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REGULARITIES OF SEMIGROUPS, CARLESON MEASURES AND THE CHARACTERIZATIONS OF BMO-TYPE SPACES ASSOCIATED WITH GENERALIZED SCHRÖDINGER OPERATORS

YUANYUAN HAO, PENGTAO LI,* and KAI ZHAO

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ABSTRACT. Let $\mathcal{L} = -\Delta + \mu$ be the generalized Schrödinger operator on $\mathbb{R}^n, n \geq 3$, where Δ is the Laplacian and $\mu \not\equiv 0$ is a nonnegative Radon measure on $\mathbb{R}^n$. In this article, we introduce two families of Carleson measures $\{d\nu_{h,k}\}$ and $\{d\nu_{P,k}\}$ generated by the heat semigroup $\{e^{-t\mathcal{L}}\}$ and the Poisson semigroup $\{e^{-t\sqrt{\mathcal{L}}}\}$, respectively. By the regularities of semigroups, we establish the Carleson measure characterizations of BMO-type spaces $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ associated with the generalized Schrödinger operators.

1. Introduction

Let $\mathcal{L} = -\Delta+\mu$ be a generalized Schrödinger operator, where μ is a nonnegative Radon measure on $\mathbb{R}^n, n \geq 3$. In this article, we will characterize the BMO-type space associated with $\mathcal{L}$ via two families of Carleson measures generated by the semigroups $\{e^{-t\mathcal{L}}\}$ and $\{e^{-t\sqrt{\mathcal{L}}}\}$, respectively.

As in [13] and [20], throughout this article we assume that μ satisfies the following conditions: there exist positive constants C_0, C_1, and δ such that, for all $x \in \mathbb{R}^n$ and $0 < r < R$,

$$\mu\big(B(x,r)\big) \leq C_0(r/R)^{n-2+\delta}\mu\big(B(x,R)\big) \tag{1.1}$$

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*Corresponding author.
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and

$$\mu\big(B(x,2r)\big) \le C_1\big\{\mu\big(B(x,r)\big) + r^{n-2}\big\}, \tag{1.2}$$

where $B(x,r)$ denotes the open ball centered at x with radius r. Shen [13] pointed out that (1.1) may be regarded as the scale-invariant Kato condition, and (1.2) says that the measure μ is a doubling measure satisfying for any ball $B(x,r) \ge cr^{n-2}$. Let $(RH)_q$ denote the set of all nonnegative locally L^q-functions on $\mathbb{R}^n$ satisfying that there exists $C > 0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)\,dy\Big)^{1/q} \le C\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy\Big)$$

holds for every ball $B \in \mathbb{R}^n$. When $d\mu = V(x)\,dx$ and $V \ge 0$ belongs to $(RH)_{n/2}$, then μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$.

The bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ was first introduced by John and Nirenberg in their study [8] of mappings from a bounded set Ω belonging to $\mathbb{R}^n$ into $\mathbb{R}^n$ and the corresponding problems arising from elasticity theory, precisely from the concept of elastic strain. In 1972, Fefferman and Stein [6] showed that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$. As an adequate substitute for the Lebesgue space $L^\infty(\mathbb{R}^n)$, the space $\mathrm{BMO}(\mathbb{R}^n)$ is widely used in various fields of analysis and partial differential equations. Since the 1960s, based on a similar idea, various BMO-type spaces were introduced by many mathematicians in different settings. (We refer the reader to [12], [17], [18], and [19] for further information.)

Let $\mathcal{L}$ be a Schrödinger operator with nonnegative potential. In recent years, the BMO-type space associated with $\mathcal{L}$ has become one of the hot issues in harmonic analysis. As the dual of the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ (see [5]), Dziubański, Garrigós, Martínez, Torrea, and Zienkiewicz [4] introduced the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ related to $\mathcal{L}$ under the assumption that the potential $V \in (RH)_q, q > n/2$. Wu and Yan [20] studied the BMO-type spaces associated with the generalized Schrödinger operators, where the potential is a nonnegative Radon measure on $\mathbb{R}^n$. (For further information on BMO-type spaces associated with operators, we refer the reader to [2], [3], [9], [21], [22] and the references therein.)

Our motivation is inspired by the following observation. A positive measure ν on $\mathbb{R}^{n+1}_+$ is called a *Carleson measure* if

$$\|\nu\|_{\mathcal{C}} =: \sup_{x\in\mathbb{R}^n, r>0} \frac{\nu(B(x,r)\times(0,r))}{|B(x,r)|} < \infty.$$

It is well known that Carleson measures and their generalizations are important tools for the characterization of function spaces. Fefferman and Stein [6] established the Carleson measure characterization of $\mathrm{BMO}(\mathbb{R}^n)$. From then on, this characterization was extended to other function spaces (see [1], [2], [4], [14] and the references therein). Let $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int \psi\,dx = 0$. For such a function ψ, set $\psi_t(x) = t^{-n}\psi(x/t)$. The following Carleson measure characterization of $\mathrm{BMO}(\mathbb{R}^n)$ is well known.

Theorem 1.1 ([16, Sections 4.3, 4.4.3])**.**

(i) *Suppose that* $f \in \mathrm{BMO}(\mathbb{R}^n)$*, and let* $d\nu = |f * \psi_t(x)|\, dx\, dt/t$*. Then* $d\nu$ *is a Carleson measure.*

(ii) *Conversely, suppose that* ψ *is a function mentioned above. If* $f \in L^1(dx/(1+|x|^{n+1}))$ *and* $d\nu = |f * \psi_t(x)|\, dx\, dt/t$ *is a Carleson measure, then* $f \in \mathrm{BMO}(\mathbb{R}^n)$*.*

In particular, in Theorem 1.1, if we take $\psi(x) = \frac{\partial h_t(x)}{\partial t}|_{t=1}$ and $\psi(x) = \frac{\partial P_t(x)}{\partial t}|_{t=1}$, where $h_t(\cdot)$ and $p_t(\cdot)$ are the heat kernel and the Poisson kernel, that is,

$$\begin{cases} h_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \\ p_t(x) = \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}, \quad c_n = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2}, \end{cases}$$

respectively, then we can obtain the Carleson measure characterizations of $\mathrm{BMO}(\mathbb{R}^n)$ associated with the semigroups $\{e^{-t(-\Delta)}\}_{t>0}$ and $\{e^{-t\sqrt{-\Delta}}\}_{t>0}$, respectively. This observation prompted us to investigate analogous characterizations of the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ for the generalized Schrödinger operator $\mathcal{L}$. Denote by $\mathbb{Z}^+$ the set of all positive integers. For $k \in \mathbb{Z}^+$, we introduce two families of operators:

$$\begin{cases} Q^{\mathcal{L}}_{t,k}(f) =: t^{2k}(\frac{d^k}{ds^k} e^{-s\mathcal{L}}|_{s=t^2}) f, \\ D^{\mathcal{L}}_{t,k}(f) =: t^{k}(\frac{d^k}{dt^k} e^{-t\sqrt{\mathcal{L}}}) f. \end{cases} \tag{1.3}$$

Let $f \in L^1(dx/(1+|x|^{n+1}))$. The Carleson measures with respect to $Q^{\mathcal{L}}_{t,k}$ and $D^{\mathcal{L}}_{t,k}$ are defined as

$$d\nu_{h,k}(x,t) =: \left|Q^{\mathcal{L}}_{t,k}(f)(y)\right|^2 dy\, dt/t \quad \forall (x,t) \in \mathbb{R}^{n+1}_+ \tag{1.4}$$

and

$$d\nu_{P,k}(x,t) =: \left|D^{\mathcal{L}}_{t,k}(f)(y)\right|^2 dy\, dt/t \quad \forall (x,t) \in \mathbb{R}^{n+1}_+. \tag{1.5}$$

Our aim is to establish the Carleson measure characterizations of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ via $\{d\nu_{h,k}\}$ and $\{d\nu_{P,k}\}$, respectively. For this purpose, we first introduce some regularity estimates of $\{e^{-t\mathcal{L}}\}$ and $\{e^{-t\sqrt{\mathcal{L}}}\}$ (see Propositions 2.15, 2.18). Such regularity estimates indicate that the kernels of $Q^{\mathcal{L}}_{t,k}$ and $D^{\mathcal{L}}_{t,k}$ have good decay properties. We can prove that if $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, then $d\nu_{h,k}$ and $d\nu_{P,k}$ are Carleson measures.

