Big arithmetic divisors on the projective spaces over $\mathbb{Z}$

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Abstract In this paper, we observe several properties of an arithmetic divisor $\overline{D}$ on $\mathbb{P}^n_{\mathbb{Z}}$ and give the exact form of the Zariski decomposition of $\overline{D}$ on $\mathbb{P}^1_{\mathbb{Z}}$. Further, we show that, if $n \geq 2$ and $\overline{D}$ is big and non-nef, then for any birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth, and normal arithmetic varieties, we cannot expect a suitable Zariski decomposition of $f^*(\overline{D})$. We also give a concrete construction of Fujita's approximation of $\overline{D}$.

0. Introduction

Let $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1, \dots, T_n])$, $H_i = \{T_i = 0\}$, and $z_i = T_i/T_0$ for $i = 0, 1, \dots, n$. Let us fix a sequence $\boldsymbol{a} = (a_0, a_1, \dots, a_n)$ of positive numbers. We define an H_0-Green function $g_{\boldsymbol{a}}$ of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n(\mathbb{C})$ and an arithmetic divisor $\overline{D}_{\boldsymbol{a}}$ of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n_{\mathbb{Z}}$ to be

$$g_{\boldsymbol{a}} := \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2) \qquad \text{and} \qquad \overline{D}_{\boldsymbol{a}} := (H_0, g_{\boldsymbol{a}}).$$

In this paper, we observe several properties of $\overline{D}_{\boldsymbol{a}}$ and give the exact form of the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$. Further, we show that, if $n \geq 2$ and $\overline{D}_{\boldsymbol{a}}$ is big and not nef, then for any birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth, and normal arithmetic varieties, we cannot expect a suitable Zariski decomposition of $f^*(\overline{D}_{\boldsymbol{a}})$. In this sense, the results in [10] are nothing short of miraculous, and arithmetic linear series are very complicated and have a richer structure than we expected. We also give a concrete construction of Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$. The following is a list of the main results of this paper.

MAIN RESULTS

Let $\varphi_{\boldsymbol{a}} : \mathbb{R}^{n+1}_{\geq 0} \to \mathbb{R}$ *be a function given by*

$$\varphi_{\boldsymbol{a}}(x_0, x_1, \dots, x_n) := -\sum_{i=0}^n x_i \log x_i + \sum_{i=0}^n x_i \log a_i,$$

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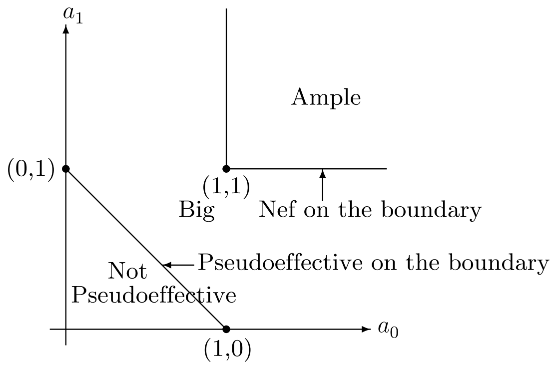

Figure 1. Geography of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$

and let

$$\Theta_{\boldsymbol{a}} := \left\{(x_1, \dots, x_n) \in \Delta_n \;\middle|\; \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \dots, x_n) \geq 0\right\},$$

where $\Delta_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\}$*. Then the following properties hold for* $\overline{D}_{\boldsymbol{a}}$*.*

(1) $\overline{D}_{\boldsymbol{a}}$ *is ample if and only if* $a_0 > 1, a_1 > 1, \dots, a_n > 1$*.*
(2) $\overline{D}_{\boldsymbol{a}}$ *is nef if and only if* $a_0 \geq 1, a_1 \geq 1, \dots, a_n \geq 1$*.*
(3) $\overline{D}_{\boldsymbol{a}}$ *is big if and only if* $a_0 + a_1 + \cdots + a_n > 1$*.*
(4) $\overline{D}_{\boldsymbol{a}}$ *is pseudoeffective if and only if* $a_0 + a_1 + \cdots + a_n \geq 1$*.*
(5) *We have* $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \neq \{0\}$ *if and only if* $l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n \neq \emptyset$*. As a consequence, we have the following.*

(5.1) *We assume that* $a_0 + a_1 + \cdots + a_n = 1$*. For a positive integer* l*,*

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \begin{cases} \{0, \pm z_1^{la_1} \cdots z_n^{la_n}\} & \text{if } la_1, \dots, la_n \in \mathbb{Z}, \\ \{0\} & \text{otherwise.} \end{cases}$$

In particular, if $\boldsymbol{a} \notin \mathbb{Q}^{n+1}$*, then* $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \{0\}$ *for all* $l \geq 1$*.*

(5.2) *For any positive integer* l*, there exists* $\boldsymbol{a} \in \mathbb{Q}^{n+1}_{>0}$ *such that* $\overline{D}_{\boldsymbol{a}}$ *is big and*

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) = \{0\}$$

for all k *with* $1 \leq k \leq l$*.*

(6) *We have* $\langle \hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \rangle_{\mathbb{Z}} = \bigoplus_{(e_1, \dots, e_n) \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \mathbb{Z} z_1^{e_1} \cdots z_n^{e_n}$ *if* $l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n \neq \emptyset$*.*

(7) *(Integral formula) The following formulae hold:*

$$\widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}}) = \frac{(n+1)!}{2} \int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

and

$$\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{\,n+1}) = \frac{(n+1)!}{2} \int_{\Delta_n} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \dots, x_n)\, dx_1 \cdots dx_n.$$

In particular, $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})$ *if and only if* $\overline{D}_{\boldsymbol{a}}$ *is nef. Yuan points out that* $\varphi_{\boldsymbol{a}}$ *coincides with the concave function* $G_{(V_\bullet,\mathcal{F})}$ *in [2] due to Boucksom and Chen, so that the above two formulae are the special cases of the main results of [2]. Yuan also constructs a similar function in [12].*

(8) *(Zariski decomposition for* $n=1$*) We assume that* $n=1$*. The Zariski decomposition of* $\overline{D}_{\boldsymbol{a}}$ *exists if and only if* $a_0+a_1\geq 1$*. Moreover, the positive part of* $\overline{D}_{\boldsymbol{a}}$ *is given by* $(\theta_{\boldsymbol{a}}H_0 - \vartheta_{\boldsymbol{a}}H_1, p_{\boldsymbol{a}})$*, where* $\vartheta_{\boldsymbol{a}} = \inf\Theta_{\boldsymbol{a}}$*,* $\theta_{\boldsymbol{a}} = \sup\Theta_{\boldsymbol{a}}$*, and*

$$p_{\boldsymbol{a}}(z_1) = \begin{cases} \vartheta_{\boldsymbol{a}} \log|z_1|^2 & \text{if } |z_1| < \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ \log(a_0 + a_1|z_1|^2) & \text{if } \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z_1| \leq \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log|z_1|^2 & \text{if } |z_1| > \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

In particular, if $a_0+a_1=1$*, then the positive part is* $-a_1\widehat{(z_1)}$*.*

(9) *(Impossibility of Zariski decomposition for* $n\geq 2$*) We assume that* $n\geq 2$*. If* $\overline{D}_{\boldsymbol{a}}$ *is big and not nef (i.e.,* $a_0+\cdots+a_n>1$ *and* $a_i<1$ *for some* i*), then for any birational morphism* $f: X\to\mathbb{P}^n_{\mathbb{Z}}$ *of projective, generically smooth, and normal arithmetic varieties, there is no decomposition* $f^*(\overline{D}_{\boldsymbol{a}}) = \overline{P}+\overline{N}$ *with the following properties.*

(9.1) $\overline{P}$ *is a nef and big arithmetic* $\mathbb{R}$*-divisor of* $(C^0\cap\mathrm{PSH})$*-type on* X*.*

(9.2) $\overline{N}$ *is an effective arithmetic* $\mathbb{R}$*-divisor of* C^0*-type on* X*.*

(9.3) *For any horizontal prime divisor* Γ *on* X *(i.e.,* Γ *is a reduced and irreducible divisor on* X *such that* Γ *is flat over* $\mathbb{Z}$*),*

$$\begin{aligned}&\mathrm{mult}_\Gamma(N)\\ &\quad\leq \inf\bigl\{\mathrm{mult}_\Gamma\bigl(f^*(H_0)+(1/l)(\phi)\bigr)\bigm| l\in\mathbb{Z}_{>0}, \phi\in\hat{H}^0\bigl(lf^*(\overline{D}_{\boldsymbol{a}})\bigr)\setminus\{0\}\bigr\}.\end{aligned}$$

(10) *(Fujita's approximation) We assume that* $\overline{D}_{\boldsymbol{a}}$ *is big. Let* $\mathrm{Int}(\Theta_{\boldsymbol{a}})$ *be the set of interior points of* $\Theta_{\boldsymbol{a}}$*. We choose* $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r\in\mathrm{Int}(\Theta_{\boldsymbol{a}})\cap\mathbb{Q}^n$ *such that*

$$\frac{(n+1)!}{2}\int_\Theta \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\,d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})-\epsilon,$$

where $\Theta := \mathrm{Conv}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r\}$ *and*

$$\begin{aligned}&\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\\ &\quad:= \max\bigl\{t\in\mathbb{R}\bigm| (\boldsymbol{x},t)\in\mathrm{Conv}\bigl\{\bigl(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)\bigr),\ldots,\bigl(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r)\bigr)\bigr\}\subseteq\mathbb{R}^n\times\mathbb{R}\bigr\}\end{aligned}$$

for $\boldsymbol{x}\in\Theta$ *(see Conventions and terminology (2) for the definition of* $\widetilde{\boldsymbol{x}}_1,\ldots,\widetilde{\boldsymbol{x}}_r$*). Using the above points* $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r$*, we can construct a birational morphism* $\mu: Y\to\mathbb{P}^n_{\mathbb{Z}}$ *of projective, generically smooth, and normal arithmetic varieties, and a nef arithmetic* $\mathbb{Q}$*-divisor* $\overline{P}$ *of* $(C^\infty\cap\mathrm{PSH})$*-type on* Y *such that*

$$\overline{P}\leq\mu^*(\overline{D}_{\boldsymbol{a}}) \qquad \text{and} \qquad \widehat{\mathrm{vol}}(\overline{P}) > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})-\epsilon$$

(for details, see Section 6).

Conventions and terminology

(1) For $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, the ith entry x_i of $\boldsymbol{x}$ is denoted by $\boldsymbol{x}(i)$. We define $|\boldsymbol{x}|$ to be $|\boldsymbol{x}| := x_1 + \cdots + x_r$.

(2) For $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$ and $m \in \mathbb{R}$, we define $\widetilde{\boldsymbol{x}}^m \in \mathbb{R}^{r+1}$ to be

$$\widetilde{\boldsymbol{x}}^m = (m - x_1 - \cdots - x_r, x_1, \ldots, x_r).$$

Note that $|\widetilde{\boldsymbol{x}}^m| = m$. For simplicity, in the case where $m = 1$, we denote $\widetilde{\boldsymbol{x}}^m$ by $\widetilde{\boldsymbol{x}}$.

(3) Let $\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r_{\geq 0}$ and $l = |\boldsymbol{e}|$. A monomial $z_1^{e_1} \cdots z_r^{e_r}$ is denoted by $z^{\boldsymbol{e}}$. The multinomial coefficient $l!/(e_1! \cdots e_r!)$ is denoted by $\binom{l}{\boldsymbol{e}}$.

(4) We freely use the notation in [10].

1. Fundamental properties of the characteristic function

Let $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1, \ldots, T_n])$, $H_i = \{T_i = 0\}$, and $z_i = T_i/T_0$ for $i = 0, \ldots, n$. Let us fix $\boldsymbol{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}_{>0}$. We set

$$h_{\boldsymbol{a}} = a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2, \qquad g_{\boldsymbol{a}} = \log h_{\boldsymbol{a}}, \qquad \text{and} \qquad \omega_{\boldsymbol{a}} = dd^c(g_{\boldsymbol{a}})$$

on $\mathbb{P}^n(\mathbb{C})$, that is,

$$g_{\boldsymbol{a}} = -\log|T_0|^2 + \log(a_0|T_0|^2 + \cdots + a_n|T_n|^2).$$

PROPOSITION 1.1

(1) *The form $\omega_{\boldsymbol{a}}$ is positive. In particular, $g_{\boldsymbol{a}}$ is an H_0-Green function of $(C^\infty \cap \mathrm{PSH})$-type.*

(2) *If we set $\Phi_{\boldsymbol{a}} = \omega_{\boldsymbol{a}}^{\wedge n}$, then*

$$\Phi_{\boldsymbol{a}} = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{n! a_0 \cdots a_n}{h_{\boldsymbol{a}}^{n+1}} \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

and

$$\int_{\mathbb{P}^n(\mathbb{C})} \Phi_{\boldsymbol{a}} = 1.$$

Proof

(1) Note that

$$\omega_{\boldsymbol{a}} = \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^n \frac{a_i}{h_{\boldsymbol{a}}(z)} \, dz_i \wedge d\bar{z}_i - \sum_{i,j} \frac{a_i a_j \bar{z}_i z_j}{h_{\boldsymbol{a}}(z)^2} \, dz_i \wedge d\bar{z}_j\right).$$

If we set

$$A = \left(\delta_{ij} \frac{a_i}{h_{\boldsymbol{a}}(z)} - \frac{a_i a_j \bar{z}_i z_j}{h_{\boldsymbol{a}}(z)^2}\right)_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq n}},$$

then it is easy to see that

$$(\bar{\lambda}_1 \cdots \bar{\lambda}_n) A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \frac{a_0 \sum_{i=1}^n a_i |\lambda_i|^2 + \sum_{i<j} a_i a_j |z_i \bar{\lambda}_j - z_j \bar{\lambda}_i|^2}{h_{\boldsymbol{a}}(z)^2}.$$

Thus $\omega_{\boldsymbol{a}}$ is positive definite.

(2) The first assertion follows from the following claim.

