathbb{G})$ is an ideal in $M^u(\mathbb{G})$. Moreover, $R_\mu$ is normal.

We are ready to apply the results of the earlier sections to the construction of extended Poisson boundaries for contractive (not necessarily positive) quantum measures. We say that $\mu \in M^u(\mathbb{G})$ is *contractive* if $\|\mu\| \le 1$.

THEOREM 4.1

*Let $\mu \in M^u(\mathbb{G})$ be contractive. Then the fixed-point space* $\mathrm{Fix}\, R_\mu := \{x \in L^\infty(\mathbb{G}) : R_\mu(x) = x\}$ *has a unique (up to a weak*-continuous complete isometry) structure of a W*-TRO, which we will denote $\mathsf{X}_\mu$.*

*Proof*

This is an immediate consequence of Theorem 1.6, as $R_\mu : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is a complete contraction. □

We will call the space $\mathrm{Fix}\, R_\mu$ with the W*-TRO structure induced via Theorem 4.1 an *extended Poisson boundary associated* to $\mu$.

LEMMA 4.2

*Let $\mu \in M^u(\mathbb{G})$ be contractive. The extended Poisson boundary* $\mathrm{Fix}\, R_\mu$ *is a unital subspace of $L^\infty(\mathbb{G})$ if and only if $\mu$ is a state.*

*Proof*

The statement follows from the equivalences $R_\mu(I_{L^\infty(\mathbb{G})}) = I_{L^\infty(\mathbb{G})}$ if and only if $\mu(I_{MC_0^u(\mathbb{G})}) = 1$ if and only if $\mu$ is a state. (Here $MC_0^u(\mathbb{G})$ denotes the multiplier algebra of $C_0^u(\mathbb{G})$.) □

COROLLARY 4.3

*A locally compact quantum group $\mathbb{G}$ is amenable if and only if there exists a contractive $\mu \in M^u(\mathbb{G})$ such that* $\mathrm{Fix}\, R_\mu = \mathbb{C} I_{L^\infty(\mathbb{G})}$.

*Proof*

This follows from the last lemma and [11, Theorem 4.2]. □

Given a contractive $\mu \in M^u(\mathbb{G})$ we can also consider an associated convolution operator $\Theta_\mu$ acting on $B(L^2(\mathbb{G}))$, as defined in [10] (see also [12]): it is a unique, normal, completely bounded map such that

$$\widetilde{\Delta} \circ \Theta_\mu = (\mathrm{id}_{B(L^2(\mathbb{G}))} \otimes R_\mu) \circ \widetilde{\Delta}. \tag{4.1}$$

THEOREM 4.4

*Let $\mu \in M^u(\mathbb{G})$ be a contractive. Then the fixed-point space* $\mathrm{Fix}\, \Theta_\mu := \{x \in B(L^2(\mathbb{G})) : \Theta_\mu(x) = x\}$ *has a unique (up to a weak*-continuous complete isometry) structure of a W*-TRO, which we will denote $\mathsf{Y}_\mu$.*

*Proof*

This is an immediate consequence of Theorem 1.6, as $\Theta_\mu : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))$ is a normal complete contraction. □

Let $\mu \in M^u(\mathbb{G})$ be contractive. Fix a free ultrafilter $\beta$, and use it as in Theorem 1.6 to construct completely contractive projections $P$ from $L^\infty(\mathbb{G})$ onto $\operatorname{Fix} R_\mu$ and $P_\Theta$ from $B(L^2(\mathbb{G}))$ onto $\operatorname{Fix} \Theta_\mu$. It is then not difficult to see that due to (4.1) we also have

$$\widetilde{\Delta} \circ P_\Theta = (\mathrm{id}_{B(L^2(\mathbb{G}))} \otimes_F P) \circ \widetilde{\Delta}. \tag{4.2}$$