Conversely, let $f \in L^1(dx/(1+|x|^{n+1}))$. Assume that $d\nu_{h,k}$ and $d\nu_{P,k}$ are Carleson measures. For any $H^1_{\mathcal{L}}$-atom a, we get that $S^{\mathcal{L}}_{h,k}(a) \in L^1(\mathbb{R}^n)$ and $S^{\mathcal{L}}_{P,k}(a) \in L^1(\mathbb{R}^n)$, uniformly (see Lemmas 3.2, 4.2). With the help of tent spaces, the identities (3.5) and (4.3) enable us to deduce that $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ (see Theorems 3.3, 4.3).

Remark 1.2.

(i) Theorems 3.3 and 4.3 show that the Carleson measure characterizations associated with $\{e^{-t\mathcal{L}}\}$ and $\{e^{-t\sqrt{\mathcal{L}}}\}$ are equivalent. In particular, let $\mathcal{L} = -\Delta$. Theorems 3.3 and 4.3 go back to Theorem 1.1 with $\psi = \frac{\partial h_t}{\partial t}|_{t=1}$ and $\psi = \frac{\partial P_t}{\partial t}|_{t=1}$, respectively. Philosophically speaking, our results reveal that for $k \in \mathbb{Z}_+$, the families of measures $\{d\nu_{h,k}\}$ and $\{d\nu_{P,k}\}$, induced by

$\{Q_{t,k}\}_{k\in\mathbb{Z}_+}$ and $\{D_{t,k}\}_{k\in\mathbb{Z}_+}$, play the same role in the characterization of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$.

(ii) For the Schrödinger operator $\mathcal{L} = -\Delta + \mu$, where $d\mu = V\,dx$ with $V \in (RH)_q$, the authors in [4] obtained a Carleson measure characterization of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$. For the case of the generalized Schrödinger operator $\mathcal{L} = -\Delta + \mu$, letting $\mu = V \in (RH)_q$ and $k = 1$, Theorem 3.3 coincides with [4, Theorem 2]. Hence our result is a generalization of [4, Theorem 2]. Moreover, for the special case $\mu = V \in B_q$, the Carleson measure characterization related to $\{e^{-t\sqrt{\mathcal{L}}}\}$ obtained in Theorem 4.3 partly generalizes the result of [10, Theorem 1.5].

We give the following notation.

- $\mathsf{U} \approx \mathsf{V}$ represents that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \le \mathsf{U} \le c\mathsf{V}$, whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \ge c\mathsf{U}$.
- For convenience, the positive constants C may change from one line to another and usually depend on the dimensions n, α, β and other fixed parameters.
- Let B be a ball with radius r. In the rest of this article, we denote by B_{2r} the ball with the same center and radius $2r$.

2. Preliminaries

2.1. Notation and function spaces associated with $\mathcal{L}$. Let μ be a Radon measure satisfying (1.1) and (1.2). The generalized Schrödinger operator $\mathcal{L} = -\Delta + \mu$ is defined as follows (see [13]). Consider the quadratic form

$$q[\phi, \psi] = \int_{\mathbb{R}^n} \langle \nabla\phi, \nabla\psi \rangle\, dx + \int_{\mathbb{R}^n} \langle \phi, \psi \rangle\, d\mu$$

with domain $W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$. Shen [13] pointed out that $q[\cdot,\cdot]$ is a semibound, symmetric closed form and that there exists a unique self-adjoint operator designated $-\Delta + \mu$ such that

$$q[\phi, \psi] = \big\langle (-\Delta + \mu)\phi, \psi \big\rangle_{L^2(\mathbb{R}^n, dx)}$$

for any $\phi \in \mathrm{Domain}(-\Delta + \mu)$ and $\psi \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$ (see [13, p. 528] for the details; we also refer the reader to [11] for more information on Schrödinger operators involving singular potentials and measure data).

The auxiliary function $m(x, \mu)$ is defined by

$$\frac{1}{m(x,\mu)} =: \sup\Big\{ r > 0 : \frac{\mu(B(x,r))}{r^{n-2}} \le C_1 \Big\}.$$

We recall some basic properties of $m(x, \mu)$.

Lemma 2.1 ([13, Proposition 1.8, Remark 1.9])**.** *Suppose that μ satisfies (1.1) and (1.2). Then the following hold.*

(i) *We have that $0 < m(x, \mu) < \infty$ for every $x \in \mathbb{R}^n$.*

(ii) *If $r = m(x, \mu)^{-1}$, then $r^{n-2} \le \mu(B(x, r)) \le C_1 r^{n-2}$.*

(iii) *If* $|x-y|\le Cm(x,\mu)^{-1}$, *then* $m(x,\mu)\approx m(y,\mu)$.

(iv) *There exist constants* $c, C>0$ *such that for* $x,y\in\mathbb{R}^n$,

$$\frac{cm(y,\mu)}{\{1+|x-y|m(y,\mu)\}^{k_0/(1+k_0)}}\le m(x,\mu)\le Cm(y,\mu)\big\{1+|x-y|m(y,\mu)\big\}^{k_0}$$

with $k_0=C_2/\delta>0$ *and* $C_2=\log_2(C_1+2^{n-2})$.

With the modified Agmon metric

$$ds^2=m(x,\mu)\{dx_1^2+\cdots+dx_n^2\},$$

the distance function $d(x,y,\mu)$ is given by

$$d(x,y,\mu)=\inf_{\gamma}\int_0^1 m\big(\gamma(\tau),\mu\big)\big|\gamma'(\tau)\big|\,d\tau, \tag{2.1}$$

where $\gamma:[0,1]\to\mathbb{R}^n$ is absolutely continuous and $\gamma(0)=x$, $\gamma(1)=y$. A parabolic-type distance function associated to $m(x,\mu)$ is defined by

$$d_\mu(x,y,t)=\inf_{\gamma}\int_0^1 m\big(\tilde{\gamma}(\tau),\mu\big)\max\big\{\big|(\tilde{\gamma})'(\tau)\big|,\big|(\gamma_{n+1})'(\tau)\big|\big\}\,d\tau, \tag{2.2}$$

where $\gamma(\tau)=(\gamma_1(\tau),\dots,\gamma_n(\tau))=(\tilde{\gamma}(\tau),\gamma_{n+1}(\tau)):[0,1]\to\mathbb{R}^n\times\mathbb{R}_+$ is absolutely continuous with $\gamma(0)=(x,0)$ and $\gamma(1)=(y,\sqrt{t})$.

Lemma 2.2. *For the distance function* $d(x,y,\mu)$ *in (2.1), we have that*

(i) *for every* $x,y,z\in\mathbb{R}^n$,

$$d(x,y,\mu)\le d(x,z,\mu)+d(z,y,\mu);$$

(ii) *there are two positive constants* c *and* C *such that for any* $x,y\in\mathbb{R}^n$,

$$c\Big\{\big[1+|x-y|m(x,\mu)\big]^{1/(k_0+1)}-1\Big\}\le d(x,y,\mu)\le C\big\{1+|x-y|m(x,\mu)\big\}^{k_0+1}.$$

Lemma 2.3 ([20, Lemma 2.3])**.** *For the distance function* $d_\mu(x,y,t)$ *defined by (2.2), there exist two positive constants* c *and* C *such that for any* $x,y\in\mathbb{R}^n$, $x\ne y$, *and* $t>0$,

$$d_\mu(x,y,y)\ge c\Big\{\big\{1+\max\{|x-y|,\sqrt{t}\}m(x,\mu)\big\}^{1/(k_0+1)}-1\Big\}$$

and

$$d_\mu(x,y,t)\le C\big\{1+\max\{|x-y|,\sqrt{t}\}m(x,\mu)\big\}^{k_0+1}.$$

It follows from (1.1), (1.2), and Lemma 2.1 that there exists a constant $C>0$ such that for every $x\in\mathbb{R}^n$ (see [20, (2.1)]),

$$\mu\big(B(x,r)\big)\le\begin{cases}C(rm(x,\mu))^{\delta}r^{n-2}, & r<m(x,\mu)^{-1},\\ C(rm(x,\mu))^{C_2}m(x,\mu)^{2-n}, & r\ge m(x,\mu)^{-1}.\end{cases} \tag{2.3}$$

Let $\mathcal{L}$ be a generalized Schrödinger operator. Denote by $\{T_t^{\mathcal{L}}\}_{t>0}:=\{e^{-t\mathcal{L}}\}_{t>0}$ the heat semigroup generated by $-\mathcal{L}$. The kernel of $\{T_t^{\mathcal{L}}\}$ is denoted by $K_t^{\mathcal{L}}(\cdot,\cdot)$; that is,

$$T_t^{\mathcal{L}}f(x)=\int_{\mathbb{R}^n}K_t^{\mathcal{L}}(x,y)f(y)\,d\mu(y).$$

Wu and Yan [20] introduced the following Hardy space associated with $\mathcal{L}$.

