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CARLESON MEASURES FOR THE GENERALIZED SCHRÖDINGER OPERATOR

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ABSTRACT. Let $\mathcal{L} = -\Delta + \mu$ be the generalized Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where Δ is the Laplacian and $\mu \not\equiv 0$ is a nonnegative Radon measure on $\mathbb{R}^n$. In this article, we give a characterization of $\mathrm{BMO}_{\mathcal{L}}$ in terms of Carleson measures, where $\mathrm{BMO}_{\mathcal{L}}$ is the BMO-type space associated with the generalized Schrödinger operator.

1. Introduction and preliminaries

Let $\mathcal{L} = -\Delta+\mu$ be the generalized Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where Δ is the Laplacian and $\mu \not\equiv 0$ is a nonnegative Radon measure on $\mathbb{R}^n$. We note that only a handful of authors have studied the harmonic analysis problems related to the generalized Schrödinger operator. Research of the generalized Schrödinger operator was motivated by Christ [1]. After that, Shen [5] established the bounds for the fundamental solution of $-\Delta + \mu$ in $\mathbb{R}^n$ and studied the boundedness of the corresponding Riesz transform $\nabla(-\Delta + \mu)^{-1/2}$ on $L^p(\mathbb{R}^n)$. Sun [7] proved a uniform Harnack inequality for nonnegative solutions of $-\Delta_G u + \mu u = 0$, where $-\Delta_G$ is a sub-Laplacian on the stratified Lie group. Moreover, Wu and Yan [8] recently studied the Hardy space $H^1_{\mathcal{L}}$ by means of a maximal function associated with the heat semigroup $e^{-t\mathcal{L}}$ generated by $\mathcal{L}$, and they obtained its characterizations via atomic decomposition and Riesz transforms. They also investigated the $\mathrm{BMO}_{\mathcal{L}}$ space, which is the dual space of $H^1_{\mathcal{L}}$. As a continuation of [8], we

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will characterize $\mathrm{BMO}_{\mathcal{L}}$ in terms of Carleson measures. This problem has been investigated in [2] and [4] for the case of the Schrödinger operator.

As in [5] and [8], we will assume throughout this article that μ satisfies the following conditions. There exist positive constants C_0, C_1, and δ such that

$$\mu\big(B(x,r)\big) \le C_0\Big(\frac{r}{R}\Big)^{n-2+\delta}\mu\big(B(x,R)\big) \tag{1.1}$$

and

$$\mu\big(B(x,r)\big) \le C_1\big\{\mu\big(B(x,r)\big)+r^{n-2}\big\} \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$, where $B(x,r)$ denotes the (open) ball centered at x with radius r. Shen [5] has proved that condition (1.1) is equivalent to the condition

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{n-2}} \le C\frac{\mu(B(x,R))}{R^{n-2}}.$$

Condition (1.1) may be regarded as the scale-invariant Kato condition, and condition (1.2) says that the measure μ is doubling on balls satisfying $\mu(B(x,r)) \ge cr^{n-2}$. As pointed out in [5], when $d\mu = V(x)\,dx$ and $V(x) \ge 0$ is in the reverse Hölder class $(\mathrm{RH})_{n/2}$, that is,

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^{n/2}\,dy\Big)^{2/n} \le C\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy\Big),$$

then μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$. However, in general, measures which satisfy (1.1) and (1.2) need not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. For instance, if $d\mu = d\sigma(x_1, x_2)\,dx_3\cdots dx_n$, where σ is a doubling measure on $\mathbb{R}^2$, then μ satisfies (1.1) and (1.2) for some $\delta > 0$.

To state our results, we recall the following definition of the auxiliary function $m(x,\mu)$ (see [5, p. 522]):

$$\frac{1}{m(x,\mu)} = \sup\Big\{r > 0 : \frac{\mu(B(x,r))}{r^{n-2}} \le C_1\Big\},$$

where C_1 is the constant in (1.2). With the modified Agmon metric

$$ds^2 = m(x,\mu)\{dx_1^2 + \cdots + dx_1^n\},$$

we define the distance function

$$d(x,y,\mu) = \inf_{\gamma}\int_0^1 m\big(\gamma(t),\mu\big)\big|\gamma'(t)\big|\,dt,$$

where $\gamma : [0,1] \to \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = x$, $\gamma(1) = y$.

We next recall some basic facts regarding Hardy and BMO spaces associated with the generalized Schrödinger operator $\mathcal{L}$, which has been studied by Wu and Yan [8]. The Hardy space was introduced in [3], where $d\mu = V(x)\,dx$ and $V \in (\mathrm{RH})_{n/2}$. Since μ is nonnegative on $\mathbb{R}^n$, the Feynman–Kac formula implies that the kernel $\mathcal{K}_t(x,y)$ of the semigroup

$$T_t^{\mathcal{L}}f(x) = e^{-t\mathcal{L}}f(x) = \int_{\mathbb{R}^n}\mathcal{K}_t(x,y)f(y)\,dy$$

has a Gaussian upper bound. We will use the notational conventions

$$M^{\mathcal{L}} f(x) = \sup_{t>0} \big| T_t^{\mathcal{L}} f(x) \big|,$$

$$s_{\mathcal{Q}} f(x) = \Big(\int_0^\infty \big| \mathcal{Q}_t f(x) \big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}},$$

which correspond to the Hardy–Littlewood maximal function and the $\mathcal{L}$-square function, respectively.

Definition 1.1. A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}}$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by $\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^1(\mathbb{R}^n)}$.

