Scaling distances on finitely ramified fractals

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Abstract In this article we study two problems about the existence of a distance d on a given fractal having certain properties. In the first problem, we require that the maps ψ_i defining the fractal be Lipschitz of prescribed constants less than 1 with respect to the distance d, and in the second one, we require that arbitrary compositions of the maps ψ_i be uniformly bi-Lipschitz of related constants. Both problems have been investigated previously by other authors. In this article, on a large class of finitely ramified fractals, we prove that these two problems are equivalent and give a necessary and sufficient condition for the existence of such a distance. Such a condition is expressed in terms of asymptotic behavior of the product of certain matrices associated to the fractal.

1. Introduction

Fractals are very irregular mathematical objects. Over the past three decades, they have been investigated rather intensively. The notion of fractal is very general, and in this article a specific but rather general class of fractals is considered. More or less, we consider the so-called *self-similar fractals*, that is, geometric objects having the property of containing copies of themselves at arbitrarily small scales. A concrete and relatively general class of self-similar fractals is the following. We are given finitely many contractive similarities $\psi_1, \dots, \psi_k$ in $\mathbb{R}^n$; that is,

$$d\big(\psi_i(x), \psi_i(x')\big) = \widetilde{\alpha}_i d(x, x') \tag{1.1}$$

for every $x, x' \in K$, where the $\widetilde{\alpha}_i$'s are constants lying in $]0,1[$. The self-similar fractal K (generated by the similarities ψ_i) is the set invariant with respect to such a set of similarities. More precisely, K is the only nonempty compact subset K of $\mathbb{R}^n$ such that (see, e.g., [1])

$$K = \bigcup_{i=1}^{k} \psi_i(K). \tag{1.2}$$

More generally, we could define a self-similar fractal as a compact metric space or as a compact topological space satisfying (1.2), where the ψ_i's are maps satisfying certain conditions (see [3], [4]). Examples of self-similar fractals are the

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Cantor set, the Koch curve, the (Sierpinski) gasket, the (Sierpinski) carpet, the Vicsek set, and the (Lindstrøm) snowflake. See, for example, [5] and [7] for the description of such fractals.

Here, following [4], we will say that $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k)$ is a *polyratio* if $\alpha_i \in]0,1[$ for every $i = 1, \dots, k$. When the self-similar fractal is defined by (1.2) in a metric space where the ψ_i's satisfy (1.1), by definition, the maps ψ_i are $\widetilde{\alpha}_i$-Lipschitz. More generally, one could ask for what polyratios $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k)$ there exists a distance d on K such that the maps ψ_i are α_i-Lipschitz with respect to d. This problem is discussed by A. Kameyama in [3] and [4]. When this happens, following [4], we will say that α is a *metric polyratio* (on K), and we will say that d is an *$\boldsymbol{\alpha}$-self-similar distance*.

In [4] a pseudodistance $D_{\boldsymbol{\alpha}}$ is constructed on a general class of self-similar fractals for every polyratio, and it is proved that if $\boldsymbol{\alpha}$ is a metric polyratio, in other words if an $\boldsymbol{\alpha}$-self-similar distance exists, then $D_{\boldsymbol{\alpha}}$ is an α-self-similar distance and, moreover, $D_{\boldsymbol{\alpha}}$ induces on K the same topology as the original one. Furthermore, in [3], a necessary condition for α being a metric polyratio is given in terms of a matrix related to the notion of *frame* introduced there.

In this article, another problem naturally related to the previous one is also studied. The problem is, roughly speaking, whether there exists a distance such that arbitrary compositions of the maps ψ_i are uniformly bi-Lipschitz. More precisely, we will say that $\boldsymbol{\alpha}$ is an *asymptotic metric polyratio* (on K) if there exist a distance d on K and positive constants $c_{1,\alpha}, c_{2,\alpha}$ such that

$$(1.3) \qquad c_{1,\alpha} \le \frac{d(\psi_{i_1,\dots,i_m}(Q), \psi_{i_1,\dots,i_m}(Q'))}{\alpha_{i_1} \cdots \alpha_{i_m} d(Q,Q')} \le c_{2,\alpha} \quad \forall Q, Q' \in K : Q \ne Q',$$

where $\psi_{i_1,\dots,i_m} := \psi_{i_1} \circ \cdots \circ \psi_{i_m}$, for every $i_1, \dots, i_m = 1, \dots, k$. We say in such a case that the distance d is an *α-scaling distance*. Now the problem is what polyratios are asymptotic metric. A strictly related problem has been studied by J. Kigami [6]. However, the problem in [6] is not exactly the same. In fact, instead of formula (1.3), it is required that the distance is *adapted to a scale*. This notion is rather technical, and it will not be described here (see [6, Definition 2.3.4]).[1] The results in [6] are discussed in relation to problems concerning Dirichlet forms and heat kernels.

There is rather extensive literature about Dirichlet forms on fractals and related notions and more generally about analysis on fractals. Analysis on fractals has been developed mainly on the *finitely ramified fractals* and more specifically on the postcritically finite (PCF) self-similar sets. The gasket, the Vicsek set, and the snowflake are examples of PCF self-similar sets, while the carpet is not a PCF self-similar set nor is it finitely ramified. However, on certain infinitely ramified fractals, for example, the carpet analysis has been developed. Standard

[1] In [6], emphasis is put mainly on the question of whether for a given polyratio α there exists a distance for which the contraction ratio of ψ_{w^*} is asymptotic to $\alpha_{w^*}^a$ for some a. J. Kigami considered, during a workshop, the question of what polyratios α satisfy the property given in Theorem 5.2(iv) in the present article.

textbooks on analysis on fractals are [5] and [7], where the precise definition of PCF self-similar sets is given.

In [6], a rather general class of self-similar fractals (not only finitely ramified) is discussed, and some conditions are given. In particular, it is proved that in a class of fractals, including that considered in this article, for many polyratios there exists a distance having a property similar to (1.3). However, no conditions given in [6] are both necessary and sufficient.

In this article, I restrict the class of fractals. (More or less, I consider the connected PCF self-similar sets.) See Section 2 for the details. For such a class of fractals, I prove that both the notion of metric polyratio and that of asymptotic metric polyratio are equivalent to a notion expressed in terms of some special matrices introduced in Section 4. In particular, the notions of metric polyratio and of asymptotic metric polyratio are in fact the same. These matrices are related to paths on the fractal. The condition is that this set of matrices satisfies a special property, which is strictly related to the notions of *joint spectral radius* or, better, of *joint spectral subradius*. Joint spectral radius and joint spectral subradius are notions that generalize the notion of spectral radius to the case of a finite set of matrices. More precisely, given a finite set E of $(n \times n)$-matrices, the joint spectral radius (resp., joint spectral subradius) is defined as

$$\lim_{h\to+\infty} \max\bigl\{\|A_{i_1}\cdots A_{i_h}\| : A_{i_1},\dots,A_{i_h} \in E\bigr\}$$

$$\Bigl(\text{resp., } \lim_{h\to+\infty} \min\bigl\{\|A_{i_1}\cdots A_{i_h}\| : A_{i_1},\dots,A_{i_h} \in E\bigr\}\Bigr).$$

A textbook on joint spectral radius and joint spectral subradius is [2].

Section 5 is devoted to proving the condition given here (see Theorem 5.2). The condition is strictly related to the statement that such spectral objects are greater than or equal to 1. However, it is not equivalent. Namely, we require that a set E of matrices with nonnegative entries satisfy

$$\bigl\|A_{i_1}\cdots A_{i_h}(e_j)\bigr\| \ge c,$$

where c is a positive constant independent of h, of the matrices $A_{i_1},\dots,A_{i_h} \in E$, and of the vector e_j of the canonical basis. Note that usually, at least to my knowledge, it is difficult to evaluate the joint spectral radius and the joint spectral subradius of a finite set of matrices. So, I expect that in the general case an explicit and effective condition for a polyratio being metric on the given fractal could be hard to find. However, this can be done for some specific fractals. In Section 6, I give explicit necessary and sufficient conditions on the gasket and on the Vicsek set. I expect that similar explicit conditions can be given for fractals having a simple structure and for more complicated fractals if the factors α_i have some good symmetry properties. Moreover, note that it can be easily seen that α is always a metric polyratio if $\alpha_i \ge \frac{1}{2}$ for every $i = 1,\dots,k$ (see Remark 5.8).

2. Fractals

In this section, we describe the construction of a fractal, following more or less the approach of [5]. The results of this section are standard (either known or simple consequences of known results), but I prefer to recall them. Let k be an integer greater than 1. Define

$$W_m = \{1,\dots,k\}^m, \quad \widetilde{W}_m := \bigcup_{l\le m} W_l, \qquad W^* := \bigcup_{m\in\mathbb{N}} W_m,$$

$$W(=W_\infty) := \{1,\dots,k\}^{\mathbb{N}}.$$

If $w^*\in W^*$, let $|w^*| := m$ if $w^*\in W_m$, and we say that w is a word and m is the length of w. We equip W with the product topology $\{1,\dots,k\}^{\mathbb{N}}$, where on $\{1,\dots,k\}$ we put the discrete distance. Note that the unique word in W_0 is the empty word $\emptyset$.

For $i=1,\dots,k$, let $\sigma_i: W\to W$ be defined by $\sigma_i(w)=iw$, where if $w=(i_1,i_2,i_3,\dots)$, we set $iw=(i,i_1,i_2,i_3,\dots)$. If $w\in W$ or $w\in W_m$ with $m\ge\overline{m}$, $w=(i_1,i_2,i_3,\dots)$, let $w_{(\overline{m})}\in W_{\overline{m}}$ be defined by $w_{(\overline{m})}=(i_1,\dots,i_{\overline{m}})$. If $w^*\in W^*$ and $m\le|w^*|$, we say that $w^*_{(m)}$ is a *segment* of w^*. We say that two words w^* and $w^{*\prime}$ are *incomparable* if neither w^* is a segment of $w^{*\prime}$ nor $w^{*\prime}$ is a segment of w^*. Denote by $i^{(m)}$ the element of W_m of the form $(i,\dots,i)$ where i is repeated m times, for $m\in\mathbb{N}\cup\{\infty\}$, $i=1,\dots,k$.

Let $(K,\overline{d})$ be a compact metric space. We say that K is a self-similar fractal is there exists a continuous map π from W onto K and continuous one-to-one maps ψ_i, $i=1,\dots,k$, from K into itself such that

$$\psi_i\circ\pi=\pi\circ\sigma_i \quad \forall i=1,\dots,k. \tag{2.1}$$

If $w^*=(i_1,\dots,i_m)$, let $\sigma_{w^*}=\sigma_{i_1}\circ\cdots\circ\sigma_{i_m}$, $\psi_{w^*}=\psi_{i_1}\circ\cdots\circ\psi_{i_m}$. It follows that

$$\psi_{w^*}\circ\pi=\pi\circ\sigma_{w^*} \quad \forall w^*\in W^*. \tag{2.2}$$

For every $w\in W$ we have

$$\{\pi(w)\} = \bigcap_{m=1}^{\infty}\psi_{w_{(m)}}(K). \tag{2.3}$$

In fact, $w=\sigma_{w_{(m)}}(w')$ for some $w'\in W$; thus, for every $m=1,2,\dots$, we have

$$\pi(w)=\pi\big(\sigma_{w_{(m)}}(w')\big)=\psi_{w_{(m)}}\big(\pi(w')\big)\in\psi_{w_{(m)}}(K),$$

and the inclusion $\subseteq$ in (2.3) is proved. On the other hand, if $y\in\bigcap_{m=1}^\infty\psi_{w_{(m)}}(K)$, then for every $m=1,2,\dots$, there exists $w'_m\in W$ such that $y=\psi_{w_{(m)}}(\pi(w'_m))=\pi(\sigma_{w_{(m)}}(w'_m))$. Since, by the definition of the topology on W, we have $\sigma_{w_{(m)}}(w'_m)\longrightarrow_{m\to+\infty} w$, in view of the continuity of π, we have $y=\pi(w)$, and (2.3) is completely proved.

Now let

$$E_{w^*}=\psi_{w^*}(E) \quad \forall w^*\in W^*, E\subseteq K.$$

By (2.1) we have

$$\bigcup_{i=1}^{k} K_i = \bigcup_{i=1}^{k} \psi_i\big(\pi(W)\big) = \bigcup_{i=1}^{k} \pi\big(\sigma_i(W)\big) = \pi\Big(\bigcup_{i=1}^{k} \sigma_i(W)\Big) = \pi(W) = K.$$

More generally, for every $m = 1, 2, 3, \ldots$, we have

$$K = \bigcup_{w^* \in W_m} K_{w^*}, \tag{2.4}$$

$$K_{i_1,\ldots,i_{m-1}} = \bigcup_{i_m=1}^{k} K_{i_1,\ldots,i_{m-1},i_m}. \tag{2.4$'$}$$

As a consequence, we have

$$K_{i_1,\ldots,i_m} \supseteq K_{i_1,\ldots,i_{m-1}} \quad \forall i_1,\ldots,i_m = 1,\ldots,k. \tag{2.4$''$}$$

In the following, we will require additional properties on the fractal. We require that the fractal be a PCF self-similar set with a little additional property similar to that required in [7]. Suppose that ψ_i has a unique fixed point which we denote by P_i, and let $\widetilde{V} = \{P_i, i = 1, \ldots, k\}$. Assume that there exists a subset $V = V^{(0)} = \{P_1, \ldots, P_N\}$ of $\widetilde{V}$ of N elements with $2 \le N \le k$ such that if $w^*, w^{*\prime} \in W^*$, $w^* \ne w^{*\prime}$, $|w^*| = |w^{*\prime}|$, then

$$K_{w^*} \cap K_{w^{*\prime}} = V_{w^*} \cap V_{w^{*\prime}}, \tag{2.7}$$

$$\#(K_{w^*} \cap K_{w^{*\prime}}) \le 1. \tag{2.7$'$}$$

Requirement (2.7) is more or less the finite ramification property. Note that the Sierpinski carpet does satisfy (2.7). Requirement (2.7′) is possibly not strictly necessary, but simplifies many arguments and is satisfied by almost all the finitely ramified fractals considered in the literature. Moreover, we require that

$$\text{if } i = 1, \ldots, k, j, h = 1, \ldots, N, \text{ and } \psi_i(P_j) = P_h, \text{then } i = j = h. \tag{2.8}$$

Note that (2.8), in particular, implies that $P_j \ne P_{j'}$ if $j \ne j'$. Let

$$\widehat{J} := \big\{(j_1, j_2) : j_1, j_2 = 1, \ldots, N, j_1 \ne j_2\big\},$$

and note that $\#(\widehat{J}) = N(N-1)$. The sets of the form $V_{i_1,\ldots,i_m}$ will be called *m-cells*. Let

$$V^{(m)} = \bigcup_{i_1,\ldots,i_m=1}^{k} V_{i_1,\ldots,i_m},$$

$$V^{(\infty)} = \bigcup_{m=0}^{\infty} V^{(m)}.$$

Note that $V^{(m)} \subseteq V^{(m+1)}$ for every positive integer m. I now prove some lemmas useful in the following.