CLAIM 1.1.1

For $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$,

$$\det(\delta_{ij} t_i - \alpha_i \bar{\alpha}_j)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} = t_1 \cdots t_n - \sum_{i=1}^{n} |\alpha_i|^2 t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_n.$$

Proof

We denote $(\delta_{ij} t_i - \alpha_i \bar{\alpha}_j)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ by B. If $t_i = t_j = 0$ for $i \neq j$, then the ith column and the jth column of B are linearly dependent, so that $\det B = 0$. Therefore we can set

$$\det B = t_1 \cdots t_n - \sum_{i=1}^{n} c_i t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_n$$

for some $c_1, \ldots, c_n \in \mathbb{C}$. It is easy to see that $\det B = -|\alpha_i|^2$ if $t_i = 0$ and $t_1 = \cdots = t_{i-1} = t_{i+1} = \cdots = t_n = 1$. Thus $c_i = |\alpha_i|^2$. □

Let $|\cdot|_{\boldsymbol{a}}$ be a C^∞-Hermitian metric of $\mathcal{O}(1)$ given by

$$|T_i|_{\boldsymbol{a}} = \frac{|T_i|}{\sqrt{a_0|T_1|^2 + a_1|T_1|^2 + \cdots + a_n|T_n|^2}}$$

for $i = 0, \ldots, n$. Then $c_1(\mathcal{O}(1), |\cdot|_{\boldsymbol{a}}) = \omega_{\boldsymbol{a}}$. Thus the second assertion of Proposition 1.1 follows. □

We define a function $\varphi_{\boldsymbol{a}} : \mathbb{R}_{\geq 0}^{n+1} \to \mathbb{R}$ to be

$$\varphi_{\boldsymbol{a}}(x_0, \ldots, x_n) = -\sum_{i=0}^{n} x_i \log x_i + \sum_{i=0}^{n} x_i \log a_i,$$

which is called the *characteristic function of* $g_{\boldsymbol{a}}$. The function $\varphi_{\boldsymbol{a}}$ plays a key role in this paper. Here note that $\varphi_{\boldsymbol{a}}(0, \ldots, \overset{\overset{i}{\vee}}{1}, \ldots, 0) = \log a_i$ for $i = 0, \ldots, n$. Moreover, $\varphi_{\boldsymbol{a}}$ is concave because $x \log x$ is convex. Notably, the characteristic function is very similar to the entropy function in coding theory.

LEMMA 1.2

For $(x_0, \ldots, x_n) \in \mathbb{R}_{\geq 0}^{n+1}$ *with* $x_0 + x_1 + \cdots + x_n = 1$,

$$\varphi_{\boldsymbol{a}}(x_0, \ldots, x_n) \leq \log(a_0 + a_1 + \cdots + a_n),$$

and the equality holds if and only if

$$x_0 = a_0/(a_0 + a_1 + \cdots + a_n), \ldots, x_n = a_n/(a_0 + a_1 + \cdots + a_n).$$

Proof

Let us begin with the following claim.

CLAIM 1.2.1

For $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_r,t_1,\dots,t_r\in\mathbb{R}_{>0}$ *with* $\alpha_1+\cdots+\alpha_r=1$,

$$\sum_{i=1}^{r}\alpha_i\log t_i\le\log\Big(\sum_{i=1}^{r}\beta_i t_i\Big)+\sum_{i=1}^{r}\alpha_i\log\frac{\alpha_i}{\beta_i},$$

and the equality holds if and only if $(\beta_1/\alpha_1)t_1=\cdots=(\beta_r/\alpha_r)t_r$.

Proof

Note that if we set $t_i'=(\beta_i/\alpha_i)t_i$ for $i=1,\dots,r$, then

$$\sum_{i=1}^{r}\alpha_i\log t_i-\log\Big(\sum_{i=1}^{r}\beta_i t_i\Big)=\sum_{i=1}^{r}\alpha_i\log t_i'-\log\Big(\sum_{i=1}^{r}\alpha_i t_i'\Big)+\sum_{i=1}^{r}\alpha_i\log\frac{\alpha_i}{\beta_i}.$$

Thus we may assume that $\alpha_i=\beta_i$ for all i. In this case, the inequality is nothing more than Jensen's inequality for the strictly concave function log. □

We set $I=\{i\mid x_i\neq 0\}$. Then, using the above claim, we have

$$\sum_{i\in I}x_i\log a_i\le\log\Big(\sum_{i\in I}a_i\Big)+\sum_{i\in I}x_i\log x_i,$$

and hence

$$\begin{aligned}\varphi_{\boldsymbol{a}}(x_0,\dots,x_n)&=\sum_{i\in I}-x_i\log x_i+\sum_{i\in I}x_i\log a_i\\&\le\log\Big(\sum_{i\in I}a_i\Big)\le\log(a_0+\cdots+a_n).\end{aligned}$$

In addition, the equality holds if and only if $a_i/x_i=a_j/x_j$ for all $i,j\in I$ and $a_i=0$ for all $i\notin I$. Thus the assertion of Lemma 1.2 follows. □

Note that

$$H^0(\mathbb{P}^n_{\mathbb{Z}},lH_0)=\bigoplus_{\boldsymbol{e}\in\mathbb{Z}^n_{\ge 0},|\boldsymbol{e}|\le l}\mathbb{Z}z^{\boldsymbol{e}}$$

(for the definition of $|\boldsymbol{e}|$ and $z^{\boldsymbol{e}}$, see Conventions and terminology (1), (3)). According to [10], $|\cdot|_{lg_{\boldsymbol{a}}}$, $\|\cdot\|_{lg_{\boldsymbol{a}}}$, and $\langle\cdot,\cdot\rangle_{lg_{\boldsymbol{a}}}$ are defined by

$$|\phi|_{lg_{\boldsymbol{a}}}:=|\phi|\exp(-lg_{\boldsymbol{a}}/2),\qquad \|\phi\|_{lg_{\boldsymbol{a}}}:=\sup\big\{|\phi|_{lg_{\boldsymbol{a}}}(x)\;\big|\;x\in\mathbb{P}^n(\mathbb{C})\big\}$$

and

$$\langle\phi,\psi\rangle_{lg_{\boldsymbol{a}}}:=\int_{\mathbb{P}^n(\mathbb{C})}\phi\bar{\psi}\exp(-lg_{\boldsymbol{a}})\Phi_{\boldsymbol{a}},$$

where $\phi,\psi\in H^0(\mathbb{P}^n(\mathbb{C}),lH_0)$.

PROPOSITION 1.3

Let l *be a positive integer, and let* $\boldsymbol{e}=(e_1,\dots,e_n),\boldsymbol{e}'=(e_1',\dots,e_n')\in\mathbb{Z}^n_{\ge 0}$ *with* $|\boldsymbol{e}|,|\boldsymbol{e}'|\le l$.

(1) *We have* $\|z^{\boldsymbol{e}}\|_{lg_{\boldsymbol{a}}}^2 = \exp(-l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^l/l))$. *(For the definition of* $\widetilde{\boldsymbol{e}}^l$*, see Conventions and terminology (2).)*

(2) *We have*

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \begin{cases} 0 & \text{if } \boldsymbol{e} \neq \boldsymbol{e}', \\ \frac{1}{\binom{n+l}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}} & \text{if } \boldsymbol{e} = \boldsymbol{e}' \end{cases}$$

(for the definition of $\binom{l}{\widetilde{\boldsymbol{e}}^l}$*, see Conventions and terminology (3)).*

Proof

(1) By the definition of $|z^{\boldsymbol{e}}|_{lg_{\boldsymbol{a}}}$, we can see

$$\log|z^{\boldsymbol{e}}|_{lg_{\boldsymbol{a}}}^2 = e_0\log|T_0|^2 + \cdots + e_n\log|T_n|^2 - l\log(a_0|T_0|^2 + \cdots + a_n|T_n|^2),$$

where $e_0 = l - e_1 - \cdots - e_n$ and where $(T_0 : \cdots : T_n)$ is a homogeneous coordinate of $\mathbb{P}^n(\mathbb{C})$ such that $z_i = T_i/T_0$. Here we set $e_i' = e_i/l$ for $i = 0, \ldots, l$ and $I = \{i \mid e_i \neq 0\}$. Then by using Claim 1.2.1,

$$\frac{1}{l}\log|z^{\boldsymbol{e}}|_{lg_{\boldsymbol{a}}}^2 \leq \sum_{i\in I} e_i'\log|T_i|^2 - \log\Bigl(\sum_{i\in I} a_i|T_i|^2\Bigr) \leq -\varphi_{\boldsymbol{a}}(e_0', \ldots, e_n').$$

Moreover, if we set $T_i = \sqrt{e_i'/a_i}$ for $i = 0, \ldots, n$, then the equality holds. Thus (1) follows.

(2) First of all, by Proposition 1.1,

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^n \int_{\mathbb{P}^n(\mathbb{C})} \frac{n!a_0\cdots a_n z^{\boldsymbol{e}}\bar{z}^{\boldsymbol{e}'}\,dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n}{(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)^{n+l+1}}.$$

If we set $z_i = x_i^{1/2}\exp(2\pi\sqrt{-1}\theta_i)$, then the above integral is equal to

$$\int_{\mathbb{R}^n_{\geq 0}\times[0,1]^n} \frac{n!a_0\cdots a_n\prod_{i=1}^n x_i^{(e_i+e_i')/2}\exp(2\pi\sqrt{-1}(e_i - e_i'))}{(a_0 + a_1x_1 + \cdots + a_nx_n)^{n+l+1}}\,dx_1\cdots dx_n\,d\theta_1\cdots d\theta_n,$$

and hence

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \begin{cases} 0 & \text{if } \boldsymbol{e} \neq \boldsymbol{e}', \\ \int_{\mathbb{R}^n} \frac{n!a_0\cdots a_n x_1^{e_1}\cdots x_n^{e_n}}{(a_0+a_1x_1+\cdots+a_nx_n)^{n+l+1}}\,dx_1\cdots dx_n & \text{if } \boldsymbol{e} = \boldsymbol{e}'. \end{cases}$$

It is easy to see that

$$\int_0^\infty \frac{ax^m}{(ax+b)^n}dx = \frac{m!}{a^m b^{n-m-1}(n-1)(n-2)\cdots(n-m)(n-m-1)}$$

for $a, b \in \mathbb{R}_{>0}$ and $n, m \in \mathbb{Z}_{\geq 0}$ with $n - m \geq 2$. Thus we can see that

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}}\rangle_{lg_{\boldsymbol{a}}} = \frac{n!e_n!\cdots e_1!}{(n+l)(n+l-1)\cdots(e_0+1)a_n^{e_n}\cdots a_1^{e_1}a_0^{e_0}},$$

where $e_0 = l - e_1 - \cdots - e_n$. Therefore the assertion follows. □

Next, we observe the following lemma.

LEMMA 1.4

If we set $A_n = (n+2)/2$ *and* $B_n = (n+2)\log\sqrt{2\pi} + (n+2)/12$, *then*

$$\left|\frac{1}{l}\log\Big(\frac{l!}{k_0!\cdots k_n!}a_0^{k_0}\cdots a_n^{k_n}\Big) - \varphi_{\boldsymbol{a}}\Big(\frac{k_0}{l},\dots,\frac{k_n}{l}\Big)\right| \le \frac{1}{l}(A_n\log l + B_n)$$

holds for all $l \ge 1$ *and* $(k_0,\dots,k_n) \in \mathbb{Z}_{\ge 0}^{n+1}$ *with* $k_0 + \cdots + k_n = l$.

Proof

First of all, note that, for $m \ge 1$,

$$m! = \sqrt{2\pi m}\frac{m^m}{e^m}e^{\theta_m/12m} \quad (0 < \theta_m < 1)$$

by Stirling's formula. We set $I = \{i \mid k_i \ne 0\}$. Then

$$\log(l!) = \log(\sqrt{2\pi l}) + l\log l - l + \frac{\theta_l}{12l},$$

$$\log(k_i!) = \log(\sqrt{2\pi k_i}) + k_i\log k_i - k_i + \frac{\theta_{k_i}}{12k_i} \quad (i \in I).$$

Therefore,

$$\begin{aligned}&\frac{1}{l}\log\Big(\frac{l!}{k_0!\cdots k_n!}a_0^{k_0}\cdots a_n^{k_n}\Big)\\ &\quad = \varphi_{\boldsymbol{a}}\Big(\frac{k_0}{l},\dots,\frac{k_n}{l}\Big) + \frac{1}{l}\log(\sqrt{2\pi l}) + \frac{\theta_l}{12l^2} - \sum_{i\in I}\Big(\frac{1}{l}\log(\sqrt{2\pi k_i}) + \frac{\theta_{k_i}}{12lk_i}\Big),\end{aligned}$$

which yields the assertion. □

Let $\overline{D}_{\boldsymbol{a}}$ be an arithmetic divisor of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n_{\mathbb{Z}}$ given by

$$\overline{D}_{\boldsymbol{a}} := (H_0, g_{\boldsymbol{a}}) = \big(H_0, \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)\big).$$

Moreover, for $\lambda \in \mathbb{R}$, $\Theta_{\boldsymbol{a},\lambda}$ is defined to be

$$\Theta_{\boldsymbol{a},\lambda} := \big\{(x_1,\dots,x_n) \in \Delta_n \bigm| \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1,\dots,x_n) \ge \lambda\big\},$$

where $\Delta_n = \{(x_1,\dots,x_n) \in \mathbb{R}^n_{\ge 0} \mid x_1 + \cdots + x_n \le 1\}$. Note that $\Theta_{\boldsymbol{a},\lambda}$ is a compact convex set because $\varphi_{\boldsymbol{a}}$ is concave. For simplicity, we denote $\Theta_{\boldsymbol{a},0}$ by $\Theta_{\boldsymbol{a}}$; that is,

$$\Theta_{\boldsymbol{a}} = \big\{(x_1,\dots,x_n) \in \Delta_n \bigm| \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1,\dots,x_n) \ge 0\big\}.$$

Finally, we consider the following proposition.