Definition 1.2. Let $1 \le q \le \infty$. A function $a \in L^q(\mathbb{R}^n)$ is called an $H^1_{\mathcal{L}}$*-atom* if $r < \frac{4}{m(x_0,\mu)}$ and the following conditions hold:

(1) $\operatorname{supp} a \subset B(x_0, r)$;
(2) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(x_0,r)|^{\frac{1}{q}-1}$;
(3) if $r < \frac{1}{m(x_0,\mu)}$, then $\int_{B(x_0,r)} a(x)\, dx = 0$.

Wu and Yan [8, Theorem 1.2] gave the following atomic decomposition for the space $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Proposition 1.3. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \ge 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where the a_j's are $H^{1,\infty}_{\mathcal{L}}(\mathbb{R}^n)$-atoms, $\sum_j |\lambda_j| < \infty$, and the sum converges in the $H^1_{\mathcal{L}}(\mathbb{R}^n)$ quasinorm. Moreover,*

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf \Big\{ \sum_j |\lambda_j| \Big\},$$

where the infimum is taken over all atomic decompositions of f into $H^{1,\infty}_{\mathcal{L}}$-atoms.

The dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ (cf. [8]). Let f be a locally integrable function on $\mathbb{R}^n$, and let $B = B(x,r)$ be a ball. Set

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

and

$$f(B,\mu) = \begin{cases} f_B & \text{if } r < m(x,\mu)^{-1}, \\ 0 & \text{if } r \ge m(x,\mu)^{-1}. \end{cases}$$

Definition 1.4. Let f be a locally integrable function on $\mathbb{R}^n$. We say that $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}_{\mathcal{L}}} \triangleq \sup_B \frac{1}{|B|} \int_B \big| f(y) - f(B,\mu) \big|\, dy < \infty.$$

Remark 1.5. We can easily get the fact that $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ and $\|f\|_{\mathrm{BMO}} \le C \|f\|_{\mathrm{BMO}_{\mathcal{L}}}$. By a simple deduction, we obtain

$$\sup_B\Big(\frac{1}{|B|}\int_B\big|f(y)-f(B,\mu)\big|^p\,dy\Big)^{\frac{1}{p}}\le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}}.$$

Definition 1.6. A positive measure μ on $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times(0,\infty)$ is said to be a *Carleson measure* if

$$\|\mu\|\triangleq\sup_{x\in\mathbb{R}^n,r>0}\frac{\mu(B(x,r)\times(0,r))}{|B(x,r)|}<\infty.$$

Let $(\mathcal{Q}_tf)(x)=t^2(\frac{dT_s^{\mathcal{L}}}{ds}|_{s=t^2}f)(x),(x,t)\in\mathbb{R}^{n+1}_+$. Then our result is given as follows.

Theorem 1.7. *Suppose that μ satisfies (1.1) and (1.2) for some $\delta>0$. Then we have the following.*

(1) *If $f\in\mathrm{BMO}_{\mathcal{L}}$, then $d\mu_f(x,t)\triangleq|\mathcal{Q}_tf(x)|^2\,dx\,dt/t$ is a Carleson measure.*
(2) *Conversely, if $f\in L^1((1+|x|)^{-(n+1)}\,dx)$ and $d\mu_f(x,t)$ is a Carleson measure, then $f\in\mathrm{BMO}_{\mathcal{L}}$.*

Moreover, in either case, there exists $C>0$ such that

$$\frac{1}{C}\|f\|^2_{\mathrm{BMO}_C}\le\|d\mu_f\|\le C\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.$$

Throughout the article, the letters c and C denote (possibly different) constants that are independent of the essential variables. By $A\sim B$, we mean that there exists a positive constant C such that $\frac{1}{C}\le\frac{A}{B}\le C$. By $\mathsf{U}\lesssim\mathsf{V}$, we mean that there is a constant $C>0$ such that $\mathsf{U}\le C\mathsf{V}$. Given a ball B, we denote by B^* the ball with same center and twice the radius.

2. Estimates for kernels

We begin by recalling some basic properties of the semigroup kernel $\mathcal{K}_t(x,y)$ associated with $T_t^{\mathcal{L}}=e^{-t\mathcal{L}}$. From the Feynman–Kac formula, it is well known that the kernel $\mathcal{K}_t(x,y)$ satisfies the estimates

$$0\le\mathcal{K}_t(x,y)\le h_t(x-y)\triangleq(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}}.\tag{2.1}$$

Denote by $\Gamma_\mu(x,y)$ the fundamental solution of $-\Delta+\mu$. Then we have the following estimate for the fundamental solution (cf. [5, Theorem 0.8]).

Proposition 2.1. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n\ge3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta>0$. Then*

$$\frac{ce^{-\varepsilon_2d(x,y,\mu)}}{|x-y|^{n-2}}\le\Gamma_\mu(x,y)\le\frac{Ce^{-\varepsilon_1d(x,y,\mu)}}{|x-y|^{n-2}},$$

where ε_1, ε_2, C, and c are positive constants depending only on n and constants C_0, C_1, δ in (1.1) and (1.2).

We can obtain the following proposition by using (2.1), Theorem 1.1 in [8], and the symmetry of $\mathcal{K}_t(x,y)$, which can be deduced from the symmetry of $\Gamma_\mu(x,y)$.