LEMMA 2.1

For every $w^* \in W^*$, *the map* ψ_{w^*} *sends* $K \setminus V^{(0)}$ *into itself. More precisely, if* $w^* \in W_m$, $m > 0$, *and* $\psi_{w^*}(Q) = P_j \in V^{(0)}$, $Q \in K$, *then* $Q = P_j$ *and* $w^* = j^{(m)}$.

Proof

Let $Q \in K$, and suppose that $\psi_{w^*}(Q) = P_j \in V^{(0)}$. Then $w^* = j^{(m)}$, and consequently $Q = P_j$. In fact, in the opposite case, as $P_j = \psi_{j^{(m)}}(P_j)$, by (2.7) we have $Q \in V^{(0)}$, and by (2.8) and an inductive argument we have $w^* = j^{(m)}$, which is a contradiction. □

Note that (2.7) holds under the hypothesis $|w^*| = |w^{*\prime}|$ and that (2.7) is no longer valid if w^*, $w^{*\prime}$ are two arbitrary different words. In fact, if $w^{*\prime}$ is a segment of w^*, then $K_{w^*} \cap K_{w^{*\prime}} = K_{w^*}$ by (2.4″). However, as we see now, this is the only case in which (2.7) does not hold.

LEMMA 2.2

If w^* *and* $w^{*\prime}$ *are two incomparable words, then*

$$K_{w^*} \cap K_{w^{*\prime}} = V_{w^*} \cap V_{w^{*\prime}}.$$

Proof

Let $w^* = (i_1, \dots, i_m)$, $w^{*\prime} = (i'_1, \dots, i'_{m'})$. Since the case $m = m'$ follows at once from (2.7), we can and do assume that $m' > m$. As we have also assumed that w^* and $w^{*\prime}$ are incomparable, we have $(i_1, \dots, i_m) \neq (i'_1, \dots, i'_m)$. Thus, if $Q \in K_{w^*} \cap K_{w^{*\prime}}$, then

$$Q \in K_{i_1,\dots,i_m} \cap K_{i'_1,\dots,i'_m} = V_{i_1,\dots,i_m} \cap V_{i'_1,\dots,i'_m}.$$

It remains to prove that $Q \in V_{i'_1,\dots,i'_{m'}}$. To see this, note that there exist $Q' \in K$ and $P_j \in V^{(0)}$ such that

$$Q = \psi_{i'_1,\dots,i'_m}\big(\psi_{i'_{m+1},\dots,i'_{m'}}(Q')\big) = \psi_{i'_1,\dots,i'_m}(P_j).$$

Hence, by Lemma 2.1, we have $Q' = P_j$ (and $i'_{m+1} = \dots = i'_{m'} = j$). Thus, $Q \in V_{i'_1,\dots,i'_{m'}}$. □

LEMMA 2.3

For every $m' \geq m$ *we have*

$$K_{i_1,\dots,i_{m'}} \cap V^{(m)} \subseteq V_{i_1,\dots,i_m}. \tag{2.9}$$

Hence, $K_{i_1,\dots,i_{m'}} \cap V^{(m)}$ *has at most one element if* $m' > m$.

Proof

Let $Q \in K_{i_1,\dots,i_{m'}} \cap V^{(m)}$, and let $i'_1, \dots, i'_m$ be such that $Q \in V_{i'_1,\dots,i'_m}$. Then, if $(i_1, \dots, i_m) = (i'_1, \dots, i'_m)$, we have $Q \in V_{i_1,\dots,i_m}$. If, on the contrary, $(i_1, \dots, i_m) \neq (i'_1, \dots, i'_m)$, we have

$$Q \in K_{i_1,\dots,i_m} \cap K_{i'_1,\dots,i'_m} \subseteq V_{i_1,\dots,i_m},$$

and (2.9) is proved. Since every point in $K_{i_1,\dots,i_{m'}} \cap V^{(m)}$ has the form $\psi_{i_1,\dots,i_{m'}}(Q) = \psi_{i_1,\dots,i_m}(P_j)$ with $P_j \in V^{(0)}$, $Q \in K$, we have $\psi_{i_{m+1},\dots,i_{m'}}(Q) = P_j$. Thus, by Lemma 2.1, $j = i_{m+1} = \cdots = i_{m'}$, $Q = P_j = P_{i_{m+1}}$. $\square$

COROLLARY 2.4

The set $V_w \cap V_{w'}$ has at most one point whenever $w, w' \in W^*$, $w \neq w'$.

Proof

Let $m = |w|$, $m' = |w'|$. If $m = m'$, this follows from (2.7′). If, for example, $m' > m$, let $w = (i_1, \dots, i_m)$, $w' = (i'_1, \dots, i'_{m'})$. Then $V_w \cap V_{w'} \subseteq V_{i'_1,\dots,i'_{m'}} \cap V^{(m)}$, and we conclude by Lemma 2.3. $\square$

COROLLARY 2.5

We have $\psi_i(V^{(m+1)} \setminus V^{(m)}) \subseteq V^{(m+2)} \setminus V^{(m+1)}$ for every $m \in \mathbb{N}$ and every $i = 1, \dots, k$.

Proof

Let $Q \in V^{(m+1)} \setminus V^{(m)}$; namely, $Q = \psi_{i_1,\dots,i_{m+1}}(P_j)$. Then, clearly, $\psi_i(Q) \in V^{(m+2)}$. Also, if $\psi_i(Q) \in V^{(m+1)}$, then $\psi_i(Q) \in K_{i,i_1,\dots,i_m} \cap V^{(m+1)} \subseteq V_{i,i_1,\dots,i_m}$, by Lemma 2.3. Thus, $\psi_i(Q) = \psi_i(\psi_{i_1,\dots,i_m}(P_{j'}))$ for some $j' = 1, \dots, N$; hence, $Q = \psi_{i_1,\dots,i_m}(P_{j'}) \in V^{(m)}$, which is a contradiction. $\square$

In the following we will often use, with no mention, the simple consequence of (2.7).

LEMMA 2.6

We have that $V^{(\infty)} \cap K_{w^*} = V^{(\infty)}_{w^*}$ $\forall w^* \in W^*$.

Proof

To prove the $\subseteq$ part, note that if $Q \in V^{(\infty)} \cap K_{w^*}$, then $Q \in V^{(\infty)}_{w^{*\prime}}$ for some $w^{*\prime} \in W^*$ with $|w^{*\prime}| = |w^*|$. Thus, if $w^{*\prime} \neq w^*$ by (2.7), we have $Q \in V_{w^*} \subseteq V^{(\infty)}_{w^*}$. The $\supseteq$ part is trivial. $\square$

LEMMA 2.7

We have that $V^{(\infty)}$ is dense in K.

Proof

Let $Q \in K$. By (2.2), $Q = \pi(w)$ for some $w \in W$. Note that, by the definition of the topology on W, we have $\sigma_{w_{(m)}}(1^{(\infty)}) \longrightarrow_{m\to+\infty} w$; hence, $\pi(\sigma_{w_{(m)}}(1^{(\infty)})) \longrightarrow_{m\to+\infty} \pi(w) = Q$. On the other hand,

$$\pi\big(\sigma_{w_{(m)}}(1^{(\infty)})\big) = \psi_{w_{(m)}}\big(\pi(1^{(\infty)})\big), \tag{2.10}$$

and since $\psi_1(P_1) = P_1$, we have $\psi_{1^{(m)}}(P_1) = P_1$. Hence, by (2.3), $\pi(1^{(\infty)}) = P_1$, and thus, in view of (2.10), $\pi(\sigma_{w_{(m)}}(1^{(\infty)})) \in V^{(\infty)}$. $\square$

Finally, we require that the fractal be *connected*. By this we mean that for every $Q, Q' \in V^{(1)}$ there exist $Q_0, \dots, Q_n \in V^{(1)}$ such that $Q_0 = Q$, $Q_n = Q'$, and for every $h = 1, \dots, n$ there exists $i(h) = 1, \dots, k$ such that $Q_{h-1}, Q_h \in V_{i(h)}$. Note that, for example, the Cantor set is not connected.

From now on, all fractals are meant to have all the properties required in this section, including, in particular, (2.7), (2.7′), and (2.8).

I now introduce the problems discussed in this article. Following Section 1, we say that $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k)$ is a *polyratio* if $\alpha_i \in]0,1[$ for every $i = 1, \dots, k$. Here, we put $\alpha_{w^*} := \alpha_{i_1} \cdots \alpha_{i_m}$, $\alpha_{\emptyset} = 1$. Put $\overline{\alpha}_{\min} = \min\{\alpha_i\}$, $\overline{\alpha}_{\max} = \max\{\alpha_i\}$. Given a polyratio $\boldsymbol{\alpha}$ as above we say that a distance d on K is $\boldsymbol{\alpha}$*-self-similar* if the maps ψ_i, $i = 1, \dots, k$, are α_i-Lipschitz with respect to d. If an $\boldsymbol{\alpha}$-self-similar distance exists on K, we say that α is a *metric polyratio* (on K). We say that a distance d is an α-scaling distance if there exist positive constants $c_{1,\alpha}, c_{2,\alpha}$ such that, for every $i_1, \dots, i_m = 1, \dots, k$, (1.3) holds. We say that $\boldsymbol{\alpha}$ is an *asymptotic metric polyratio* (on K) if there exists an α-scaling distance on K. The problems discussed in this article are what polyratios are metric and what polyratios are asymptotic metric. We will treat such problems in Section 5. Sections 3 and 4 are devoted to introducing preparatory notions.

3. Graphs on the fractal

In this section, we first define a suitable graph on $V^{(\infty)}$ and, then, based on it, the notion of a path on $V^{(\infty)}$. We define a graph on $V^{(\infty)}$ by putting $Q \sim Q'$ for $Q, Q' \in V^{(\infty)}$ if $Q \neq Q'$ and there exist $P, P' \in V^{(0)}$ and $w^* \in W^*$ such that $Q = \psi_{w^*}(P)$, $Q' = \psi_{w^*}(P')$. Put also $Q \simeq Q'$ if either $Q \sim Q'$ or $Q = Q'$. Given $\iota = (j_1, j_2) \in \widehat{J}$, let

$$P_\iota := (P_{j_1}, P_{j_2}), \qquad \psi_{w^*}(P_\iota) = \big(\psi_{w^*}(P_{j_1}), \psi_{w^*}(P_{j_2})\big).$$

Let $\widetilde{Y}$ be the set of $(Q, Q') \in V^{(\infty)} \times V^{(\infty)}$ such that $Q \sim Q'$. The following function will be useful in the sequel. Let $\alpha_{\overline{\mathbf{w}}} : \widetilde{Y} \to \mathbb{R}$ be defined as

$$\alpha_{\overline{\mathbf{w}}}\big(\psi_{w^*}(P), \psi_{w^*}(P')\big) = \alpha_{w^*} \quad \forall P, P' \in V^{(0)}\ \forall w^* \in W^*.$$

Note that, in view of Corollary 2.4, such a definition is correct; that is, the pair (Q, Q') with $Q, Q' \in V^{(\infty)}$, $Q \neq Q'$, can be represented uniquely as $(\psi_w(P), \psi_w(P'))$. The following lemma is a consequence of the assumptions on the fractal (in particular, (2.7)).

LEMMA 3.1
If $Q, Q' \in V^{(\infty)}$, $Q \sim Q'$, and $Q \in K_{i_1,\dots,i_m}$, $Q' \notin K_{i_1,\dots,i_m}$, then $Q \in V_{i_1,\dots,i_m}$.

Proof
We have $Q, Q' \in V_{i'_1,\dots,i'_l}$ for some $i'_1, \dots, i'_l$. Let $h = \min\{l, m\}$. If

$$(i'_1, \dots, i'_h) = (i_1, \dots, i_h),$$

then

$$Q' \in K_{i_1,\dots,i_h} \setminus K_{i_1,\dots,i_m}.$$

Thus, $m > h = l$ and

$$Q \in V_{i_1,\dots,i_l} \cap K_{i_1,\dots,i_m} \subseteq V_{i_1,\dots,i_m},$$

where the inclusion holds since if

$$Q = \psi_{i_1,\dots,i_l} \circ \psi_{i_{l+1},\dots,i_m}(\widetilde{Q}) = \psi_{i_1,\dots,i_l}(P)$$

with $\widetilde{Q} \in K$, $P \in V^{(0)}$, we have $\psi_{i_{l+1},\dots,i_m}(\widetilde{Q}) = P$. Thus, by Lemma 2.1, $\widetilde{Q} \in V^{(0)}$.

If instead $(i'_1,\dots,i'_h) \neq (i_1,\dots,i_h)$, then either $l \geq m$ and $Q \in K_{i_1,\dots,i_m} \cap K_{i'_1,\dots,i'_m}$ or $l < m$ and $Q \in K_{i'_1,\dots,i'_l}$. Thus, by (2.4), $Q \in K_{i'_1,\dots,i'_m}$ for some $i'_{l+1},\dots,i'_m$. In both cases $(i_1,\dots,i_m) \neq (i'_1,\dots,i'_m)$. Thus, in view of (2.7), $Q \in V_{i_1,\dots,i_m}$. □

A $V^{(\infty)}$-path (or simply a path) Π is a sequence of the form

$$(Q_0,\dots,Q_n) = (Q_{0,\Pi},\dots,Q_{n(\Pi),\Pi})$$

such that $Q_h \in V^{(\infty)}$ for every $h = 0,\dots,n$ and, moreover, $Q_{h-1} \sim Q_h$ for every $h = 1,\dots,n$. Thus, there exist $\widetilde{w}(h,\Pi) \in W^*$, $\widetilde{\iota}(h,\Pi) \in \widehat{J}$ such that

$$(Q_{h-1,\Pi}, Q_{h,\Pi}) = \psi_{\widetilde{w}(h,\Pi)}(P_{\widetilde{\iota}(h,\Pi)}).$$

More generally we say that a *weak path* is a sequence $(Q_0,\dots,Q_n)$ such that $Q_h \in V^{(\infty)}$ for every $h = 0,\dots,n$ and, moreover, $Q_{h-1} \simeq Q_h$ for every $h = 1,\dots,n$.

Here, we say that $Q_{h,\Pi}$, $h = 0,\dots,n(\Pi)$, are the *vertices* of Π and that the pairs $(Q_{h-1,\Pi}, Q_{h,\Pi})$, $h = 1,\dots,n(\Pi)$, are the *edges* of Π. We say that $n(\Pi)+1$ is the *length* of Π.