Definition 2.4. Let $\mathcal{L}$ be the generalized Schrödinger operator. The Hardy space associated with $\mathcal{L}$, $H^1_{\mathcal{L}}(\mathbb{R}^n)$, is defined as the set of all functions $f \in L^1(\mathbb{R}^n)$ satisfying

$$\mathcal{M}_{\mathcal{L}}(f)(x) =: \sup_{t>0} \left|T_t^{\mathcal{L}} f(x)\right| \in L^1(\mathbb{R}^n)$$

with the norm $\|f\|_{H^1_{\mathcal{L}}} =: \|\mathcal{M}_{\mathcal{L}}(f)\|_{L^1}$.

The $H^1_{\mathcal{L}}$-atoms were introduced by [20].

Definition 2.5. A function $a : \mathbb{R}^n \to \mathbb{C}$ is an $H^1_{\mathcal{L}}$-*atom* associated with a ball $B(x_0, r)$ if the following properties hold:

(i) $\operatorname{supp} a \subset B(x_0, r)$ with $r < 4/m(x_0, \mu)$,
(ii) $\|a\|_\infty \leq |B(x_0, r)|^{-1}$,
(iii) if $r \leq 1/m(x_0, \mu)$, then $\int a(x)\, dx = 0$.

Wu and Yan [20] obtained the following atomic decomposition for $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Theorem 2.6 ([20, Theorem 1.2])**.** *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assume that μ satisfies (1.1) and (1.2) for some $\delta > 0$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^1_{\mathcal{L}}$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathcal{L}}} \leq \inf\Big\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\Big\} \leq C\|f\|_{H^1_{\mathcal{L}}},$$

where the infimum is taken over all atomic decompositions of f into $H^1_{\mathcal{L}}$-atoms.

As the dual of $H^1_{\mathcal{L}}(\mathbb{R}^n)$, the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ was introduced by Wu and Yan [20]. Let f be a locally integrable function on $\mathbb{R}^n$, and let $B = B(x, r)$ be a ball. Denote by f_B the mean of f on B; that is, $f_B =: |B|^{-1} \int_B f(y)\, dy$. Let

$$f(B, \mu) = \begin{cases} f_B, & r < m(x, \mu)^{-1}, \\ 0, & r \geq m(x, \mu)^{-1}. \end{cases}$$

Definition 2.7. Let f be a locally integrable function on $\mathbb{R}^n$. We say that $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}_{\mathcal{L}}} =: \sup_B \frac{1}{|B|} \int_B \left|f(y) - f(B, \mu)\right| dy < \infty,$$

where the supremum is taken over all cubes with edges parallel to the axis.

Corollary 2.8. *It is easy to see that $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ and $\|f\|_{\mathrm{BMO}} \leq c\|f\|_{\mathrm{BMO}_{\mathcal{L}}}$. A simple deduction gives*

$$\sup_B \Big(\frac{1}{|B|} \int_B \left|f(y) - f(B, \mu)\right|^p dy\Big)^{1/p} \leq c\|f\|_{\mathrm{BMO}_{\mathcal{L}}}.$$

Given a ball B, denote by B^* the ball with the same center and twice the radius. We obtain the following covering lemma from [20, Lemmas 2.1, 2.7].

Proposition 2.9. *There exists a sequence of points* $\{x_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}^n$ *such that the family of critical balls* $\mathcal{B} = \{\mathcal{B}_k\}_{k=1}^{\infty}$ *defined by* $\mathcal{B}_k = \{x : |x - x_k| < 1/m(x_k, \mu)\}$ *satisfy the following.*

(i) *We have* $\bigcup_k \mathcal{B}_k = \mathbb{R}^n$.

(ii) *There exists* $N = N(\rho)$ *such that* $\operatorname{card}\{j : \mathcal{B}_j^{**} \cap \mathcal{B}_k^{**} \neq \varnothing\} \leq N$ *for all* $k \geq 1$. *Moreover, we have*

$$\big|B(x,R)\big| \leq \sum_{\mathcal{B}_k \cap B(x,R) \neq \varnothing} |\mathcal{B}_k| \leq c\big|B(x,R)\big|,$$

where $c = c(\delta)$ *and* $R > m(x,\mu)^{-1}$.

The following lemma can be easily deduced from the proofs of [20, Theorem 1.2] and [4, Theorem 4].

Lemma 2.10. *The correspondence*

$$\mathrm{BMO}_{\mathcal{L}} \ni f \to \Phi_f \in (H^1_{\mathcal{L}})^*$$

is a linear isomorphism of Banach spaces.

Similar to [4], the following lemma is also valid for the case of the generalized Schrödinger operator.

Lemma 2.11. *There exists* $c > 0$ *such that, for all* $f \in \mathrm{BMO}_{\mathcal{L}}$ *and* $B = B(x,r)$ *with* $r < m(x,\mu)^{-1}$, *we have*

$$|f_{B_{2r}}| \leq c\big(1 + \log\big(rm(x,\mu)\big)^{-1}\big)\|f\|_{\mathrm{BMO}_{\mathcal{L}}}.$$

The following result is well known.

Lemma 2.12 ([16, p. 162])**.** *Let* $F(\cdot,\cdot)$ *and* $G(\cdot,\cdot)$ *be two measurable functions on* $\mathbb{R}^{n+1}_+$ *satisfying*

$$\mathcal{I}(F)(x) =: \sup_{x\in B}\Big(\frac{1}{|B|}\int_0^{r(B)}\int_B \big|F(y,t)\big|^2 \frac{dy\,dt}{t}\Big)^{1/2} \in L^\infty(\mathbb{R}^n)$$

and

$$\mathcal{G}(G)(x) =: \Big(\iint_{\Gamma(x)} \big|G(y,t)\big|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2} \in L^1(\mathbb{R}^n),$$

where $r(B)$ *denotes the radius of* B *and* $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-t| < t\}$. *Then there is a universal* $c > 0$ *so that*

$$\int_{\mathbb{R}^{n+1}_+} \big|F(y,t)G(y,t)\big| \frac{dy\,dt}{t} \lesssim \int_{\mathbb{R}^n} \mathcal{I}(F)(x)\mathcal{G}(G)(x)\,dx \lesssim \big\|\mathcal{I}(F)\big\|_{L^\infty}\big\|\mathcal{G}(G)\big\|_{L^1}.$$

Lastly, we give a technical lemma.

Lemma 2.13. *Let* $S(\cdot,\cdot)$ *be a function satisfying for arbitrary* N, N',

$$\big|S(x,y)\big| \leq C_N t^{-n}\big(1 + |x-y|/t\big)^{-N'}\big(1 + tm(x,\mu) + tm(y,\mu)\big)^{-N}.$$

Then there is $C_{y_0,r} > 0$ *such that, for every* $H^1_{\mathcal{L}}$*-atom* a *supported on* $B(y_0,r)$,

$$\mathcal{M}_s a(x) = \sup_{t>0}\Big|\int_{\mathbb{R}^n} S(x,y)a(y)\,dy\Big| \lesssim C_{y_0,r}\big(1+|x|\big)^{-(n+1)}, \quad x \in \mathbb{R}^n.$$

Proof. The case $N = N'$ has been proved in [4, Lemma 7]. Without loss of generality, we assume that $r < 2m(y_0,\mu)^{-1}$. We consider two cases.

