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STABILITY OF FUNCTIONAL EQUATIONS ARISING FROM NUMBER THEORY AND DETERMINANT OF MATRICES

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Abstract. In this paper, we consider the Ulam–Hyers stability of the functional equations

$$f(ux - vy, uy - vx) = f(x, y)f(u, v),$$
$$f(ux + vy, uy - vx) = f(x, y)f(u, v),$$
$$f(ux + vy, uy + vx) = f(x, y)f(u, v),$$
$$f(ux - vy, uy + vx) = f(x, y)f(u, v)$$

for all $x, y, u, v \in \mathbb{R}$, where $f : \mathbb{R}^2 \to \mathbb{R}$, which arise from number theory and are connected with the characterizations of the determinant and permanent of two-by-two matrices.

1. Introduction

Throughout this article, we denote by $\mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{Q}, \mathbb{R}^n$ the set of real numbers, nonnegative real numbers, complex numbers, rational numbers, and the n-dimensional Euclidean space (resp., $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_0^2 = \mathbb{R}^2 \setminus \{(0,0)\}$, $f, g : \mathbb{R}^2 \to \mathbb{R}$, and $\phi : \mathbb{R}^2 \to \mathbb{R}^+$). A function $M : \mathbb{R} \to \mathbb{R}$ is called *a multiplicative function* provided that $M(xy) = M(x)M(y)$ for all $x, y \in \mathbb{R}$, and $E : \mathbb{R} \to \mathbb{R}$ is called *an exponential function* provided that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

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In the problems column of the *Newsletter of the European Mathematical Society*, Sahoo in [11] posed the problem of how to determine the general solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equations

$$g(ux - vy, uy - vx) = g(x, y) + g(u, v) + g(x, y)g(u, v), \tag{1.1}$$

$$g(ux + vy, uy - vx) = g(x, y) + g(u, v) + g(x, y)g(u, v) \tag{1.2}$$

for all $x, y, u, v \in \mathbb{R}$. Houston and Sahoo in [8] showed that the general solutions of equations (1.1) and (1.2) are given by

$$g(x, y) = M(x^2 - y^2) - 1,$$

and

$$g(x, y) = M(x^2 + y^2) - 1$$

for all $x, y \in \mathbb{R}$, respectively, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function. Replacing $g(x, y)$ by $f(x, y) - 1$ in (1.1) and (1.2), equations (1.1) and (1.2) are reduced to

$$f(ux - vy, uy - vx) = f(x, y)f(u, v), \tag{1.3}$$

$$f(ux + vy, uy - vx) = f(x, y)f(u, v) \tag{1.4}$$

for all $x, y, u, v \in \mathbb{R}$. In 2002, the functional equation

$$f(ux + vy, uy + vx) = f(x, y)f(u, v) \tag{1.5}$$

for all $x, y, u, v \in \mathbb{R}$ was studied by Chung and Sahoo in [6]. The functional equations (1.3), (1.4), and (1.5) are connected with the characterizations of the determinant and permanent of two-by-two matrices. Furthermore, equation (1.4) arises from number theory (see [9] or [5]). In addition to the last three functional equations, we also introduce one more similar functional equation

$$f(ux - vy, uy + vx) = f(x, y)f(u, v) \tag{1.6}$$

for all $x, y, u, v \in \mathbb{R}$, and it will be shown that all nonconstant solutions of (1.6) are given by

$$f(x, y) = M(x^2 + y^2)E\big(\tan^{-1}(y/x)\big)$$

for all $x, y \in \mathbb{R}$ with $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function and $E : \mathbb{R} \to \mathbb{R}$ is an exponential function satisfying $E(x + r\pi) = E(x)$ for all $x \in \mathbb{R}, r \in \mathbb{Q}$.