An E-path Π is a path whose vertices belong to E whenever $E \subseteq V^{(\infty)}$. In particular, we will use the terms $V^{(1)}$-paths and $V^{(0)}$-paths. So, Π is a $V^{(1)}$-path if, for every $h = 1,\dots,n(\Pi)$, $Q_{h,\Pi} \neq Q_{h-1,\Pi}$ and $Q_{h,\Pi}$ and $Q_{h-1,\Pi}$ either both lie in a common 1-cell or both lie in $V^{(0)}$ (see Lemma 2.3).

A path Π is *strong* if $\widetilde{w}(h,\Pi) \neq \emptyset$ for every $h = 1,\dots,n(\Pi)$, in other words if two consecutive vertices of the path are not both in $V^{(0)}$. We say that Π is a ι-path if $P_\iota = (Q_{0,\Pi}, Q_{n(\Pi),\Pi})$. If $E \subseteq V^{(\infty)}$, we say that Π is a (ι, E)-path if it is both an E-path and a ι-path. We say that a path is a strong ι-path if it is both a strong path and a ι-path. We say that Π connects Q to Q' if $Q_{0,\Pi} = Q$ and $Q_{n(\Pi),\Pi} = Q'$. We say that Π is a strict path if $Q_{h,\Pi} \neq Q_{h',\Pi}$ when $h \neq h'$.

When Π is a path and $w^* \in W^*$, we define the path $\psi_{w^*}(\Pi) = (\psi_{w^*}(Q_{0,\Pi}),\dots, \psi_{w^*}(Q_{n(\Pi),\Pi}))$. If $\Pi = (Q_0,\dots,Q_n)$ and $\Pi' = (Q'_0,\dots,Q'_m)$ with $Q_n = Q'_0$, we put

$$\Pi \circ \Pi' = (Q_0,\dots,Q_n,Q'_1,\dots,Q'_m).$$

A *subpath* of a path Π is a path of the form

$$\Pi' := (Q_{n_1,\Pi}, Q_{n_1+1,\Pi},\dots,Q_{n_2-1,\Pi},Q_{n_2,\Pi})$$

with $0 \le n_1 \le n_2 \le n(\Pi)$, where $n_1 =: n_1(\Pi,\Pi')$, $n_2 =: n_2(\Pi,\Pi')$. Note the following simple properties of the graphs.

LEMMA 3.2

(i) *If $Q, Q' \in K$, $w^* \in W^*$, then $Q \sim Q'$ if and only if $\psi_{w^*}(Q) \sim \psi_{w^*}(Q')$.*

(ii) *If $w^* \in W^*$, then the sequence $(Q_0, \dots, Q_n)$ is a path if and only if $(\psi_{w^*}(Q_0), \dots, \psi_{w^*}(Q_n))$ is a path.*

Proof

(i) The $\Rightarrow$ part is trivial. We prove $\Leftarrow$. Suppose $\psi_{w^*}(Q) \sim \psi_{w^*}(Q')$, so that there exist $w_1^* \in W^*$ and $P_j, P_{j'} \in V^{(0)}$ such that $\psi_{w^*}(Q) = \psi_{w_1^*}(P_j)$, $\psi_{w^*}(Q') = \psi_{w_1^*}(P_{j'})$. Let $w_1^* := (i_1, \dots, i_m)$, $w^* := (i'_1, \dots, i'_{m'})$. Let $\widetilde{Q} = \psi_{w^*}(Q)$, $\widetilde{Q}' = \psi_{w^*}(Q')$. We have $\widetilde{Q} \neq \widetilde{Q}'$ and

$$\widetilde{Q}, \widetilde{Q}' \in K_{i'_1,\dots,i'_{m'}} \cap K_{i_1,\dots,i_m} \cap V^{(m)}.$$

Thus, on one hand, by Lemma 2.3 we have $m' \le m$; on the other hand, by (2.7′) and (2.4″), we have $(i_1, \dots, i_{m'}) = (i'_1, \dots, i'_{m'})$. It follows that

$$\psi_{i'_1,\dots,i'_{m'}}(Q) = \psi_{i'_1,\dots,i'_{m'}}\big(\psi_{i_{m'+1},\dots,i_m}(P_j)\big),$$

$$\psi_{i'_1,\dots,i'_{m'}}(Q') = \psi_{i'_1,\dots,i'_{m'}}\big(\psi_{i_{m'+1},\dots,i_m}(P_{j'})\big),$$

and $Q = \psi_{i_{m'+1},\dots,i_m}(P_j) \sim \psi_{i_{m'+1},\dots,i_m}(P_{j'}) = Q'$. Thus, (i) is proved, and (ii) is a simple consequence of (i). □

In the next lemmas we investigate some properties of the graphs connecting points lying in some specific subsets of $V^{(\infty)}$. In particular, the statement of Lemma 3.3 corresponds to the intuitive idea that $V_{i_1,\dots,i_m}$ is a sort of boundary of $K_{i_1,\dots,i_m}$. Hence, a path connecting a point of $K_{i_1,\dots,i_m}$ to a point not in $K_{i_1,\dots,i_m}$ necessarily passes through $V_{i_1,\dots,i_m}$.

LEMMA 3.3

If Π is a path connecting $Q \notin K_{i_1,\dots,i_m}$ to $Q' \in K_{i_1,\dots,i_m}$, then there exists $\overline{n} \le n(\Pi)$ such that $Q_{n,\Pi} \notin K_{i_1,\dots,i_m}$ for every $n \le \overline{n}$ and $Q_{\overline{n}+1,\Pi} \in V_{i_1,\dots,i_m}$.

Proof

Let $\overline{n}$ be the maximum n such that

$$Q_{n',\Pi} \notin K_{i_1,\dots,i_m} \quad \forall n' = 0, \dots, n.$$

Then $Q_{\overline{n},\Pi} \notin K_{i_1,\dots,i_m}$, $Q_{\overline{n}+1,\Pi} \in K_{i_1,\dots,i_m}$, and $Q_{\overline{n},\Pi} \sim Q_{\overline{n}+1,\Pi}$. Thus, by Lemma 3.1, $Q_{\overline{n}+1,\Pi} \in V_{i_1,\dots,i_m}$. □

LEMMA 3.4

Suppose that Π is a path connecting $Q \in K_{i_1,\dots,i_m} \setminus V^{(m)}$ to $Q' \in V^{(m)}$. Then there exists $\overline{n}$ such that $Q_{\overline{n},\Pi} \in V_{i_1,\dots,i_m}$ and $Q_{n,\Pi} \in K_{i_1,\dots,i_m}$ for every $n \le \overline{n}$.

Proof
The proof is similar to that of Lemma 3.3, but a bit more complicated. We have $Q_{0,\Pi} = Q$, $Q_{n(\Pi),\Pi} = Q'$. Let $\overline{n}$ be the maximum $n \in [0, n(\Pi)]$ such that

$$Q_{n',\Pi} \in K_{i_1,\dots,i_m} \quad \forall n' = 0, \dots, n.$$

We will prove that $Q_{\overline{n},\Pi} \in V_{i_1,\dots,i_m}$. If $\overline{n} = n(\Pi)$, then, in view of Lemma 2.3, $Q_{\overline{n},\Pi} \in K_{i_1,\dots,i_m} \cap V^{(m)} \subseteq V_{i_1,\dots,i_m}$.

If $\overline{n} < n(\Pi)$, then we have $Q_{\overline{n},\Pi} \sim Q_{\overline{n}+1,\Pi}$ by the definition of a path and $Q_{\overline{n},\Pi} \in K_{i_1,\dots,i_m}$, $Q_{\overline{n}+1,\Pi} \notin K_{i_1,\dots,i_m}$, by the definition of $\overline{n}$. Thus, we have $Q_{\overline{n},\Pi} \in V_{i_1,\dots,i_m}$ by Lemma 3.1. □

Note that, given a path $(Q_0, \dots, Q_n)$ connecting Q_0 to Q_n, we can associate to it a sort of reverse path connecting Q_n to Q_0, that is, $(Q_n, \dots, Q_0)$. Thus, we can use Lemmas 3.3 and 3.4 (and other similar lemmas) also in the reverse direction. This is what we will do in the proof of Lemma 3.5.

LEMMA 3.5
If Π is a path connecting two points in $K_{i_1,\dots,i_m}$ and Π is not entirely contained in $K_{i_1,\dots,i_m}$, then there exist $l < m$ and a subpath of Π of length greater than 2, connecting two points in $K_{i_1,\dots,i_{l+1}}$, entirely contained in $K_{i_1,\dots,i_l}$.

Proof
Since every point of Π lies in $K = K_{\emptyset}$, by our hypothesis there exists a natural $l < m$ (possibly $l = 0$) such that there exists a point $Q_{\overline{n},\Pi} \in K_{i_1,\dots,i_l} \setminus K_{i_1,\dots,i_{l+1}}$. We can assume that l is the minimum natural number satisfying such a property. By Lemma 3.3 there exist n_1 and n_2 with $n_1 < \overline{n} < n_2$ such that $Q_{n,\Pi} \notin K_{i_1,\dots,i_{l+1}}$ for every $n = n_1 + 1, \dots, n_2 - 1$ and $Q_{n_1,\Pi}, Q_{n_2,\Pi} \in V_{i_1,\dots,i_{l+1}} \subseteq K_{i_1,\dots,i_l}$. But by the definition of l, we then have $Q_{n,\Pi} \in K_{i_1,\dots,i_l}$ for every $n = n_1 + 1, \dots, n_2 - 1$. □

Suppose we are given a path Π. We now describe a way to construct a longer path by inserting new paths between any pair of consecutive vertices of Π, and in Lemma 3.6 we will prove that any path connecting two given points Q and Q' of $V^{(m)}$ can be obtained by repeating this process starting from a $V^{(m)}$-path connecting Q and Q'.

More formally, let Γ be the set of functions γ from $\widehat{J}$ to the set of strict paths such that $\gamma(\iota)$ is a $(\iota, V^{(1)})$-path. Let $\overline{\Gamma}$ be the set of functions $\overline{\gamma}$ from $\widehat{J} \times W^*$ to the set of strict paths such that $\overline{\gamma}(\iota, w)$ is a $(\iota, V^{(1)})$-path. For $\overline{\gamma} \in \overline{\Gamma}$, let

$$\overline{D}(\overline{\gamma})\big(\psi_w(P_{j_1}), \psi_w(P_{j_2})\big) = \psi_w\big(\overline{\gamma}\big((j_1, j_2), w\big)\big),$$
$$\overline{D}(\overline{\gamma})(Q_0, \dots, Q_n) = \overline{D}(\overline{\gamma})(Q_0, Q_1) \circ \dots \circ \overline{D}(\overline{\gamma})(Q_{n-1}, Q_n),$$

when $\Pi := (Q_0, \dots, Q_n)$ is a path. When this happens, clearly, $\overline{D}(\overline{\gamma})(\Pi)$ is a path as well. We will say that $\overline{D}(\overline{\gamma})(\Pi)$ is the $\overline{\gamma}$-*insertion* of Π. Moreover, note that

$Q_{0,\Pi} = Q_{0,\overline{D}(\overline{\gamma})(\Pi)}$, $Q_{n(\Pi),\Pi} = Q_{n(\overline{D}(\overline{\gamma})(\Pi)),\overline{D}(\overline{\gamma})(\Pi)}$; in other words, Π and $\overline{D}(\overline{\gamma})(\Pi)$ have the same endpoints.

LEMMA 3.6
If Π' is a strict path connecting two points Q and Q' of $V^{(m)}$, then there exist a strict path Π lying in $V^{(m)}$ connecting Q and Q' and $\overline{\gamma}_1,\dots,\overline{\gamma}_r \in \overline{\Gamma}$ such that

$$\Pi' = \overline{D}(\overline{\gamma}_1) \circ \cdots \circ \overline{D}(\overline{\gamma}_r)(\Pi).$$

Proof
We proceed by induction on the number of points $s(\Pi')$ of Π' not lying in $V^{(m)}$. The lemma is trivial if $s(\Pi') = 0$. Suppose that $s(\Pi') > 0$, and let $Q_{\overline{n},\Pi'} \notin V^{(m)}$. Thus, $Q_{\overline{n},\Pi'} \in V^{(\overline{m})} \setminus V^{(\overline{m}-1)}$ for some $\overline{m} > m$. We can assume that $\overline{m}$ is the maximum index with such a property. In other words,

$$Q_{n,\Pi'} \in V^{(\overline{m})} \quad \forall n = 0,\dots,n(\Pi'). \tag{3.1}$$

Since $Q, Q' \in V^{(m)} \subseteq V^{(\overline{m}-1)}$, we have $0 < \overline{n} < n(\Pi')$. Moreover, there exist $\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}$ such that $Q_{\overline{n},\Pi'} \in K_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}}$. Thus, $Q_{\overline{n},\Pi'} \in K_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}} \setminus V^{(\overline{m}-1)}$. By Lemma 3.4 there exist n_1, n_2 such that $n_1 < \overline{n} < n_2$, $Q_{n_1,\Pi'}, Q_{n_2,\Pi'} \in V_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}}$, and

$$Q_{n,\Pi'} \in K_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}} \quad \text{if } n_1 \le n \le n_2. \tag{3.2}$$

Let

$$\Pi'' = (Q_{0,\Pi'},\dots,Q_{n_1,\Pi'},Q_{n_2,\Pi'},\dots,Q_{n(\Pi'),\Pi'}).$$

Then, Π'' is a strict path and $s(\Pi'') < s(\Pi')$. By the inductive hypothesis, there exist a strict path Π lying in $V^{(m)}$ connecting Q and Q' and $\overline{\gamma}_1,\dots,\overline{\gamma}_r \in \overline{\Gamma}$ such that

$$\Pi'' = \overline{D}(\overline{\gamma}_1) \circ \cdots \circ \overline{D}(\overline{\gamma}_r)(\Pi).$$