Case 1: $x \in B(y_0, 2r)$. For this case, $|x - y_0| < 2r < 4m(y_0,\mu)^{-1}$. We have

$$\Big|\int_{\mathbb{R}^n} S(x,y)a(y)\,dy\Big| \lesssim \|a\|_\infty \int_{B(y_0,r)} \frac{1}{t^n}\frac{dy}{(1+|x-y|/t)^M} \lesssim c\|a\|_\infty. \tag{2.4}$$

Note that $1 + |x| \le 1 + |y_0| + 2r$. We apply (2.4) to get

$$\mathcal{M}_s a(x) \lesssim c\|a\|_\infty\big(1+|y_0|+2r\big)^{n+1}\big(1+|x|\big)^{-(n+1)} =: C_{y_0,r}\big(1+|x|\big)^{-(n+1)}.$$

Case 2: $x \notin B(y_0, 2r)$. Then for $y \in B(y_0, 2r)$, we have $|x - y| \sim |x - y_0|$ and $m(y_0,\mu)^{-1} \sim m(y,\mu)^{-1}$. We divide the proof into the following two situations. For simplicity, let

$$A = t^{-n}\big(1+tm(y_0,\mu)\big)^{-N}\big(1+|x-y_0|/t\big)^{-N'}.$$

Case I: $t > |x - y_0|$. Let $N' = N$. Then

$$A \lesssim t^n\big(1+tm(y_0,\mu)\big)^{-N}\big(|x-y_0|/t\big)^{-N} \lesssim m(y_0,\mu)^{-N}|x-y_0|^{-n-N}.$$

Case II: $t \le |x - y_0|$. Let $N' = N + n$. Then

$$A \lesssim t^{-n}\big(1+tm(y_0,\mu)\big)^{-N}\big(|x-y_0|/t\big)^{-N-n} \lesssim m(y_0,\mu)^{-N}|x-y_0|^{-(n+N)}.$$

Thus, we obtain that, for arbitrary N,

$$\begin{aligned}\Big|\int_{R^n} S(x,y)a(y)\,dy\Big| &\lesssim \|a\|_1 t^{-n}\big(1+|x-y_0|/t\big)^{n+N}\big(1+tm(y_0,\mu)\big)^{-N}\\ &\lesssim |x-y_0|^{-n-N}m(y_0,\mu)^{-N}. \end{aligned}\tag{2.5}$$

It is easy to see that $[(1 + |x|)/|x - y_0|] \le (1/2r + |y_0|/2r + 1)$. Taking $N = 1$ in (2.5), we can get

$$\mathcal{M}_s a(x) \lesssim \big(1/2r+|y_0|/2r+1\big)^{n+1}\frac{m(y_0,\mu)}{(1+|x|)^{(n+1)}} =: C_{y_0,r}\big(1+|x|\big)^{-(n+1)}. \qquad \square$$

2.2. Regularity properties of semigroups. We begin with some basic properties of the kernels $K_t^{\mathcal{L}}(\cdot,\cdot)$. By the Feynman–Kac formula, it is well known that the kernel $K_t^{\mathcal{L}}(\cdot,\cdot)$ satisfies the following estimates:

$$0 \le K_t^{\mathcal{L}}(x,y) \le h_t(x-y) =: (4\pi t)^{-n/2}e^{-|x-y|^2/4t}.$$

Denote by $\Gamma_\mu(\cdot,\cdot)$ the fundamental solution of $-\Delta + \mu$. Shen [13] showed that Γ_μ satisfies the following optimal upper and lower bounds.

Proposition 2.14 ([13, Theorem 0.8])**.** *Let μ be a nonnegative Radon measure in $\mathbb{R}^n, n \ge 3$. Assume that μ satisfies the conditions (1.1) and (1.2) for some $\delta > 0$. Then*

$$\frac{ce^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \le \Gamma_\mu(x,y) \le \frac{Ce^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}},$$

where $\varepsilon_1, \varepsilon_2, C, c$ are positive constants depending only on n and constants C_0, C_1, δ in (1.1) and (1.2).

From the symmetry of Γ_μ, we can see that the kernel $K_t^{\mathcal{L}}(\cdot,\cdot)$ is symmetric. The following proposition can be deduced from (2.3), [4, Theorem 1.1], and the symmetry of $K_t^{\mathcal{L}}(\cdot,\cdot)$. (We refer the reader to [20, (1.6)] and [20, Lemma 3.7] for the details.)

Proposition 2.15.

(i) *For every M, there is a constant C_N such that*

$$0 \le K_t^{\mathcal{L}}(x,y) \le \frac{C_N}{t^{n/2}} \frac{e^{-c|x-y|^2/t}}{[1+\sqrt{t}m(x,\mu)+\sqrt{t}m(y,\mu)]^M}.$$

(ii) *For every $0<\delta'<\delta_0=\min\{\alpha,\delta,\nu\}$, there exists a constant C such that for every $M>0$ there exists a constant $C>0$ such that for $|h|<\sqrt{t}$ we have*

$$\left|K_t^{\mathcal{L}}(x+h,y)-K_t^{\mathcal{L}}(x,y)\right| \le C_M\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta'} \frac{1}{t^{n/2}} \frac{e^{-c|x-y|^2/t}}{[1+\sqrt{t}m(x,\mu)+\sqrt{t}m(y,\mu)]^M}.$$

Let $Q_{t,k}^{\mathcal{L}}(\cdot,\cdot)$ denote the integral kernel of $Q_{t,k}^{\mathcal{L}}$ defined in (1.3); that is,

$$Q_{t,k}^{\mathcal{L}}(x,y) =: t^{2k}\frac{d^k K_s^{\mathcal{L}}}{ds^k}\Big|_{s=t^2}(x,y).$$

Following the method of [20, Lemma 3.8], we can obtain the following results by Proposition 2.15.

Proposition 2.16. *The kernel $Q_{t,k}^{\mathcal{L}}(\cdot,\cdot)$ satisfies the following estimates.*

(i) *For $M>0$, there exists a constant $C_M>0$ such that*

$$\left|Q_{t,k}^{\mathcal{L}}(x,y)\right| \le C_M t^{-n} e^{-|x-y|^2/2t^2}\left[1+tm(x,\mu)+tm(y,\mu)\right]^{-M}.$$

(ii) *Let $0<\delta'<\min\{1,\delta\}$. For any $M>0$, there exists a constant $C_M>0$ such that for all $|h|<\sqrt{t}$,*

$$\left|Q_{t,k}^{\mathcal{L}}(x+h,y)-Q_{t,k}^{\mathcal{L}}(x,y)\right| \le C_M t^{-n}\Big(\frac{|h|}{t}\Big)^{\delta'} \frac{e^{-|x-y|^2/t^2}}{[1+tm(x,\mu)+tm(y,\mu)]^M}.$$

(iii) *For any $N>0$, there exists a constant $C_M>0$ such that*

$$\Big|\int_{\mathbb{R}^n} Q_{t,k}^{\mathcal{L}}(x,y)\,dy\Big| \le \big(tm(x,\mu)\big)^{\delta}\frac{C_M}{[1+tm(x,\mu)]^M}.$$

Let $\{e^{-t\sqrt{\mathcal{L}}}\}_{t>0}$ be the Poisson semigroup generated by $-\sqrt{\mathcal{L}}$. Denote by $P_t^{\mathcal{L}}(\cdot,\cdot)$ the integral kernel of $e^{-t\sqrt{\mathcal{L}}}$. Wu and Yan [20] proved that the kernel $P_t^{\mathcal{L}}(\cdot,\cdot)$ satisfies the following estimate.

Proposition 2.17 ([20, Proposition 3.2])**.** *Let $\{e^{-t\sqrt{\mathcal{L}}}\}_{t>0}$ be the Poisson semigroup generated by $-\sqrt{\mathcal{L}}$. Let $P_t^{\mathcal{L}}(x,y)$ be the integral kernel of $e^{-t\sqrt{\mathcal{L}}}$. We have*

$$\left|P_t^{\mathcal{L}}(x,y)\right| \le \frac{C_M t}{(t^2+4|x-y|^2)^{(n+1)/2}}\big(1+tm(x,\mu)\big)^{-M}\big(1+tm(y,\mu)\big)^{-M}.$$

By functional calculus and Proposition 2.15(ii), we can prove a regularity estimate of the kernel $P_t^{\mathcal{L}}(\cdot,\cdot)$. We omit the proof and refer the reader to [7, Proposition 3.5].

Proposition 2.18. *For every $0<\delta'<\delta_0=\min\{0,\delta\}$ there exists a constant C such that for every $N>0$ there exists a constant $C>0$ such that for $|h|<t$ we have*

$$\left|P_t^{\mathcal{L}}(x,y+h)-P_t^{\mathcal{L}}(x,y)\right| \le \frac{C_M t(|h|/t)^{\delta'}}{(t^2+|x-y|^2)^{(n+1)/2}}\left[1+tm(x,\mu)\right]^{-N}\left[1+tm(y,\mu)\right]^{-N}.$$

For $k\in\mathbb{Z}^+$, let $D_{t,k}^{\mathcal{L}}$ be the family of operators defined by (1.3). The kernels of the family $\{D_{t,k}^{\mathcal{L}}\}_{t>0}$ are defined as

$$D_{t,k}^{\mathcal{L}}(x,y)=:t^k\frac{\partial^k}{\partial t^k}P_t^{\mathcal{L}}(x,y). \tag{2.6}$$

With the help of Propositions 2.17 and 2.18, by imitating the procedure of [7, Proposition 3.9], we can obtain the following proposition for the kernel $D_{t,k}^{\mathcal{L}}(\cdot,\cdot)$.