Proposition 2.2. *For every N, there is a constant C_N such that*

$$0\le\mathcal{K}_t(x,y)\le C_Nt^{-\frac{n}{2}}e^{-\frac{c|x-y|^2}{t}}\big\{1+\sqrt{t}m(x,\mu)+\sqrt{t}m(y,\mu)\big\}^{-N}.$$

By using Proposition 2.2 and arguments similar to those in the proof of Lemma 3.8 in [8], we obtain the following estimate for the integral kernels of the operators $\mathcal{Q}_t$:

$$\mathcal{Q}_t(x,y) = t^2 \frac{\partial \mathcal{K}_s(x,y)}{\partial s}\Big|_{s=t^2}.$$

Proposition 2.3. *The kernel $\mathcal{Q}_t(x,y)$ satisfies the following estimates.*

(1) *For every $N \in \mathbb{Z}_+$, there is a constant C_N such that*

$$\big|\mathcal{Q}_t(x,y)\big| \le C_N t^{-n} e^{-\frac{c|x-y|^2}{t}} \big\{1 + tm(x,\mu) + tm(y,\mu)\big\}^{-N}.$$

(2) *For every $0 < \delta' < \min\{1,\delta\}$, there exists a constant $c > 0$ such that for all $|h| \le \sqrt{t}$, we have*

$$\begin{aligned}&\big|\mathcal{Q}_t(x+h,y) - \mathcal{Q}_t(x,y)\big| \\ &\quad \le C_N \Big(\frac{|h|}{t}\Big)^{\delta'} t^{-n} e^{-\frac{c|x-y|^2}{t}} \big\{1 + tm(x,\mu) + tm(y,\mu)\big\}^{-N}.\end{aligned}$$

(3) *We have* $\big|\int_{\mathbb{R}^n} \mathcal{Q}_t(x,y)\,dy\big| \le C_N \frac{(tm(x,\mu))^\delta}{(1+tm(x,\mu))^N}$.

From Lemmas 2.1 and 2.7 in [8], we obtain the following.

Proposition 2.4. *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$ such that the family $\mathcal{B} = \{B_k \mid B_k \triangleq B(x_k, m(x_k,\mu)^{-1}), k = 1,2,\ldots\}$ satisfies the following conditions:*

(1) $\bigcup_k B_k = \mathbb{R}^n$,
(2) *there exists $N = N(\delta)$ such that* $\operatorname{card}\{j \mid B_j^{**} \cap B_k^{**} \ne \emptyset\} \le N$ *for all $k \ge 1$.*

Moreover, we have

$$\big|B(x,R)\big| \le \sum_{B_k \cap B(x,R) \ne \emptyset} |B_k| \le c\big|B(x,R)\big|,$$

where $c = c(\delta)$ and $R > m(x,\mu)^{-1}$.

By the proof of Theorem 1.2 in [8] and the proof of Theorem 4 in [2], we can easily obtain the following lemma.

Lemma 2.5. *The correspondence*

$$\mathrm{BMO}_{\mathcal{L}} \ni f \mapsto \Phi_f \in (H^1_{\mathcal{L}})^*$$

is a linear isomorphism of Banach spaces.

Similar to [2, Lemma 2], the following lemma is also valid for the case of the generalized Schrödinger operator.

Lemma 2.6. *There exists $c > 0$ such that for all $f \in \mathrm{BMO}_{\mathcal{L}}$ and $B = B(x,r)$ with $r < m(x,\mu)^{-1}$, we have*

$$|f_{B^*}| \le c\big(1 + \log\big(rm(x,\mu)\big)^{-1}\big)\|f\|_{\mathrm{BMO}_{\mathcal{L}}}.$$

3. Proofs of main results

In this section, we prove Theorem 1.7.

Lemma 3.1. *For all $f \in L^2(\mathbb{R}^n)$, we have $\|s_{\mathcal{Q}} f\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$. Moreover, we have, in $L^2(\mathbb{R}^n)$,*

$$f(x) = 8 \lim_{\varepsilon\to 0, N\to\infty} \int_\varepsilon^N \mathcal{Q}_t^2 f(x) \frac{dt}{t}.$$

We can prove the above lemma by using spectral techniques and the method of the proof of Lemma 3 in [2]. We omit the details.

3.1. Proof of Theorem 1.7(1). Noting the kernel decay in Proposition 2.3 and the integrability of $(1+|y|)^{-n-1}|f(y)|$ (see [6, p. 141]), we can conclude that

$$\mathcal{Q}_t f(x) = \int_{\mathbb{R}^n} \mathcal{Q}_t(x,y) f(y)\, dy$$

is a well-defined absolutely convergent integral for all $(x,t) \in \mathbb{R}^{n+1}_+$. Let $B = B(x_0, r)$. We wish to show that

$$\frac{1}{|B|}\int_0^r \int_B \big|\mathcal{Q}_t f(x)\big|^2 \frac{dx\,dt}{t} \le C\|f\|^2_{\mathrm{BMO}_\mathcal{L}}. \tag{3.1}$$

To do this, we write

$$\begin{aligned} f &= (f - f_{B^*})\chi_{B^*} + (f - f_{B^*})\chi_{(B^*)^c} + f_{B^*} \\ &= f_1 + f_2 + f_{B^*}. \end{aligned}$$

For f_1, using Lemma 3.1 we have

$$\begin{aligned} \frac{1}{|B|}\int_0^r \int_B \big|\mathcal{Q}_t f_1(x)\big|^2 \frac{dx\,dt}{t} &\le \frac{C}{|B|}\int_B \big|s_\mathcal{Q} f_1(x)\big|^2\, dx \\ &\le \frac{C}{|B|}\|f_1\|_2^2 = \frac{C}{|B|}\int_{B^*} |f - f_{B^*}|^2\, dx \\ &\le C\|f\|^2_{\mathrm{BMO}_\mathcal{L}}, \end{aligned}$$

where we have used Remark 1.5 in the last step.