By (3.1), (3.2), and (2.4′), for every n such that $n_1 \le n \le n_2$ there exists $\overline{i}_{\overline{m}} = 1,\dots,k$ (depending on n) such that we have

$$Q_{n,\Pi'} \in K_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1},\overline{i}_{\overline{m}}} \cap V^{(\overline{m})} \subseteq V_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1},\overline{i}_{\overline{m}}} \subseteq V^{(1)}{}_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}},$$

where the first inclusion follows from Lemma 2.3. Hence, in view of Lemma 3.2(ii), for some $\overline{\iota} \in \widehat{J}$, the path $\overline{\Pi}$ defined by

$$\overline{\Pi} := \big(\psi^{-1}_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}}(Q_{n_1,\Pi'}),\dots,\psi^{-1}_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}}(Q_{n_2,\Pi'})\big)$$

is a strict $(\overline{\iota}, V^{(1)})$-path. Thus, $(Q_{n_1,\Pi'}, Q_{n_2,\Pi'}) = \psi_{\overline{i}_1,\dots,\overline{i}_{\overline{m}-1}}(P_{\overline{\iota}})$. Let $\overline{\gamma} \in \overline{\Gamma}$ be defined by

$$\overline{\gamma}(\iota, w^*) = \begin{cases} \overline{\Pi} & \text{if } (\iota, w^*) = (\overline{\iota}, \overline{i}_1,\dots,\overline{i}_{\overline{m}-1}), \\ P_\iota & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned}\overline{D}(\overline{\gamma})(Q_{n_1,\Pi'},Q_{n_2,\Pi'}) &= \overline{D}(\overline{\gamma})\big(\psi_{\bar{i}_1,\dots,\bar{i}_{\overline{m}-1}}(P_{\bar{\iota}})\big)\\ &= \psi_{\bar{i}_1,\dots,\bar{i}_{\overline{m}-1}}\big(\overline{\gamma}(\bar{\iota},\bar{i}_1,\dots,\bar{i}_{\overline{m}-1})\big)\\ &= \psi_{\bar{i}_1,\dots,\bar{i}_{\overline{m}-1}}(\overline{\Pi}) = (Q_{n_1,\Pi'},\dots,Q_{n_2,\Pi'}).\end{aligned}$$

Moreover, since Π'' is strict, if $h \neq n_1$, we have $(Q_{h,\Pi''},Q_{h+1,\Pi''}) \neq (Q_{n_1,\Pi''}, Q_{n_1+1,\Pi''})$. Thus, by the definition of $\overline{D}(\overline{\Gamma})$ we have

$$\begin{aligned}&\overline{D}(\overline{\gamma})(Q_{h,\Pi''},Q_{h+1,\Pi''})\\ &\quad= \begin{cases}\overline{D}(\overline{\gamma})(Q_{n_1,\Pi'},Q_{n_2,\Pi'}) = (Q_{n_1,\Pi'},\dots,Q_{n_2,\Pi'}) & \text{if } h = n_1,\\ (Q_{h,\Pi''},Q_{h+1,\Pi''}) & \text{otherwise,}\end{cases}\end{aligned}$$

and letting Q_n denote $Q_{n,\Pi'}$, we have

$$\begin{aligned}\overline{D}(\overline{\gamma})(\Pi'') &= \overline{D}(\overline{\gamma})(Q_0,Q_1)\circ\cdots\circ\overline{D}(\overline{\gamma})(Q_{n_1},Q_{n_2})\circ\cdots\circ\overline{D}(\overline{\gamma})(Q_{n(\Pi')-1},Q_{n(\Pi')})\\ &= (Q_0,Q_1,\dots,Q_{n_1},\dots,Q_{n_2},\dots,Q_{n(\Pi')-1},Q_{n(\Pi')}) = \Pi',\end{aligned}$$

so that $\overline{D}(\overline{\gamma})\circ\overline{D}(\overline{\gamma}_1)\circ\cdots\circ\overline{D}(\overline{\gamma}_r)(\Pi) = \overline{D}(\overline{\gamma})(\Pi'') = \Pi'$. □

4. Linear operators related to the paths

In this section, we associate some special linear operators (or equivalently matrices) to every element of Γ or of $\overline{\Gamma}$. Basically, we interpret the edges of the paths as the basis of a linear space. Recall that we have defined the function $\alpha_{\overline{\mathbf{w}}}$ on $\widetilde{Y}$ with real values. Then, we associate to the given path the vector having as the (Q,Q')-component the value $\alpha_{\overline{\mathbf{w}}}(Q,Q')$. More precisely, let e_ι be the elements of the canonical basis in $\mathbb{R}^{\widehat{J}}$; that is, $(e_\iota)_{\iota'} = \delta_{\iota,\iota'}$. For every $\gamma\in\Gamma$ we define a linear operator $T^{(\alpha)}_\gamma$ from $\mathbb{R}^{\widehat{J}}$ to $\mathbb{R}^{\widehat{J}}$ by

$$T^{(\alpha)}_\gamma(e_\iota) = \sum_{\iota'\in\widehat{J}} \widetilde{\alpha}_{\gamma,\iota,\iota'} e_{\iota'}, \qquad \widetilde{\alpha}_{\gamma,\iota,\iota'} := \sum_{h:\widetilde{\iota}(h,\gamma(\iota))=\iota'} \alpha_{\widetilde{w}(h,\gamma(\iota))},$$

or in other words,

$$T^{(\alpha)}_\gamma(e_\iota) = \sum_{h=1}^{n(\gamma(\iota))} \alpha_{\widetilde{w}(h,\gamma(\iota))} e_{\widetilde{\iota}(h,\gamma(\iota))}.$$

Now, we introduce similar notions related to $\overline{\Gamma}$ instead of related to Γ. The reason for which we introduce these notions both in Γ and in $\overline{\Gamma}$ is that, on one hand, we have a closer relationship between $\overline{D}(\overline{\gamma})$ and the linear operators related to $\overline{\Gamma}$, and on the other hand, the linear operators related to Γ are simpler to handle. However, for our purposes the two notions are equivalent; more precisely, the asymptotic behaviors of the composition of linear operators are in some sense the same in both situations (see Corollary 4.4).

Let e_{ι,w^*} be the elements of the canonical basis of $\overline{Z}$, where $\overline{Z}$ is the set of $z\in\mathbb{R}^{\widehat{J}\times W^*}$ such that $z_{\iota,w^*}=0$ for almost all (ι,w^*) (i.e., for all (ι,w^*) but finitely many). That is, $(e_{\iota,w^*})_{\iota',w^{*\prime}} = \delta_{(\iota,w^*),(\iota',w^{*\prime})}$. Let $\overline{Z}_m$ be the set of z of

$\mathbb{R}^{\widehat{J}\times W^*}$ such that $z_{\iota,w^*}=0$ for every $w^*\notin\widetilde{W}_m$. Of course, we have $\overline{Z}_m\subseteq\overline{Z}$. We also set $e_\iota:=e_{\iota,\emptyset}$. For every $\overline{\gamma}\in\overline{\Gamma}$ we define a linear operator $\overline{T}^{(\alpha)}_{\overline{\gamma}}$ from $\overline{Z}$ into itself by

$$\overline{T}^{(\alpha)}_{\overline{\gamma}}(e_{\iota,w^*})=\sum_{h=1}^{n(\overline{\gamma}(\iota,w^*))}\alpha_{\widetilde{w}(h,\overline{\gamma}(\iota,w^*))}e_{\widetilde{\iota}(h,\overline{\gamma}(\iota,w^*)),w^*\widetilde{w}(h,\overline{\gamma}(\iota,w^*))}.$$

Note that, for every $m\in\mathbb{N}$, $\overline{T}^{(\alpha)}_{\overline{\gamma}}$ maps $\overline{Z}_m$ into $\overline{Z}_{m+1}$. Let $H:\mathbb{R}^{\widehat{J}}\to\mathbb{R}$, $\overline{H}:\overline{Z}\to\mathbb{R}$ be defined by

$$H(x)=\sum_{\iota\in\widehat{J}}x_\iota,\qquad \overline{H}(x)=\sum_{(\iota,w^*)\in\widehat{J}\times W^*}x_{\iota,w^*}.$$

Let $f:\widetilde{Y}\to\mathbb{R}$. We now define the *sum operators along a path*, that is, some kind of sum of f along a given path. We will use such notions in the sequel, specifically when $f=\alpha_{\overline{\mathbf{w}}}$, but also other f's will be considered in Section 5. Namely, if Π is a path, we define

$$\widetilde{\Sigma}_\Pi(f)=\sum_{h=1}^{n(\Pi)}f(Q_{h-1,\Pi},Q_{h,\Pi})e_{\widetilde{\iota}(h,\Pi),\widetilde{w}(h,\Pi)}\in\overline{Z},$$

$$\widehat{\Sigma}_\Pi(f)=\sum_{h=1}^{n(\Pi)}f(Q_{h-1,\Pi},Q_{h,\Pi})e_{\widetilde{\iota}(h,\Pi)}\in\mathbb{R}^{\widehat{J}},$$

$$\Sigma_\Pi(f)=\sum_{h=1}^{n(\Pi)}f(Q_{h-1,\Pi},Q_{h,\Pi}).$$

For paths Π, Π', we will occasionally use such definitions (more specifically, the definition of $\Sigma_\Pi(f)$) also when Π is a weak path, with the convention $f(Q,Q)=0$. We will use the following simple properties in the sequel without mention:

$$\Sigma_\Pi(f)=\overline{H}\big(\widetilde{\Sigma}_\Pi(f)\big)=H\big(\widehat{\Sigma}_\Pi(f)\big),$$
$$\widetilde{\Sigma}_{\Pi\circ\Pi'}(f)=\widetilde{\Sigma}_\Pi(f)+\widetilde{\Sigma}_{\Pi'}(f),\qquad \widehat{\Sigma}_{\Pi\circ\Pi'}(f)=\widehat{\Sigma}_\Pi(f)+\widehat{\Sigma}_{\Pi'}(f).$$

Recall that, given $\iota=(j_1,j_2)\in\widehat{J}$, the symbol P_ι denotes the pair (P_{j_1},P_{j_2}), which of course can be interpreted as a path of length 1. We immediately have

$$\widehat{\Sigma}_{P_\iota}(\alpha_{\overline{\mathbf{w}}})=e_\iota,\qquad \widetilde{\Sigma}_{P_\iota}(\alpha_{\overline{\mathbf{w}}})=e_\iota. \tag{4.1}$$

REMARK 4.1

When Π is a path, the following formulas are immediate consequences of the definition of $\widetilde{w}(h,\Pi)$:

$$\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi},Q_{h,\Pi})=\alpha_{\widetilde{w}(h,\Pi)}, \tag{4.2}$$

$$\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}})=\sum_{h=1}^{n(\Pi)}\alpha_{\widetilde{w}(h,\Pi)}. \tag{4.2$'$}$$

Note that (4.2) and (4.2′) are also valid when Π is a weak path if we use the convention $\alpha_{\widetilde{w}(h,\Pi)} = 0$ when $Q_{h-1,\Pi} = Q_{h,\Pi}$.

For every $\gamma \in \Gamma$, let $I(\gamma) \in \overline{\Gamma}$ be defined by $I(\gamma)(\iota, w^*) = \gamma(\iota)$. Let $D(\gamma) = \overline{D}(I(\gamma))$. The following properties will be useful in the sequel. For every $\iota, \iota' \in \widehat{J}$ and every $w^* \in W^*$, we have

$$(4.3) \qquad \big(T_\gamma^{(\alpha)}(e_\iota)\big)_{\iota'} = \widetilde{\alpha}_{\gamma,\iota,\iota'} = \sum_{w^{*\prime} \in W^*} \big(\overline{T}_{I(\gamma)}^{(\alpha)}(e_{\iota,w^*})\big)_{\iota',w^{*\prime}},$$

$$(4.4) \qquad H\big(T_\gamma^{(\alpha)}(e_\iota)\big) = \sum_{h=1}^{n(\gamma(\iota))} \alpha_{\widetilde{w}(h,\gamma(\iota))} = \overline{H}\big(\overline{T}_{I(\gamma)}^{(\alpha)}(e_{\iota,w^*})\big).$$

The next lemma shows that the linear operators $T_\gamma^{(\alpha)}$ and $\overline{T}_{\overline{\gamma}}^{(\alpha)}$ allow us to evaluate the sum operators along insertion paths.

LEMMA 4.2

For every path Π *we have*

(i) $\widetilde{\Sigma}_{\overline{D}(\overline{\gamma}_1)\circ\cdots\circ\overline{D}(\overline{\gamma}_n)(\Pi)}(\alpha_{\overline{\mathbf{w}}}) = \overline{T}_{\overline{\gamma}_1}^{(\alpha)} \circ \cdots \circ \overline{T}_{\overline{\gamma}_n}^{(\alpha)}(\widetilde{\Sigma}_\Pi(\alpha_{\overline{\mathbf{w}}}))\ \forall \overline{\gamma}_1, \ldots, \overline{\gamma}_n \in \overline{\Gamma}$;

(ii) $\widehat{\Sigma}_{D(\gamma_1)\circ\cdots\circ D(\gamma_n)(\Pi)}(\alpha_{\overline{\mathbf{w}}}) = T_{\gamma_1}^{(\alpha)} \circ \cdots \circ T_{\gamma_n}^{(\alpha)}(\widehat{\Sigma}_\Pi(\alpha_{\overline{\mathbf{w}}}))\ \forall \gamma_1, \ldots, \gamma_n \in \Gamma$.

Proof

It suffices to prove the lemma when $n = 1$. Note that, for every $\iota \in \widehat{J}$, every $w^* \in W^*$, and every $\overline{\gamma} \in \overline{\Gamma}$, we have

$$\begin{aligned}\widetilde{\Sigma}_{\psi_{w^*}(\overline{\gamma}(\iota,w^*))}(\alpha_{\overline{\mathbf{w}}}) &= \sum_{h=1}^{n(\overline{\gamma}(\iota,w^*))} \alpha_{w^*\widetilde{\iota}(h,\overline{\gamma}(\iota,w^*))} e_{\widetilde{w}(h,\overline{\gamma}(\iota,w^*)),w^*\widetilde{w}(h,\overline{\gamma}(\iota,w^*))} \\ &= \alpha_{w^*}\overline{T}_{\overline{\gamma}}^{(\alpha)}(e_{\iota,w^*}).\end{aligned}$$

Let $\overline{\gamma} \in \overline{\Gamma}$. We thus have

$$\begin{aligned}\widetilde{\Sigma}_{\overline{D}(\overline{\gamma})(\Pi)}(\alpha_{\overline{\mathbf{w}}}) &= \sum_{h=1}^{n(\Pi)} \widetilde{\Sigma}_{\overline{D}(\overline{\gamma})(Q_{h-1,\Pi},Q_{h,\Pi})}(\alpha_{\overline{\mathbf{w}}}) \\ &= \sum_{h=1}^{n(\Pi)} \widetilde{\Sigma}_{D(\overline{\gamma})(\psi_{\widetilde{w}(h,\Pi)}(P_{\widetilde{\iota}(h,\Pi)}))}(\alpha_{\overline{\mathbf{w}}}) \\ &= \sum_{h=1}^{n(\Pi)} \widetilde{\Sigma}_{\psi_{\widetilde{w}(h,\Pi)}(\overline{\gamma}(\widetilde{\iota}(h,\Pi),\widetilde{w}(h,\Pi)))}(\alpha_{\overline{\mathbf{w}}}) \\ &= \sum_{h=1}^{n(\Pi)} \alpha_{\widetilde{w}(h,\Pi)}\overline{T}_{\overline{\gamma}}^{(\alpha)}(e_{\widetilde{\iota}(h,\Pi),\widetilde{w}(h,\Pi)})\end{aligned}$$

$$= \overline{T}^{(\alpha)}_{\overline{\gamma}}\Big(\sum_{h=1}^{n(\Pi)} \alpha_{\widetilde{w}(h,\Pi)} e_{\widetilde{\iota}(h,\Pi),\widetilde{w}(h,\Pi)}\Big)$$
$$= \overline{T}^{(\alpha)}_{\overline{\gamma}}\big(\widetilde{\Sigma}_{\Pi}(\alpha_{\overline{\mathbf{w}}})\big).$$

This proves (i), and (ii) can be proved similarly. □

We are now going to introduce a notion for polyratios which will turn out to be equivalent to the notions of metric polyratios and of asymptotic metric polyratios.