PROPOSITION 1.5

Let us fix a positive integer l. *Then we have the following:*

(1) $l\Theta_{\boldsymbol{a},\lambda} \cap \mathbb{Z}^n \ne \emptyset$ *if and only if there is a nonzero rational function* ϕ *on* $\mathbb{P}^n_{\mathbb{Z}}$ *such that* $lH_0 + (\phi) \ge 0$ *and* $\|\phi\|_{lg_{\boldsymbol{a}}} \le e^{-l\lambda}$;

(2) *if* $l\Theta_{\boldsymbol{a},\lambda} \cap \mathbb{Z} \ne \emptyset$, *then*

$$\big\langle\{\phi \in \mathrm{Rat}(\mathbb{P}^n_{\mathbb{Z}})^\times \bigm| lH_0 + (\phi) \ge 0, \|\phi\|_{lg_{\boldsymbol{a}}} \le e^{-l\lambda}\}\big\rangle_{\mathbb{Z}} = \bigoplus_{\boldsymbol{e}\in l\Theta_{\boldsymbol{a},\lambda}\cap\mathbb{Z}^n} \mathbb{Z}z^{\boldsymbol{e}}.$$

Proof
Let us begin with the following claim.

CLAIM 1.5.1
Let ϕ be a nonzero rational function on $\mathbb{P}^n_{\mathbb{Z}}$ such that $lH_0 + (\phi) \geq 0$ and $\|\phi\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}$. If we write

$$\phi = \sum_{\boldsymbol{e}\in\mathbb{Z}^n_{\geq 0}, |\boldsymbol{e}|\leq l} c_{\boldsymbol{e}} z^{\boldsymbol{e}} \quad (c_{\boldsymbol{e}} \in \mathbb{Z}),$$

then $\{\boldsymbol{e} \mid c_{\boldsymbol{e}} \neq 0\} \subseteq l\Theta_{\boldsymbol{a},\lambda}$.

Proof
The following proof, which is easier than the original, was pointed out by the referees. Let N be a positive integer with $N > l$, and let ζ be the primitive Nth root of unity, that is, $\zeta = \exp(2\pi\sqrt{-1}/N)$. For $\psi \in \mathbb{C}[z_1, \dots, z_n]$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we define $\psi_{\boldsymbol{\alpha}} \in \mathbb{C}[z_1, \dots, z_n]$ to be $\psi_{\boldsymbol{\alpha}}(z_1, \dots, z_n) = \psi(\zeta^{\alpha_1} z_1, \dots, \zeta^{\alpha_n} z_n)$; that is, if we set $\psi = \sum_{\boldsymbol{e}\in\mathbb{Z}^n_{\geq 0}} b_{\boldsymbol{e}} z^{\boldsymbol{e}}$, then $\psi_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{e}\in\mathbb{Z}^n_{\geq 0}} \zeta^{\langle \boldsymbol{\alpha}, \boldsymbol{e}\rangle} b_{\boldsymbol{e}} z^{\boldsymbol{e}}$, where $\langle\,,\rangle$ is the standard inner product. Note that, for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ with $-N < \beta_1 < N, \dots, -N < \beta_n < N$,

$$\sum_{\boldsymbol{\alpha}\in\{0,1,\dots,N-1\}^n} \zeta^{\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle} = \begin{cases} N^n & \text{if } \boldsymbol{\beta} = (0, \dots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $\boldsymbol{e}_0 \in \mathbb{Z}^n_{\geq 0}$ with $|\boldsymbol{e}_0| \leq l$,

$$\sum_{\boldsymbol{\alpha}\in\{0,\dots,N-1\}^n} \zeta^{-\langle \boldsymbol{\alpha}, \boldsymbol{e}_0\rangle} \phi_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{e}\in\mathbb{Z}^n_{\geq 0}, |\boldsymbol{e}|\leq l} \Bigl(\sum_{\boldsymbol{\alpha}\in\{0,\dots,N-1\}^n} \zeta^{\langle \boldsymbol{\alpha}, \boldsymbol{e}-\boldsymbol{e}_0\rangle} \Bigr) c_{\boldsymbol{e}} z^{\boldsymbol{e}} = N^n c_{\boldsymbol{e}_0} z^{\boldsymbol{e}_0},$$

which yields

$$\|c_{\boldsymbol{e}_0} z^{\boldsymbol{e}_0}\|_{lg_{\boldsymbol{a}}} \leq \frac{1}{N^n} \sum_{\boldsymbol{\alpha}\in\{0,1,\dots,N-1\}^n} \|\zeta^{-\langle \boldsymbol{\alpha}, \boldsymbol{e}_0\rangle} \phi_{\boldsymbol{\alpha}}\|_{lg_{\boldsymbol{a}}} = \|\phi\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}.$$

Therefore, if $c_{\boldsymbol{e}_0} \neq 0$, then $\|z^{\boldsymbol{e}_0}\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}$. Thus the claim follows by Proposition 1.3. $\square$

Let us go back to the proof of Proposition 1.5(1), (2). By Proposition 1.3,

$$\|z^{\boldsymbol{e}}\|_{lg_{\boldsymbol{a}}} = \exp\Bigl(-l\varphi_{\boldsymbol{a}}\Bigl(\frac{\widetilde{\boldsymbol{e}}^l}{l}\Bigr)\Bigr).$$

Thus (1) and (2) follow from the above claim. $\square$

REMARK 1.6

Let $\tilde{\rho}_{\boldsymbol{a}}$ be a Hermitian inner product of $H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n}(1))$ given by

$$\left(\tilde{\rho}_{\boldsymbol{a}}(T_i, T_j)\right)_{0 \le i,j \le n} = \begin{pmatrix} 1/a_0 & 0 & \cdots & 0 & 0 \\ 0 & 1/a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1/a_n \end{pmatrix}.$$

Let $\rho_{\boldsymbol{a}}$ be the quotient C^∞-Hermitian metric of $\mathcal{O}_{\mathbb{P}^n}(1)$ induced by $\tilde{\rho}_{\boldsymbol{a}}$ and the canonical surjective homomorphism

$$H^0\big(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n}(1)\big) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1).$$

Then $g_{\boldsymbol{a}} = -\log \rho_{\boldsymbol{a}}(T_0, T_0)$.

REMARK 1.7

Hajli [7] pointed out that, for $(x_1, \ldots, x_n) \in \Delta_n$,

$$-\varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n)$$

is the Legendre-Fenchel transform of $\log(a_0 + a_1 e^{u_1} + \cdots + a_n e^{u_n})$; that is,

$$\begin{aligned} &-\varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \\ &\quad = \sup\big\{u_1 x_1 + \cdots + u_n x_n - \log(a_0 + a_1 e^{u_1} + \cdots + a_n e^{u_n}) \bigm| (u_1, \ldots, u_n) \in \mathbb{R}^n\big\}. \end{aligned}$$

This can be easily checked by Claim 1.2.1.

2. Integral formula and Geography of $\overline{D}_{\boldsymbol{a}}$

Let X be a d-dimensional, generically smooth, normal and projective arithmetic variety. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X. Let Φ be an F_∞-invariant volume form on $X(\mathbb{C})$ with $\int_{X(\mathbb{C})} \Phi = 1$. Recall that $\langle \phi, \psi \rangle_g$ and $\|\phi\|_{g,L^2}$ are given by

$$\langle \phi, \psi \rangle_g := \int_{X(\mathbb{C})} \phi \bar{\psi} \exp(-g) \Phi \qquad \text{and} \qquad \|\phi\|_{g,L^2} := \sqrt{\langle \phi, \phi \rangle_g}$$

for $\phi, \psi \in H^0(X, D)$. We set

$$\hat{H}^0_{L^2}(X, \overline{D}) := \big\{\phi \in H^0(X, D) \bigm| \|\phi\|_{g,L^2} \le 1\big\}.$$

Let us begin with the following lemmas.

LEMMA 2.1

We have $\widehat{\mathrm{vol}}(\overline{D}) = \lim_{l \to \infty} \frac{\log \# \hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!}$.

Proof

First of all, note that

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{l \to \infty} \frac{\log \# \hat{H}^0(X, l\overline{D})}{l^d/d!}$$

(see [10, Theorem 5.2.2]). Since $\hat{H}^0(X, l\overline{D}) \subseteq \hat{H}^0_{L^2}(X, l\overline{D})$, we have

$$\widehat{\mathrm{vol}}(\overline{D}) \leq \liminf_{l\to\infty} \frac{\log \#\hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!}.$$

On the other hand, by using Gromov's inequality (see [10, Proposition 3.1.1]), there is a constant C such that $\|\cdot\|_{\sup} \leq Cl^{d-1}\|\cdot\|_{L^2}$ on $H^0(X, lD)$. Thus for any positive number ϵ, $\|\cdot\|_{\sup} \leq \exp(l\epsilon/2)\|\cdot\|_{L^2}$ holds for $l \gg 1$. This implies that

$$\hat{H}^0_{L^2}(X, l\overline{D}) \subseteq \hat{H}^0\big(X, l(\overline{D} + (0,\epsilon))\big)$$

for $l \gg 1$, which yields

$$\limsup_{l\to\infty} \frac{\log \#\hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!} \leq \widehat{\mathrm{vol}}\big(\overline{D} + (0,\epsilon)\big).$$

Therefore, by virtue of the continuity of $\widehat{\mathrm{vol}}$, we have

$$\limsup_{l\to\infty} \frac{\log \#\hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!} \leq \widehat{\mathrm{vol}}(\overline{D}),$$

and hence the lemma follows. □

LEMMA 2.2

Let Θ be a compact convex set in $\mathbb{R}^n$ such that $\mathrm{vol}(\Theta) > 0$. *For each $l \in \mathbb{Z}_{\geq 1}$, let $A_l = (a_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ be a positive definite symmetric real matrix indexed by $l\Theta \cap \mathbb{Z}^n$, and let K_l be a subset of $\mathbb{R}^{l\Theta\cap\mathbb{Z}^n} \simeq \mathbb{R}^{\#(l\Theta\cap\mathbb{Z}^n)}$ given by*

$$K_l = \Big\{(x_{\boldsymbol{e}}) \in \mathbb{R}^{l\Theta\cap\mathbb{Z}^n} \;\Big|\; \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq 1\Big\}.$$

We assume that there are positive constants C and D and a continuous function $\varphi : \Theta \to \mathbb{R}$ such that

$$\left|\log\Big(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\Big) - l\varphi\Big(\frac{\boldsymbol{e}}{l}\Big)\right| \leq C\log(l) + D$$

for all $l \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n$. Then we have

$$\liminf_{l\to\infty} \frac{\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} \geq \frac{1}{2}\int_\Theta \varphi(\boldsymbol{x})\,d\boldsymbol{x}.$$

Moreover, if A_l is diagonal and all entries of A_l are less than or equal to 1 *(i.e., $a_{\boldsymbol{e},\boldsymbol{e}'} \leq 1$, $\forall \boldsymbol{e}, \boldsymbol{e}' \in l\Theta \cap \mathbb{Z}^n$) for each l, then*

$$\lim_{l\to\infty} \frac{\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} = \frac{1}{2}\int_\Theta \varphi(\boldsymbol{x})\,d\boldsymbol{x}.$$

Proof

By Minkowski's theorem,

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n}) \geq \log\big(\mathrm{vol}(K_l)\big) - m_l \log(2),$$

where $m_l = \#(l\Theta \cap \mathbb{Z}^n)$. Note that

$$\log\bigl(\mathrm{vol}(K_l)\bigr) = -\frac{1}{2}\log\bigl(\det(A_l)\bigr) + \log V_{m_l},$$

where $V_r = \mathrm{vol}(\{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1^2 + \cdots + x_r^2 \le 1\})$. Moreover, by Hadamard's inequality,

$$\det(A_l) \le \prod_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}}.$$

Thus

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta \cap \mathbb{Z}^n}) \ge \frac{1}{2} \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \log\Bigl(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\Bigr) + \log V_{m_l} - m_l \log(2).$$

Further, there is a positive constant c_1 such that $m_l \le c_1 l^n$ for $l \ge 1$. Thus we can see

$$\lim_{l \to \infty} \log(V_{m_l})/l^{n+1} = 0.$$

Therefore, it is sufficient to show that

$$\lim_{l \to \infty} \frac{1}{l^{n+1}} \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \log\Bigl(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\Bigr) = \int_\Theta \varphi(\boldsymbol{x})\, d\boldsymbol{x}.$$

By our assumption, we have

$$\varphi\Bigl(\frac{\boldsymbol{e}}{l}\Bigr) - \frac{1}{l}(C \log l + D) \le \frac{1}{l} \log\Bigl(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\Bigr) \le \varphi\Bigl(\frac{\boldsymbol{e}}{l}\Bigr) + \frac{1}{l}(C \log l + D).$$

Note that

$$\lim_{l \to \infty} \frac{1}{l^n} \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \varphi\Bigl(\frac{\boldsymbol{e}}{l}\Bigr) = \lim_{l \to \infty} \sum_{\boldsymbol{x} \in \Theta \cap (1/l)\mathbb{Z}^n} \varphi(\boldsymbol{x}) \frac{1}{l^n} = \int_\Theta \varphi(\boldsymbol{x})\, d\boldsymbol{x}.$$

On the other hand, since $m_l \le c_1 l^n$, we can see

$$\lim_{l \to \infty} \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \frac{1}{l^{n+1}}(C \log l + D) = 0.$$

Thus the first assertion follows.

Next, we assume that A_l is diagonal for each l. Then, since

$$K_l \subseteq \prod_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \Bigl[-\sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}}, \sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}}\Bigr],$$

we have

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta \cap \mathbb{Z}^n}) \le \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \log\Bigl(2\sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}} + 1\Bigr).$$

Thus

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta \cap \mathbb{Z}^n}) \le \frac{1}{2} \sum_{\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n} \log\Bigl(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\Bigr) + m_l \log(3)$$

because $a_{\boldsymbol{e},\boldsymbol{e}} \leq 1$ and $2t+1 \leq 3t$ for $t \geq 1$. Therefore, as before,

$$\limsup_{l\to\infty} \frac{\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} \leq \frac{1}{2}\int_{\Theta} \varphi(\boldsymbol{x})\, d\boldsymbol{x}. \qquad \square$$

From now on, we use the same notation as in Section 1. The purpose of this section is to prove the following theorem.