Proposition 2.19. *For $k\in\mathbb{Z}^+$, the kernel $D_{t,k}^{\mathcal{L}}(\cdot,\cdot)$ defined as in (2.6) satisfies the following estimates.*

(i) *For every $M>0$ there exists a constant $C_M>0$ such that*

$$\left|D_{t,k}^{\mathcal{L}}(x,y)\right| \le \frac{C_M t}{(t^2+|x-y|^2)^{(n+1)/2}}\frac{1}{[1+tm(x,\mu)+tm(y,\mu)]^M}.$$

(ii) *For every $0<\delta'<\min\{1,\delta\}$ and every $M>0$ there exists a constant $C_M>0$ such that for all $|h|<\sqrt{t}$,*

$$\left|D_{t,k}^{\mathcal{L}}(x+h,y)-D_{t,k}^{\mathcal{L}}(x,y)\right| \le \frac{C_M(|h|/t)^{\delta'}t}{(t^2+|x-y|^2)^{(n+1)/2}}\frac{1}{[1+tm(x,\mu)+tm(y,\mu)]^M}.$$

(iii) *For every $M>0$ and k even there exists a constant $C_M>0$ such that*

$$\left|\int_{\mathbb{R}^n}D_{t,k}^{\mathcal{L}}(x,y)\,dy\right| \le \frac{C_M(tm(x,\mu))^{\delta}}{[1+tm(y,\mu)]^M}.$$

3. Carleson measure characterization associated with the heat semigroup

3.1. Reproducing formula generated by the heat kernel. Similar to [4], in this section, we first give a reproducing formula associated with $\{Q_{t,k}\}$ in the sense of L^2. For $\mu=V\in B_q$ and $k=1$, our result goes back to [4, Lemma 3]. For $k\in\mathbb{Z}_+$, define the Littlewood–Paley g-function associated with the heat semigroup as

$$g_{h,k}^{\mathcal{L}}(f)(x)=\left(\int_0^\infty\left|Q_{t,k}^{\mathcal{L}}f(x)\right|^2\frac{dt}{t}\right)^{1/2}.$$

Lemma 3.1. *For all $f\in L^2(\mathbb{R}^n)$, we have $\|g_{h,k}^{\mathcal{L}}(f)\|_2=\frac{1}{\sqrt{8}}\|f\|_2$. Moreover,*

$$f(x)=8\lim_{\varepsilon\to0,N\to\infty}\int_\varepsilon^N(Q_{t,k}^{\mathcal{L}})^2f(x)\frac{dt}{t}\quad in\ L^2(\mathbb{R}^n). \tag{3.1}$$

Proof. The proof of this lemma is similar to that of [4, Lemma 3]. By the spectral theorem, we can write the operator $T_t^{\mathcal{L}}$ in the form

$$T_t^{\mathcal{L}} f = e^{-t\mathcal{L}} f = \int_0^\infty e^{-t\lambda}\, dE(\lambda) f,$$

where $\{E(\lambda)\}$ is a resolution of the identity (see [15, Section 3.3, p. 74]). Hence,

$$t\frac{dT_t^{\mathcal{L}}}{dt} f = -t\mathcal{L}T_t^{\mathcal{L}} f = -\int_0^\infty t\lambda e^{-t\lambda}\, dE(\lambda) f.$$

Hence, for all $f \in L^2(\mathbb{R}^n)$, the self-adjointness of $Q_{t,k}$ implies that

$$\begin{aligned}
\|g_{h,k}^{\mathcal{L}} f\|_2^2 &= \int_{\mathbb{R}^n} \int_0^\infty \left|Q_{t,k}^{\mathcal{L}} f(x)\right|^2 \frac{dt}{t}\, dx \\
&= \int_0^\infty \Big\langle t^{4k}\Big(\frac{d^k}{ds^k} e^{-s\mathcal{L}}\Big|_{s=t^2}\Big)^2 f, f\Big\rangle \frac{dt}{t} \\
&= \int_0^\infty \Big[\int_0^\infty t^{4k}\lambda^{2k} e^{-2t^2\lambda}\frac{dt}{t}\Big]\, dE_{f,f}(\lambda) = \frac{1}{8}\|f\|_2^2.
\end{aligned}$$

For (3.1), we only need to prove that, for every pair of sequences $(\{n_l\},\{\varepsilon_l\})$ satisfying $n_l \nearrow \infty$ and $\varepsilon_l \searrow 0$,

$$\lim_{l\to\infty} \int_{n_l}^{n_{l+m}} (Q_{t,k}^{\mathcal{L}})^2 f \frac{dt}{t} = \lim_{l\to\infty} \int_{\varepsilon_{l+m}}^{\varepsilon_l} (Q_{t,k}^{\mathcal{L}})^2 f \frac{dt}{t} = 0 \quad \forall m \geq 1. \tag{3.2}$$

If (3.2) holds, then we can find $h \in L^2(\mathbb{R}^n)$ such that $\lim_{l\to\infty} \int_{\varepsilon_l}^{n_l} (Q_{t,k}^{\mathcal{L}})^2 f \frac{dt}{t} = h$. Using a polarized version of the first part, we obtain that for $g \in L^2(\mathbb{R}^n)$,

$$\langle h, g\rangle = \lim_{l\to\infty} \int_{\varepsilon_{l+m}}^{\varepsilon_l} \langle Q_{t,k}^{\mathcal{L}} f, Q_{t,k}^{\mathcal{L}} g\rangle \frac{dt}{t} = \int_0^\infty \langle Q_{t,k}^{\mathcal{L}} f, Q_{t,k}^{\mathcal{L}} g\rangle \frac{dt}{t} = \frac{1}{8}\langle f, g\rangle,$$

which implies that $h = \frac{1}{8} f$. To prove (3.2), we apply functional calculus to get

$$\Big\| \int_{n_l}^{n_{l+m}} (Q_{t,k}^{\mathcal{L}})^2 f \frac{dt}{t}\Big\|^2 \leq \int_0^\infty \Big| \int_{n_l}^{n_{l+m}} t^{4k}\lambda^{2k} e^{-2t^2\lambda} \frac{dt}{t}\Big|^2 dE_{f,f}(\lambda).$$

Computing the integral inside, one is led to the estimate

$$\int_0^\infty \Big[\sum_{j=1}^{2k-1} \frac{1}{2^{j+1}} \frac{(2k-1)!}{(2k-j)!} (n_l^2\lambda)^{2k-j} + \frac{(2k-1)!}{2^{2k+1}}\Big] e^{-2n_l^2\lambda}\, dE_{f,f}(\lambda), \quad \text{as } n_l \to \infty,$$

which by dominated convergence tends to zero. Because $\mu > 0$ for almost every x, $\langle \mathcal{L} f, f\rangle \geq \langle \mu f, f\rangle > 0$ (unless $f \equiv 0$). This means that zero is not an eigenvalue of $\mathcal{L}$. We can use a similar procedure to deal with the limit $\varepsilon_l \to 0$. This completes the proof of Lemma 3.1. ∎

Define the area function associated with the heat semigroup as

$$S_{h,k}^{\mathcal{L}}(f)(x) =: \Big(\int_0^\infty \int_{|x-y|<t} \left|Q_{t,k}^{\mathcal{L}} f(y)\right|^2 \frac{dy\, dt}{t^{n+1}}\Big)^{1/2}, \quad x \in \mathbb{R}^n.$$

Lemma 3.2. *Let f be a finite linear combination of $H^1_{\mathcal{L}}$-atoms. There exists $c > 0$ such that $\|S_h(f)\|_{\mathcal{L}^1} \le c\|f\|_{H^1_{\mathcal{L}}}$.*

Proof. By Theorem 2.6, it is enough to consider sums of atoms associated to balls $B(x_0, r)$ with $r \lesssim m(x_0, \mu)^{-1}$. Let a be an $H^1_{\mathcal{L}}$-atom with the support $B = B(x_0, r)$. Then we can apply Lemma 3.1 to deduce that

$$\begin{aligned}\big\|S^{\mathcal{L}}_{h,k}(a)\big\|^2_{L^2} &\lesssim \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^{n+1}_+}\big|Q^{\mathcal{L}}_{t,k}a(y)\big|^2\chi_{\Gamma(x)}(y,t)\frac{dy\,dt}{t^{n+1}}\Big]\,dx\\ &\lesssim \int_{\mathbb{R}^{n+1}_+}\big|Q^{\mathcal{L}}_{t,k}a(y)\big|^2\frac{dy\,dt}{t} \lesssim \big\|g^{\mathcal{L}}_{h,k}(a)\big\|^2_{L^2} \lesssim \frac{1}{8}\|a\|^2_{L^2}.\end{aligned}$$

Thus, using Hölder's inequality, we have

$$\begin{aligned}\int_{B_{8r}} S^{\mathcal{L}}_{h,k}a(x)\,dx &\lesssim |B_{8r}|^{1/2}\Big(\int_{B_{8r}} S^{\mathcal{L}}_{h,k}a(x)^2\,dx\Big)^{1/2}\\ &\lesssim |B|^{1/2}\|a\|_{L^2} = |B|^{1/2}\Big(\int_{B(x_0,r)}\frac{1}{|B(x_0,r)|^2}\,dx\Big)^{1/2} \lesssim 1,\end{aligned}$$

where in the last step we have used the fact that $\|a\|_\infty \le |B(x_0, r)|^{-1}$.