In the present article, we consider the Ulam–Hyers stability of functional equations (1.3)–(1.6); that is, we investigate both bounded functions and unbounded functions satisfying the functional inequalities

$$\big|f(ux - vy, uy - vx) - f(x, y)f(u, v)\big| \leq \phi(x, y) \text{ or } \phi(u, v), \tag{1.7}$$

$$\big|f(ux + vy, uy - vx) - f(x, y)f(u, v)\big| \leq \phi(x, y) \text{ or } \phi(u, v), \tag{1.8}$$

$$\big|f(ux + vy, uy + vx) - f(x, y)f(u, v)\big| \leq \phi(x, y) \text{ or } \phi(u, v), \tag{1.9}$$

$$\big|f(ux - vy, uy + vx) - f(x, y)f(u, v)\big| \leq \phi(x, y) \text{ or } \phi(u, v) \tag{1.10}$$

for all $x, y, u, v \in \mathbb{R}$. In particular, we give a refined description of bounded functions satisfying the inequalities (see Albert and Baker [1] and Chung [4] for a refined result for bounded solutions of exponential functional equations; for some related results, see [2], [3], [7], [10], and [12])

2. General solutions of the equations

In this section, we present the general solutions of the equations (1.3)–(1.6).

The following result can be found in [8, Theorem 1].

Theorem 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.3). Then f has the form*

$$f(x,y) = M(x^2 - y^2) \tag{2.1}$$

for all $x, y \in \mathbb{R}$, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function.

The following result can be found in [5, Theorem 1] and [8, Theorem 2], which is a revised version of [9].

Theorem 2.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.4). Then f has the form*

$$f(x,y) = M(x^2 + y^2) \tag{2.2}$$

for all $x, y \in \mathbb{R}$, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function.

The following theorem was established in [6, Theorem 1].

Theorem 2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.5). Then f has the form*

$$f(x,y) = M_1(x+y)M_2(x-y) \tag{2.3}$$

for all $x, y \in \mathbb{R}$, where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

Now, we determine general solutions of equation (1.6).

Theorem 2.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.6). Then $f = 1$ or it has the form*

$$f(x,y) = M(x^2 + y^2)E\big(\tan^{-1}(y/x)\big) \tag{2.4}$$

for all $x, y \in \mathbb{R}$, where $M : [0,\infty) \to \mathbb{R}$ is a multiplicative function and $E : \mathbb{R} \to \mathbb{R}$ is an exponential function satisfying $E(x + r\pi) = E(x)$ for all $x \in \mathbb{R}, r \in \mathbb{Q}$.

Proof. Define $m : \mathbb{C} \to \mathbb{R}$ by $m(x+iy) = f(x,y)$. Then functional equation (1.6) is reduced to

$$m(zw) = m(z)m(w) \tag{2.5}$$

for all $z, w \in \mathbb{C}$. If $m(0) \neq 0$, then, by putting $w = 0$ in (2.5), we have $m(z) = 1$ for all $z \in \mathbb{C}$, and hence $f(x,y) = 1$ for all $x, y \in \mathbb{R}$. Assume that $m(0) = 0$. For each complex number $z = x + iy \neq 0$, using polar form $z = re^{i\theta}, r > 0, 0 \leq \theta \in \mathbb{R}$, we define $\beta : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ by

$$\beta(r,\theta) = m(re^{i\theta}) \tag{2.6}$$

for all $r > 0, \theta \in \mathbb{R}$. Then, from (2.5) and (2.6), we have

$$\beta(r_1,\theta_1)\beta(r_2,\theta_2) = \beta(r_1r_2, \theta_1+\theta_2), \qquad \beta(r,\theta) = \beta(r, \theta+2\pi) \tag{2.7}$$

for all $r_1, r_2 > 0, \theta_1, \theta_2 \in \mathbb{R}$. Let $M(r) = \beta(\sqrt{r}, 0), M(0) = 0, E(\theta) = \beta(1,\theta)$ for all $r > 0, \theta \in \mathbb{R}$. Then, since E is a nonzero exponential function, we have $E(0) = 1$, and from (2.7) we have $E(2\pi) = E(0) = 1$. Thus we have $E(r\pi) = E(2\pi)^{r/2} = 1$

for all $r \in \mathbb{Q}$. Now, we can write

$$\begin{aligned} f(x,y) &= m(x+iy) = \beta(r,\theta) = \beta(r,0)\beta(1,\theta) = M(r^2)E(\theta) \\ &= M(x^2+y^2)E\big(\arg(x+iy)\big) = M(x^2+y^2)E\big(\tan^{-1}(y/x)\big) \end{aligned} \tag{2.8}$$

for all $x, y \in \mathbb{R}$ with $(x,y) \neq (0,0)$. Since $f(0,0) = m(0) = 0$ and $M(0) = 0$, the equality (2.8) holds for all $(x,y) \in \mathbb{R}^2$. Thus, from (2.8), we get (2.4). This completes the proof. □