We say that the polyratio α is (T,Γ)-uniformly positive ((T,Γ)-UP) (on K) if there exists $c_{3,\alpha} > 0$ such that

$$H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_\iota)\big) \geq c_{3,\alpha} \quad \forall m \in \mathbb{N}, \forall \gamma_1,\ldots,\gamma_m \in \Gamma, \forall \iota \in \widehat{J}. \tag{4.5}$$

Given a polyratio $\boldsymbol{\alpha}$ and $c_{3,\alpha} > 0$, we will also consider the following variant of (4.5):

$$\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_\iota)\big) \geq c_{3,\alpha} \quad \forall m \in \mathbb{N}, \forall \overline{\gamma}_1,\ldots,\overline{\gamma}_m \in \overline{\Gamma}, \forall \iota \in \widehat{J}. \tag{4.5$'$}$$

Note that, since $\widehat{J}$ and $V^{(1)}$ are finite sets and the elements of Γ are by definition *strict* $V^{(1)}$-paths, Γ is a finite set as well. Thus, as hinted in Section 1, (4.5) is related to the notion of joint spectral radius and to that of joint spectral subradius of a finite set of matrices. On the contrary, $\overline{\Gamma}$ is an infinite set. Thus, (4.5$'$) is more complicated to verify. However, as we will see in Corollary 4.4, (4.5) and (4.5$'$) are equivalent. We need the following lemma.

LEMMA 4.3

(i) *For every* $\gamma_1,\ldots,\gamma_{m+1} \in \Gamma$, *we have*

$$H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)\big) = \sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})\big).$$

(ii) *For every* $\overline{\gamma}_1,\ldots,\overline{\gamma}_{m+1} \in \overline{\Gamma}$, *we have*

$$\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{\gamma}_{m+1}}(e_{\iota,w^*})\big)$$
$$= \sum_{(\iota',w^{*\prime}) \in \widehat{J}\times W^*} \overline{T}^{(\alpha)}_{\overline{\gamma}_{m+1}}(e_{\iota,w^*})_{\iota',w^{*\prime}} \overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_{\iota',w^{*\prime}})\big)$$
$$= \sum_{h=1}^{n(\overline{\gamma}_{m+1}(\iota,w^*))} \widetilde{\alpha}_h \overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_{\widetilde{\iota}(h,\overline{\gamma}_{m+1}(\iota,w^*)),w^*\widetilde{w}(h,\overline{\gamma}_{m+1}(\iota,w^*))})\big),$$

$\widetilde{\alpha}_h := \alpha_{\widetilde{w}(h,\overline{\gamma}_{m+1}(\iota,w^*))}$.

(iii) *Given* $\gamma_l \in \Gamma$, $l = 1,2,\ldots,m$, $\iota \in \widehat{J}$, *and* $w^* \in W^*$, *we have*

$$H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_\iota)\big) = \overline{H}\big(\overline{T}^{(\alpha)}_{I(\gamma_1)} \circ \cdots \circ \overline{T}^{(\alpha)}_{I(\gamma_m)}(e_{\iota,w^*})\big).$$

(iv) *Given* $\overline{\gamma}_l \in \overline{\Gamma}$, $l = 1, 2, \ldots, m$, *there exist* $\gamma_l \in \Gamma$ *such that for every* $\iota \in \widehat{J}$, *we have*

$$H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_\iota)\big) \le \overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_\iota)\big). \tag{4.6}$$

Proof

(i) Note that

$$\begin{aligned}
&H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)\big)\\
&\quad= \sum_{\iota'' \in \widehat{J}} \big((T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}) \circ T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)\big)_{\iota''}\\
&\quad= \sum_{\iota'' \in \widehat{J}} \Big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}\Big(\sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} e_{\iota'}\Big)\Big)_{\iota''}\\
&\quad= \sum_{\iota'' \in \widehat{J}} \Big(\sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})\Big)_{\iota''}\\
&\quad= \sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} \Big(\sum_{\iota'' \in \widehat{J}} T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})_{\iota''}\Big)\\
&\quad= \sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})\big),
\end{aligned}$$

and (i) is proved. For (ii), the proof of the first equality is very similar and is omitted. The second equality follows from the definition of $\overline{T}^{(\alpha)}_{\overline{\gamma}_{m+1}}$.

(iii) The case $m = 1$ follows from (4.4). The general case follows by induction. In fact, if (iii) holds for m, by (i), (ii), (4.3), and the definitions of $T(\gamma)$, $\overline{T}(\overline{\gamma})$, H, and $\overline{H}$, we have

$$\begin{aligned}
&H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)\big) = \sum_{\iota' \in \widehat{J}} T^{(\alpha)}_{\gamma_{m+1}}(e_\iota)_{\iota'} H\big(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})\big)\\
&= \sum_{\iota' \in \widehat{J}} \Big(\sum_{w^{*\prime} \in W^*} \overline{T}^{(\alpha)}_{I(\gamma_{m+1})}(e_{\iota,w^*})_{\iota',w^{*\prime}} \overline{H}\big(\overline{T}^{(\alpha)}_{I(\gamma_1)} \circ \cdots \circ \overline{T}^{(\alpha)}_{I(\gamma_m)}(e_{\iota',w^{*\prime}})\big)\Big)\\
&= \sum_{(\iota',w^{*\prime}) \in \widehat{J} \times W^*} \overline{T}^{(\alpha)}_{I(\gamma_{m+1})}(e_{\iota,w^*})_{\iota',w^{*\prime}} \overline{H}\big(\overline{T}^{(\alpha)}_{I(\gamma_1)} \circ \cdots \circ \overline{T}^{(\alpha)}_{I(\gamma_m)}(e_{\iota',w^{*\prime}})\big)\\
&= \overline{H}\big(\overline{T}^{(\alpha)}_{I(\gamma_1)} \circ \cdots \circ \overline{T}^{(\alpha)}_{I(\gamma_{m+1})}(e_{\iota,w^*})\big),
\end{aligned}$$

and (iii) holds for $m + 1$.

(iv) Note that, for every $\iota, \iota' \in \widehat{J}$, every $w^* \in \widetilde{W}_m$, and every $\overline{\gamma} \in \overline{\Gamma}$, we have the following simple analogue of (4.3):

$$\begin{gathered}
\sum_{w^{*\prime} \in W^*} \big(\overline{T}^{(\alpha)}_{\overline{\gamma}}(e_{\iota,w^*})\big)_{\iota',w^{*\prime}} = \sum_{w^{*\prime} \in \widetilde{W}_{m+1}} \big(\overline{T}^{(\alpha)}_{\overline{\gamma}}(e_{\iota,w^*})\big)_{\iota',w^{*\prime}} = \widetilde{\alpha}_{\overline{\gamma},\iota,\iota',w^*},\\
\widetilde{\alpha}_{\overline{\gamma},\iota,\iota',w^*} := \sum_{h : \widetilde{\iota}(h,\overline{\gamma}(\iota,w^*)) = \iota'} \alpha_{\widetilde{w}(h,\overline{\gamma}(\iota,w^*))}.
\end{gathered} \tag{4.7}$$

We define γ_l by induction. Suppose that we have defined $\gamma_1,\dots,\gamma_l$, $l<m$, satisfying

$$H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_l}(e_\iota)\big)\le \overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ \overline{T}^{(\alpha)}_{\overline{\gamma}_l}(e_{\iota,w^*})\big)\quad \forall w^*\in \widetilde{W}_{m-l}, \tag{4.8}$$

and define γ_{l+1} in this way. Let

$$\theta_\iota(w^*):=\sum_{\iota'\in\widehat{J}}\widetilde{\alpha}_{\overline{\gamma}_{l+1},\iota,\iota',w^*}H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_l}(e_{\iota'})\big),$$
$$\gamma_{l+1}(\iota)=\overline{\gamma}_{l+1}\big(\iota,w^*(\iota)\big),$$

where, for every given $\iota\in\widehat{J}$, $w^*(\iota)$ is an element of $\widetilde{W}_{m-l-1}$ minimizing θ_ι, that is, such that

$$\theta_\iota\big(w^*(\iota)\big)\le\theta_\iota(w^*)\quad\forall w^*\in\widetilde{W}_{m-l-1}.$$

Note that, in view of (4.3) and (4.7), we have

$$\sum_{w^{*\prime}\in\widetilde{W}_{m-l}}\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*(\iota)})_{\iota',w^{*\prime}}=T^{(\alpha)}_{\gamma_{l+1}}(e_\iota)_{\iota'} \tag{4.9}$$

for every $\iota,\iota'\in\widehat{J}$. Thus, for every $w^*\in\widetilde{W}_{m-l-1}$, we have

$$\begin{aligned}
&\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*})\big)\\
&\quad=\sum_{(\iota',w^{*\prime})\in\widehat{J}\times W^*}\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*})_{\iota',w^{*\prime}}\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_l}(e_{\iota',w^{*\prime}})\big)\quad\big(\text{by (ii)}\big)\\
&\quad=\sum_{\iota'\in\widehat{J}}\sum_{w^{*\prime}\in\widetilde{W}_{m-l}}\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*})_{\iota',w^{*\prime}}\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_l}(e_{\iota',w^{*\prime}})\big)\\
&\quad\ge\sum_{\iota'\in\widehat{J}}\sum_{w^{*\prime}\in\widetilde{W}_{m-l}}\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*})_{\iota',w^{*\prime}}H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_l}(e_{\iota'})\big)\quad\big(\text{by (4.8)}\big)\\
&\quad=\theta_\iota(w^*)\ge\theta_\iota\big(w^*(\iota)\big)\\
&\quad=\sum_{\iota'\in\widehat{J}}\sum_{w^{*\prime}\in\widetilde{W}_{m-l}}\overline{T}^{(\alpha)}_{\overline{\gamma}_{l+1}}(e_{\iota,w^*(\iota)})_{\iota',w^{*\prime}}H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_l}(e_{\iota'})\big)\\
&\quad=\sum_{\iota'\in\widehat{J}}T^{(\alpha)}_{\gamma_{l+1}}(e_\iota)_{\iota'}H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_l}(e_{\iota'})\big)\quad\big(\text{by (4.9)}\big)\\
&\quad=H\big(T^{(\alpha)}_{\gamma_1}\circ\cdots\circ T^{(\alpha)}_{\gamma_{l+1}}(e_\iota)\big)\quad\big(\text{by (i)}\big)
\end{aligned}$$

and (4.8) holds for $l+1$. The inductive step is completed. Now, (iv) follows from the case $l=m$. □

COROLLARY 4.4

A polyratio α is (T,Γ)-UP if and only if there exists $c_{3,\alpha}>0$ such that (4.5′) holds.

Proof
This immediately follows from Lemmas 4.3(iii) and 4.3(iv). □

5. Distances on the fractal

In this section, we prove that, for a given polyratio α, α being (T,Γ)-UP is a necessary and sufficient condition both for being metric and for being asymptotic metric (see Theorem 5.2). In the sequel we will always implicitly use Corollary 4.4. Moreover, an explicit distance which is both α-self-similar and α-scaling is described when α is (T,Γ)-UP. This distance was introduced by A. Kameyama [4] in a more general context, including fractals that are not finitely ramified. It is denoted by D_α and is called the *standard distance* in [4]. Recall the definition of D_α. If $Q,Q'\in K$ and $w_1^*,\dots,w_n^*\in W^*$, we say that $(w_1^*,\dots,w_n^*)$ is a prechain if $K_{w_h^*}\cap K_{w_{h+1}^*}\neq\emptyset$ for every $h=1,\dots,n-1$. We say that $(w_1^*,\dots,w_n^*)$ is between Q and Q' if $Q\in K_{w_1^*}$ and $Q'\in K_{w_n^*}$. We say that a prechain $(w_1^*,\dots,w_n^*)$ is a chain if $w_1^*,\dots,w_n^*$ are pairwise incomparable. Denote by $G'(Q,Q')$ the set of prechains between Q and Q', and denote by $G(Q,Q')$ the set of chains between Q and Q'. Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ be a polyratio, and let

$$A(\mathcal{C}) = \sum_{h=1}^{n}\alpha_{w_h^*}\quad \text{if } \mathcal{C}\in G'(Q,Q'),$$

$$D_\alpha(Q,Q') := \inf\{A(\mathcal{C}):\mathcal{C}\in G(Q,Q')\} = \inf\{A(\mathcal{C}):\mathcal{C}\in G'(Q,Q')\}.$$

Note that a standard argument shows that D_α is in any case a *pseudodistance*, in the sense that it satisfies all properties of a distance, except possibly for the fact that $D_\alpha(Q,Q')=0$ could also occur when $Q\neq Q'$. It will turn out to be a distance when α is (T,Γ)-UP.

LEMMA 5.1

(i) *For every polyratio $\boldsymbol{\alpha}$, D_α is a pseudodistance, and if $D_\alpha(Q,Q')>0$ for every $Q,Q'\in V^{(\infty)}$ with $Q\neq Q'$, then D_α is an $\boldsymbol{\alpha}$-self-similar distance.*

(ii) *If $\boldsymbol{\alpha}$ is a metric polyratio, then D_α is an $\boldsymbol{\alpha}$-self-similar distance, which induces the same topology as the original distance.*

Proof
(i) This follows from [4, Theorem 1.33, Proposition 1.12]. (ii) See [4, Propositions 1.13, 1.11]. Note that, according to the definitions in [4], which are equivalent to the definitions here but slightly differ from them, in the hypothesis of [4, Proposition 1.13] it is assumed that d induces the same topology on K as the original distance, but the proof does not use this fact. □

In this section, we will prove the following theorem.

THEOREM 5.2

Given a polyratio $\boldsymbol{\alpha}$, the following are equivalent.

(i) *There exist a distance d on K and $c_{2,\alpha}>0$ such that, for every $w^*\in W^*$, we have* $\operatorname{diam}_d(K_{w^*})\le c_{2,\alpha}\alpha_{w^*}\operatorname{diam}_d(K)$.