Next we prove that the integral

$$I =: \int_{|x-x_0|>8r} S^{\mathcal{L}}_{h,k}(a)(x)\,dx$$

is bounded for all $H^1_{\mathcal{L}}$-atoms a uniformly. We divide the proof into two cases.

Case 1: $r < m(x_0, \mu)^{-1}$. By the cancelation condition of a, we have

$$S^{\mathcal{L}}_{h,k}a(x) \lesssim T_1(x) + T_2(x),$$

where

$$T_1(x) =: \Big[\int_0^{|x-x_0|/2}\int_{|x-y|<t}\Big(\int_B\big|Q^{\mathcal{L}}_{t,k}(y,x') - Q^{\mathcal{L}}_{t,k}(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$

and

$$T_2(x) =: \Big[\int_{|x-x_0|/2}^{\infty}\int_{|x-y|<t}\Big(\int_B\big|Q^{\mathcal{L}}_{t,k}(y,x') - Q^{\mathcal{L}}_{t,k}(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}.$$

For T_1, if $x' \in B$, then $|y - x'| \sim |y - x_0| \sim |x - x_0|$ and $|x' - x_0| < |y - x_0|/4$. Applying Proposition 2.16(ii), we obtain the following estimate:

$$\begin{aligned}T_1(x) &\lesssim \Big\{\int_0^{|x-x_0|/2}\int_{|x-y|<t}\Big[\int_B\Big(\frac{|x-x_0|}{t}\Big)^{\delta'}\frac{1}{t^n}\frac{|B|^{-1}dx'}{(1+|y-x_0|/t)^{(n+1)}}\Big]^2\frac{dy\,dt}{t^{n+1}}\Big\}^{1/2}\\ &\lesssim \Big[\int_0^{|x-x_0|/2}\Big(\frac{r}{t}\Big)^{2\delta'}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)}\frac{dt}{t^{2n+1}}\Big]^{1/2} \lesssim \frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.\end{aligned}$$

For T_2, we can see that $|x'-x_0| \le r < |x-x_0|/2 \le t$ for $x' \in B$. Similar to the arguments of T_1, we can utilize Proposition 2.16(ii) again to get

$$T_2(x) \lesssim \Big[\int_{|x-x_0|/2}^{\infty} \int_{|x-y|<t} \Big(\int_B \frac{|x'-x_0|^{\delta'}}{t^{n+\delta'}} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$
$$\lesssim \Big[\int_{|x-x_0|/2}^{\infty} \Big(\frac{r}{t}\Big)^{2\delta'} t^{-2n} \frac{dt}{t}\Big]^{1/2} \lesssim \frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.$$

Then integrating $S^{\mathcal{L}}_{h,k}(a)$ over $(B_{8r})^c$ gives

$$\int_{|x-x_0|>8r} S^{\mathcal{L}}_{h,k}a(x)\,dx \lesssim \int_{|x-x_0|>8r} \big[T_1(x)+T_2(x)\big]\,dx$$
$$\lesssim \int_{|x-x_0|>8r} \frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}\,dx = 1.$$

Case 2: $m(x_0,\mu)^{-1} \le r < 4m(x_0,\mu)^{-1}$. Similar to Case 1 above, we divide the integral in $t>0$ defining $S^{\mathcal{L}}_{h,k}a$ into three parts: $S^{\mathcal{L}}_{h,k}(a)(x) \lesssim T_1'(x)+T_2'(x)+T_3'(x)$, where

$$T_1'(x) =: \Big[\int_0^{r/2} \int_{|x-y|<t} \Big(\int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2},$$
$$T_2'(x) =: \Big[\int_{r/2}^{|x-x_0|/4} \int_{|x-y|<t} \Big(\int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2},$$
$$T_3'(x) =: \Big[\int_{|x-x_0|/4}^{\infty} \int_{|x-y|<t} \Big(\int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}.$$

For T_1', it is easy to see that $|x'-y| \backsim |x-x_0|$. Using Proposition 2.16(i), we get

$$T_1'(x) \lesssim \Big[\int_0^{r/2} \int_{|x-y|<t} \Big(\int_B t^{-n}\Big(1+\frac{|y-x'|}{t}\Big)^{-(n+1)} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$
$$\lesssim \Big[\int_0^{r/2} \int_{|x-y|<t} t^{-2n}\Big(1+\frac{|x-x_0|}{t}\Big)^{-2(n+1)} \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$
$$\lesssim \Big[\int_0^{r/2} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)} \frac{dt}{t}\Big]^{1/2} \lesssim \frac{r}{|x-x_0|^{n+1}}.$$

For T_2', note that the fact $|x'-y| \backsim |x-x_0|$ implies $m(x',\mu)^{-1} \backsim m(x_0,\mu)^{-1} \backsim r$. Applying Proposition 2.16(i), we obtain

$$T_2'(x) \lesssim \Big[\int_{r/2}^{|x-x_0|/4} \int_{|x-y|<t} \Big(\int_{|B|} \frac{t^{-n}(1+|x-x_0|/t)^{-(n+M+1)}}{(1+tm(x_0,\mu))^M} \frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$
$$\lesssim \Big[\int_{r/2}^{|x-x_0|} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+M+1)} \Big(\frac{1}{tm(x_0,\mu)}\Big)^{2M} \frac{dt}{t}\Big]^{1/2} \lesssim \frac{r^M}{|x-x_0|^{n+M}}.$$

For T_3', a direct computation gives

$$T_3'(x) \lesssim \Big[\int_{|x-x_0|/4}^{\infty} \int_{|x-y|<t} \Big(\int_{|B|} t^{-n}\big(1+tm(x_0,\mu)\big)^{-M}\frac{dx'}{|B|}\Big)^2 \frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$
$$\lesssim \Big[\int_{|x-x_0|/4}^{\infty} t^{-2n}\Big(\frac{1}{tm(x_0,\mu)}\Big)^{2M}\frac{dt}{t}\Big]^{1/2} \lesssim \frac{r^M}{|x-x_0|^{n+M}}.$$

The estimates for $T_i', i=1,2,3$, indicate that

$$I \le \int_{|x-x_0|>8r} \big[T_1'(x)+T_2'(x)+T_3'(x)\big]\,dx$$
$$\lesssim \int_{|x-x_0|>8r} \frac{r^M}{|x-x_0|^{n+M}}\,dx \lesssim 1.$$

This completes the proof of Lemma 3.2. □

3.2. Characterization associated with $e^{-t\mathcal{L}}$. In this section, we establish the Carleson measure characterization via the operator family $\{Q^{\mathcal{L}}_{t,k}\}$. Precisely, we have the following.