Remark 2.5. It is possible to find a nonconstant exponential function $E : \mathbb{R} \to \mathbb{R}$ satisfying $E(\theta + r\pi) = E(\theta)$ for all $\theta \in \mathbb{R}, r \in \mathbb{Q}$. Indeed, let H be a basis (Hamel basis) of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers such that $\pi \in H$. Let $E_0 : H \to \mathbb{R}$ be a nonconstant function such that $E_0(\pi) = 1$. Then there exists a unique extension $E : \mathbb{R} \to \mathbb{R}$ of E_0 such that $E(\theta_1 + \theta_2) = E(\theta_1)E(\theta_2)$ for all $\theta_1, \theta_2 \in \mathbb{R}$. Since $E(\pi) = 1$, we have $E(r\pi) = E(\pi)^r = 1$ for all $r \in \mathbb{Q}$, and hence $E(\theta + r\pi) = E(\theta)E(r\pi) = E(\theta)$ for all $\theta \in \mathbb{R}$.

Remark 2.6. Let $\mathbb{F}$ be a field. Using the same methods as in [8], we obtain that all solutions $f : \mathbb{F}^2 \to \mathbb{R}$ of the equations (1.3) and (1.5) have the same forms as (2.1) and (2.3), respectively. However, the solutions of the equations (1.4) and (1.6) are not known when f is defined on $\mathbb{F}^2$ and taking values on $\mathbb{R}$. It can be verified that the solutions of (1.4) are of the same form as (2.2) if $\mathbb{F}$ is algebraically closed or, more generally, if $\{(x^2 - y^2, 2xy) : x, y \in \mathbb{F}\} = \mathbb{F}^2$.

3. Ulam–Hyers stability of the equations

In this section, we investigate bounded functions and unbounded functions satisfying each of the functional inequalities (1.7)–(1.10). We first present a preliminary lemma which will be useful to describe bounded functions f satisfying (1.7)–(1.10). In the following lemma, let G be a group, and let $\tau : G \times G \to G$ and $\phi : G \times G \to \mathbb{R}^+$ be given. We denote by S_1 the set of all $x \in G$ such that $\tau(x, \cdot) : G \to G$ is surjective, and we denote by S_2 the set of all $y \in G$ such that $\tau(\cdot, y) : G \to G$ is surjective, and $B = \{x \in G : \phi(x) < 1/4\}$.

Lemma 3.1. *Let $f : G \to \mathbb{R}$ be a bounded function satisfying*

$$\big|f \circ \tau(x,y) - f(x)f(y)\big| \leq \phi(x) \quad \big(\textit{resp.}, \phi(y)\big) \tag{3.1}$$

for all $x, y \in G$. Then f satisfies

$$\big|f(x)\big| \leq \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x)}\big) \tag{3.2}$$

for all $x \in G$. In particular, if $S_1 \cap B \neq \emptyset$ (resp., $S_2 \cap B \neq \emptyset$), then f satisfies either

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x)}\big) \leq \big|f(x)\big| \leq \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x)}\big) \tag{3.3}$$

for all $x \in S_1 \cap B$ (resp., $S_2 \cap B$) or

$$\big|f(x)\big| \leq \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x)}\big) \tag{3.4}$$

for all $x \in S_1 \cap B$ (resp., $S_2 \cap B$).