Let $x \in B(x_0, r)$ and $t < r$. Then via Proposition 2.3(1), we get

$$\begin{aligned} \big|\mathcal{Q}_t f_2(x)\big| &\lesssim \int_{\mathbb{R}^n} \big|f_2(y)\big| \frac{t^{-n}}{(1+\frac{|x-y|}{t})^{n+1}}\, dy \\ &\lesssim \int_{(B^*)^c} \big|f(y) - f_{B^*}\big| \frac{t}{|x_0 - y|^{n+1}}\, dy \\ &\lesssim \sum_{k=1}^\infty \frac{t}{(2^k r)^{n+1}} \Big[\int_{2^k r \le |y-x_0| < 2^{k+1} r} \big|f(y) - f_{B_{2^{k+1}r}}\big|\, dy \\ &\quad + (2^{k+1} r)^n \big|f_{B_{2^{k+1}r}} - f_{B^*}\big|\Big] \\ &\lesssim \frac{t}{r}\sum_{k=1}^\infty 2^{-k}\big[\|f\|_{\mathrm{BMO}} + k\|f\|_{\mathrm{BMO}}\big] \lesssim \frac{t}{r}\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Thus, by integrating over $B \times (0, r)$, we obtain

$$\begin{aligned}\frac{1}{|B|} \int_0^r \int_B \big|\mathcal{Q}_t f_2(x)\big|^2 \frac{dx\,dt}{t} &\lesssim \int_0^r \frac{t^2}{r^2} \frac{dt}{t} \|f\|_{\mathrm{BMO}}^2 \\ &= \frac{C}{2} \|f\|_{\mathrm{BMO}}^2 \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2.\end{aligned}$$

It remains to estimate the third term. At first, we assume that $r < m(x_0, \mu)^{-1}$. Using $m(x, \mu)^{-1} \sim m(x_0, \mu)^{-1}$ for $x \in B$ (cf. [5, Proposition 1.8]), we have

$$\begin{aligned}\frac{1}{|B|} \int_0^r \int_B \big|\mathcal{Q}_t(f_{B^*})(x)\big|^2 \frac{dx\,dt}{t} &= \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \Big|\int_{\mathbb{R}^n} \mathcal{Q}_t(x, y)\,dy\Big|^2 \frac{dx\,dt}{t} \\ &\lesssim \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \big(tm(x, \mu)\big)^{2\delta} \frac{dx\,dt}{t} \\ &\lesssim |f_{B^*}|^2 \big(rm(x_0, \mu)\big)^{2\delta} \\ &\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2 \big(1 + \log\big(rm(x_0, \mu)\big)^{-1}\big)^2 \big(rm(x_0, \mu)\big)^{2\delta} \\ &\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2,\end{aligned}$$

where the second line follows from Proposition 2.3(3), and we have used Lemma 2.6 in the last step.

Finally, suppose that $r \geq m(x_0, \mu)^{-1}$, and choose from Proposition 2.4 a finite family of critical balls $\{B_k\}$ such that $B \subset \cup B_k$ and $\sum |B_k| \lesssim |B|$. Via Proposition 2.3 and the fact that $|f_{B^*}| \leq \|f\|_{\mathrm{BMO}_{\mathcal{L}}}$, we have

$$\begin{aligned}&\frac{1}{|B|} \int_0^r \int_B \big|\mathcal{Q}_t(f_{B^*})(x)\big|^2 \frac{dx\,dt}{t} \\ &\quad= \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \Big|\int_{\mathbb{R}^n} \mathcal{Q}_t(x, y)\,dy\Big|^2 \frac{dx\,dt}{t} \\ &\quad\lesssim \frac{\|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2}{|B|} \sum_k \Big(\int_0^{m(x_k,\mu)^{-1}} \int_{B_k} \big(tm(x_k, \mu)\big)^{2\delta} \frac{dx\,dt}{t} \\ &\qquad+ \int_{m(x_k,\mu)^{-1}}^{\infty} \int_{B_k} \frac{dx}{(1 + rm(x_0, \mu))^{2N-2\delta}} \frac{dt}{t}\Big) \\ &\quad\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2 |B|^{-1} \sum |B_k| \\ &\quad\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2.\end{aligned}$$

The above argument implies that (3.1) holds. This establishes the first part of Theorem 1.7.

3.2. Proof of Theorem 1.7(2). Let us fix $f \in L^1((1 + |x|)^{-n-1}\,dx)$ such that $\mu_f \triangleq |\mathcal{Q}_t f(x)|^2\,dx\,dt/t$ is a Carleson measure. In what follows, we show that such f must belong to $\mathrm{BMO}_{\mathcal{L}}$. By Lemma 2.5, it suffices to show that the linear functional

$$H^1_{\mathcal{L}} \ni g \mapsto \Phi_f(g) \triangleq \int_{\mathbb{R}^n} f(x) g(x)\,dx,$$

defined at least over finite linear combinations of $H^1_{\mathcal{L}}$-atoms, satisfies the estimate

$$\big|\Phi_f(g)\big| \le c\|\mu_f\|^{\frac{1}{2}}\|g\|_{H^1_{\mathcal{L}}}. \tag{3.2}$$

We list some notation as follows:

$$\begin{aligned}
&F(x,t) \triangleq \mathcal{Q}_t f(x), \quad (x,t) \in \mathbb{R}^{n+1}_+,\\
&G(x,t) \triangleq \mathcal{Q}_t g(x), \quad (x,t) \in \mathbb{R}^{n+1}_+,\\
&S_{\mathcal{Q}} g(x) \triangleq \Big(\int_0^\infty \int_{|x-y|<t} \big|\mathcal{Q}_t g(y)\big|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.
\end{aligned}$$

From [6] we obtain the following.