Theorem 3.3. *Suppose that μ satisfies (1.1) and (1.2) for some $\delta>0$. Let $d\nu_{h,k}$ be the measure defined by (1.4).*

(1) *If $f\in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, then $d\nu_{h,k}$ is a Carleson measure.*
(2) *Conversely, if $f\in L^1((1+|x|)^{-(n+1)}\,dx)$ and $d\nu_{h,k}$ is a Carleson measure, then $f\in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$.*

Moreover, in either case there exists $C>0$ such that

$$\frac{1}{C}\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}} \le \|d\nu_{h,k}\|_{\mathcal{C}} \le C\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.$$

Proof. Because of Proposition 2.16 and the integrability of $(1+|y|)^{-n-1}|f(y)|$, we can get that

$$Q^{\mathcal{L}}_{t,k}f(x) = \int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k}(x,y)f(y)\,dy$$

is a well-defined absolutely convergent integral for all $(x,t)\in\mathbb{R}^{n+1}_+$. Fix a ball $B=B(x_0,r)$. We wish to show that

$$\frac{1}{|B|}\int_0^r\int_B \big|Q^{\mathcal{L}}_{t,k}f(x)\big|^2\frac{dx\,dt}{t} \le c\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}. \tag{3.3}$$

We split the function f into three parts:

$$f = (f-f_{B_{2r}})\chi_{B_{2r}} + (f-f_{B_{2r}})\chi_{(B_{2r})^c} + f_{B_{2r}} = f_1+f_2+f_{B_{2r}}.$$

This notation corresponds, respectively, to the local, global, and constant parts. For f_1, using Lemma 3.1, we have

$$\frac{1}{|B|}\int_0^r\int_B \big|Q^{\mathcal{L}}_{t,k}f_1(x)\big|^2\frac{dx\,dt}{t} \lesssim \frac{1}{|B|}\int_B\int_0^{\infty}\big|Q^{\mathcal{L}}_{t,k}(f_1)(x)\big|^2\,dx$$
$$\lesssim \frac{1}{|B|}\int_B \big|g^{\mathcal{L}}_{h,k}(f_1)(x)\big|^2\,dx \lesssim \frac{1}{|B|}\|f_1\|_2^2 \lesssim \|f\|^2_{\mathrm{BMO}_{\mathcal{L}}},$$

where we have used Corollary 2.8 in the last step. Next we estimate $|Q^{\mathcal{L}}_{t,k} f_2(x)|$. Let $x \in B = B(x_0, r)$ and $t < r$. Then we have

$$
\begin{aligned}
\big|Q^{\mathcal{L}}_{t,k} f_2(x)\big| &\lesssim \int_{\mathbb{R}^n} \frac{1}{t^n} \frac{e^{-|x-y|^2/t^2}}{(1+tm(x,u)+tm(y,u))^M} \big|f_2(y)\big|\, dy \\
&\lesssim \int_{(B_{2r})^c} \big|f(y) - f_{B_{2r}}\big| \frac{t}{|x_0-y|^{n+1}}\, dy \qquad (3.4) \\
&\lesssim \sum_{k=1}^{\infty} \frac{t}{(2^k r)^{n+1}} \int_{|y-x_0|\sim 2^k r} \big[\big|f(y) - f_{B_{2^{k+1}r}}\big| + |f_{B_{2^{k+1}r}} - f_{B_{2^k r}}| \\
&\qquad + \cdots + |f_{B_{4r}} - f_{B_{2r}}|\big]\, dy \\
&\lesssim \frac{t}{r} \sum_{k=1}^{\infty} 2^{-k} \big[\|f\|_{\mathrm{BMO}} + k\|f\|_{\mathrm{BMO}}\big] \lesssim \frac{t}{r}\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

Thus, integrating over $B \times (0,r)$, we obtain that

$$
\frac{1}{|B|}\int_0^r \big|Q^{\mathcal{L}}_{t,k}(f_{B_2})(x)\big|^2 \frac{dx\, dt}{t} \lesssim c\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}} \frac{1}{|B|}\sum_r |Q_k| \lesssim \|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.
$$

It remains to estimate the constant term $f_{B_{2r}}$. At first, we assume that $r < m(x_0,\mu)^{-1}$. For this case, it follows from [13, Proposition 1.8] that $m(x,\mu)^{-1} \sim m(x_0,\mu)^{-1}$ for $x \in B$. By Lemma 2.11 and Proposition 2.16(iii), we have

$$
\begin{aligned}
\frac{1}{|B|}\int_0^r \int_B \big|Q^{\mathcal{L}}_{t,k}(f_{B_{2r}})(x)\big|^2 \frac{dx\, dt}{t} &\lesssim \frac{|f_{B_{2r}}|^2}{|B|}\int_0^r \int_B \big(tm(x,\mu)\big)^{2\delta} \frac{dx\, dt}{t} \\
&\lesssim |f_{B_{2r}}|^2 \big(rm(x_0,\mu)\big)^{2\delta} \\
&\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}} \big(1 + \log\big(rm(x_0,\mu)\big)^{-1}\big)^2 \big(rm(x_0,\mu)\big)^{2\delta} \\
&\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}.
\end{aligned}
$$

Then we deal with the case $r \geq m(x_0,\mu)^{-1}$. By Proposition 2.9, we can choose a finite family of critical balls $\{\mathcal{B}_k\}$ such that $B \subset \cup \mathcal{B}_k$ and $\sum |\mathcal{B}_k| \lesssim |B|$. By Proposition 2.16(iii) and the fact that $|f_{B_{2r}}| \leq \|f\|_{\mathrm{BMO}_{\mathcal{L}}}$, we obtain

$$
\begin{aligned}
\frac{1}{|B|}\int_0^r \int_B \big|Q^{\mathcal{L}}_{t,k}(f_{B_{2r}})(x)\big|^2 \frac{dx\, dt}{t} &= \frac{|f_{B_{2r}}|^2}{|B|}\int_0^r \int_B \big|Q^{\mathcal{L}}_{t,k}(x,y)\, dy\big|^2 \frac{dx\, dt}{t} \\
&= \frac{\|f\|_{\mathrm{BMO}_{\mathcal{L}}}}{|B|}\sum_k (A_k + B_k),
\end{aligned}
$$

where

$$
\begin{cases}
A_k =: \int_0^{1/m(x_k,\mu)} \int_{\mathcal{B}_k} (tm(x_k,\mu))^{2\delta} \frac{dx\, dt}{t}, \\
B_k =: \int_{1/m(x_k,\mu)}^{\infty} \int_{\mathcal{B}_k} \frac{dx}{[1+rm(x_k,\mu)]^{2M-2\delta}} \frac{dt}{t}.
\end{cases}
$$

A direct computation gives

$$
A_k \lesssim |\mathcal{B}_k| \int_0^{1/m(x_k,\mu)} \frac{t^{2\delta-1}}{m(x_k,\mu)^{2\delta}} \frac{dx\, dt}{t} \lesssim |\mathcal{B}_k|
$$

and

$$B_k \lesssim \int_{1/m(x_k,\mu)}^{\infty} \int_{\mathcal{B}_k} \frac{dx}{(tm(x_k,\mu)^{2M-2\delta})} \frac{dt}{t} = |\mathcal{B}_k| \int_{1/m(x_k,u)}^{\infty} \frac{m(x_k,\mu)^{2\delta-2M}}{t^{2M-2\delta+1}}\, dt \lesssim |\mathcal{B}_k|.$$

The arguments above imply that (3.3) holds. Thus we have $\|\nu_{h,k}\|_{\mathcal{C}} < \infty$. This establishes Theorem 3.3(i).

Now we prove (ii). Fix $f \in L^1((1+|x|)^{-n-1}\, dx)$ such that

$$d\nu_{h,k}(x,t) =: \left|Q^{\mathcal{L}}_{t,k} f(x)\right|^2 \frac{dx\, dt}{t} \quad \forall (x,t) \in \mathbb{R}^{n+1}_+$$

is a Carleson measure. We want to prove that such an f belongs to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$. By Lemma 2.10, it suffices to show that the linear functional

$$H^1_{\mathcal{L}} \ni a \to \Phi_f(a) =: \int_{\mathbb{R}^n} f(x)a(x)\, dx,$$

which is defined at least over finite linear combinations of $H^1_{\mathcal{L}}$-atoms, satisfies the estimate

$$\left|\Phi_f(a)\right| \le c\|\nu_{h,k}\|^{1/2}_{\mathcal{C}} \|a\|_{H^1_{\mathcal{L}}}.$$

For this purpose, let

$$\begin{cases} F(x,t) =: Q^{\mathcal{L}}_{t,k} f(x), & (x,t) \in \mathbb{R}^{n+1}_+, \\ G(x,t) =: Q^{\mathcal{L}}_{t,k} a(x), & (x,t) \in \mathbb{R}^{n+1}_+. \end{cases}$$

We only need to prove the following identity:

$$\frac{1}{8} \int_{\mathbb{R}^n} f(x)\overline{a(x)}\, dx = \int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)} \frac{dx\, dt}{t}. \tag{3.5}$$

Note that (3.5) is clearly valid when $f, a \in L^2(\mathbb{R}^n)$. Hence we should justify the convergence of the integrals in the case when $f \in L^1((1+|x|)^{-(n+1)}\, dx)$ and a is an $H^1_{\mathcal{L}}$-atom.