Proof. Let $M_f = \sup_{x\in G} |f(x)|$. Using the triangle inequality with (3.1), we have

$$\big|f(x)f(y)\big| \le \big|f\circ\tau(x,y)\big| + \phi(y) \le M_f + \phi(y) \tag{3.5}$$

for all $x, y \in G$. Taking the supremum of the left-hand side of (3.5) with respect to $x \in G$, we get $M_f|f(y)| \le M_f + \phi(y)$ for all $y \in G$. Thus we have

$$M_f\big(\big|f(y)\big| - 1\big) \le \phi(y) \tag{3.6}$$

for all $y \in G$, which implies that

$$\big|f(x)\big|\big(\big|f(x)\big| - 1\big) \le \phi(x) \tag{3.7}$$

for all $x \in G$, and (3.2) follows. For every $y \in S_2, z \in G$, there exists $x = \lambda(y,z)$ such that $\tau(\lambda(y,z), y) = z$. Putting $x = \lambda(y,z)$ in (3.1) and using the triangle inequality, we have

$$\big|f(z)\big| \le \big|f\big(\lambda(y,z)\big)f(y)\big| + \phi(y) \le M_f\big|f(y)\big| + \phi(y) \tag{3.8}$$

for all $y \in S_2, z \in G$. Taking the supremum of the left-hand side of (3.8) with respect to $z \in G$, we get $M_f \le M_f|f(y)| + \phi(y)$ for all $y \in G$. Thus we have

$$M_f\big(1 - \big|f(y)\big|\big) \le \phi(y) \tag{3.9}$$

for all $y \in S_2$, which implies that

$$\big|f(x)\big|\big(1 - \big|f(x)\big|\big) \le \phi(x) \tag{3.10}$$

for all $x \in S_2$. For each fixed $x \in S_2 \cap B$, solving the inequalities (3.7) and (3.10) simultaneously, we have

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x)}\big) \le \big|f(x)\big| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x)}\big) \tag{3.11}$$

or

$$\big|f(x)\big| \le \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x)}\big). \tag{3.12}$$

Now, assume that there exist $x_1, x_2 \in S_2 \cap B$ such that

$$\big|f(x_1)\big| \le \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x_1)}\big) \quad \text{and} \quad \big|f(x_2)\big| \ge \frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x_2)}\big). \tag{3.13}$$

Then from (3.9) we have

$$\big|f(x_2)\big|\big(1 - \big|f(x_1)\big|\big) \le M_f\big(1 - \big|f(x_1)\big|\big) \le \phi(x_1). \tag{3.14}$$

On the other hand, from (3.13) we have

$$\begin{aligned}\big|f(x_2)\big|\big(1 - \big|f(x_1)\big|\big) &\ge \frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x_2)}\big)\Big(1 - \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x_1)}\big)\Big) \\ &> \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x_1)}\big)\Big(1 - \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x_1)}\big)\Big) = \phi(x_1),\end{aligned}$$

which contradicts (3.14). Thus f satisfies either (3.3) for all $x \in S_2 \cap B$ or satisfies (3.4) for all $x \in S_2 \cap B$. Since the left-hand side of (3.1) is symmetric in x and y, then the proof is complete. □

Theorem 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.7). If f is bounded, then we have*

$$\big|f(x,y)\big| \leq \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x,y)}\big) \tag{3.15}$$

for all $x, y \in \mathbb{R}$. Furthermore, let $K = \{(x,y) \in \mathbb{R}^2 : x^2 \neq y^2, \phi(x,y) < \frac{1}{4}\}$. Then f satisfies either

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x,y)}\big) \leq \big|f(x,y)\big| \leq \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x,y)}\big) \tag{3.16}$$

for all $(x,y) \in K$ or

$$\big|f(x,y)\big| \leq \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x,y)}\big) \tag{3.17}$$

for all $(x,y) \in K$. If f is unbounded, then there exists a multiplicative function $M : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = M(x^2 - y^2) \tag{3.18}$$

for all $x, y \in \mathbb{R}$.

Proof. If f is bounded, applying Lemma 3.1, since $S_2 = S_1 = \{(x,y) \in \mathbb{R}^2 : x^2 \neq y^2\}$, we get (3.15), (3.16), and (3.17). Assume that f is unbounded. We first consider the case when f satisfies

$$\big|f(ux - vy, uy - vx) - f(x,y)f(u,v)\big| \leq \phi(u,v) \tag{3.19}$$

for all $x, y, u, v \in \mathbb{R}$. Choose a sequence $(p_n, q_n) \in \mathbb{R}^2, n = 1, 2, 3, \ldots,$ such that $|f(p_n, q_n)| \to \infty$ as $n \to \infty$. Replacing (x,y) by (p_n, q_n) in (3.19) and dividing the result by $|f(p_n, q_n)|$, we have

$$\Big|f(u,v) - \frac{f(up_n - vq_n, uq_n - vp_n)}{f(p_n, q_n)}\Big| \leq \frac{\phi(u,v)}{|f(p_n, q_n)|}. \tag{3.20}$$