Lemma 3.2 ([6, p. 162])**.** *Let $F(x,t)$ and $G(x,t)$ be measurable functions on $\mathbb{R}^{n+1}_+$ satisfying*

$$\begin{aligned}
&\mathcal{I}(F)(x) \triangleq \sup_{x\in B}\Big(\frac{1}{|B|}\int_0^{r(B)}\int_B \big|F(y,t)\big|^2 \frac{dy\,dt}{t}\Big)^{\frac{1}{2}} \in L^\infty(\mathbb{R}^n),\\
&\mathcal{G}(G)(x) \triangleq \Big(\int\int_{\Gamma(x)} \big|G(y,t)\big|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{\frac{1}{2}} \in L^1(\mathbb{R}^n),
\end{aligned}$$

where $r(B)$ denotes the radius of B and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$. Then there exists a constant $c > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} \big|F(y,t)G(y,t)\big| \frac{dy\,dt}{t} &\le c\int_{\mathbb{R}^n} \mathcal{I}(F)(x)\mathcal{G}(G)(x)\,dx\\
&\le c\big\|\mathcal{I}(F)\big\|_{L^\infty}\big\|\mathcal{G}(G)\big\|_{L^1}.
\end{aligned}$$

We temporarily assume that the following two lemmas are true.

Lemma 3.3. *Suppose that $f \in L^1((1+|x|)^{-(n+1)}\,dx)$, and let g be an $H^1_{\mathcal{L}}$-atom. Then*

$$\frac{1}{8}\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)}\frac{dx\,dt}{t}. \tag{3.3}$$

Lemma 3.4. *If g is a finite linear combination of $H^1_{\mathcal{L}}$-atoms, then there exists $c > 0$ such that $\|S_{\mathcal{Q}} g\|_{L^1} \le c\|g\|_{H^1_{\mathcal{L}}}$.*

Noting that $\|\mu_f\| = \|\mathcal{I}(F)\|^2_{L^\infty}$ and $\mathcal{G}(G)(x) = S_{\mathcal{Q}} g(x)$, then by the above three lemmas we obtain

$$\begin{aligned}
\big|\Phi_f(g)\big| &\le \int_{\mathbb{R}^n} \big|f(x)\overline{g(x)}\big|\,dx\\
&\le c\int_{\mathbb{R}^{n+1}_+} \big|F(x,t)\overline{G(x,t)}\big|\frac{dx\,dt}{t}\\
&\le c\big\|\mathcal{I}(F)\big\|_{L^\infty}\big\|\mathcal{G}(G)\big\|_{L^1}\\
&\le c\|\mu_f\|^{\frac{1}{2}}\|g\|_{H^1_{\mathcal{L}}},
\end{aligned}$$

which establishes (3.2). To complete the proof of Theorem 1.7, it only remains to prove Lemmas 3.3 and 3.4.

Proof of Lemma 3.4. By Proposition 1.3 and Definition 1.2, it suffices to consider sums of atoms associated to balls $B(x_0,r)$ with $r \lesssim m(x_0,\mu)^{-1}$. Suppose that $g(x)$ is an $H^1_{\mathcal{L}}$-atom associated with a ball $B = B(x_0,r)$. Then we have

$$\begin{aligned}
\|S_{\mathcal{Q}}g\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^{n+1}_+}\big|\mathcal{Q}_t g(y)\big|^2\chi_{\Gamma(x)}(y,t)\frac{dy\,dt}{t^{n+1}}\Big]\,dx\\
&= \int_{\mathbb{R}^{n+1}_+}\big|\mathcal{Q}_t g(y)\big|^2\big|B(y,t)\big|\frac{dy\,dt}{t^{n+1}}\\
&= C_n\int_{\mathbb{R}^{n+1}_+}\big|\mathcal{Q}_t g(y)\big|^2\frac{dy\,dt}{t}\\
&= C_n\|s_{\mathcal{Q}}g\|^2_{L^2(\mathbb{R}^n)} = \frac{C_n}{8}\|g\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we have used Lemma 3.1 in the last step. Thus, by Hölder's inequality we obtain

$$\begin{aligned}
\int_{B^{***}} S_{\mathcal{Q}}g(x)\,dx &\le |B^{***}|^{\frac12}\Big(\int_{B^{***}} S_{\mathcal{Q}}g(x)^2\,dx\Big)^{\frac12}\\
&\lesssim |B|^{\frac12}\|g\|_{L^2} \lesssim 1.
\end{aligned}$$

In order to complete the proof of Lemma 3.4, it remains to find a uniform bound for

$$I = \int_{(B^{***})^c} S_{\mathcal{Q}}g(x)\,dx.$$

We first assume that $r < m(x_0,\mu)^{-1}$. Then by the moment condition on g, we have

$$\begin{aligned}
S_{\mathcal{Q}}g(x) &= \Big[\int_0^\infty\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n}\big(\mathcal{Q}_t(y,x')-\mathcal{Q}_t(y,x_0)\big)g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\le \Big[\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\Big(\int_B\big|\mathcal{Q}_t(y,x')-\mathcal{Q}_t(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\quad + \Big[\int_{\frac{|x-x_0|}{2}}^\infty\int_{|x-y|<t}\Big(\int_B\big|\mathcal{Q}_t(y,x')-\mathcal{Q}_t(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&= W_1(x)+W_2(x).
\end{aligned}$$