THEOREM 2.3

(1) (*Integral formula*) *The following formulae hold.*

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = \frac{(n+1)!}{2}\int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(1-x_1-\cdots-x_n, x_1,\ldots,x_n)\, dx_1\cdots dx_n$$

and

$$\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \frac{(n+1)!}{2}\int_{\Delta_n} \varphi_{\boldsymbol{a}}(1-x_1-\cdots-x_n, x_1,\ldots,x_n)\, dx_1\cdots dx_n$$

(2) $\overline{D}_{\boldsymbol{a}}$ *is ample if and only if* $\boldsymbol{a}(i) > 1$ *for all* $i = 0,\ldots,n$.
(3) $\overline{D}_{\boldsymbol{a}}$ *is nef if and only if* $\boldsymbol{a}(i) \geq 1$ *for all* $i = 0,\ldots,n$.
(4) $\overline{D}_{\boldsymbol{a}}$ *is big if and only if* $|\boldsymbol{a}| > 1$.
(5) $\overline{D}_{\boldsymbol{a}}$ *is pseudoeffective if and only if* $|\boldsymbol{a}| \geq 1$.
(6) *If* $|\boldsymbol{a}| = 1$, *then*

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \begin{cases} \{0, \pm z_1^{l\boldsymbol{a}(1)}\cdots z_n^{l\boldsymbol{a}(n)}\} & \text{if } l\boldsymbol{a} \in \mathbb{Z}^{n+1}, \\ \{0\} & \text{if } l\boldsymbol{a} \notin \mathbb{Z}^{n+1}. \end{cases}$$

(7) *We have* $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})$ *if and only if* $\overline{D}_{\boldsymbol{a}}$ *is nef.*

Proof
First, let us consider the essential case (1).

CLAIM 2.3.1
If $|\boldsymbol{a}| > 1$, *then* $\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = \frac{(n+1)!}{2}\int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{t}})\, d\boldsymbol{t}$.

Proof
In this case, $\mathrm{vol}(\Theta_{\boldsymbol{a}}) > 0$. By using Proposition 1.5,

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \subseteq \Big\{\phi \in \bigoplus_{\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n} \mathbb{Z}z^{\boldsymbol{e}} \;\Big|\; \langle\phi,\phi\rangle_{lg_{\boldsymbol{a}}} \leq 1\Big\} \subseteq \hat{H}^0_{L^2}(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}),$$

which yields

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = (n+1)!\lim_{l\to\infty} \frac{\log \#\{\phi \in \bigoplus_{\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n} \mathbb{Z}z^{\boldsymbol{e}} | \langle\phi,\phi\rangle_{lg_{\boldsymbol{e}}} \leq 1\}}{l^{n+1}}$$

by Lemma 2.1. If we set

$$K_l = \Big\{(x_{\boldsymbol{e}}) \in \mathbb{R}^{l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n} \;\Big|\; \sum_{\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n} \frac{x_{\boldsymbol{e}}^2}{\binom{l+n}{n}\binom{l}{\tilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\tilde{\boldsymbol{e}}^l}} \leq 1\Big\},$$

then by Proposition 1.3,

$$\#\Big\{\phi \in \bigoplus_{\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n} \mathbb{Z}z^{\boldsymbol{e}} \;\Big|\; \langle\phi,\phi\rangle_{lg_{\boldsymbol{a}}} \le 1\Big\} = \#(K_l \cap \mathbb{Z}^{l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n}).$$

On the other hand, for $\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n$,

$$\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l} = \frac{1}{\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}}\rangle_{lg_{\boldsymbol{a}}}} \ge \exp\Big(l\varphi_{\boldsymbol{a}}\Big(\frac{\widetilde{\boldsymbol{e}}^l}{l}\Big)\Big) \ge 1.$$

Moreover, by Lemma 1.4, there are positive constants A and B such that

$$\left|\log\left(\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}\right) - l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^l/l)\right| \le A\log l + B$$

holds for all $l\in\mathbb{Z}_{\ge 1}$ and $\boldsymbol{e}\in l\Theta_{\boldsymbol{a}}\cap\mathbb{Z}^n$. Thus the assertion follows from Lemma 2.2. □

Next, let us consider the following claim.

CLAIM 2.3.2

If $s,t\in\mathbb{R}_{>0}$ and $\alpha,\beta\in\mathbb{R}$ with $\alpha+\beta\ne 0$, then

$$\alpha\overline{D}_{t\boldsymbol{a}} + \beta\overline{D}_{s\boldsymbol{a}} = (\alpha+\beta)\overline{D}_{(t^\alpha s^\beta)^{1/(\alpha+\beta)}\boldsymbol{a}}.$$

Proof

This is a straightforward calculation. □

We now prove cases (2) and (3). First of all, $\omega_{\boldsymbol{a}}$ is positive by Proposition 1.1. Let γ_i be a 1-dimensional closed subscheme given by $H_0\cap\cdots\cap H_{i-1}\cap H_{i+1}\cap\cdots\cap H_n$. Then it is easy to see that $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}|_{\gamma_i}) = (1/2)\log(\boldsymbol{a}(i))$. Therefore we have the "only if" part of (2) and (3).

We assume that $\boldsymbol{a}(i) > 1$ for all i. Then $\varphi_{\boldsymbol{a}}$ is positive on

$$\big\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1}_{\ge 0} \;\big|\; x_0+\cdots+x_n = 1\big\}.$$

Thus, for $\boldsymbol{e}\in\mathbb{Z}^n_{\ge 0}$ with $|\boldsymbol{e}|\le 1$, $z^{\boldsymbol{e}}$ is a strictly small section by Proposition 1.3, which shows that $\overline{D}_{\boldsymbol{a}}$ is ample; that is, H_0 is ample, the first Chern form of $\overline{D}_{\boldsymbol{a}}$ is positive, and $H^0(\mathbb{P}^n_{\mathbb{Z}}, H_0)$ is generated by strictly small sections.

Next, we assume that $\boldsymbol{a}(i)\ge 1$ for all i. Let γ be a 1-dimensional closed integral subscheme of $\mathbb{P}^n_{\mathbb{Z}}$. Then we can find H_i such that $\gamma\not\subseteq H_i$. Note that

$$\overline{D}_{\boldsymbol{a}} + \widehat{(z_i)} = \big(H_i, \log(\boldsymbol{a}(0)|w_0|^2+\cdots+\boldsymbol{a}(n)|w_n|^2)\big),$$

where $w_k = T_k/T_i$ $(k=0,\dots,n)$. Therefore $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}|_\gamma)\ge 0$ because

$$\log\big(\boldsymbol{a}(0)|w_0|^2+\cdots+\boldsymbol{a}(n)|w_n|^2\big)\ge 0.$$

In case (6), $\Theta_{\boldsymbol{a}} = \{(\boldsymbol{a}(1),\dots,\boldsymbol{a}(n))\}$ and $\varphi_{\boldsymbol{a}}(\boldsymbol{a}) = 0$ by Lemma 1.2. Moreover, if $l\boldsymbol{a}\in\mathbb{Z}^{n+1}$, then

$$\|z^{l(\boldsymbol{a}(1),\dots,\boldsymbol{a}(n))}\|^2_{lg_{\boldsymbol{a}}} = \exp\big(-l\varphi_{\boldsymbol{a}}(\boldsymbol{a})\big) = 1$$

by Proposition 1.3. Thus the assertion follows from Proposition 1.5.

In order to see (4) and (5) using (6), it is sufficient to show the following:

(i) $\overline{D}_{\boldsymbol{a}}$ is big if $|\boldsymbol{a}| > 1$;
(ii) $\overline{D}_{\boldsymbol{a}}$ is pseudoeffective if $|\boldsymbol{a}| \geq 1$;
(iii) $\overline{D}_{\boldsymbol{a}}$ is not pseudoeffective if $|\boldsymbol{a}| < 1$.

(i) It follows from Claim 2.3.1 because $\mathrm{vol}(\Theta_{\boldsymbol{a}}) > 0$.

(ii) We choose a real number t such that $t > 1$ and $\overline{D}_{t\boldsymbol{a}}$ is ample. By Claim 2.3.2,

$$\overline{D}_{\boldsymbol{a}} + \epsilon \overline{D}_{t\boldsymbol{a}} = (1+\epsilon)\overline{D}_{t^{\epsilon/(1+\epsilon)}\boldsymbol{a}}.$$

For any $\epsilon > 0$, since $t^{\epsilon/(1+\epsilon)}|\boldsymbol{a}| > 1$, $(1+\epsilon)\overline{D}_{t^{\epsilon/(1+\epsilon)}\boldsymbol{a}}$ is big by (i), which shows that $\overline{D}_{\boldsymbol{a}}$ is pseudoeffective.

(iii) Let us choose a positive real number t such that $\overline{D}_{t\boldsymbol{a}}$ is ample. We also choose a positive number ϵ such that if we set $\boldsymbol{a}' = t^{\epsilon/(1+\epsilon)}\boldsymbol{a}$, then $|\boldsymbol{a}'| < 1$. We assume that $\overline{D}_{\boldsymbol{a}}$ is pseudoeffective. Then

$$\overline{D}_{\boldsymbol{a}} + \epsilon \overline{D}_{t\boldsymbol{a}} = (1+\epsilon)\overline{D}_{\boldsymbol{a}'}$$

is big by [10, Proposition 6.3.2], which means that $\overline{D}_{\boldsymbol{a}'}$ is big. On the other hand, as $|\boldsymbol{a}'| < 1$, we have $\Theta_{\boldsymbol{a}'} = \emptyset$. Thus $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, n\overline{D}_{\boldsymbol{a}'}) = \{0\}$ for all $n \geq 1$ by Proposition 1.5. This is a contradiction.

For the first formula of case (1), we may assume that $|\boldsymbol{a}| \leq 1$ by Claim 2.3.1. In this case, $\overline{D}_{\boldsymbol{a}}$ is not big by (4) and $\Theta_{\boldsymbol{a}}$ is either $\emptyset$ or $\{(a_1, \ldots, a_n)\}$. Thus the assertion follows. For the second formula, the arithmetic Hilbert-Samuel formula (see [5], [1]) yields

$$\frac{\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1})}{(n+1)!} = \lim_{l \to \infty} \frac{\widehat{\chi}(H^0(\mathbb{P}^n_{\mathbb{Z}}, lH_0), \langle , \rangle_{lg_a})}{l^{n+1}}.$$

On the other hand,

$$\widehat{\chi}\big(H^0(\mathbb{P}^n_{\mathbb{Z}}, lH_0), \langle\ ,\ \rangle_{lg_a}\big) = \sum_{\boldsymbol{e} \in l\Delta_n \cap \mathbb{Z}^n} \log\left(\sqrt{\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}}\right) + \log V_{\#(l\Delta_n \cap \mathbb{Z}^n)}.$$

Thus, in the same way as the proofs of Lemma 2.2 and Claim 2.3.1, we can see the second formula.

Case (7) follows from (1) and (3). This concludes the proof of Theorem 2.3. □

Finally, let us consider the following proposition.

PROPOSITION 2.4

For any positive integer l, there exists $\boldsymbol{a} \in \mathbb{Q}^{n+1}_{>0}$ such that $|\boldsymbol{a}| > 1$ and $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) = \{0\}$ for $k = 1, \ldots, l$.

Proof
Let us choose positive rational numbers $a'_1, \ldots, a'_n$ such that $a'_1 + \cdots + a'_n < 1$ and $a'_1 < 1/l, \ldots, a'_n < 1/l$. We set $a'_0 = 1 - a'_1 - \cdots - a'_n$ and $\boldsymbol{a}' = (a'_0, \ldots, a'_n)$. Moreover, for a rational number $\lambda > 1$, we set

$$K_\lambda = \left\{ \boldsymbol{x} \in \Delta_n \;\middle|\; \varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}) + \log \lambda \geq 0 \right\},$$

where $\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\}$.

CLAIM 2.4.1
We can find a rational number $\lambda > 1$ such that $K_\lambda \subseteq (0, 1/l)^n$.

Proof
We assume that $K_{1+(1/m)} \not\subseteq (0, 1/l)^n$ for all $m \in \mathbb{Z}_{\geq 1}$; that is, we can find $\boldsymbol{x}_m \in K_{1+(1/m)} \setminus (0, 1/l)^n$ for each $m \geq 1$. Since Δ_n is compact, there is a subsequence $\{\boldsymbol{x}_{m_i}\}$ of $\{\boldsymbol{x}_m\}$ such that $\boldsymbol{x} = \lim_{i \to \infty} \boldsymbol{x}_{m_i}$ exists. Note that $\boldsymbol{x} \notin (0, 1/l)^n$ because $\boldsymbol{x}_{m_i} \notin (0, 1/l)^n$ for all i. On the other hand, since $\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_{m_i}) + \log(1 + (1/m_i)) \geq 0$ for all i, we have $\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}) \geq 0$, and hence $\boldsymbol{x} = (a'_1, \ldots, a'_n)$ by Lemma 1.2. This is a contradiction. $\square$

We choose a rational number $\lambda > 1$ as in the above claim. Here we set $\boldsymbol{a} = \lambda \boldsymbol{a}'$. Then, as $\varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{a}'} + \log \lambda$, we have $\Theta_{\boldsymbol{a}} \subseteq (0, 1/l)^n$. We assume that $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) \neq \{0\}$ for some k with $1 \leq k \leq l$. Then, by Proposition 1.5, there is $\boldsymbol{e} = (e_1, \ldots, e_n) \in k\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$; that is, $\boldsymbol{e}/k \in \Theta_{\boldsymbol{a}}$. Thus $0 < e_i/k < 1/l$ for all i. This is a contradiction. $\square$

3. Asymptotic multiplicity

Let X be a d-dimensional, projective, generically smooth and normal arithmetic variety. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X. We set

$$N(\overline{D}) = \left\{ l \in \mathbb{Z}_{>0} \;\middle|\; \hat{H}^0(X, l\overline{D}) \neq \{0\} \right\}.$$

We assume that $N(\overline{D}) \neq \emptyset$. Then $\mu_x(\overline{D})$ for $x \in X$ is defined to be

$$\mu_x(\overline{D}) := \inf \left\{ \operatorname{mult}_x(D + (1/l)(\phi)) \;\middle|\; l \in N(\overline{D}), \phi \in \hat{H}^0(X, l\overline{D}) \setminus \{0\} \right\},$$

which is called the *asymptotic multiplicity of $\overline{D}$ at x*. The following proposition states the fundamental properties of the asymptotic multiplicity.