Letting $n \to \infty$ in (3.20), we have

$$f(u,v) = \lim_{n\to\infty} \frac{f(up_n - vq_n, uq_n - vp_n)}{f(p_n, q_n)} \tag{3.21}$$

for all $u, v \in \mathbb{R}$. Multiplying both sides of (3.21) by $f(x,y)$ and using (3.19) and (3.21), we have

$$\begin{aligned} f(u,v)f(x,y) &= \lim_{n\to\infty} \frac{f(up_n - vq_n, uq_n - vp_n)f(x,y)}{f(p_n, q_n)} \\ &= \lim_{n\to\infty} \frac{f((ux+vy)p_n - (uy+vx)q_n, (ux+vy)q_n - (uy+vx)p_n)}{f(p_n, q_n)} \\ &= f(ux + vy, uy + vx) \end{aligned} \tag{3.22}$$

for all $x, y, u, v \in \mathbb{R}$. Replacing v by $-v$ in (3.22), we have

$$f(u,-v)f(x,y) = f(ux - vy, uy - vx) \tag{3.23}$$

for all $u, v, x, y \in \mathbb{R}$. From (3.19) and (3.23) we have

$$\big|f(x,y)\big|\big|f(u,v) - f(u,-v)\big| \leq \phi(u,v) \tag{3.24}$$

for all $x, y, u, v \in \mathbb{R}$. Since f is unbounded, we have

$$f(u, v) = f(u, -v) \tag{3.25}$$

for all $u, v \in \mathbb{R}$. Thus, from (3.23) and (3.25), we get the equation

$$f(ux - vy, uy - vx) = f(x, y)f(u, v) \tag{3.26}$$

for all $x, y, u, v \in \mathbb{R}$. Second, we consider the case when f satisfies

$$\big|f(ux - vy, uy - vx) - f(x, y)f(u, v)\big| \le \phi(x, y) \tag{3.27}$$

for all $x, y, u, v \in \mathbb{R}$. Putting $(x, y) = (1, 0)$ in (3.27), we have

$$\big|f(u, -v) - f(1, 0)f(u, v)\big| \le \phi(1, 0) \tag{3.28}$$

for all $u, v \in \mathbb{R}$. Choose a sequence $(p_n, q_n) \in \mathbb{R}^2, n = 1, 2, 3, \ldots,$ such that $|f(p_n, q_n)| \to \infty$ as $n \to \infty$. Replacing (u, v) by (p_n, q_n) in (3.27), dividing the result by $|f(p_n, q_n)|$, and letting $n \to \infty$, we have

$$f(x, y) = \lim_{n\to\infty} \frac{f(p_n x - q_n y, p_n y - q_n x)}{f(p_n, q_n)} \tag{3.29}$$

for all $x, y \in \mathbb{R}$. Multiplying both sides of (3.29) by $f(u, v)$ and using (3.27), (3.28), and (3.29), we have

$$\begin{aligned}
&f(u, v)f(x, y)\\
&\quad= \lim_{n\to\infty} \frac{f(u, v)f(p_n x - q_n y, p_n y - q_n x)}{f(p_n, q_n)}\\
&\quad= \lim_{n\to\infty} \frac{f(p_n(ux - vy) - q_n(uy - vx), p_n(vx - uy) - q_n(vy - ux))}{f(p_n, q_n)}\\
&\quad= \lim_{n\to\infty} \frac{f(1, 0)f(p_n(ux - vy) - q_n(uy - vx), p_n(uy - vx) - q_n(ux - vy))}{f(p_n, q_n)}\\
&\quad= f(1, 0)f(ux - vy, uy - vx)
\end{aligned} \tag{3.30}$$

for all $x, y, u, v \in \mathbb{R}$. Putting $(x, y) = (1, 0)$ in (3.30), we have