PROPOSITION 4.5

*Let $\mathbb{G}$ and $\mu$ be as above, and let $\iota : \operatorname{Fix} R_\mu \to \mathsf{X}_\mu$, $\kappa : \operatorname{Fix} \Theta_\mu \to \mathsf{Y}_\mu$ denote the respective weak* homeomorphisms. Then the formula*

$$\gamma = (\mathrm{id}_{B(L^2(\mathbb{G}))} \otimes \iota) \circ \widetilde{\Delta} \circ \kappa^{-1}$$

*defines an injective normal TRO morphism $\gamma : \mathsf{Y}_\mu \to B(L^2(\mathbb{G})) \overline{\otimes} \mathsf{X}_\mu$.*

*Proof*

This is proved similarly as Proposition 3.8, using the intertwining relation (4.2). □

In the case in which $G$ is a classical group and $\mu \in M(G)$ is contractive, we can in fact identify the image of the map $\gamma$. First of all we can show via Theorem 3.10 that there is a natural action of $G$ on the W*-TRO arising from $\operatorname{Fix} R_\mu$.

LEMMA 4.6

*Let $G$ be a locally compact group, and let $\mu \in M(G)$ be contractive. Then there is a natural action $\alpha$ of $G$ on the W*-TRO $\mathsf{X}_\mu$, given essentially by left multiplication.*

*Proof*

Consider the action $\beta$ of $G$ on $L^\infty(G)$ given by the formula

$$\big(\beta_g(f)\big)(h) = f(g^{-1}h), \quad f \in L^\infty(G),\ g, h \in G.$$

It is then easy to check that we have $\beta_g \circ R_\mu = R_\mu \circ \beta_g$, and so also $\beta_g \circ P = P \circ \beta_g$, where $P$ is a completely contractive projection given by the limit (along some ultrafilter) of iterates of $R_\mu$. Theorem 3.10 ends the proof. □

We are now ready to establish the connection between the W*-TROs $\mathsf{X}_\mu$ and $\mathsf{Y}_\mu$.

THEOREM 4.7

*Let $G$ be a locally compact group, and let $\mu \in M(G)$ be contractive. Let $\alpha$ be the action of $G$ on the W*-TRO $\mathsf{X}_\mu$ introduced in Lemma 4.6. We then have a natural isomorphism*

$$\mathsf{Y}_\mu \cong G \ltimes_\alpha \mathsf{X}_\mu,$$

*given by the map $\gamma$ introduced in Proposition 4.5.*

*Proof*
We begin by showing that $\gamma(\mathsf{Y}_\mu)$ is contained in $G \ltimes_\alpha \mathsf{X}_\mu$. By Proposition 2.12 it suffices to show that for each $x \in \widetilde{\mathsf{Y}}_\mu$ and $g \in G$ we have $(\mathrm{Ad}_\rho \otimes \alpha)_g(\gamma(\kappa(x))) = \gamma(\kappa(x))$. Recall that by the definition of the action constructed in Lemma 4.6 we have $\alpha_g \circ \iota = \iota \circ \beta_g$. Recall also that if we view $L^\infty(G)$ as a subalgebra of $B(L^2(G))$, then the map $\beta_g$ is equal to $(\mathrm{Ad}_\lambda)_g$, where $\lambda$ denotes again the left regular representation. This means that

$$\big((\mathrm{Ad}_\rho)_g \otimes \alpha_g\big)\big((\mathrm{id} \otimes \iota) \circ \widetilde{\Delta}(x)\big) = (\mathrm{id} \otimes \iota)\big((\rho_g \otimes \lambda_g)V(x \otimes 1)V^*(\rho_g^* \otimes \lambda_g^*)\big).$$

Recall once again that the right multiplicative unitary for $G$ is given by the formula (2.8). Then an explicit calculation shows that for any $y \in B(L^2(G))$ we have

$$(\rho_g \otimes \lambda_g)V(y \otimes 1)V^*(\rho_g^* \otimes \lambda_g^*) = V(y \otimes 1)V^*.$$

Thus, $(\mathrm{Ad}_\rho \otimes \alpha)_g(\gamma(\kappa(x)))$ does not in fact depend on $g$, and the first part of the theorem is proved.