PROPOSITION 3.1 ([10, PROPOSITIONS 6.5.2, 6.5.3])
Let $\overline{D}$ and $\overline{E}$ be arithmetic $\mathbb{R}$-divisors of C^0-type such that $N(\overline{D}) \neq \emptyset$ and $N(\overline{E}) \neq \emptyset$. Then we have the following:

(1) $\mu_x(\overline{D} + \overline{E}) \leq \mu_x(\overline{D}) + \mu_x(\overline{E})$;
(2) *if* $\overline{D} \leq \overline{E}$, *then* $\mu_x(\overline{E}) \leq \mu_x(\overline{D}) + \operatorname{mult}_x(E - D)$;
(3) $\mu_x(\overline{D} + \widehat{(\phi)}) = \mu_x(\overline{D})$ *for* $\phi \in \operatorname{Rat}(X)^\times$;
(4) $\mu_x(a\overline{D}) = a\mu_x(\overline{D})$ *for* $a \in \mathbb{Q}_{>0}$;
(5) *if* $\overline{D}$ *is nef and big, then* $\mu_x(\overline{D}) = 0$.

Moreover, we have the following lemma.

LEMMA 3.2

For each $l \in N(\overline{D})$, let $\{\phi_{l,1}, \dots, \phi_{l,r_l}\}$ be a subset of $\hat{H}^0(X, l\overline{D}) \setminus \{0\}$ such that $\hat{H}^0(X, l\overline{D}) \subseteq \langle \phi_{l,1}, \dots, \phi_{l,r_l} \rangle_{\mathbb{Z}}$. Let x be a point of X such that the Zariski closure $\overline{\{x\}}$ of $\{x\}$ is flat over $\mathbb{Z}$. Then

$$\mu_x(\overline{D}) = \inf\left\{ \operatorname{mult}_x\left(D + \left(\frac{1}{l}\right)(\phi_{l,i})\right) \,\middle|\, l \in N(D), i = 1, \dots, r_l \right\}.$$

Proof

Clearly,

$$\mu_x(\overline{D}) \leq \inf\left\{ \operatorname{mult}_x\left(D + \left(\frac{1}{l}\right)(\phi_{l,i})\right) \,\middle|\, l \in N(D), i = 1, \dots, r_l \right\}.$$

Let us consider the converse inequality. For $l \in N(\overline{D})$ and $\phi \in \hat{H}^0(X, l\overline{D}) \setminus \{0\}$, we set $\phi = \sum_{i=1}^{r_l} c_i \phi_{l,i}$ for some $c_1, \dots, c_{r_l} \in \mathbb{Z}$. Note that

$$\operatorname{mult}_x\big((\phi + \psi)\big) \geq \min\{\operatorname{mult}_x((\phi)), \operatorname{mult}_x((\psi))\} \qquad \text{and} \qquad \operatorname{mult}_x\big((a)\big) = 0$$

for $\phi, \psi \in \operatorname{Rat}(X)^\times$ and $a \in \mathbb{Q}^\times$ with $\phi + \psi \neq 0$. Thus we can find i such that

$$\operatorname{mult}_x\big((\phi)\big) \geq \operatorname{mult}_x\big((\phi_{l,i})\big),$$

and hence the converse inequality holds. □

4. Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$

We use the same notation as in Section 1. We assume that $n = 1$. In this section, we consider the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1])$. Note that $\Theta_{\boldsymbol{a}}$ is a closed interval in $[0, 1]$. For simplicity, we denote the affine coordinate z_1 by z; that is, $z = T_1/T_0$.

THEOREM 4.1

The Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ exists if and only if $a_0 + a_1 \geq 1$. Moreover, if we set $\vartheta_{\boldsymbol{a}} = \inf \Theta_{\boldsymbol{a}}$, $\theta_{\boldsymbol{a}} = \sup \Theta_{\boldsymbol{a}}$, $P_{\boldsymbol{a}} = \theta_{\boldsymbol{a}} H_0 - \vartheta_{\boldsymbol{a}} H_1$, and

$$p_{\boldsymbol{a}}(z) = \begin{cases} \vartheta_{\boldsymbol{a}} \log |z|^2 & \text{if } |z| < \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z| \leq \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log |z|^2 & \text{if } |z| > \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \end{cases}$$

then the positive part of $\overline{D}_{\boldsymbol{a}}$ is $\overline{P}_{\boldsymbol{a}} = (P_{\boldsymbol{a}}, p_{\boldsymbol{a}})$, where $\sqrt{(a_0\theta_{\boldsymbol{a}})/(a_1(1-\theta_{\boldsymbol{a}}))}$ is treated as ∞ if $\theta_{\boldsymbol{a}} = 1$.

Proof

First, we consider the case where $\overline{D}_{\boldsymbol{a}}$ is big, that is, where $a_0 + a_1 > 1$ by Theorem 2.3. In this case, $0 \leq \vartheta_{\boldsymbol{a}} < \theta_{\boldsymbol{a}} \leq 1$. The existence of the Zariski decomposition follows from [10, Theorem 9.2.1]. Here we consider functions

$$r_1 : \left\{ z \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; |z| < \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}} \right\} \to \mathbb{R}$$

and

$$r_2 : \left\{ z \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; |z| > \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \right\} \to \mathbb{R}$$

given by

$$r_1(z) = \begin{cases} 0 & \text{if } |z| < \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ -\vartheta_{\boldsymbol{a}} \log |z|^2 + \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \le |z| < \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}} \end{cases}$$

and

$$r_2(z) = \begin{cases} -\theta_{\boldsymbol{a}} \log |z|^2 + \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} < |z| \le \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ 0 & \text{if } |z| > \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

Note that

$$p_{\boldsymbol{a}}(z) = \begin{cases} \vartheta_{\boldsymbol{a}} \log |z|^2 + r_1(z) & \text{on } |z| < \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log |z|^2 + r_2(z) & \text{on } |z| > \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}. \end{cases}$$

Thus, to see that $p_{\boldsymbol{a}}$ is a $P_{\boldsymbol{a}}$-Green function of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1(\mathbb{C})$, it is sufficient to check that r_1 and r_2 are continuous and subharmonic on each area. Let us show that r_1 is continuous and subharmonic. If $\vartheta_{\boldsymbol{a}} = 0$, then the assertion is obvious, so that we may assume that $\vartheta_{\boldsymbol{a}} > 0$. First of all, as $\varphi_{\boldsymbol{a}}(1-\vartheta_{\boldsymbol{a}}, \vartheta_{\boldsymbol{a}}) = 0$, we have $r_1(z) = 0$ if $|z| = \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$, and hence r_1 is continuous. It is obvious that r_1 is subharmonic on

$$\left\{ z \in \mathbb{C} \;\middle|\; |z| < \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \right\} \cup \left\{ z \in \mathbb{C} \;\middle|\; \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} < |z| < \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}} \right\}.$$

By using Claim 1.2.1,

$$\begin{aligned} \vartheta_{\boldsymbol{a}} \log |z|^2 &= (1-\vartheta_{\boldsymbol{a}}) \log(1) + \vartheta_{\boldsymbol{a}} \log |z|^2 \\ &\le \log(a_0 + a_1|z|^2) + \varphi_{\boldsymbol{a}}(1-\vartheta_{\boldsymbol{a}}, \vartheta_{\boldsymbol{a}}) = \log(a_0 + a_1|z|^2). \end{aligned}$$

Thus $r_1 \ge 0$. Therefore, if $|z| = \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$, then

$$r_1(z) = 0 \le \frac{1}{2\pi} \int_0^{2\pi} r_1(z + \epsilon e^{\sqrt{-1}t})\, dt$$

for a small positive real number ϵ, and hence r_1 is subharmonic at z with $|z| = \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$. In a similar way, we can check that r_2 is continuous and subharmonic.

Next, let us show that $\overline{P}_{\boldsymbol{a}}$ is nef. As $r_1(0) = 0$ and $r_2(\infty) = 0$, we have

$$\widehat{\deg}(\overline{P}_{\boldsymbol{a}}|_{H_0}) = \widehat{\deg}(\overline{P}_{\boldsymbol{a}}|_{H_1}) = 0.$$

Note that

$$\overline{P}_{\boldsymbol{a}} + \vartheta_{\boldsymbol{a}}\widehat{(z)} = \big((\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}})H_0, p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log |z|^2\big)$$

and

$$p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log |z|^2 = \begin{cases} r_1(z) & \text{if } |z| \le \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ (\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}}) \log |z|^2 & \text{if } |z| > \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

Therefore, $p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log |z|^2 \ge 0$ on $\mathbb{P}^1(\mathbb{C})$, which means that $\overline{P}_{\boldsymbol{a}} + \vartheta_{\boldsymbol{a}}\widehat{(z)}$ is effective. Let γ be a 1-dimensional closed integral subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ with $\gamma \ne H_0, H_1$. Then

$$\widehat{\deg}(\overline{P}_{\boldsymbol{a}}|_{\gamma}) = \widehat{\deg}\big(((\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}})H_0, p_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}} \log |z|^2)|_{\gamma}\big) \ge 0.$$

By using Proposition 1.5, we have $\mu_{H_0}(\overline{D}_{\boldsymbol{a}}) = 1 - \theta_{\boldsymbol{a}}$ and $\mu_{H_1}(\overline{D}_{\boldsymbol{a}}) = \vartheta_{\boldsymbol{a}}$. Thus the positive part of $\overline{D}_{\boldsymbol{a}}$ can be written as a form $(P_{\boldsymbol{a}}, q)$, where q is a $P_{\boldsymbol{a}}$-Green function of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1(\mathbb{C})$ (see [10, Claim 9.3.5.1, Proposition 9.3.1]). Note that $\overline{P}_{\boldsymbol{a}}$ is nef and $\overline{P}_{\boldsymbol{a}} \le \overline{D}_{\boldsymbol{a}}$, so that

$$p_{\boldsymbol{a}}(z) \le q(z) \le \log(a_0 + a_1|z|^2).$$

We choose a continuous function u such that $p_{\boldsymbol{a}} + u = q$. Then $u(z) = 0$ on

$$\sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \le |z| \le \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}.$$

Moreover, since $q(z) = \vartheta_{\boldsymbol{a}} \log |z|^2 + u(z)$ on $|z| \le \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$, u is subharmonic on $|z| \le \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$. On the other hand, $u(0) = 0$ because

$$\widehat{\deg}\big((P_{\boldsymbol{a}}, q)|_{H_1}\big) = u(0) = 0.$$

Therefore, $u = 0$ on $|z| \le \sqrt{(a_0\vartheta_{\boldsymbol{a}})/(a_1(1-\vartheta_{\boldsymbol{a}}))}$ by the maximal principle. In a similar way, we can see that $u = 0$ on $|z| \ge \sqrt{(a_0\theta_{\boldsymbol{a}})/(a_1(1-\theta_{\boldsymbol{a}}))}$.

Next, we consider the case where $a_0 + a_1 = 1$. By Claim 1.2.1,

$$a_1 \log |z|^2 \le \log(a_0 + a_1|z|^2)$$

on $\mathbb{P}^1(\mathbb{C})$. Thus $-a_1\widehat{(z)} \le \overline{D}_{\boldsymbol{a}}$, and hence the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ exists by [10, Theorem 9.2.1]. Let $\overline{P}$ be the positive part of $\overline{D}_{\boldsymbol{a}}$. Then $-a_1\widehat{(z)} \le \overline{P}$.

Let us consider the converse inequality. Let t be a real number with $t > 1$. Since $\overline{P} \le \overline{D}_{\boldsymbol{a}} \le \overline{D}_{t\boldsymbol{a}}$, we have $\overline{P} \le \overline{P}_{t\boldsymbol{a}}$ because $\overline{P}_{t\boldsymbol{a}}$ is the positive part of $\overline{D}_{t\boldsymbol{a}}$ by the previous observation. Since $\varphi_{t\boldsymbol{a}} = \varphi_{\boldsymbol{a}} + \log(t)$, we have $\lim_{t\to 1} \vartheta_{t\boldsymbol{a}} = \lim_{t\to 1} \theta_{t\boldsymbol{a}} = a_1$. Therefore, we can see

$$\lim_{t\to 1} \overline{P}_{t\boldsymbol{a}} = \overline{P}_{\boldsymbol{a}} = -a_1\widehat{(z)}.$$

Thus $\overline{P} \le -a_1\widehat{(z)}$.

Finally, we consider the case where $a_0 + a_1 < 1$. Then, by Theorem 2.3, $\overline{D}_{\boldsymbol{a}}$ is not pseudoeffective. Thus the Zariski decomposition does not exist by [10, Proposition 9.3.2]. □

5. Weak Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$

Let X be a d-dimensional, projective, generically smooth, and normal arithmetic variety. Let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on X. A decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *weak Zariski decomposition of* $\overline{D}$ if the following conditions are satisfied:

(1) $\overline{P}$ is a nef and big arithmetic $\mathbb{R}$-divisor of $(C^0 \cap \mathrm{PSH})$-type;
(2) $\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of C^0-type;
(3) $\mathrm{mult}_\Gamma(N) \leq \mu_\Gamma(\overline{D})$ for any horizontal prime divisor Γ on X; that is, Γ is a reduced and irreducible divisor Γ on X such that Γ is flat over $\mathbb{Z}$.