(ii) $\boldsymbol{\alpha}$ *is a metric polyratio.*

(iii) $\boldsymbol{\alpha}$ *is an asymptotic metric polyratio.*

(iv) *There exist a distance d on K and two positive constants $c_{1,\alpha}$, $c_{2,\alpha}$ such that, for every $w^*\in W^*$, we have*

$$c_{1,\alpha}\alpha_{w^*}\operatorname{diam}_d(K)\le\operatorname{diam}_d(K_{w^*})\le c_{2,\alpha}\alpha_{w^*}\operatorname{diam}_d(K).$$

(v) $\boldsymbol{\alpha}$ *is (T,Γ)-UP.*

If such equivalent conditions hold, D_α is both an α-self-similar distance and an α-scaling distance. More precisely, D_α satisfies (1.3) and (v) with $c_{2,\alpha}=1$.

The proof will be given later. Note that [6, Theorem 2.3.16] states that if α is a metric polyratio, then D_α is a distance adapted to a scale, which, as seen in Section 1, is a property similar to that of being an α-scaling distance.

LEMMA 5.3

In the statement of Theorem 5.2, (i) $\Rightarrow$ (v).

Proof

Suppose that Theorem 5.2(i) holds. Note that for every path Π connecting P_{j_1} to P_{j_2} and every $h=1,\dots,n(\Pi)$ we have

$$d(P_{j_1},P_{j_2})\le\sum_{h=1}^{n(\Pi)}d(Q_{h-1,\Pi},Q_{h,\Pi})=\Sigma_\Pi(d). \tag{5.1}$$

Moreover, for some j, j' (depending on h), by (i) we have

$$\begin{aligned}d(Q_{h-1,\Pi},Q_{h,\Pi})&=d\big(\psi_{\widetilde{w}(h,\Pi)}(P_j),\psi_{\widetilde{w}(h,\Pi)}(P_{j'})\big)\\&\le\operatorname{diam}_d(K_{\widetilde{w}(h,\Pi)})\le c_{2,\alpha}\alpha_{\widetilde{w}(h,\Pi)}\operatorname{diam}_d(K)\\&=c_{2,\alpha}\operatorname{diam}_d(K)\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi},Q_{h,\Pi}).\end{aligned}$$

Thus, summing the previous inequalities, we obtain

$$\begin{aligned}\Sigma_\Pi(d)&=\sum_{h=1}^{n(\Pi)}d(Q_{h-1,\Pi},Q_{h,\Pi})\\&\le c_{2,\alpha}\operatorname{diam}_d(K)\sum_{h=1}^{n(\Pi)}\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi},Q_{h,\Pi})\\&=c_{2,\alpha}\operatorname{diam}_d(K)\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}),\end{aligned}$$

and in view of (5.1), we have

$$d(P_{j_1},P_{j_2})\le c_{2,\alpha}\operatorname{diam}_d(K)\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}). \tag{5.2}$$

Now let $\overline{\gamma}_1,\dots,\overline{\gamma}_l\in\overline{\Gamma}$, and let $(j_1,j_2)\in\widehat{J}$. Set $\Pi:=(P_{j_1},P_{j_2})$, so that $\widetilde{\Sigma}_\Pi(\alpha_{\overline{\mathbf{w}}})=e_{(j_1,j_2)}$ (see (4.1)). Recall that the path $\overline{D}(\overline{\gamma}_1)\circ\cdots\circ\overline{D}(\overline{\gamma}_l)(\Pi)$ connects P_{j_1} to P_{j_2}, so we can use (5.2) with this path in place of Π, and in view of Lemma 4.2(i), we have

$$\begin{aligned}\min\big\{d(P_{h_1},P_{h_2}):(h_1,h_2)\in\widehat{J}\big\}&\le d(P_{j_1},P_{j_2})\\&\le c_{2,\alpha}\operatorname{diam}_d(K)\Sigma_{\overline{D}(\overline{\gamma}_1)\circ\cdots\circ\overline{D}(\overline{\gamma}_l)(\Pi)}(\alpha_{\overline{\mathbf{w}}})\\&=c_{2,\alpha}\operatorname{diam}_d(K)\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_l}(e_{(j_1,j_2)})\big).\end{aligned}$$

Hence, α is (T,Γ)-UP, since (4.5′) is satisfied with

$$c_{3,\alpha}=\frac{\min\{d(P_{h_1},P_{h_2}):(h_1,h_2)\in\widehat{J}\}}{c_{2,\alpha}\operatorname{diam}_d(K)}.\qquad\square$$

Now, we are going to prove the other implications in Theorem 5.2. To do this, we will introduce a pseudodistance $\widetilde{d}_\alpha$ on $V^{(\infty)}$ that is more strictly related to the notion of (T,Γ)-UP than D_α. However, in Lemma 5.7 we will prove that D_α and $\widetilde{d}_\alpha$ are equivalent on $V^{(\infty)}$. A similar distance is discussed in [4, Theorem 1.34]. The context in [4] was a bit different; that is, it was assumed that there exists a distance on a set like $V^{(0)}$. Note that we could extend $\widetilde{d}_\alpha$ on K and use such a distance on K in place of D_α, but I have preferred to use D_α in order to relate it to results already existing in the literature. We will use in the sequel the following trivial statement. For every path Π and every $w^*\in W^*$, we have

$$\Sigma_{\psi_{w^*}(\Pi)}(\alpha_{\overline{\mathbf{w}}})=\alpha_{w^*}\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}).\tag{5.3}$$

Let $(\mathcal{P}a)'_{Q,Q'}$ be the set of weak paths connecting Q to Q', let $(\mathcal{P}a)_{Q,Q'}$ be the set of paths connecting Q to Q', and let $(\widetilde{\mathcal{P}a})_{Q,Q'}$ be the set of strict paths connecting Q to Q'. We now define the function $\widetilde{d}_\alpha$ on $V^{(\infty)}$ by

$$\begin{aligned}\widetilde{d}_\alpha(Q,Q')&=\inf\big\{\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}):\Pi\in(\mathcal{P}a)_{Q,Q'}\big\}\\&=\inf\big\{\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}):\Pi\in(\widetilde{\mathcal{P}a})_{Q,Q'}\big\}\\&=\inf\big\{\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}):\Pi\in(\mathcal{P}a)'_{Q,Q'}\big\}.\end{aligned}$$

It can be easily proved that $\widetilde{d}_\alpha$ is a pseudodistance on $V^{(\infty)}$. The only nontrivial point for proving this is that for every $Q,Q'\in V^{(\infty)}$ we have $(\mathcal{P}a)_{Q,Q'}\neq\emptyset$, so that $\widetilde{d}_\alpha(Q,Q')<+\infty$. This will follow from the next lemma. There, $\operatorname{diam}_{\widetilde{d}_\alpha}(E)$, of course, denotes $\sup\{\widetilde{d}_\alpha(Q,Q'):Q,Q'\in E\}$ for every $E\subseteq V^{(\infty)}$.

LEMMA 5.4

There exists a constant $C_{1,\alpha}\ge 1$ such that

$$\operatorname{diam}_{\widetilde{d}_\alpha}(V^{(\infty)}_{w^*})\le C_{1,\alpha}\alpha_{w^*}$$

for every $w^*\in W^*$. Thus, in particular, $\operatorname{diam}_{\widetilde{d}_\alpha}(V^{(\infty)}_{w^*})$ is finite for every $w^*\in W^*$.

Proof
Since the fractal is connected, there exists $C_\alpha \geq 1$ such that for every $\widehat{Q}, \widehat{Q}' \in V^{(1)}$ there exists a path Π connecting them such that $\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}) \leq C_\alpha$. Thus, for every $w^* \in W^*$, in view of (5.3), the path $\Pi' := \psi_{w^*}(\Pi)$ satisfies $\Sigma_{\Pi'}(\alpha_{\overline{\mathbf{w}}}) \leq C_\alpha \alpha_{w^*}$. Since Π' connects $\psi_{w^*}(\widehat{Q})$ to $\psi_{w^*}(\widehat{Q}')$, we then have

$$\widetilde{d}_\alpha\big(\psi_{w^*}(\widehat{Q}), \psi_{w^*}(\widehat{Q}')\big) \leq C_\alpha \alpha_{w^*}. \tag{5.4}$$

Now, suppose that $w^* = (i_1, \dots, i_m)$. If $Q \in V_{w^*}^{(\infty)}$, then there exist $i_{m+1}, \dots, i_{m'}$ and $P \in V^{(0)}$ such that $Q = \psi_{i_1,\dots,i_m,i_{m+1},\dots,i_{m'}}(P)$. Let $\overline{\Pi} := (Q_m, \dots, Q_{m'})$, where $Q_h = \psi_{i_1,\dots,i_m,\dots,i_h}(P)$. Then $\overline{\Pi}$ is a path that connects $\widetilde{Q} := \psi_{i_1,\dots,i_m}(P) \in V_{i_1,\dots,i_m}$ to Q. Therefore,

$$\begin{aligned}
\widetilde{d}_\alpha(Q, \widetilde{Q}) &\leq \sum_{h=m+1}^{m'} \widetilde{d}_\alpha(Q_{h-1}, Q_h) \\
&= \sum_{h=m+1}^{m'} \widetilde{d}_\alpha\big(\psi_{i_1,\dots,i_m,\dots,i_{h-1}}(P), \psi_{i_1,\dots,i_m,\dots,i_{h-1}}(\psi_{i_h}(P))\big) \\
&\leq C_\alpha \sum_{h=m+1}^{m'} \alpha_{i_1,\dots,i_m,\dots,i_{h-1}} \quad \big(\text{by } (5.4)\big) \\
&= C_\alpha \alpha_{w^*} \sum_{h=m+1}^{m'} \alpha_{i_{m+1}} \cdots \alpha_{i_{h-1}} \\
&\leq C_\alpha \alpha_{w^*} \sum_{h=m+1}^{m'} \overline{\alpha}_{\max}^{h-m-1} \\
&\leq \frac{C_\alpha}{1 - \overline{\alpha}_{\max}} \alpha_{w^*}.
\end{aligned}$$

Now, given $Q, Q' \in V_{w^*}^{(\infty)}$, let $\widetilde{Q}$, $\widetilde{Q'}$ be as above. Then, we have

$$\begin{aligned}
\widetilde{d}_\alpha(Q, Q') &\leq \widetilde{d}_\alpha(Q, \widetilde{Q}) + \widetilde{d}_\alpha(\widetilde{Q}, \widetilde{Q'}) + \widetilde{d}_\alpha(\widetilde{Q'}, Q') \\
&\leq \Big(C_\alpha + 2\frac{C_\alpha}{1 - \overline{\alpha}_{\max}}\Big)\alpha_{w^*}.
\end{aligned}$$

Thus, the lemma is proved with $C_{1,\alpha} = C_\alpha + 2\frac{C_\alpha}{1-\overline{\alpha}_{\max}}$. $\square$

To completely prove Theorem 5.2 we need some more lemmas.

LEMMA 5.5
If α is (T,Γ)-UP and Π is a strict path connecting two different points Q and Q' of $V^{(m)}$, then

$$\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}) \geq c_{3,\alpha}(\overline{\alpha}_{\min})^m \tag{5.5}$$

and, consequently, $\widetilde{d}_\alpha$ is a distance on $V^{(\infty)}$.

Proof
In view of Lemma 3.6, there exist a strict path Π' lying in $V^{(m)}$ connecting Q and Q' and $\overline{\gamma}_1,\dots,\overline{\gamma}_m\in\overline{\Gamma}$ such that

$$\Pi = D(\overline{\gamma}_1)\circ\cdots\circ D(\overline{\gamma}_m)(\Pi').$$

Note that, by Lemma 2.3, since Π' lies in $V^{(m)}$, $|w(Q_{h-1,\Pi'},Q_{h,\Pi'})|\le m$. Thus,

$$\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi'},Q_{h,\Pi'})\ge(\overline{\alpha}_{\min})^m \quad \forall h=1,\dots,n(\Pi'). \tag{5.6}$$

Therefore, by Lemma 4.2 we have

$$\begin{aligned}\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}) &= \Sigma_{D(\overline{\gamma}_1)\circ\cdots\circ D(\overline{\gamma}_m)(\Pi')}(\alpha_{\overline{\mathbf{w}}})\\ &= \overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_m}(\widetilde{\Sigma}_{\Pi'}(\alpha_{\overline{\mathbf{w}}}))\big)\\ &= \sum_{h=1}^{n(\Pi')}\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi'},Q_{h,\Pi'})\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_{\widetilde{\imath}(h,\Pi'),\widetilde{w}(h,\Pi')})\big)\\ &\ge \alpha_{\overline{\mathbf{w}}}(Q_{0,\Pi'},Q_{1,\Pi'})\overline{H}\big(\overline{T}^{(\alpha)}_{\overline{\gamma}_1}\circ\cdots\circ\overline{T}^{(\alpha)}_{\overline{\gamma}_m}(e_{\widetilde{\imath}(1,\Pi'),\widetilde{w}(1,\Pi')})\big)\\ &\ge c_{3,\alpha}(\overline{\alpha}_{\min})^m,\end{aligned}$$

where we have used (5.6), (4.5′), and the definition of $\widetilde{\Sigma}_{\Pi'}(\alpha_{\overline{\mathbf{w}}})$. Therefore, (5.5) holds. To prove that $\widetilde{d}_\alpha$ is a distance, the only nontrivial fact to prove is that if $Q,Q'\in V^{(\infty)}$, $Q\ne Q'$, then $\widetilde{d}_\alpha(Q,Q')>0$. But, since for some natural m we have $Q,Q'\in V^{(m)}$, by (5.5) we have $\widetilde{d}_\alpha(Q,Q')\ge c_{3,\alpha}(\overline{\alpha}_{\min})^m>0$. □

LEMMA 5.6
If α is (T,Γ)-UP, there exists $c_{4,\alpha}>0$ such that for every $w^*\in W^*$ and every $P,P'\in V^{(\infty)}$ we have

$$c_{4,\alpha}\alpha_{w^*}\widetilde{d}_\alpha(P,P')\le\widetilde{d}_\alpha\big(\psi_{w^*}(P),\psi_{w^*}(P')\big)\le\alpha_{w^*}\widetilde{d}_\alpha(P,P'). \tag{5.7}$$

Proof
Let $Q:=\psi_{w^*}(P)$, $Q':=\psi_{w^*}(P')$. We have

$$\begin{aligned}\widetilde{d}_\alpha(Q,Q') &= \min\{A,B\},\\ A &:= \inf_{\Pi\in\mathcal{P}}\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}), \qquad \mathcal{P}:=\psi_{w^*}\big((\widetilde{\mathcal{P}a})_{P,P'}\big),\\ B &:= \inf_{\Pi\in\mathcal{P}'}\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}), \quad \mathcal{P}'=(\widetilde{\mathcal{P}a})_{Q,Q'}\setminus\mathcal{P}.\end{aligned}$$