$$f(u, v) = f(u, -v) \tag{3.31}$$

for all $u, v \in \mathbb{R}$. From (3.28) and (3.31), we have

$$\big|f(u, v)\big|\big|1 - f(1, 0)\big| \le \phi(1, 0) \tag{3.32}$$

for all $u, v \in \mathbb{R}$. Since f is unbounded, from (3.32) we have $f(1, 0) = 1$. Thus from (3.30), we get

$$f(ux - vy, uy - vx) = f(x, y)f(u, v) \tag{3.33}$$

for all $x, y, u, v \in \mathbb{R}$. By Theorem 2.1, we get (3.18). This completes the proof. □

As a direct consequence of Theorem 3.2, we have the following.

Corollary 3.3. *Let $0 \le \epsilon < \frac{1}{4}$. Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded function satisfying*

$$\big|f(ux - vy, uy - vx) - f(x, y)f(u, v)\big| \le \epsilon \tag{3.34}$$

for all $x, y, u, v \in \mathbb{R}$. Then f satisfies

$$\big|f(x, y)\big| \le \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{3.35}$$

for all $x, y \in \mathbb{R}$. If we let $K = \{(x, y) \in \mathbb{R}^2 : x^2 \ne y^2\}$, then f also satisfies either

$$\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \le \big|f(x, y)\big| \le \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{3.36}$$

for all $(x, y) \in K$ or f satisfies

$$\big|f(x, y)\big| \le \frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) \tag{3.37}$$

for all $(x, y) \in K$.

Remark 3.4. The inequalities (3.36) and (3.37) imply that $\sup_{(x,y)\in K} |f(x, y)|$ is close to 1 or 0 provided that ϵ is small. However, it is not true in general that $\sup_{(x,y)\in\mathbb{R}^2} |f(x, y)|$ is close to 1 or 0. Indeed, choose a function f such that

$$\sup_{(x,y)\in\mathbb{R}^2\setminus K} \big|f(x, y)\big| \approx 0, \qquad \sup_{(x,y)\in K} \big|f(x, y) - 1\big| \approx 0.$$

If $(x, y) \in \mathbb{R}^2 \setminus K$ or $(u, v) \in \mathbb{R}^2 \setminus K$, then we have $(ux - vy, uy - vx) \in \mathbb{R}^2 \setminus K$. Thus it follows that $f(ux - vy, uy - vx) \approx 0, f(x, y)f(u, v) \approx 0$ and that

$$\big|f(ux - vy, uy - vx) - f(x, y)f(u, v)\big| \approx 0. \tag{3.38}$$

If $(x, y) \in K$ and $(u, v) \in K$, then we have $(ux-vy, uy-vx) \in K$. Thus it follows that $f(ux - vy, uy - vx) \approx 1, f(x, y) \approx 1, f(u, v) \approx 1$ and that

$$\big|f(ux - vy, uy - vx) - f(x, y)f(u, v)\big| \approx 0. \tag{3.39}$$

From (3.38) and (3.39), f satisfies (3.34) for all $x, y, u, v \in \mathbb{R}$, but $f(x, y)$ satisfies (3.36) for all $(x, y) \in K$ and satisfies (3.37) for all $(x, y) \in \mathbb{R}^2 \setminus K$.

The stability theorems of (1.8) are considered in [5, Theorem 3.6] and [9]. Here, we state the result from [5].

Theorem 3.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.8). If f is bounded, then f satisfies*

$$\big|f(x, y)\big| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x, y)}\big) \tag{3.40}$$

for all $x, y \in \mathbb{R}$. Let $K = \{(x, y) \in \mathbb{R}^2 : (x, y) \ne (0, 0), \phi(x, y) < \frac{1}{4}\}$. Then f satisfies

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x, y)}\big) \le \big|f(x, y)\big| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x, y)}\big) \tag{3.41}$$

for all $(x, y) \in K$ or

$$\big|f(x, y)\big| \le \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x, y)}\big) \tag{3.42}$$

for all $(x, y) \in K$. If f is unbounded, then there exists a multiplicative function $M : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, y) = M(x^2 + y^2) \tag{3.43}$$

for all $x, y \in \mathbb{R}$.