Note that the Zariski decomposition of a big arithmetic $\mathbb{R}$-divisor of C^0-type on an arithmetic surface is a weak Zariski decomposition (see [10, Claim 9.3.5.1]). Property (3) above implies that $\mathrm{mult}_\Gamma(N) = \mu_\Gamma(\overline{D})$ for any horizontal prime divisor Γ on X. Indeed, by Proposition 3.1(2) and (5),

$$\mu_\Gamma(\overline{D}) \leq \mu_\Gamma(\overline{P}) + \mathrm{mult}_\Gamma(N) = \mathrm{mult}_\Gamma(N) \leq \mu_\Gamma(\overline{D}).$$

From now on, we use the same notation as in Section 1. Let us begin with the following lemma.

LEMMA 5.1

Let $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ and $g : Y \to X$ be birational morphisms of projective, generically smooth, and normal arithmetic varieties. If $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition, then $g^*(f^*(\overline{D}_{\boldsymbol{a}}))$ also admits a weak Zariski decomposition.

Proof

Let $f^*(\overline{D}_{\boldsymbol{a}}) = \overline{P} + \overline{N}$ be a weak Zariski decomposition of $f^*(\overline{D}_{\boldsymbol{a}})$. We denote birational morphisms $X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$ and $Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ by $f_{\mathbb{Q}}$ and $g_{\mathbb{Q}}$, respectively. We set

$$\widetilde{\Theta}_{\boldsymbol{a}} = \{\widetilde{e} \in \mathbb{R}^{n+1} \mid e \in \Theta_{\boldsymbol{a}}\},$$

$f_{\mathbb{Q}}^*(H_i) = \sum_j a_{ij} D_j$ for $i = 0, \dots, n$ and $N = \sum_j b_j D_j$ on $X_{\mathbb{Q}}$, where the D_j's are reduced and irreducible divisors on $X_{\mathbb{Q}}$. Since

$$lH_0 + (z^{\boldsymbol{e}}) = \big(l - \boldsymbol{e}(1) - \cdots - \boldsymbol{e}(n)\big)H_0 + \boldsymbol{e}(1)H_1 + \cdots + \boldsymbol{e}(n)H_n$$

for $\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$, by Lemma 3.2, we have

$$\mu_{D_j}\big(f^*(\overline{D}_{\boldsymbol{a}})\big) = \min\Big\{\sum_{i=0}^n x_i a_{ij} \;\Big|\; (x_0, \dots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}}\Big\}.$$

Thus

$$b_j \leq \min\Big\{\sum_{i=0}^n x_i a_{ij} \;\Big|\; (x_0, \dots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}}\Big\}$$

for all j.

Here let us show that $g^*(f^*(\overline{D}_{\boldsymbol{a}})) = g^*(\overline{P}) + g^*(\overline{N})$ is a weak Zariski decomposition. For this purpose, it is sufficient to show that $\mathrm{mult}_\Gamma(g^*(N)) \leq$

$\mu_\Gamma\big(g^*(f^*(\overline{D}_{\boldsymbol{a}}))\big)$ for any horizontal prime divisor Γ on Y. If we set $c_j = \mathrm{mult}_\Gamma(g_{\mathbb{Q}}^*(D_j))$, then

$$d_i := \mathrm{mult}_\Gamma\big(g_{\mathbb{Q}}^*(f_{\mathbb{Q}}^*(H_i))\big) = \sum_j a_{ij} c_j.$$

For $(x_0, \dots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}}$,

$$\sum_i x_i d_i = \sum_j \Big(\sum_i x_i a_{ij}\Big) c_j \geq \sum_j b_j c_j = \mathrm{mult}_\Gamma\big(g_{\mathbb{Q}}^*(N)\big),$$

which yields $\mu_\Gamma\big(g^*(f^*(\overline{D}_{\boldsymbol{a}}))\big) \geq \mathrm{mult}_\Gamma(g^*(N))$. □

Next, let us consider the following lemma.

LEMMA 5.2

Let Θ be a compact convex set in $\mathbb{R}^n$, and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection given by $p(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then $p(\Theta)$ is a compact convex set in $\mathbb{R}^{n-1}$, and there exist a concave function θ on $p(\Theta)$ and a convex function ϑ on $p(\Theta)$ such that

$$\Theta = \left\{ (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (x_1, \dots, x_{n-1}) \in p(\Theta), \\ \vartheta(x_1, \dots, x_{n-1}) \leq x_n \leq \theta(x_1, \dots, x_{n-1}) \end{array} \right\}.$$

Proof

Obviously $p(\Theta)$ is a compact convex set in $\mathbb{R}^{n-1}$. For $(x_1, \dots, x_{n-1}) \in p(\Theta)$, we set

$$\begin{cases} \theta(x_1, \dots, x_{n-1}) := \max\{x_n \in \mathbb{R} \mid (x_1, \dots, x_{n-1}, x_n) \in \Theta\}, \\ \vartheta(x_1, \dots, x_{n-1}) := \min\{x_n \in \mathbb{R} \mid (x_1, \dots, x_{n-1}, x_n) \in \Theta\}. \end{cases}$$

Clearly,

$$\Theta = \left\{ (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (x_1, \dots, x_{n-1}) \in p(\Theta), \\ \vartheta(x_1, \dots, x_{n-1}) \leq x_n \leq \theta(x_1, \dots, x_{n-1}) \end{array} \right\}.$$

We need to show that θ (resp., ϑ) is a concave (resp., convex) function. Since

$$\big(x_1, \dots, x_{n-1}, \theta(x_1, \dots, x_{n-1})\big), \big(x_1', \dots, x_{n-1}', \theta(x_1', \dots, x_{n-1}')\big) \in \Theta$$

for $(x_1, \dots, x_{n-1}), (x_1', \dots, x_{n-1}') \in p(\Theta)$, we have

$$\lambda\big(x_1, \dots, x_{n-1}, \theta(x_1, \dots, x_{n-1})\big) + (1-\lambda)\big(x_1', \dots, x_{n-1}', \theta(x_1', \dots, x_{n-1}')\big) \in \Theta$$

for $0 \leq \lambda \leq 1$, which shows that

$$\begin{aligned} &\lambda\theta(x_1, \dots, x_{n-1}) + (1-\lambda)\theta(x_1', \dots, x_{n-1}') \\ &\quad \leq \theta\big(\lambda(x_1, \dots, x_{n-1}) + (1-\lambda)(x_1', \dots, x_{n-1}')\big). \end{aligned}$$

Thus θ is concave. Similarly, we can see that ϑ is convex. □

REMARK 5.3

If $p(\Theta)$ is a polytope in Lemma 5.2, then θ and ϑ are continuous on $p(\Theta)$ (see

[4]). In general, θ and ϑ are not necessarily continuous on $p(\Theta)$. Indeed, let us consider the following set:

$$\Theta = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le y \le 1, 0 \le z \le 1, x^2 \le yz\}.$$

Since

$$x^2 \le yz \quad \Longleftrightarrow \quad x^2 + \left(\frac{y-z}{2}\right)^2 \le \left(\frac{y+z}{2}\right)^2,$$

we can easily see that Θ is a compact convex set in $\mathbb{R}^3$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection given by $p(x,y,z) = (x,y)$. Then

$$p(\Theta) = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le 1\}.$$

Moreover, ϑ is given by

$$\vartheta(x,y) = \begin{cases} x^2/y & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

and hence ϑ is not continuous at $(0,0)$.

Note that $\Theta_{\boldsymbol{a}}$ is a compact convex set of $\mathbb{R}^n$. We say that a hyperplane $\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta$ in $\mathbb{R}^n$ is a *supporting hyperplane* of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n) \in \Theta_{\boldsymbol{a}}$ if

$$\Theta_{\boldsymbol{a}} \subseteq \{\alpha_1 x_1 + \cdots + \alpha_n x_n \ge \beta\} \qquad \text{and} \qquad \alpha_1 b_1 + \cdots + \alpha_n b_n = \beta.$$

PROPOSITION 5.4

Let $(b_1, \ldots, b_n) \in \partial(\Theta_{\boldsymbol{a}})$; that is, $(b_1, \ldots, b_n)$ is a boundary point of $\Theta_{\boldsymbol{a}}$. We set $b_0 = 1 - b_1 - \cdots - b_n$. We assume the following:

$$a_0 + a_1 + \cdots + a_n > 1 \qquad \text{and} \qquad \#\{i \mid 0 \le i \le n, b_i = 0\} \le 1.$$

Then $\Theta_{\boldsymbol{a}}$ has a unique supporting hyperplane at $(b_1, \ldots, b_n)$. Moreover, in the case where $b_i = 0$, the supporting hyperplane is given by

$$\begin{cases} x_1 + \cdots + x_n = 1 & \text{if } b_0 = 0, \\ x_i = 0 & \text{if } b_i = 0 \text{ for some } i \text{ with } 1 \le i \le n. \end{cases}$$

Proof

Here we set

$$\phi_{\boldsymbol{a}}(x_1, \ldots, x_n) = \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n)$$

on $\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\ge 0} \mid x_1 + \cdots + x_n \le 1\}$. Then

$$\Theta_{\boldsymbol{a}} = \{(x_1, \ldots, x_n) \in \Delta_n \mid \phi_{\boldsymbol{a}}(x_1, \ldots, x_n) \ge 0\}.$$

First, we assume that $(b_1, \ldots, b_n) \notin \partial(\Delta_n)$. Then $\phi_{\boldsymbol{a}}(b_1, \ldots, b_n) = 0$. Note that, for $(x_1, \ldots, x_n) \in \Delta_n \setminus \partial(\Delta_n)$,

$$\begin{aligned} &(\phi_{\boldsymbol{a}})_{x_1}(x_1, \ldots, x_n) = \cdots = (\phi_{\boldsymbol{a}})_{x_n}(x_1, \ldots, x_n) = 0 \\ \Longleftrightarrow \quad &(x_1, \ldots, x_n) = \left(\frac{a_1}{a_0 + \cdots + a_n}, \ldots, \frac{a_n}{a_0 + \cdots + a_n}\right) \end{aligned}$$

and

$$\phi_{\boldsymbol{a}}\Big(\frac{a_1}{a_0+\cdots+a_n},\dots,\frac{a_n}{a_0+\cdots+a_n}\Big)=\log(a_0+\cdots+a_n)>0.$$

Thus we have

$$\big((\phi_{\boldsymbol{a}})_{x_1}(b_1,\dots,b_n),\dots,(\phi_{\boldsymbol{a}})_{x_1}(b_1,\dots,b_n)\big)\neq(0,\dots,0),$$

which means that $\Theta_{\boldsymbol{a}}$ has a unique supporting hyperplane at $(b_1,\dots,b_n)$.

Next, we assume that $(b_1,\dots,b_n)\in\partial(\Delta_n)$. Considering the linear transformations

$$\begin{cases} x'_1=x_1, \\ \vdots \\ x'_{n-1}=x_{n-1}, \\ x'_n=1-x_1-\cdots-x_n, \end{cases} \qquad \begin{cases} x'_1=x_1, \\ \vdots \\ x'_i=x_n, \\ \vdots \\ x'_n=x_i, \end{cases}$$

we may assume that $b_n=0$. Note that $(b_1,\dots,b_{n-1})\in\Delta_{n-1}\setminus\partial(\Delta_{n-1})$. Let $p:\mathbb{R}^n\to\mathbb{R}^{n-1}$ be the projection given by $p(x_1,\dots,x_n)=(x_1,\dots,x_{n-1})$. By Lemma 5.2, there are a concave function θ on $p(\Theta_{\boldsymbol{a}})$ and a convex function ϑ on $p(\Theta_{\boldsymbol{a}})$ such that

$$\Theta_{\boldsymbol{a}}=\left\{(x_1,\dots,x_{n-1},x_n)\;\middle|\;\begin{array}{l}(x_1,\dots,x_{n-1})\in p(\Theta_{\boldsymbol{a}}),\\ \vartheta(x_1,\dots,x_{n-1})\le x_n\le\theta(x_1,\dots,x_{n-1})\end{array}\right\}.$$

CLAIM 5.4.1

The point $(b_1,\dots,b_{n-1})$ is an interior point of $p(\Theta_{\boldsymbol{a}})$. In particular, ϑ is continuous around $(b_1,\dots,b_{n-1})$ (see [6, Theorem 2.2]).

Proof

Let us consider a function $\psi:[0,1-b_1-\cdots-b_{n-1}]\to\mathbb{R}$ given by $\psi(t)=\phi_{\boldsymbol{a}}(b_1,\dots,b_{n-1},t)$. Note that

$$\psi'(t)=\log\frac{a_n}{a_0}\Big(\frac{1-b_1-\cdots-b_{n-1}}{t}-1\Big)>0$$

on $\big(0,(a_n(1-b_1-\cdots-b_{n-1}))/(a_0+a_n)\big)$. Thus

$$\phi_{\boldsymbol{a}}\Big(b_1,\dots,b_{n-1},\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}\Big)>\phi_{\boldsymbol{a}}(b_1,\dots,b_{n-1},0)\ge 0.$$

Therefore, as $(b_1,\dots,b_{n-1},(a_n(1-b_1-\cdots-b_{n-1}))/(a_0+a_n))\in\Delta_n\setminus\partial(\Delta_n)$, we can find a sufficiently small positive number ϵ such that

$$\prod_{i=1}^{n-1}(b_i-\epsilon,b_i+\epsilon)\times\Big(\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}-\epsilon,\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}+\epsilon\Big)$$

is a subset of $\Theta_{\boldsymbol{a}}$, and hence

$$(b_1,\dots,b_{n-1})\in\prod_{i=1}^{n-1}(b_i-\epsilon,b_i+\epsilon)\subseteq p(\Theta_{\boldsymbol{a}}).$$

□

We set $\boldsymbol{a}' = (a_0, \ldots, a_{n-1})$. Then

$$\Theta_{\boldsymbol{a}'} = \left\{ (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \;\middle|\; (x_1, \ldots, x_{n-1}, 0) \in \Theta_{\boldsymbol{a}} \right\}.$$

Clearly, $(b_1, \ldots, b_{n-1}) \in \Theta_{\boldsymbol{a}'}$ and $\vartheta \equiv 0$ on $\Theta_{\boldsymbol{a}'}$.