Let $w^*=(i_1,\dots,i_m)$. Since, in view of (5.3),

$$A=\alpha_{w^*}\widetilde{d}_\alpha(P,P'), \tag{5.8}$$

the second inequality in (5.7) follows at once. By Lemma 3.5, if $\Pi\in\mathcal{P}'$, then there exist $l<m$ and a subpath Π' of Π connecting different points of $V_{i_1,\dots,i_{l+1}}$ entirely contained in $V^{(\infty)}_{i_1,\dots,i_l}$. Thus, $\Pi'=\psi_{i_1,\dots,i_l}(\Pi'')$, where Π'' is a path in $V^{(\infty)}$

connecting two different points in $V^{(1)}$. By Lemma 5.5, $\Sigma_{\Pi''}(\alpha_{\overline{\mathbf{w}}}) \geq c_{3,\alpha}\overline{\alpha}_{\min}$. Hence,

$$\begin{aligned}\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) &\geq \Sigma_{\Pi'}(\alpha_{\overline{\mathbf{w}}})\\ &= \alpha_{i_1,\dots,i_l}\Sigma_{\Pi''}(\alpha_{\overline{\mathbf{w}}})\\ &\geq \alpha_{w^*}c_{3,\alpha}\overline{\alpha}_{\min}\\ &\geq c_{4,\alpha}\alpha_{w^*}\widetilde{d}_\alpha(P,P') \quad \text{if } c_{4,\alpha} \leq \frac{c_{3,\alpha}\overline{\alpha}_{\min}}{\operatorname{diam}_{\widetilde{d}_\alpha}(V^{(\infty)})}.\end{aligned}$$

If, moreover, $c_{4,\alpha} < 1$, then in view of (5.8), the first inequality in (5.7) holds. □

LEMMA 5.7

For every polyratio $\boldsymbol{\alpha}$ we have

$$D_\alpha(Q,Q') \leq \widetilde{d}_\alpha(Q,Q') \leq C_{1,\alpha}D_\alpha(Q,Q') \quad \forall Q,Q' \in V^{(\infty)}. \tag{5.9}$$

Proof

We can and do assume $Q \neq Q'$. Let $\Pi \in (\mathcal{P}a)_{Q,Q'}$. Then for every $h = 1,\dots, n(\Pi)-1$ we have $Q_{h,\Pi} \in K_{\widetilde{w}(h,\Pi)} \cap K_{\widetilde{w}(h+1,\Pi)}$. Thus, $\overline{\mathcal{C}} := (\widetilde{w}(1,\Pi),\dots, \widetilde{w}(n(\Pi),\Pi)) \in G'(Q,Q')$. Moreover,

$$A(\overline{\mathcal{C}}) = \sum_{h=1}^{n(\Pi)} \boldsymbol{\alpha}_{\widetilde{w}(h,\Pi)} = \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}),$$

so that the first inequality in (5.9) follows. Next, let $\overline{\mathcal{C}} \in G(Q,Q')$ and $\overline{\mathcal{C}} = (w_1^*,\dots,w_n^*)$. Suppose for the moment that $n > 1$. For $h = 0,\dots,n-2$, let Q_h be an element of $K_{w_{h+1}^*} \cap K_{w_{h+2}^*}$. Thus, w_{h+1}^* and w_{h+2}^* being incomparable, by Lemma 2.2 we have $Q_h \in V_{w_{h+1}^*} \cap V_{w_{h+2}^*}$. It follows that Π, defined by

$$\Pi := (Q_0,\dots,Q_{n-2}),$$

is a weak path, but not necessarily a path. In fact, for every $h = 1,\dots,n-2$ $Q_{h-1}, Q_h \in V_{w_{h+1}^*}$, so that either $Q_{h-1} \neq Q_h$ and $\widetilde{w}(h,\Pi) = w_{h+1}^*$ or $Q_{h-1} = Q_h$. Now, thanks to the convention $\boldsymbol{\alpha}_{\widetilde{w}(h,\Pi)} = 0$ (see Remark 4.1), we have

$$\widetilde{d}_\alpha(Q_0,Q_{n-2}) \leq \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}) = \sum_{h=1}^{n-2} \boldsymbol{\alpha}_{\widetilde{w}(h,\Pi)} \leq \sum_{h=2}^{n-1} \alpha_{w_h^*}.$$

Moreover, since $Q \in K_{w_1^*} \cap V^{(\infty)} = V_{w_1^*}^{(\infty)}$, we have $Q, Q_0 \in V_{w_1^*}^{(\infty)}$. Hence, by Lemma 5.4, $\widetilde{d}_\alpha(Q,Q_0) \leq C_{1,\boldsymbol{\alpha}}\boldsymbol{\alpha}_{w_1^*}$. Similarly, $\widetilde{d}_\alpha(Q_{n-2},Q') \leq C_{1,\boldsymbol{\alpha}}\boldsymbol{\alpha}_{w_n^*}$. In conclusion,

$$\begin{aligned}\widetilde{d}_\alpha(Q,Q') &\leq \widetilde{d}_\alpha(Q,Q_0) + \widetilde{d}_\alpha(Q_0,Q_{n-2}) + \widetilde{d}_\alpha(Q_{n-2},Q')\\ &\leq C_{1,\boldsymbol{\alpha}}\boldsymbol{\alpha}_{w_1^*} + \sum_{h=2}^{n-1}\alpha_{w_h^*} + C_{1,\boldsymbol{\alpha}}\boldsymbol{\alpha}_{w_n^*}\end{aligned}$$

$$\leq C_{1,\boldsymbol{\alpha}}\Big(\boldsymbol{\alpha}_{w_1^*} + \sum_{h=2}^{n-1} \alpha_{w_h^*} + \boldsymbol{\alpha}_{w_n^*}\Big)$$
$$= C_{1,\boldsymbol{\alpha}} \sum_{h=1}^{n} \alpha_{w_h^*}$$
$$= C_{1,\boldsymbol{\alpha}} A(\overline{\mathcal{C}}).$$

We have proved the inequality

$$\widetilde{d}_\alpha(Q,Q') \leq C_{1,\boldsymbol{\alpha}} A(\overline{\mathcal{C}}) \tag{5.10}$$

when $n > 1$. However, (5.10) also holds in the case $n = 1$. In fact, in this case, $Q, Q' \in V^{(\infty)} \cap K_{w_1^*} = V^{(\infty)}_{w_1^*}$. Thus,

$$\widetilde{d}_\alpha(Q,Q') \leq \operatorname{diam}_{\widetilde{d}_\alpha}(V^{(\infty)}_{w_1^*}) \leq C_{1,\boldsymbol{\alpha}} \alpha_{w_1^*} = C_{1,\boldsymbol{\alpha}} A(\overline{\mathcal{C}}).$$

Since (5.10) holds for every $\overline{\mathcal{C}} \in G(Q,Q')$, the second inequality in (5.9) easily follows. □

Proof of Theorem 5.2

The implications (iii)⇒(i), (ii)⇒(i), (iii)⇒(iv), and (iv)⇒(i) are trivial, and in fact, if (iii) holds, then (iv) holds with the same constants $c_{1,\alpha}$ and $c_{2,\alpha}$ as in (1.3). Moreover, (i)⇒(v) is proved in Lemma 5.3. It remains to prove that if (v) holds, then D_α is both an α-self-similar distance and an α-scaling distance, so that (ii) and (iii) hold, and moreover, since the maps ψ_i are α_i-Lipschitz with respect to D_α, D_α satisfies (1.3) with $c_{2,\alpha} = 1$. Suppose α is (T,Γ)-UP. By Lemmas 5.7 and 5.5, if $Q, Q' \in V^{(\infty)}$, $Q \neq Q'$, then $D_\alpha(Q,Q') \geq \frac{1}{C_{1,\boldsymbol{\alpha}}} \widetilde{d}_\alpha(Q,Q') > 0$. Thus, by Lemma 5.1, D_α is an α-self-similar distance on K and induces on K the same topology as the original distance. Therefore, in view of Lemma 2.7, $V^{(\infty)}$ is dense in K also with respect to D_α. Now, D_α is an α-scaling distance by Lemmas 5.6 and 5.7. □

REMARK 5.8

Note that it follows from Lemma 4.3(i) that if we have

$$H\big(T^{(\alpha)}_\gamma(e_\iota)\big) \geq 1 \quad \forall \iota \in \widehat{J}\ \forall \gamma \in \Gamma, \tag{5.11}$$

then α is(T,Γ)-UP. More precisely, it is not difficult to prove by induction that (4.5) holds with $c_{3,\alpha} = 1$. As a consequence, if $\alpha_i \geq \frac{1}{2}$ for every $i = 1, \ldots, k$, then α is (T,Γ)-UP. In fact, if $\gamma(\iota)$ is not a strong $V^{(1)}$-path, then $\widetilde{w}(h,\gamma(\iota)) = \emptyset$; thus, $\alpha_{\widetilde{w}(h,\gamma(\iota))} = 1$ for at least one $h = 1, \ldots, n(\gamma(\iota))$. Therefore, in view of (4.4), (5.11) holds. If on the contrary $\gamma(\iota)$ is a strong $V^{(1)}$-path, then the sum in (4.4) has at least two summands. Indeed, on one hand, $\gamma(\iota)$ connects two different points of $V^{(0)}$, and on the other, two consecutive vertices of $\gamma(\iota)$, by the definition of a strong $V^{(1)}$-path, lie in the same 1-cell, which, by (2.8), cannot contain more than one point of $V^{(0)}$. Moreover, in (4.4) we have $\widetilde{w}(h,\gamma(\iota)) = i_h$ for some

$i_h = 1, \ldots, k$, and by our assumptions, such summands are not less than $\frac{1}{2}$. Thus, (5.11) holds in any case.

6. Examples

In view of the results of Section 5, the problem of whether a given polyratio α is metric (or asymptotic metric, which is the same) on K is reduced to the problem of whether α is (T,Γ)-UP. In turn, such a problem is strictly related to the notion of *joint spectral radius* or, better, *joint spectral subradius*. More precisely, it is related to the fact that the joint spectral subradius is greater than or equal to 1. However, the notion of (T,Γ)-UP in general is not perfectly equivalent to having the joint spectral subradius greater than or equal to 1. In this section, we will discuss some explicit necessary and sufficient conditions for α being (T,Γ)-UP. However, such conditions require the existence of some special paths, and this occurs only on some fractals having a rather simple structure.

If $\mathbf{\Pi} = (\Pi(1), \ldots, \Pi(l))$ is a finite sequence of paths, we put

$$\Sigma_{\mathbf{\Pi}}(\alpha_{\overline{\mathbf{w}}}) = \sum_{\beta=1}^{l} \Sigma_{\Pi_\beta}(\alpha_{\overline{\mathbf{w}}}).$$

We say that the subpaths $\Pi(1), \ldots, \Pi(l)$ of Π are *separated* if they have length greater than 1, and moreover, the intervals

$$\big]n_1\big(\Pi, \Pi(\beta)\big), n_2\big(\Pi, \Pi(\beta)\big)\big], \quad \beta = 1, \ldots, l,$$

are mutually disjoint. Roughly speaking this means that they have no common edge. In this setting we say that $\mathbf{\Pi} := (\Pi(1), \ldots, \Pi(l))$ is a *multisubpath* of Π. We say that a sequence $\Pi' := (Q_{h_0,\Pi}, \ldots, Q_{h_s,\Pi})$ is a *refinement* of Π if $0 = h_0 < h_1 < \cdots < h_s = n(\Pi)$, and moreover, $Q_{h_{l+1},\Pi} = Q_{h_l+1,\Pi}$ for every $l = 0, \ldots, s-1$. Thus, Π' is a path. Note that if Π is a *strict* path, then every refinement of Π amounts to Π itself.

Suppose now that there exist finitely many $V^{(1)}$-paths $\Pi_1, \ldots, \Pi_r$ such that the following statements hold.

(i) For every $s = 1, \ldots, r$ there exists $\iota_s \in \widehat{J}$ such that Π_s is a strict and strong $(\iota_s, V^{(1)})$-path, and also $\widetilde{\iota}(h, \Pi_s) = \iota_s$ for every h.

(ii) There exists $\widetilde{m} \in \mathbb{N}$ such that every path Π connecting two different points of $V^{(0)}$ and such that $|\widetilde{w}(h, \Pi)| \geq \widetilde{m}$ for every $h = 1, \ldots, n(\Pi)$ also contains a subpath of the form $\psi_{w^*}(\overline{\Pi})$, where $\overline{\Pi}$ is a ι_s-path for some $s = 1, \ldots, r$ and $w^* \in W_{\widetilde{m}}$.

(iii) For every $s = 1, \ldots, r$ and every strong ι_s-path Π, there exist

(1) $s' = 1, \ldots, r$,

(2) a multisubpath $\widehat{\mathbf{\Pi}} := (\widehat{\Pi}_1, \ldots, \widehat{\Pi}_{n(\Pi_{s'})})$ of Π,

(3) refinements $\widehat{\Pi}'_\beta$ of $\widehat{\Pi}_\beta$ for every $\beta = 1, \ldots, n(\Pi_{s'})$,

(4) $s(\beta) = 1, \ldots, r$ for every $\beta = 1, \ldots, n(\Pi_{s'})$

such that $\widehat{\Pi}'_\beta = \psi_{\widetilde{w}(\beta, \Pi_{s'})}(\widehat{\Pi}''_\beta)$, where $\widehat{\Pi}''_\beta$ is a $\iota_{s(\beta)}$-path. We put $\widehat{\mathbf{\Pi}}' = (\widehat{\Pi}'_1, \ldots, \widehat{\Pi}'_{n(\Pi_{s'})})$.

Note that [4, Section 3, Condition A(2)] has some relationship with the set of conditions (i), (ii), and (iii) above. In [4], continuous curves are used and not discrete paths as in this article.