Now, we consider the functional inequality (1.9), which is symmetric with respect to (x, y) and (u, v); thus it suffices to consider

$$\big|f(ux + vy, uy + vx) - f(x, y)f(u, v)\big| \le \phi(u, v) \tag{3.44}$$

for all $x, y, u, v \in \mathbb{R}$.

Theorem 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (3.44). If f is bounded, then f satisfies (3.40) for all $x, y \in \mathbb{R}$. Let $K := \{(x, y) \in \mathbb{R}^2 : x^2 \neq y^2, \phi(x, y) < \frac{1}{4}\}$. Then f either satisfies (3.41) for all $(x, y) \in K$ or satisfies (3.42) for all $(x, y) \in K$. If f is unbounded, then there exist multiplicative functions $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x, y) = M_1(x + y)M_2(x - y) \tag{3.45}$$

for all $x, y \in \mathbb{R}$.

Proof. If f is bounded, then the result follows immediately from Lemma 3.1 since $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 \neq y^2\}$. Assume that f is unbounded. Choose a sequence $(p_n, q_n) \in \mathbb{R}^2, n = 1, 2, 3, \ldots$, such that $|f(p_n, q_n)| \to \infty$ as $n \to \infty$. Replacing (x, y) by (p_n, q_n) in (3.44), dividing the result by $|f(p_n, q_n)|$, and letting $n \to \infty$, we have

$$f(u, v) = \lim_{n\to\infty} \frac{f(up_n + vq_n, uq_n + vp_n)}{f(p_n, q_n)} \tag{3.46}$$

for all $u, v \in \mathbb{R}$. Multiplying both sides of (3.46) by $f(x, y)$ and using (3.44) and (3.46), we have

$$\begin{aligned} f(u, v)f(x, y) &= \lim_{n\to\infty} \frac{f(up_n + vq_n, uq_n + vp_n)f(x, y)}{f(p_n, q_n)} \\ &= \lim_{n\to\infty} \frac{f((ux + vy)p_n + (uy + vx)q_n, (ux + vy)q_n + (uy + vx)p_n)}{f(p_n, q_n)} \\ &= f(ux + vy, uy + vx) \end{aligned} \tag{3.47}$$

for all $u, v, x, y \in \mathbb{R}$. By Theorem 2.3 we get (3.45). This completes the proof. □

Finally, we consider the inequality (1.10), which is also symmetric with respect to (x, y) and (u, v). Thus it suffices to consider

$$\big|f(ux - vy, uy + vx) - f(x, y)f(u, v)\big| \le \phi(u, v) \tag{3.48}$$

for all $x, y, u, v \in \mathbb{R}$.

Theorem 3.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (3.48). If f is bounded, then f satisfies (3.40) for all $x, y \in \mathbb{R}$. Let $K := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0), \phi(x, y) < \frac{1}{4}\}$. Then f either satisfies (3.41) for all $(x, y) \in K$ or satisfies (3.42) for all $(x, y) \in K$. If f is unbounded, then f has the form*

$$f(x, y) = M(x^2 + y^2)E\big(\tan^{-1}(y/x)\big) \tag{3.49}$$

for all $x, y \in \mathbb{R}$, where M is a multiplicative function and $E : \mathbb{R} \to \mathbb{R}$ is an exponential function satisfying $E(x + r\pi) = E(x)$ for all $x \in \mathbb{R}, r \in \mathbb{Q}$.

Proof. If f is bounded, then the result follows immediately from Lemma 3.1 since $S_2 = \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$. Assume that f is unbounded. Choose a sequence $(p_n, q_n) \in \mathbb{R}^2, n = 1, 2, 3, \ldots$, such that $|f(p_n, q_n)| \to \infty$ as $n \to \infty$. Replacing (x, y) by (p_n, q_n) in (3.48), dividing the result by $|f(p_n, q_n)|$, and letting $n \to \infty$, we have

$$f(u, v) = \lim_{n\to\infty} \frac{f(up_n - vq_n, uq_n + vp_n)}{f(p_n, q_n)} \tag{3.50}$$