For W_1, it is obvious that $|y-x'| \sim |y-x_0| \sim |x-x_0|$ and $|x'-x_0| < |y-x_0|/4$ and we will get the following estimate from the smoothness of $\mathcal{Q}_t(x,y) = \mathcal{Q}_t(y,x)$ established in Proposition 2.3:

$$\begin{aligned}
&W_1(x)\\
&\lesssim \Big[\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\Big(\int_B\Big(\frac{|x'-x_0|}{t}\Big)^{\delta'}t^{-n}\Big(1+\frac{|y-x_0|}{t}\Big)^{-(n+1)}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\lesssim \Big[\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\Big(\frac{r}{t}\Big)^{2\delta'}t^{-2n}\Big(1+\frac{|y-x_0|}{t}\Big)^{-2(n+1)}\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \Big[\int_0^{\frac{|x-x_0|}{2}} \Big(\frac{r}{t}\Big)^{2\delta'} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)}\frac{dt}{t}\Big]^{\frac12}\\
&= C\frac{r^{\delta'}}{|x-x_0|^{n+1}}\Big[\int_0^{\frac{|x-x_0|}{2}} t^{2-2\delta'}\frac{dt}{t}\Big]^{\frac12}\\
&= C\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.
\end{aligned}$$

For W_2, we have $|x'-x_0|\le r<|x-x_0|/2\le t$. Applying Proposition 2.3 gives

$$\big|\mathcal{Q}_t(y,x')-\mathcal{Q}_t(y,x_0)\big|\lesssim \Big(\frac{|x'-x_0|}{t}\Big)^{\delta'} t^{-n}.$$

Thus, for $x\in(B^{***})^c$ a similar argument to the above deduction leads to

$$\begin{aligned}
W_2(x) &\lesssim \Big[\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\Big(\int_B\Big(\frac{|x'-x_0|}{t}\Big)^{\delta'} t^{-n}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\lesssim \Big[\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\Big(\frac{r}{t}\Big)^{2\delta'} t^{-2n}\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&= Cr^{\delta}\Big[\int_{\frac{|x-x_0|}{2}}^{\infty}\frac{dt}{t^{2n+2\delta'+1}}\Big]^{\frac12}\\
&= C\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.
\end{aligned}$$

Then integrating $S_{\mathcal{Q}}g(x)$ over $(B^{***})^c$ gives

$$\begin{aligned}
\int_{(B^{***})^c} S_{\mathcal{Q}}g(x)\,dx &\le \int_{(B^{***})^c} W_1(x)+W_2(x)\,dx\\
&\lesssim \int_{|x-x_0|>8r}\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}\,dx\lesssim 1.
\end{aligned}$$

We next estimate $\int_{(B^{***})^c} S_{\mathcal{Q}}g(x)\,dx$ with the condition that r is comparable to $m(x_0,\mu)^{-1}$. Similar to the argument before, we will obtain the pointwise estimate of $S_{\mathcal{Q}}g(x)$ for each $x\in(B^{***})^c$. To do this, we split the integral in $t>0$ defining $S_{\mathcal{Q}}g(x)$ into three parts:

$$\begin{aligned}
S_{\mathcal{Q}}g(x) &= \Big[\int_0^{\infty}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n}\mathcal{Q}_t(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\le \Big[\int_0^{\frac{r}{2}}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n}\mathcal{Q}_t(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\quad+\Big[\int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n}\mathcal{Q}_t(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&\quad+\Big[\int_{\frac{|x-x_0|}{4}}^{\infty}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n}\mathcal{Q}_t(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{\frac12}\\
&= E_1(x)+E_2(x)+E_3(x).
\end{aligned}$$

For E_1, we have $|x'-y| \sim |x-x_0|$. Applying Proposition 2.3(1), we obtain

$$\begin{aligned}
E_1(x) &\lesssim \Big[\int_0^{\frac{r}{2}} \int_{|x-y|<t} \Big(\int_B t^{-n}\Big(1+\frac{|y-x'|}{t}\Big)^{-(n+1)} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{\frac12} \\
&\lesssim \Big[\int_0^{\frac{r}{2}} \int_{|x-y|<t} t^{-2n}\Big(1+\frac{|x-x_0|}{t}\Big)^{-2(n+1)} \frac{dy\,dt}{t^{n+1}}\Big]^{\frac12} \\
&\lesssim \Big[\int_0^{\frac{r}{2}} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)} \frac{dt}{t}\Big]^{\frac12} \\
&\lesssim \frac{r}{|x-x_0|^{n+1}}.
\end{aligned}$$

For the second term, by using Proposition 2.3(1) together with $|x'-y| \sim |x-x_0|$ and $m(x',\mu)^{-1} \sim m(x_0,\mu)^{-1} \sim r$, we obtain

$$\begin{aligned}
E_2(x) &\lesssim \Big[\int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \Big(\int_B \frac{t^{-n}(1+|y-x'|/t)^{-(n+N+1)}}{(1+tm(x_0,\mu))^N} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{\frac12} \\
&\lesssim \Big[\int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+N+1)} \Big(\frac{1}{tm(x_0,\mu)}\Big)^{2N} \frac{dt}{t}\Big]^{\frac12} \\
&= C\Big[\frac{r^{2(N+1)}}{|x-x_0|^{2(n+N+1)}} \int_1^{\frac{2|x-x_0|}{r}} t\,dt\Big]^{\frac12} \\
&= C\frac{r^N}{|x-x_0|^{n+N}}.
\end{aligned}$$