As $\gamma$ is a normal TRO morphism, it has a weak*-closed image. Thus, to show that $\gamma(\mathsf{Y}_\mu)$ contains $G \ltimes_\alpha \mathsf{X}_\mu$ it suffices (by Lemma 2.8 and Definition 2.9) to show that for every $g \in G$ and $x \in \mathrm{Fix}\, R_\mu$ we have $(\lambda_g \otimes I)(\pi_\alpha(\iota(x))) \in \gamma(\mathsf{Y}_\mu)$. (Note that the symbol $I$ above can be interpreted explicitly once we fix a concrete representation of $\mathsf{X}_\mu$.) This is equivalent to proving that $(\mathrm{id} \otimes \iota^{-1})((\lambda_g \otimes I)(\pi_\alpha(\iota(x)))) \in \widetilde{\Delta}(\mathrm{Fix}\, \Theta_\mu)$. Consider the map $(\mathrm{id} \otimes \iota^{-1}) : B(L^2(G)) \overline{\otimes} \mathsf{X}_\mu \to B(L^2(G)) \overline{\otimes} \mathrm{Fix}\, R_\mu$. Note that both the domain and range spaces are in fact $B(L^2(G))$ left modules in a natural way and, moreover, that $(\mathrm{id} \otimes \iota^{-1})$ is a $B(L^2(G))$-module map. Thus, recalling how the action $\alpha$ was constructed in Lemma 4.6 we can first verify that the integrated forms of actions $\alpha$ and $\beta$ satisfy the equality $(\mathrm{id} \otimes \iota) \circ \pi_\beta = \pi_\alpha \circ \iota$ (remembering that $\iota$ is a homeomorphism for weak* topologies and by using equality (2.1)) and then see that $(\mathrm{id} \otimes \iota^{-1})((\lambda_g \otimes I)(\pi_\alpha(\iota(x)))) = (\lambda_g \otimes I)\pi_\beta(x)$. Now the integrated form of the action $\beta$ is nothing but the coproduct, so we need to show simply that $(\lambda_g \otimes I)\Delta(x) \in \widetilde{\Delta}(\mathrm{Fix}\, \Theta_\mu)$. To this end we consider $\lambda_g x \in B(L^2(G))$. As $\Theta_\mu$ is a $\mathrm{VN}(G)$-module map which extends $R_\mu$ and $x \in \mathrm{Fix}\, R_\mu$, we have $\lambda_g x \in \mathrm{Fix}\, \Theta_\mu$. Then as the first leg of $V$ commutes with $\mathrm{VN}(G)$ we have $\widetilde{\Delta}(\lambda_g x) = (\lambda_g \otimes I)\widetilde{\Delta}(x) = (\lambda_g \otimes I)\Delta(x)$. This ends the proof. □

REMARK 4.8
The analogous result for $\mu$ being a state is shown in [11] for $G$ replaced by any locally compact quantum group. Here the stumbling block in extending the last theorem to the quantum setting is precisely the fact that we are not able to deduce in general that the map $\alpha$ constructed in Proposition 3.8 is nondegenerate. (Otherwise we could use Theorem 3.7 instead of Proposition 2.12.)

*Acknowledgments.* The work on this article began during our joint Research in Pairs visit with Matthias Neufang and Nico Spronk to the Mathematisches

Forschungsinstitut Oberwolfach (MFO) in August 2012. We are very grateful to MFO for providing ideal research conditions and to both Matthias and Nico for many observations which initiated this work. The second author would also like to thank Zhong-Jin Ruan for stimulating discussions on the subject of this article. We thank the referee for a careful reading of our manuscript and useful remarks.

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