CLAIM 5.4.2

ϑ is a continuously differentiable function around $(b_1, \ldots, b_{n-1})$ such that

$$\vartheta_{x_1}(b_1, \ldots, b_{n-1}) = \cdots = \vartheta_{x_{n-1}}(b_1, \ldots, b_{n-1}) = 0.$$

Proof

By Claim 5.4.1, there is a positive number ϵ such that

$$b_1 - \epsilon > 0, \ldots, b_{n-1} - \epsilon > 0, \qquad (b_1 + \epsilon) + \cdots + (b_{n-1} + \epsilon) < 1,$$

and ϑ is continuous on $U = \prod_{i=1}^{n-1} (b_i - \epsilon, b_i + \epsilon)$. If $(x_1, \ldots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$, then $\vartheta(x_1, \ldots, x_{n-1}) > 0$, and hence

$$\phi_{\boldsymbol{a}}\big(x_1, \ldots, x_{n-1}, \vartheta(x_1, \ldots, x_{n-1})\big) = 0$$

for $(x_1, \ldots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$. Note that

$$(\phi_{\boldsymbol{a}})_{x_i} = \log \frac{a_i}{a_0} \Big(\frac{1 - x_1 - \cdots - x_n}{x_i} \Big). \tag{1}$$

Since $\vartheta(b_1, \ldots, b_{n-1}) = 0$ and ϑ is continuous at $(b_1, \ldots, b_{n-1})$, choosing a smaller ϵ if necessary, we may assume that

$$(\phi_{\boldsymbol{a}})_{x_n}\big(x_1, \ldots, x_{n-1}, \vartheta(x_1, \ldots, x_{n-1})\big) > 0$$

for all $(x_1, \ldots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$. Thus, by using the implicit function theorem, ϑ is a C^∞-function on $U \setminus \Theta_{\boldsymbol{a}'}$ and

$$\vartheta_{x_i}(x_1, \ldots, x_{n-1}) = - \frac{(\phi_{\boldsymbol{a}})_{x_i}(x_1, \ldots, x_{n-1}, \vartheta(x_1, \ldots, x_{n-1}))}{(\phi_{\boldsymbol{a}})_{x_n}(x_1, \ldots, x_{n-1}, \vartheta(x_1, \ldots, x_{n-1}))}. \tag{2}$$

Let us consider a function γ_i on U given by

$$\gamma_i(x_1, \ldots, x_{n-1}) = \begin{cases} 0 & \text{if } (x_1, \ldots, x_{n-1}) \in U \cap \Theta_{\boldsymbol{a}'}, \\ \vartheta_{x_i}(x_1, \ldots, x_{n-1}) & \text{if } (x_1, \ldots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}. \end{cases}$$

Then, by using (1) and (2), it is easy to see that γ_i is continuous on U. Thus the claim follows. $\square$

Claim 5.4.2 shows that $\Theta_{\boldsymbol{a}}$ has the unique supporting hyperplane at $(b_1, \ldots, b_n)$ and that it is given by $x_n = 0$. This proves Proposition 5.4. $\square$

COROLLARY 5.5

We assume that $a_0 < 1$ and $a_0 + a_1 + \cdots + a_n \geq 1$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{>0}$ and $(b_1, \ldots, b_n) \in \Theta_{\boldsymbol{a}}$ be such that

$$\alpha_1 b_1 + \cdots + \alpha_n b_n = \min\left\{ \alpha_1 x_1 + \cdots + \alpha_n x_n \;\middle|\; (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}} \right\}.$$

Then $(b_1, \ldots, b_n) \notin \partial(\Delta_n)$.

Proof
We prove it by induction on n. If $n=1$, then the assertion is obvious, so that we may assume that $n>1$. If $a_0+\cdots+a_n=1$, then

$$\Theta_{\boldsymbol{a}}=\Big\{\Big(\frac{a_1}{a_0+\cdots+a_n},\dots,\frac{a_n}{a_0+\cdots+a_n}\Big)\Big\}.$$

In this case, the assertion is also obvious. Thus we may assume that $a_0+\cdots+a_n>1$.

We assume that $b_i=0$ for some $1\le i\le n$. Then, since $\Theta_{\boldsymbol{a}}\cap\{x_i=0\}\neq\emptyset$, we have

$$a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n\ge 1.$$

Thus by the hypothesis of induction,

$$b_1\neq 0,\dots,b_{i-1}\neq 0,\qquad b_{i+1}\neq 0,\dots,b_n\neq 0,\qquad b_1+\cdots+b_n\neq 1.$$

Therefore, by Proposition 5.4, we have the unique supporting hyperplane $x_i=0$ of $\Theta_{\boldsymbol{a}}$ at $(b_1,\dots,b_n)$. On the other hand, $\alpha_1x_1+\cdots+\alpha_nx_n=\alpha_1b_1+\cdots+\alpha_nb_n$ is also a supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1,\dots,b_n)$. This is a contradiction.

Next, we assume that $b_1+\cdots+b_n=1$. Since $b_i\neq 0$ for all i, by Proposition 5.4, the unique supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1,\dots,b_n)$ is $x_1+\cdots+x_n=1$, which yields $\alpha_1=\cdots=\alpha_n$, and hence $\Theta_{\boldsymbol{a}}\subseteq\{x_1+\cdots+x_n=1\}$. This is a contradiction because

$$\Big(\frac{a_1}{a_0+\cdots+a_n},\dots,\frac{a_n}{a_0+\cdots+a_n}\Big)\in\Theta_{\boldsymbol{a}},$$

as required. □

THEOREM 5.6
We assume that $n\ge 2$ and $\overline{D}_{\boldsymbol{a}}$ is big. Then $\overline{D}_{\boldsymbol{a}}$ is nef if and only if there is a birational morphism $f:X\to\mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth, and normal arithmetic varieties such that $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition on X.

Proof
If $\overline{D}_{\boldsymbol{a}}$ is nef, then $\overline{D}_{\boldsymbol{a}}=\overline{D}_{\boldsymbol{a}}+(0,0)$ is a weak Zariski decomposition. Next, we assume that $\overline{D}_{\boldsymbol{a}}$ is not nef and there is a birational morphism $f:X\to\mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth, and normal arithmetic varieties such that $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition $f^*(\overline{D}_{\boldsymbol{a}})=\overline{P}+\overline{N}$ on X. By our assumptions, $a_0+\cdots+a_n>1$ and $a_i<1$ for some i. Renumbering the homogeneous coordinate $T_0,\dots,T_n$, we may assume that $a_0<1$. Let ξ be the generic point of $H_1\cap\cdots\cap H_n$; that is, $\xi=(1:0:\cdots:0)\in\mathbb{P}^n(\mathbb{Q})$. Let L_i be the strict transform of H_i by f for $i=0,\dots,n$. We denote the birational morphism $X_{\mathbb{Q}}\to\mathbb{P}^n_{\mathbb{Q}}$ by $f_{\mathbb{Q}}$. Let $f':X'\to\mathbb{P}^n_{\mathbb{Z}}$ be the blowup along $H_1\cap\cdots\cap H_n$. By using Lemma 5.1 and [8], we may assume the following.

(1) Let Σ be the exceptional set of $f_{\mathbb{Q}}:X_{\mathbb{Q}}\to\mathbb{P}^n_{\mathbb{Q}}$. Then Σ is a divisor on $X_{\mathbb{Q}}$ and $(\Sigma+(L_0)_{\mathbb{Q}}+\cdots+(L_n)_{\mathbb{Q}})_{\mathrm{red}}$ is a normal crossing divisor on $X_{\mathbb{Q}}$.

(2) There is a birational morphism $g : X \to X'$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "g"] & \\ & X' \arrow[dl, "f'"] \\ \mathbb{P}^n_{\mathbb{Z}} & \end{tikzcd}$$

CLAIM 5.6.1

There are $\xi' \in X(\mathbb{Q})$ and a reduced and irreducible divisor E on $X_{\mathbb{Q}}$ with the following properties:

(a) $f_{\mathbb{Q}}(\xi') = \xi$ *and* $\xi' \in E \cap (L_n)_{\mathbb{Q}}$*;*
(b) E *and* $(L_n)_{\mathbb{Q}}$ *are nonsingular at* ξ'*;*
(c) E *is exceptional with respect to* $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$*;*
(d) *there are positive integers* $\alpha_1, \ldots, \alpha_n$ *such that*

$$f^*_{\mathbb{Q}}(H_i) = \alpha_i E + (\textit{the sum of divisors which do not pass through } \xi')$$

for $i = 1, \ldots, n-1$ *and*

$$f^*_{\mathbb{Q}}(H_n) = (L_n)_{\mathbb{Q}} + \alpha_n E + (\textit{the sum of divisors which do not pass through } \xi').$$

Proof

Let L'_n be the strict transform of H_n by f', and let Σ' be the exceptional set of $f'_{\mathbb{Q}} : X'_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$. Then $\Sigma' = \mathbb{P}^{n-1}_{\mathbb{Q}}$ and $D' := (L'_n)_{\mathbb{Q}} \cap \Sigma' = \mathbb{P}^{n-2}_{\mathbb{Q}}$. Let $h : L_n \to L'_n$ and $h_{\mathbb{Q}} : (L_n)_{\mathbb{Q}} \to (L'_n)_{\mathbb{Q}}$ be the birational morphisms induced by $g : X \to X'$ and $g_{\mathbb{Q}} : X_{\mathbb{Q}} \to X'_{\mathbb{Q}}$, respectively. Let D be the strict transformation of D' by $h_{\mathbb{Q}}$. As before, let Σ be the exceptional set of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$. Let

$$\big(\Sigma + (L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}}\big)_{\mathrm{red}} = (L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}} + E_0 + \cdots + E_l$$

be the irreducible decomposition such that the E_i's are exceptional with respect to $f_{\mathbb{Q}}$. Since $D \subseteq (L_n)_{\mathbb{Q}} \cap \Sigma$, there is E_i such that $D \subseteq (L_n)_{\mathbb{Q}} \cap E_i$. Renumbering $E_0, \ldots, E_l$, we may assume that $E_i = E_l$. As $(L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}} + E_0 + \cdots + E_l$ is a normal crossing divisor on $X_{\mathbb{Q}}$, we have

$$\begin{cases} D \cap \mathrm{Sing}((L_n)_{\mathbb{Q}}) \subsetneq D, D \cap \mathrm{Sing}(E) \subsetneq D, \\ D \cap (L_i)_{\mathbb{Q}} \subsetneq D \quad (i = 0, \ldots, n-1), \\ D \cap E_j \subsetneq D \qquad (j = 0, \ldots, l-1). \end{cases}$$

Note that $D(\mathbb{Q})$ is dense in D because $D \to D'$ is birational. Thus we can find $\xi' \in D(\mathbb{Q})$ such that

$$\xi' \notin \big(D \cap \mathrm{Sing}((L_n)_{\mathbb{Q}})\big) \cup \big(D \cap \mathrm{Sing}(E)\big) \cup \bigcup_{i=0}^{n-1} \big(D \cap (L_i)_{\mathbb{Q}}\big) \cup \bigcup_{j=0}^{l-1} (D \cap E_j).$$

Therefore the claim follows. □

Note that

$$\begin{aligned} f_{\mathbb{Q}}^*\big(lH_0 + (z_1^{e_1} \cdots z_n^{e_n})\big) &= f_{\mathbb{Q}}^*\big((l - e_1 - \cdots - e_n)H_0 + e_1 H_1 + \cdots + e_n H_n\big) \\ &= e_n (L_n)_{\mathbb{Q}} + (\alpha_1 e_1 + \cdots + \alpha_n e_n) E \\ &\qquad + (\text{the sum of divisors which do not pass through } \xi'). \end{aligned}$$

Therefore, by Lemma 3.2,

$$\begin{cases} \mu_{\xi'}(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + (\alpha_n + 1) x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ \mu_E(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ \mu_{L_n}(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}. \end{cases}$$

Further, as $0 \leq \mathrm{mult}_\Gamma(N) = \mu_\Gamma(f^*(\overline{D}_{\boldsymbol{a}})) = 0$ for any horizontal prime divisor Γ with $\Gamma \not\subseteq \mathrm{Supp}(f^*(H_0 + \cdots + H_n))$ we have

$$\mathrm{mult}_{\xi'}(N) = \mathrm{mult}_E(N) + \mathrm{mult}_{L_n}(N) \leq \mu_E\big(f^*(\overline{D}_{\boldsymbol{a}})\big) + \mu_{L_n}\big(f^*(\overline{D}_{\boldsymbol{a}})\big).$$

By Proposition 3.1(2) and (5),

$$0 = \mu_{\xi'}(\overline{P}) \geq \mu_{\xi'}\big(f^*(\overline{D}_{\boldsymbol{a}})\big) - \mathrm{mult}_{\xi'}(N).$$

Therefore, if we set

$$\begin{cases} A = \min\{\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + (\alpha_n + 1) x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ B = \min\{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ C = \min\{x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \end{cases}$$

then we have $0 \geq A - B - C$. We choose $(b_1, \ldots, b_n) \in \Theta_{\boldsymbol{a}}$ such that

$$A = \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1} + (\alpha_n + 1) b_n.$$

Thus, as $\alpha_1 b_1 + \cdots + \alpha_n b_n \geq B$ and $b_n \geq C$, we have

$$\begin{aligned} 0 &\geq A - B - C \\ &\geq \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1} + (\alpha_n + 1) b_n - (\alpha_1 b_1 + \cdots + \alpha_n b_n) - b_n = 0, \end{aligned}$$

which implies that $\alpha_1 b_1 + \cdots + \alpha_n b_n = B$ and $b_n = C$. On the other hand, by Corollary 5.5, $(b_1, \ldots, b_n) \notin \partial(\Delta_n)$, and hence there is a unique supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$ by Proposition 5.4. This is a contradiction because

$$\begin{cases} \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + (\alpha_n + 1) x_n = A, \\ \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \alpha_n x_n = B, \\ x_n = C \end{cases}$$

are distinct supporting hyperplanes of $\Theta_{\boldsymbol{a}}$ at $(b_1,\dots,b_n)$. This proves Theorem 5.6. $\square$

6. Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$

Fujita's approximation of arithmetic divisors was established by Chen and Yuan (see [3], [11], [9], [10]). In this section, we consider Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$ in terms of rational interior points of $\Theta_{\boldsymbol{a}}$.