Finally, for the last term the extra decay just gives

$$\begin{aligned}
E_3(x) &\lesssim \Big[\int_{\frac{|x-x_0|}{4}}^{\infty} \int_{|x-y|<t} \Big(\int_B t^{-n}\big(1+tm(x_0,\mu)\big)^{-N} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{\frac12} \\
&\lesssim \Big[\int_{\frac{|x-x_0|}{4}}^{\infty} t^{-2n}\Big(\frac{1}{tm(x_0,\mu)}\Big)^{2N} \frac{dt}{t}\Big]^{\frac12} \\
&= C\frac{r^N}{|x-x_0|^{n+N}}.
\end{aligned}$$

Thus, by integrating $S_{\mathcal{Q}}g(x)$ over $(B^{***})^c$, we also have

$$\begin{aligned}
\int_{(B^{***})^c} S_{\mathcal{Q}}g(x)\,dx &\le \int_{(B^{***})^c} E_1(x)+E_2(x)+E_3(x)\,dx \\
&\lesssim \int_{|x-x_0|>8r} \frac{r^N}{|x-x_0|^{n+N}}\,dx \lesssim 1.
\end{aligned}$$

This completes the proof of Lemma 3.4. □

We observe that (3.3) is clearly valid when $f, g \in L^2(\mathbb{R}^n)$, while we must justify the convergence of the integrals in the case when $f \in L^1((1+|x|)^{-(n+1)}\,dx)$ and g is an $H^1_{\mathcal{L}}$-atom.

Proof of Lemma 3.3. It should be noted that, by Lemmas 3.2 and 3.4 and the dominated convergence theorem, the following integral is absolutely convergent and satisfies

$$\begin{aligned} J &= \int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)}\frac{dx\,dt}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \int_{\mathbb{R}^n} \mathcal{Q}_t f(x)\overline{\mathcal{Q}_t g(x)}\frac{dx\,dt}{t}. \end{aligned}$$

Then, for each $t>0$, via Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{Q}_t f(x)\overline{\mathcal{Q}_t g(x)}\,dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{Q}_t(x,y)f(y)\overline{\mathcal{Q}_t g(x)}\,dy\,dx \\ &= \int_{\mathbb{R}^n} f(y)\overline{\mathcal{Q}_t^2 g(y)}\,dy, \end{aligned}$$

and then

$$\begin{aligned} J &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \Big[\int_{\mathbb{R}^n} f(y)\overline{\mathcal{Q}_t^2 g(y)}\,dy\Big]\frac{dt}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} f(y)\Big[\int_\varepsilon^N \overline{\mathcal{Q}_t^2 g(y)}\frac{dt}{t}\Big]\,dy. \end{aligned} \tag{3.4}$$

It is easy to prove the absolute integrability in these steps. We can obtain the following lemma by combining the hypothesis $f\in L^1((1+|x|)^{-(n+1)}\,dx)$, the kernel decay $|\mathcal{Q}_t(x,y)|\lesssim t^{-n}(1+|x-y|/t)^{-N}$, and the following general estimate on $H^1_{\mathcal{L}}$-atoms.

Lemma 3.5. *Let $V_t(x,y)$ be a function satisfying*

$$\big|V_t(x,y)\big| \le C_N t^{-n}\Big(1+\frac{|x-y|}{t}\Big)^{-N}\big(1+tm(x,\mu)+tm(y,\mu)\big)^{-N}. \tag{3.5}$$

Then there exists $C_{y_0,r}>0$ such that for each $H^1_{\mathcal{L}}$-atom g supported by $B(y_0,r)$, we have

$$\Psi_V g(x) \triangleq \sup_{t>0}\Big|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy\Big| \le C_{y_0,r}\big(1+|x|\big)^{-(n+1)}, \quad x\in\mathbb{R}^n. \tag{3.6}$$

Proof. By Definition 1.2, we know that $r<4m(y_0,\mu)^{-1}$. We use (3.5) to obtain

$$\Big|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy\Big| \le \Big|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy\Big| \le C\|g\|_{L^\infty} \le Cr^{-n}. \tag{3.7}$$

If $x\in B(y_0,2r)$, it is easy to see that $1\le 1+|x|\le 1+|y_0|+2r$. Combining with (3.7), we obtain

$$\Big|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy\Big| \le Cr^{-n}\frac{(1+|y_0|+2r)^{n+1}}{(1+|x|)^{n+1}} \le C_{y_0,r}\big(1+|x|\big)^{-(n+1)},$$

where $C_{y_0,r}=Cr^{-n}(1+|y_0|+2r)^{n+1}$.