LEMMA 6.1

Under assumption (iii), if Π *is a strong* ι_s*-path,* $s = 1, \dots, r$*, and* $\Sigma_{\widehat{\Pi}''_\beta}(\alpha_{\overline{\mathbf{w}}}) \geq 1$ *for every* $\beta = 1, \dots, n(\Pi_{s'})$*, then*

$$\Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}) \geq \Sigma_{\Pi_{s'}}(\alpha_{\overline{\mathbf{w}}}).$$

Proof

Note that if Π' is a refinement of Π, we have

$$\Sigma_{\Pi'}(\alpha_{\overline{\mathbf{w}}}) \leq \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}). \tag{6.1}$$

In fact, using the previous notation we have

$$\begin{aligned}
\Sigma_{\Pi'}(\alpha_{\overline{\mathbf{w}}}) &= \sum_{l=1}^{s} \alpha_{\overline{\mathbf{w}}}(Q_{h_{l-1},\Pi}, Q_{h_l,\Pi}) \\
&= \sum_{l=1}^{s} \alpha_{\overline{\mathbf{w}}}(Q_{h_{l-1},\Pi}, Q_{h_{l-1}+1,\Pi}) \\
&\leq \sum_{h=1}^{n(\Pi)} \alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi}, Q_{h,\Pi}) = \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}).
\end{aligned}$$

Moreover,

$$\Sigma_{\widehat{\mathbf{\Pi}}'}(\alpha_{\overline{\mathbf{w}}}) \leq \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}). \tag{6.2}$$

In fact, on one hand, by (6.1) we have

$$\Sigma_{\widehat{\mathbf{\Pi}}'}(\alpha_{\overline{\mathbf{w}}}) = \sum_{\beta=1}^{n(\Pi_{s'})} \Sigma_{\widehat{\Pi}'_\beta}(\alpha_{\overline{\mathbf{w}}}) \leq \sum_{\beta=1}^{n(\Pi_{s'})} \Sigma_{\widehat{\Pi}_\beta}(\alpha_{\overline{\mathbf{w}}}) = \Sigma_{\widehat{\mathbf{\Pi}}}(\alpha_{\overline{\mathbf{w}}}).$$

On the other hand, by (4.2′) we have

$$\begin{aligned}
\Sigma_{\widehat{\mathbf{\Pi}}}(\alpha_{\overline{\mathbf{w}}}) &= \sum_{\beta=1}^{n(\Pi_{s'})} \Sigma_{\widehat{\Pi}_\beta}(\alpha_{\overline{\mathbf{w}}}) = \sum_{\beta=1}^{n(\Pi_{s'})} \sum_{h=1}^{n(\widehat{\Pi}_\beta)} \alpha_{\widetilde{w}(h,\widehat{\Pi}_\beta)} \\
&= \sum_{\beta=1}^{n(\Pi_{s'})} \sum_{h=1}^{n(\widehat{\Pi}_\beta)} \alpha_{\widetilde{w}(h+n_1(\Pi,\widehat{\Pi}_\beta),\Pi)} \leq \sum_{h=1}^{n(\Pi)} \alpha_{\widetilde{w}(h,\Pi)} = \Sigma_\Pi(\alpha_{\overline{\mathbf{w}}}).
\end{aligned}$$

In fact, we can easily verify that $n(\widehat{\Pi}_\beta) = n_2(\Pi, \widehat{\Pi}_\beta) - n_1(\Pi, \widehat{\Pi}_\beta)$, and, for every $h = 1, \dots, n(\widehat{\Pi}_\beta)$, we have $\widetilde{w}(h, \widehat{\Pi}_\beta) = \widetilde{w}(h + n_1(\Pi, \widehat{\Pi}_\beta), \Pi)$, $h + n_1(\Pi, \widehat{\Pi}_\beta) \in\,]n_1(\Pi, \widehat{\Pi}_\beta), n_2(\Pi, \widehat{\Pi}_\beta)]$, and moreover, the intervals $]n_1(\Pi, \widehat{\Pi}_\beta), n_2(\Pi, \widehat{\Pi}_\beta)]$ are mutually disjoint. Thus, (6.2) is proved. By (6.2) and (5.3), we have

$$\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq \Sigma_{\widehat{\mathbf{\Pi}}'}(\alpha_{\overline{\mathbf{w}}}) = \sum_{\beta=1}^{n(\Pi_{s'})} \Sigma_{\widehat{\Pi}'_\beta}(\alpha_{\overline{\mathbf{w}}})$$

$$= \sum_{\beta=1}^{n(\Pi_{s'})} \alpha_{\widetilde{w}(\beta,\Pi_{s'})} \Sigma_{\widehat{\Pi}''_\beta}(\alpha_{\overline{\mathbf{w}}}) \geq \sum_{\beta=1}^{n(\Pi_{s'})} \alpha_{\widetilde{w}(\beta,\Pi_{s'})} = \Sigma_{\Pi_{s'}}(\alpha_{\overline{\mathbf{w}}}). \qquad \square$$

LEMMA 6.2

Under the previous assumptions, suppose moreover that for every $s' = 1, \dots, r$ we have $\Sigma_{\Pi_{s'}}(\alpha_{\overline{\mathbf{w}}}) \geq 1$. Let Π be a path connecting two different points of $V^{(0)}$. Then

$$\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq \overline{\alpha}^{\widetilde{m}}_{\min}. \tag{6.3}$$

Proof

Suppose for the moment that Π is a ι_s-path for some $s = 1, \dots, r$, and prove

$$\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq 1. \tag{6.4}$$

Let $\overline{m} = \overline{m}(\Pi)$ be the maximum m such that there exists a point in Π that lies in $V^{(m)} \setminus V^{(m-1)}$. (Here $V^{(-1)} := \emptyset$.) Note that we have $Q_{h,\Pi} \in V^{(\overline{m})}$ for every h by the definition of $\overline{m}$.

We prove (6.4) by induction on $\overline{m}$. If $\overline{m} = 0$, then Π is a $V^{(0)}$-path. Thus, $\alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi}, Q_{h,\Pi}) = 1$ for every $h = 1, \dots, n(\Pi)$, and (6.4) is trivial.

Suppose now that (6.4) holds for $\overline{m} \leq m_1 \in \mathbb{N}$, and prove that it holds also for $m_1 + 1$. So, assume $\overline{m} = m_1 + 1$. If Π is not strong, then there exists $\overline{h} = 1, \dots, n(\Pi)$ such that $Q_{\overline{h}-1,\Pi}, Q_{\overline{h},\Pi} \in V^{(0)}$. Thus, $\alpha_{\overline{\mathbf{w}}}(Q_{\overline{h}-1,\Pi}, Q_{\overline{h},\Pi}) = 1$ and (6.4) is trivial. Thus, we can and do assume Π is strong. For every $\beta = 1, \dots, n(\Pi_{s'})$ and $h = 1, \dots, n(\widehat{\Pi}''_\beta)$, let $m_{\beta,h} \in \mathbb{N}$ be such that

$$Q_{h,\widehat{\Pi}''_\beta} \in V^{(m_{\beta,h})} \setminus V^{(m_{\beta,h}-1)}.$$

If $m_{\beta,h} > 0$, we have

$$Q_{h,\widehat{\Pi}'_\beta} = \psi_{\widetilde{w}(\beta,\Pi_{s'})}(Q_{h,\widehat{\Pi}''_\beta}) \in V^{(m_{\beta,h}+1)} \setminus V^{(m_{\beta,h})}.$$

In fact, since $\Pi_{s'}$ is a strong $V^{(1)}$-path, $\widetilde{w}(\beta, \Pi_{s'})$ has length 1, and we can use Corollary 2.5. Now, since $Q_{h,\widehat{\Pi}'_\beta}$ is also a vertex of Π, by definition we have $m_{\beta,h} + 1 \leq \overline{m} = m_1 + 1$. Thus, $m_{\beta,h} \leq m_1$, and this is trivially valid also if $m_{\beta,h} = 0$. By definition, $\overline{m}(\widehat{\Pi}''_\beta) \leq m_1$, and by our inductive hypothesis, $\widehat{\Pi}''_\beta$ satisfies (6.4). By Lemma 6.1, $\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq \Sigma_{\Pi_{s'}}(\alpha_{\overline{\mathbf{w}}}) \geq 1$. Thus, Π satisfies (6.4).

Now, we prove (6.3). If there exists $\overline{h} = 1, \dots, n(\Pi)$ such that $|\widetilde{w}(\overline{h}, \Pi)| < \widetilde{m}$, then $\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq \alpha_{\widetilde{w}(\overline{h},\Pi)} \geq \overline{\alpha}^{\widetilde{m}}_{\min}$, and (6.3) holds. In the opposite case, by assumption (ii), using notation thereof, we have $\alpha_{w^*} \geq \overline{\alpha}^{\widetilde{m}}_{\min}$ and, by (6.4), $\Sigma_{\overline{\Pi}}(\alpha_{\overline{\mathbf{w}}}) \geq 1$ since $\overline{\Pi}$ is a ι_s-path. Also using (5.3), we obtain $\Sigma_{\Pi}(\alpha_{\overline{\mathbf{w}}}) \geq \Sigma_{\psi_{w^*}(\overline{\Pi})}(\alpha_{\overline{\mathbf{w}}}) = \alpha_{w^*} \Sigma_{\overline{\Pi}}(\alpha_{\overline{\mathbf{w}}}) \geq \overline{\alpha}^{\widetilde{m}}_{\min}$. $\square$

THEOREM 6.3

Under the previous assumptions, the polyratio α *is* (T,Γ)*-UP on* K *if and only if* $\Sigma_{\Pi_s}(\alpha_{\overline{\mathbf{w}}}) \geq 1$ *for every* $s = 1,\dots,r$.

Proof

Suppose α is (T,Γ)-UP. Let $\gamma \in \Gamma$ be such that $\gamma(\iota_s) = \Pi_s$. Putting $\Pi = D(\gamma)(P_{\iota_s})$ and $\iota_s = (j_1, j_2)$, we obtain

$$\Pi = D(\gamma)(P_{\iota_s}) = D(\gamma)(P_{j_1}, P_{j_2}) = \gamma(j_1, j_2) = \Pi_s.$$

Hence, in view of (4.1) and Lemma 4.2, we get

$$\begin{aligned} T_\gamma^{(\alpha)}(e_{\iota_s}) &= T_\gamma^{(\alpha)}\big(\widehat{\Sigma}_{\iota_s}(\alpha_{\overline{\mathbf{w}}})\big) = \widehat{\Sigma}_{D(\gamma)(P_{\iota_s})}(\alpha_{\overline{\mathbf{w}}}) \\ &= \sum_{h=1}^{n(\Pi)} \alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi}, Q_{h,\Pi}) e_{\widetilde{\iota}(h,\Pi)} = \sum_{h=1}^{n(\Pi)} \alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi}, Q_{h,\Pi}) e_{\widetilde{\iota}(h,\Pi_s)} \\ &= \Big(\sum_{h=1}^{n(\Pi_s)} \alpha_{\overline{\mathbf{w}}}(Q_{h-1,\Pi_s}, Q_{h,\Pi_s})\Big) e_{\iota_s} = \big(\Sigma_{\Pi_s}(\alpha_{\overline{\mathbf{w}}})\big) e_{\iota_s}. \end{aligned}$$

Consequently,

$$(T_\gamma^{(\alpha)})^n(e_{\iota_s}) = \big(\Sigma_{\Pi_s}(\alpha_{\overline{\mathbf{w}}})\big)^n e_{\iota_s}$$

for every positive integer n. Thus, since α is (T,Γ)-UP, we must have $\Sigma_{\Pi_s}(\alpha_{\overline{\mathbf{w}}}) \geq 1$. For the converse, suppose $\Sigma_{\Pi_s}(\alpha_{\overline{\mathbf{w}}}) \geq 1$ for every $s = 1,\dots,r$, and prove that α is (T,Γ)-UP. Let $\gamma_1,\dots,\gamma_n \in \Gamma$, and let $\iota \in \widehat{J}$. Of course, the path $D(\gamma_1) \circ \cdots \circ D(\gamma_n)(P_\iota)$ connects two points of $V^{(0)}$. Thus, by Lemma 6.2, we have

$$\Sigma_{D(\gamma_1)\circ\cdots\circ D(\gamma_n)(P_\iota)}(\alpha_{\overline{\mathbf{w}}}) \geq \overline{\alpha}_{\min}^{\widetilde{m}}.$$

By Lemma 4.2(ii) we have

$$\begin{aligned} H\big(T_{\gamma_1}^{(\alpha)} \circ \cdots \circ T_{\gamma_n}^{(\alpha)}(e_\iota)\big) &= H\big(T_{\gamma_1}^{(\alpha)} \circ \cdots \circ T_{\gamma_n}^{(\alpha)}\big(\widehat{\Sigma}_{P_\iota}(\alpha_{\overline{\mathbf{w}}})\big)\big) \\ &= H\big(\widehat{\Sigma}_{D(\gamma_1)\circ\cdots\circ D(\gamma_n)(P_\iota)}(\alpha_{\overline{\mathbf{w}}})\big) \\ &= \Sigma_{D(\gamma_1)\circ\cdots\circ D(\gamma_n)(P_\iota)}(\alpha_{\overline{\mathbf{w}}}) \geq \overline{\alpha}_{\min}^{\widetilde{m}}, \end{aligned}$$

and thus, α is (T,Γ)-UP. □

We now apply Theorem 6.3 to two examples. For the gasket with three 1-cells V_1, V_2, V_3, let P_1, P_2, P_3 be the three fixed points of the maps, and let $Q_{j_1,j_2} = Q_{j_2,j_1} := \psi_{j_1}(P_{j_2}) = \psi_{j_2}(P_{j_1})$ when $j_1, j_2 = 1,2,3$, $j_1 \neq j_2$. It is simple to verify that the six paths of the form $(P_{j_1}, Q_{j_1,j_2}, P_{j_2})$, $j_1, j_2 = 1,2,3$, $j_1 \neq j_2$, satisfy (i), (ii), and (iii) with $\widetilde{m} = 0$. Thus, in view of Theorem 6.3, α is (T,Γ)-UP on the gasket if and only if we have

$$\alpha_1 + \alpha_2 \geq 1, \qquad \alpha_1 + \alpha_3 \geq 1, \qquad \alpha_2 + \alpha_3 \geq 1. \tag{6.5}$$

In the Vicsek set, let ψ_i, $i = 1,2,3,4,5$, be the contractions defining it, and we order them in such a way that ψ_5 is the contraction that fixes the center.

Let P_j be the fixed points of V_j for $j = 1,2,3,4$. Choose the order so that P_1 is opposite to P_3 and P_2 is opposite to P_4. Let Q_j be the only point in $V_j \cap V_5$. Now, we take the four paths of the form $\Pi_j := (P_j, Q_j, Q_{j'}, P_{j'})$ when $j = 1,2,3,4$ and $P_{j'}$ is opposite to P_j. It is simple to verify that such paths satisfy (i), (ii), and (iii) with $\widetilde{m} = 1$. Thus, by Theorem 6.3, α is (T,Γ)-UP on the Vicsek set if and only if we have

$$\alpha_1 + \alpha_3 + \alpha_5 \geq 1, \qquad \alpha_2 + \alpha_4 + \alpha_5 \geq 1. \tag{6.6}$$

Similar considerations could be extended to other fractals. However, to have simple conditions like (6.5) or (6.6) the structure of the fractal should be simple. In most cases, I expect that it could be hard to give simple necessary and sufficient conditions for α to be (T,Γ)-UP.

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