First of all, we fix notation. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\in\mathbb{R}^n$ and $\phi_1,\dots,\phi_r\in\mathbb{R}$. We define a function $\phi_{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)}$ on $\Theta=\operatorname{Conv}\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\}$ to be

$$\phi_{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)}(\boldsymbol{x}):=\max\left\{\sum_{i=1}^r\lambda_i\phi_i\;\middle|\;\begin{array}{l}\boldsymbol{x}=\sum_{i=1}^r\lambda_i\boldsymbol{x}_i,\\ \lambda_1,\dots,\lambda_r\in\mathbb{R}_{\geq 0},\sum_{i=1}^r\lambda_i=1\end{array}\right\}.$$

In other words, $\phi_{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)}$ is given by

$$\begin{aligned}&\phi_{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)}(\boldsymbol{x})\\ &\quad=\max\bigl\{\phi\in\mathbb{R}\bigm|(\boldsymbol{x},\phi)\in\operatorname{Conv}\{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)\}\subseteq\mathbb{R}^n\times\mathbb{R}\bigr\}.\end{aligned}$$

Thus we can easily see that $\phi_{(\boldsymbol{x}_1,\phi_1),\dots,(\boldsymbol{x}_r,\phi_r)}$ is a continuous function on Θ (see [4]).

Let φ be a continuous concave function on Θ. Clearly, $\phi_{(\boldsymbol{x}_1,\varphi(\boldsymbol{x}_1)),\dots,(\boldsymbol{x}_r,\varphi(\boldsymbol{x}_r))}\leq\varphi$. Moreover, for a positive number ϵ, if we add sufficiently many points $\boldsymbol{x}_{r+1},\dots,\boldsymbol{x}_m\in\Theta$ to $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\}$, then

$$\varphi-\epsilon\leq\phi_{(\boldsymbol{x}_1,\varphi(\boldsymbol{x}_1)),\dots,(\boldsymbol{x}_r,\varphi(\boldsymbol{x}_r)),(\boldsymbol{x}_{r+1},\varphi(\boldsymbol{x}_{r+1})),\dots,(\boldsymbol{x}_m,\varphi(\boldsymbol{x}_m))}\leq\varphi.$$

From now on, we use the same notation as in Section 1. We assume that $\overline{D}_{\boldsymbol{a}}$ is big.

CLAIM 6.1

For a given positive number ϵ, we can find rational interior points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r$ of $\Theta_{\boldsymbol{a}}$, that is, $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\in\operatorname{Int}(\Theta_{\boldsymbol{a}})\cap\mathbb{Q}^n$ such that

$$\frac{(n+1)!}{2}\int_\Theta\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\,d\boldsymbol{x}>\widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}})-\epsilon,$$

where $\Theta=\operatorname{Conv}\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\}$.

Proof

First of all, we can find $\boldsymbol{x}_1,\dots,\boldsymbol{x}_{r'}\in\operatorname{Int}(\Theta_{\boldsymbol{a}})\cap\mathbb{Q}^n$ such that

$$\frac{(n+1)!}{2}\int_\Theta\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}})\,d\boldsymbol{x}>\widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}})-\epsilon,$$

where $\Theta=\operatorname{Conv}\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{r'}\}$. Thus, adding more points $\boldsymbol{x}_{r'+1},\dots,\boldsymbol{x}_r\in\Theta\cap\mathbb{Q}^n$ to $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{r'}\}$, we have

$$\frac{(n+1)!}{2}\int_\Theta\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\,d\boldsymbol{x}>\widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}})-\epsilon.\qquad\square$$

We choose a sufficiently small positive number δ such that

(a) $\Theta \subseteq \Theta_{e^{-\delta}\boldsymbol{a}}$ and

(b) $(n+1)!/2 \int_{\Theta} \phi_{(\boldsymbol{x}_1, \varphi_{e^{-\delta}\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r, \varphi_{e^{-\delta}\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\, d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon$.

We set $\boldsymbol{a}' = e^{-\delta}\boldsymbol{a}$. By virtue of [10, Theorem 3.2.3], we can find positive integer l_0 such that

(c) $\log \mathrm{dist}(H^0(lH_0) \otimes \mathbb{C}; l_0 g_{\boldsymbol{a}'}) \le l_0 \delta$ and

(d) $l_0 \boldsymbol{x}_1, \dots, l_0 \boldsymbol{x}_r \in \mathbb{Z}^n_{\ge 0}$.

Let us consider the following $\mathbb{Z}$-module:

$$V := \bigoplus_{i=1}^{r} \mathbb{Z} z^{l_0 \boldsymbol{x}_i} \subseteq H^0(\mathbb{P}^n_{\mathbb{Z}}, l_0 H_0).$$

Then we have a birational morphism $\mu : Y \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth, and normal arithmetic varieties such that the image of

$$V \otimes_{\mathbb{Z}} \mathcal{O}_Y \to \mathcal{O}_Y\big(\mu^*(l_0 H_0)\big)$$

is invertible; that is, there is an effective Cartier divisor F on Y such that

$$V \otimes_{\mathbb{Z}} \mathcal{O}_Y \to \mathcal{O}_Y\big(\mu^*(l_0 H_0) - F\big)$$

is surjective. Here we set

$$\begin{cases} Q := \mu^*(l_0 H_0) - F, \\ g_F := \mu^*(-\log \mathrm{dist}(V \otimes \mathbb{C}; l_0 g_{\boldsymbol{a}'}) + l_0 \delta), \\ g_Q := \mu^*(l_0 g_{\boldsymbol{a}'} + \log \mathrm{dist}(V \otimes \mathbb{C}; l_0 g_{\boldsymbol{a}'})). \end{cases}$$

CLAIM 6.2

(i) *We have $g_Q + g_F = \mu^*(l_0 g_{\boldsymbol{a}})$.*

(ii) *The function g_Q is a Q-Green function of $(C^\infty \cap \mathrm{PSH})$-type, and $\overline{Q} := (Q, g_Q)$ is nef.*

(iii) *The function g_F is an F-Green function of C^∞-type and $g_F \ge 0$.*

(iv) *If we set $\overline{P} = (P, g_P) = (1/l_0)\overline{Q}$, then, for $\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n$, $\mu^*(z^{\boldsymbol{e}}) \in H^0(lP)$ and*

$$|\mu^*(z^{\boldsymbol{e}})|^2_{l g_P} \le \exp\big(-l \phi_{(\boldsymbol{x}_1, \varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r, \varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{e}/l)\big).$$

Proof

(i) is obvious. (ii) is a consequence of Lemma 6.1 below. The first assertion of (iii) follows from (i) and (ii), and the second follows from the above condition (c).

(iv) Let us consider arbitrary $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that $\boldsymbol{e}/l = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_r \boldsymbol{x}_r$ and $\lambda_1 + \cdots + \lambda_r = 1$. Then, since $Q + (\mu^*(z^{l_0 \boldsymbol{x}_i})) \ge 0$ for all i,

$$\begin{aligned} lP + \big(\mu^*(z^{\boldsymbol{e}})\big) &= (l/l_0)Q + \sum_{i=1}^{r} \lambda_i (l/l_0)\big(\mu^*(z^{l_0 \boldsymbol{x}_i})\big) \\ &= \sum_{i=1}^{r} \lambda_i (l/l_0)\big(Q + (\mu^*(z^{l_0 \boldsymbol{x}_i}))\big) \ge 0, \end{aligned}$$

and hence $\mu^*(z^{\boldsymbol{e}}) \in H^0(lP)$. Moreover, by using [10, Proposition 3.2.1] and Proposition 1.3,

$$
\begin{aligned}
|\mu^*(z^{\boldsymbol{e}})|^2_{lg_P} &= |\mu^*(z^{\boldsymbol{e}})|^2 \exp\bigl(-(l/l_0)g_Q\bigr) \\
&= \prod_{i=1}^{r} \bigl(|\mu^*(z^{l_0\boldsymbol{x}_i})|^2\bigr)^{\lambda_i(l/l_0)} \frac{\exp(-l\mu^*(g_{\boldsymbol{a}'}))}{\mu^*(\operatorname{dist}(V\otimes\mathbb{C}; l_0 g_{\boldsymbol{a}'}))^{l/l_0}} \\
&= \prod_{i=1}^{r} \mu^*\Bigl(\frac{|z^{l_0\boldsymbol{x}_i}|^2_{l_0 g_{\boldsymbol{a}'}}}{\operatorname{dist}(V\otimes\mathbb{C}; l_0 g_{\boldsymbol{a}'})}\Bigr)^{\lambda_i(l/l_0)} \le \prod_{i=1}^{r} \bigl(\|z^{l_0\boldsymbol{x}_i}\|^2_{l_0 g_{\boldsymbol{a}'}}\bigr)^{\lambda_i(l/l_0)} \\
&= \prod_{i=1}^{r} \exp\bigl(-l_0\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_i)\bigr)^{\lambda_i(l/l_0)} = \exp\Bigl(-l\sum_{i=1}^{r}\lambda_i\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_i)\Bigr).
\end{aligned}
$$

Thus (iv) follows. □

LEMMA 6.1

Let $\mu : Y \to X$ be a birational morphism of projective, generically smooth, and normal arithmetic varieties. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X, and let S be a subset of $\hat{H}^0(X, \overline{D})$. We assume that there is an effective $\mathbb{R}$-divisor E on Y with the following properties:

(1) *$\mu^*(D) - E \in \operatorname{Div}(Y)$; that is, $\mu^*(D) - E$ is a Cartier divisor;*
(2) *$\mu^*(s) \in H^0(Y, \mu^*(D) - E)$ for all $s \in S$ and*

$$\bigcap_{s\in S} \operatorname{Supp}\bigl(\mu^*(D) - E + (\mu^*(s))\bigr) = \emptyset.$$

We set

$$M := \mu^*(D) - E \qquad \text{and} \qquad g_M := \mu^*\bigl(g + \log \operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g)\bigr).$$

Then g_M is an M-Green function of $(C^\infty \cap \mathrm{PSH})$-type and (M, g_M) is nef.

Proof

Let $e_1, \ldots, e_N$ be an orthonormal basis of $\langle S\rangle_{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle_g$. We fix $y \in Y(\mathbb{C})$. Let f be a local equation of $\mu^*(D) - E$ around y. We set $s_j = \mu^*(e_j)f$ for $j = 1, \ldots, N$. Then $s_1, \ldots, s_N$ are holomorphic around y and $s_j(y) \neq 0$ for some j. On the other hand,

$$g_M = \log\Bigl(\sum_{j=1}^{N} |\mu^*(e_j)|^2\Bigr) = -\log|f|^2 + \log\Bigl(\sum_{j=1}^{N} |s_j|^2\Bigr)$$

around y. Thus g_M is an M-Green function of $(C^\infty \cap \mathrm{PSH})$-type. By virtue of [10, Proposition 3.1], we have

$$|s|_g^2 \le \langle s, s\rangle_g \operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g) \le \operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g),$$

which yields $\mu^*(s) \in \hat{H}^0(Y, \overline{M})$ for all $s \in S$. Let C be a 1-dimensional closed integral subscheme on Y. Then there is $s \in S$ such that $C \not\subseteq \operatorname{Supp}\bigl(M + (\mu^*(s))\bigr)$. Thus $\widehat{\deg}((M, g_M)|_C) \ge 0$. □

Finally, let us show that $\widehat{\mathrm{vol}}(\overline{P}) > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon$. We fix an F_∞-invariant volume form Φ on Y with $\int_{Y(\mathbb{C})} \Phi = 1$. Using Φ and lg_P, we can give the inner product $\langle\, ,\rangle_{lg_P}$ on $H^0(lP)$. Then, by Claim 6.2(iv),

$$\langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}})\rangle_{lg_P} \le \exp\left(-l\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}\left(\frac{\boldsymbol{e}}{l}\right)\right).$$

Here we consider positive definite symmetric real matrices $A_l = (a_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ and $A'_l = (a'_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ given by

$$a_{\boldsymbol{e},\boldsymbol{e}'} = \langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}'})\rangle_{lg_P}$$

and

$$a'_{\boldsymbol{e},\boldsymbol{e}'} = \begin{cases} \exp(-l\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{e}/l)) & \text{if } \boldsymbol{e} = \boldsymbol{e}', \\ \langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}'})\rangle_{lg_P} & \text{if } \boldsymbol{e} \neq \boldsymbol{e}'. \end{cases}$$

Then, since

$$\sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \le \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'},$$

we have

$$\begin{aligned} \#\hat{H}^0_{L^2}(l\overline{P}) &\ge \#\Big\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \;\Big|\; \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \le 1\Big\} \\ &\ge \#\Big\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \;\Big|\; \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \le 1\Big\}. \end{aligned}$$

On the other hand, by Lemma 2.2,

$$\begin{aligned} &\liminf_{l\to\infty} \frac{\log\#\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \mid \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \le 1\}}{l^{n+1}/(n+1)!} \\ &\quad \ge \frac{(n+1)!}{2}\int_\Theta \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\dots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})\,d\boldsymbol{x}, \end{aligned}$$

and hence $\widehat{\mathrm{vol}}(\overline{P}) > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon$ by Lemma 2.1 and (b) in the proof of Claim 6.1.

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