for all $u, v \in \mathbb{R}$. Multiplying both sides of (3.50) by $f(x, y)$ and using (3.48) and (3.50), we have

$$\begin{aligned} f(u, v)f(x, y) &= \lim_{n\to\infty} \frac{f(up_n - vq_n, uq_n + vp_n)f(x, y)}{f(p_n, q_n)} \\ &= \lim_{n\to\infty} \frac{f((ux - vy)p_n - (uy + vx)q_n, (ux - vy)q_n + (uy + vx)p_n)}{f(p_n, q_n)} \\ &= f(ux - vy, uy + vx) \end{aligned} \tag{3.51}$$

for all $u, v, x, y \in \mathbb{R}$. By Theorem 2.4 we get (3.49). This completes the proof. □

In particular, if $\phi(x, y) = \epsilon < \frac{1}{4}$ for all $x, y \in \mathbb{R}$, then we can give a refined description of bounded solutions of (1.10).

Corollary 3.8. *Let $0 \leq \epsilon < \frac{1}{4}$. Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded function satisfying*

$$\left|f(ux - vy, uy + vx) - f(x, y)f(u, v)\right| \leq \epsilon \tag{3.52}$$

for all $x, y, u, v \in \mathbb{R}$. Then f either satisfies

$$\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \leq f(x, y) \leq \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{3.53}$$

for all $(x, y) \in \mathbb{R}_0^2$ or f satisfies

$$-\epsilon \leq f(x, y) \leq \frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) \tag{3.54}$$

for all $(x, y) \in \mathbb{R}_0^2$.

Proof. Since $\epsilon < \frac{1}{4}$, we have $K = \mathbb{R}_0^2$. Thus, by Theorem 3.7, we have

$$\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \leq \left|f(x, y)\right| \leq \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{3.55}$$

for all $(x, y) \in \mathbb{R}_0^2$ or

$$\left|f(x, y)\right| \leq \frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) \tag{3.56}$$

for all $(x, y) \in \mathbb{R}_0^2$. Replacing (u, v) by (x, y) in (3.52) and using the triangle inequality, we have

$$f(x^2 - y^2, 2xy) \geq f(x, y)^2 - \epsilon \geq -\epsilon \tag{3.57}$$

for all $(x,y)\in\mathbb{R}^2$. Since $\{(x^2-y^2,2xy): x,y\in\mathbb{R}\}=\mathbb{R}^2$, it follows that

$$f(x,y)\geq -\epsilon \tag{3.58}$$

for all $(x,y)\in\mathbb{R}^2$. Since $\epsilon<\frac{1}{4}$, we have $\epsilon<\frac{1}{2}(1-\sqrt{1-4\epsilon})$. Thus the inequality (3.53) follows from (3.55) and (3.58), and the inequality (3.54) follows from (3.56) and (3.58). This completes the proof. □

Remark 3.9. The inequalities (3.53) and (3.54) imply that $\sup_{(x,y)\in\mathbb{R}_0^2}|f(x,y)-1|$ is close to 0 or $\sup_{(x,y)\in\mathbb{R}_0^2}|f(x,y)|$ is close to 0 provided that ϵ is small. However, the value $f(0,0)$ may not be close to $f(\mathbb{R}_0^2)$. Indeed, choose a function f such that $f(0,0)\approx 0$, $\sup_{(x,y)\in\mathbb{R}_0^2}|f(x,y)-1|\approx 0$. If $(x,y)=(0,0)$ or $(u,v)=(0,0)$, then we have

$$\big|f(0,0)-f(x,y)f(0,0)\big|\approx 0. \tag{3.59}$$

If $(x,y)\neq(0,0)$ and $(u,v)\neq(0,0)$, then $(ux-vy,uy+vx)\neq(0,0)$. Therefore, it follows that $f(ux-vy,uy-vx)\approx 1$, $f(x,y)\approx 1$, $f(u,v)\approx 1$ and that

$$\big|f(ux-vy,uy+vx)-f(x,y)f(u,v)\big|\approx 0. \tag{3.60}$$

From (3.59) and (3.60), f satisfies (3.52), but $|f(x,y)-f(0,0)|\approx 1$ for all $(x,y)\in\mathbb{R}_0^2$.

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