If $x \notin B(y_0, 2r)$, then for $y \in B(y_0, r)$ we have $|x-y| \sim |x-y_0|, m(y_0,\mu) \sim m(y,\mu)$. Therefore, via (3.5) we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy\Big| &\le C_N\|g\|_{L^1}t^{-n}\Big(1+\frac{|x-y_0|}{t}\Big)^{-N}\big(1+tm(y_0,\mu)\big)^{-N}\\ &\le C_N|x-y_0|^{-n-N}m(y_0,\mu)^{-N}. \end{aligned}\tag{3.8}$$

Denote by I the integral $|\int_{\mathbb{R}^n} V_t(x,y)g(y)\,dy|$. Applying (3.7) and (3.8) and choosing $N=1$ in (3.8), we can easily obtain, for $|x| \ge 2|y_0|$,

$$\big(1+|y_0|\big)I^{\frac{1}{n+1}} \le Cr^{-\frac{n}{n+1}}\big(1+|y_0|\big)$$

and

$$\big(|x|-|y_0|\big)I^{\frac{1}{n+1}} \le C_N m(y_0,\mu)^{-\frac{1}{n+1}}.$$

Then

$$I \le \Big(Cr^{-\frac{n}{n+1}}\big(1+|y_0|\big)+C_N m(y_0,\mu)^{-\frac{1}{n+1}}\Big)^{n+1}\big(1+|x|\big)^{-n-1}.$$

Therefore, we conclude that (3.6) holds true by letting $C_{y_0,r} = (Cr^{-\frac{n}{n+1}}(1+|y_0|)+C_N m(y_0,\mu)^{-\frac{1}{n+1}})^{n+1}$. For $|x| < 2|y_0|$, we have

$$I \le Cr^{-n}\big(1+2|y_0|\big)^{n+1}\big(1+|x|\big)^{-n-1}.$$

We also conclude that (3.6) holds true by letting $C_{y_0,r} = Cr^{-n}(1+2|y_0|)^{n+1}$. $\square$

Finally, to complete the proof of Lemma 3.3, it remains to prove the estimate

$$\sup_{\varepsilon,N>0}\Big|\int_\varepsilon^N \mathcal{Q}_t^2 g(y)\frac{dt}{t}\Big| \le C_{y_0,r}\big(1+|y|\big)^{-(n+1)}, \quad y \in \mathbb{R}^n. \tag{3.9}$$

Thus, we define a new kernel $\mathcal{D}_\varepsilon(x,y)$ associated to the operator $\int_\varepsilon^\infty \mathcal{Q}_t^2 g(y)\frac{dt}{t}$. Then

$$\begin{aligned}\Big|\int_\varepsilon^N \mathcal{Q}_t^2 g(y)\frac{dt}{t}\Big| &= \Big|\int_\varepsilon^\infty \mathcal{Q}_t^2 g(y)\frac{dt}{t} - \int_N^\infty \mathcal{Q}_t^2 g(y)\frac{dt}{t}\Big|\\ &= \Big|\int_{\mathbb{R}^n}\mathcal{D}_\varepsilon(x,y)g(y)\,dy - \int_{\mathbb{R}^n}\mathcal{D}_N(x,y)g(y)\,dy\Big|\\ &\le \sup_{\varepsilon>0}\Big|\int_{\mathbb{R}^n}\mathcal{D}_\varepsilon(x,y)g(y)\,dy\Big| + \sup_{N>0}\Big|\int_{\mathbb{R}^n}\mathcal{D}_N(x,y)g(y)\,dy\Big|.\end{aligned}$$

By using spectral techniques, we can easily conclude that

$$\mathcal{D}_\varepsilon(x,y) = \frac{1}{8}\big(\mathcal{K}_{2\varepsilon^2}(x,y) - \mathcal{Q}_{\sqrt{2}\varepsilon}(x,y)\big).$$

So the kernel $\mathcal{D}_\varepsilon(x,y)$ satisfies the condition of Lemma 3.5. Thus, (3.9) holds.

Indeed, (3.9) allows passing the limit inside the integral in (3.4). Applying Lemma 3.1, we conclude that

$$J = \frac{1}{8}\int_{\mathbb{R}^n} f(y)\overline{g(y)}\,dy.$$

This completes the proof of Theorem 1.7. $\square$

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References

1. M. Christ, *On the $\bar{\partial}$ equation in weighted L^2 norms in $\mathbb{C}^1$*, J. Geom. Anal. **1** (1991), no. 3, 193–230. Zbl 0737.35011. MR1120680. DOI 10.1007/BF02921303. 537
2. J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, BMO *spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356. Zbl 1136.35018. MR2115447. DOI 10.1007/s00209-004-0701-9. 538, 541, 542
3. J. Dziubański and J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoam. **15** (1999), no. 2, 279–296. Zbl 0959.47028. MR1715409. DOI 10.4171/RMI/257. 538
4. C. Lin and H. Liu, $\mathrm{BMO}_L(\mathbb{H}^n)$ *spaces and Carleson measures for Schrödinger operators*, Adv. Math. **228** (2011), no. 3, 1631–1688. Zbl 1235.22012. MR2824565. DOI 10.1016/j.aim.2011.06.024. 538
5. Z. Shen, *On fundamental solutions of generalized Schrödinger operators*, J. Funct. Anal. **167** (1999), no. 2, 521–564. Zbl 0936.35051. MR1716207. DOI 10.1006/jfan.1999.3455. 537, 538, 540, 543
6. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, 1993. Zbl 0821.42001. MR1232192. 542, 544
7. L. Sun, *Harnack's inequality for generalized subelliptic Schrödinger operators*, Anal. Theory Appl. **24** (2008), no. 3, 247–259. Zbl 1199.35054. MR2452393. DOI 10.1007/s10496-008-0247-5. 537
8. L. Wu and L. Yan, *Heat kernels, upper bounds and Hardy spaces associated to the generalized Schrödinger operators*, J. Funct. Anal. **270** (2016), no. 10, 3709–3749. Zbl 1356.42016. MR3478871. DOI 10.1016/j.jfa.2015.12.016. 537, 538, 539, 540, 541

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