If (3.5) holds, then, noting that $\|\mu_{h,k}\|_{\mathcal{C}} = \|\mathcal{I}(F)\|^2_{L^\infty}$, we can deduce from Lemma 2.12 that

$$\left|\frac{1}{8} \int_{\mathbb{R}^n} f(x)\overline{a(x)}\, dx\right| \le \left\|\mathcal{I}(F)\right\|_{L^\infty} \left\|\mathcal{G}(G)\right\|_{L^1} \le \|\mu_{h,k}\|^{1/2}_{\mathcal{C}} \left\|\mathcal{G}(G)\right\|_{L^1}.$$

On the other hand, it is easy to see that $\mathcal{G}(G) = S^{\mathcal{L}}_{h,k}(a)$. It follows from Lemma 3.2 that $\|\mathcal{G}(G)\|_{L^1} \le C\|a\|_{H^1_{\mathcal{L}}}$ and

$$\left|\frac{1}{8} \int_{\mathbb{R}^n} f(x)\overline{a(x)}\, dx\right| \le C\|\nu_{h,k}\|^{1/2}_{\mathcal{C}} \|a\|_{H^1_{\mathcal{L}}},$$

which implies that f is a bounded linear functional on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Now we begin to prove (3.5). By Lemmas 2.12 and 3.2 and the dominated convergence theorem, we can deduce that the following integral is absolutely convergent and satisfies

$$V = \int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)} \frac{dx\, dt}{t} = \lim_{\varepsilon \to 0, N \to \infty} \int_{\varepsilon}^{N} \int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k} f(x) \overline{Q^{\mathcal{L}}_{t,k} a(x)} \frac{dx\, dt}{t}.$$

For each $t > 0$, using Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k} f(x)\overline{Q^{\mathcal{L}}_{t,k} a(x)}\, dx &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} Q^{\mathcal{L}}_{t,k}(x,y) f(y)\, dy \Big) \overline{Q^{\mathcal{L}}_{t,k} a(x)}\, dx \\ &= \int_{\mathbb{R}^n} f(y) \overline{(Q^{\mathcal{L}}_{t,k})^2 a(y)}\, dy.\end{aligned}$$

Then we get

$$\begin{aligned} V &= \lim_{\varepsilon\to 0, N\to\infty} \int_\varepsilon^N \Big[\int_{\mathbb{R}^n} f(y)\overline{(Q^{\mathcal{L}}_{t,k})^2 a(y)}\, dy \Big] \frac{dt}{t} \\ &= \lim_{\varepsilon\to 0, N\to\infty} \int_{\mathbb{R}^n} f(y) \Big[\int_\varepsilon^N \overline{(Q^{\mathcal{L}}_{t,k})^2 a(y)\frac{dt}{t}} \Big]\, dy. \end{aligned} \tag{3.6}$$

By Lemma 2.13 and the kernel decay $|Q^{\mathcal{L}}_{t,k}(x,y)| \lesssim t^{-n}(1+|x-y|/t)^{-M}$, we can apply the hypothesis $f \in L^1((1+|x|)^{-(n+1)}\, dx)$ to verify the absolute integrability in these steps.

Finally, to complete the proof, we also need to prove the following estimate:

$$\sup_{\varepsilon, N>0} \Big| \int_\varepsilon^N (Q^{\mathcal{L}}_{t,k})^2 a(y) \frac{dt}{t} \Big| \le C_{y_0,r} \big(1+|y|\big)^{-(n+1)}, \quad y \in \mathbb{R}^n. \tag{3.7}$$

Denote by $W_{\varepsilon,k}(\cdot,\cdot)$ the integral kernel of the operator $\int_\varepsilon^\infty (Q^{\mathcal{L}}_{t,k})^2 \frac{dt}{t}$. By a simple yet somewhat complicated calculus, we have

$$\int_\varepsilon^\infty (Q^{\mathcal{L}}_{t,k})^2 \frac{dt}{t} = \frac{1}{4}\Big(\frac{1}{2}\Big)^{2k-1} \sum_{j=1}^{2k-1} \frac{(2k-1)!}{(2k-j)!} Q^{\mathcal{L}}_{\sqrt{2}\varepsilon, 2k-j} + (2k-1)! T^{\mathcal{L}}_{\sqrt{2}\varepsilon},$$

which indicates that the kernel $W_{\varepsilon,k}(\cdot,\cdot)$ satisfies the same properties as the kernels $T^{\mathcal{L}}_{t,k}(\cdot,\cdot)$ and $Q^{\mathcal{L}}_{t,k}(\cdot,\cdot)$. This means that $W_{\varepsilon,k}(\cdot,\cdot)$ satisfies the assumption of Lemma 2.13. Note that

$$\Big| \int_\varepsilon^\infty (Q^{\mathcal{L}}_{t,k})^2 a(y) \frac{dt}{t} \Big| = \sup_{\varepsilon>0} \Big| \int_{\mathbb{R}^n} W_{\varepsilon,k}(x,y) a(y)\, dy \Big|.$$

We have

$$\begin{aligned}\Big| \int_\varepsilon^N (Q^{\mathcal{L}}_{t,k})^2 a(y) \frac{dt}{t} \Big| &= \Big| \int_\varepsilon^\infty (Q^{\mathcal{L}}_{t,k})^2 a(y) \frac{dt}{t} - \int_N^\infty (Q^{\mathcal{L}}_{t,k})^2 a(y) \frac{dt}{t} \Big| \\ &= \Big| \int_{\mathbb{R}^n} W_{\varepsilon,k}(x,y) a(y)\, dy - \int_{\mathbb{R}^n} W_{N,k}(x,y) a(y) \frac{dt}{t} \Big| \\ &\le \sup_{\varepsilon>0} \Big| \int_{\mathbb{R}^n} W_{\varepsilon,k}(x,y) a(y)\, dy \Big| + \sup_{N>0} \Big| \int_{\mathbb{R}^n} W_{N,k}(x,y) a(y)\, dy \Big|.\end{aligned}$$

It follows from Lemma 2.13 that (3.7) holds. Indeed, (3.7) allows passing to the limit inside the integral in (3.6). Combining Lemma 3.1, we have

$$V = \frac{1}{8} \int_{\mathbb{R}^n} f(y) \overline{a(y)}\, dy.$$

This completes the proof of Theorem 3.3. ∎

4. Carleson measure characterization associated with the Poisson semigroup

4.1. Reproducing formula generated by the Poisson kernel. For $k \in \mathbb{Z}_+$, define the Littlewood–Paley g-function associated with the Poisson semigroup as

$$g^{\mathcal{L}}_{P,k}(f)(x) = \Big(\int_0^\infty \big|D^{\mathcal{L}}_{t,k} f(x)\big|^2 \frac{dt}{t}\Big)^{1/2}.$$

Lemma 4.1. *For all $f \in L^2(\mathbb{R}^n)$, we have $\|g^{\mathcal{L}}_{P,k}(f)\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$. Moreover,*

$$f(x) = 8 \lim_{\varepsilon\to 0, N\to\infty} \int_\varepsilon^N (D^{\mathcal{L}}_{t,k})^2 f(x) \frac{dt}{t} \quad \text{in } L^2(\mathbb{R}^n). \tag{4.1}$$

Proof. Similar to Lemma 3.1, let $\{E(\lambda)\}$ denote a resolution of the identity. By the spectral theorem, we have

$$t\frac{d}{dt} e^{-t\sqrt{\mathcal{L}}} f = -\int_0^\infty t\sqrt{\lambda} e^{-t\sqrt{\lambda}}\, dE(\lambda) f.$$

For all $f \in L^2(\mathbb{R}^n)$, the self-adjointness of $D^{\mathcal{L}}_{t,k}$ implies that

$$\begin{aligned}\|g^{\mathcal{L}}_{P,k}(f)\|_2^2 &= \int_0^\infty \Big\langle t^{2k}\Big(\frac{d^k e^{-t\sqrt{\mathcal{L}}}}{dt^k}\Big)^2 f, f\Big\rangle \frac{dt}{t}\\ &= \int_0^\infty \Big[\int_0^\infty t^{2k}\lambda^k e^{-2t\sqrt{\lambda}} \frac{dt}{t}\Big]\, dE_{f,f}(\lambda) = \frac{1}{8}\|f\|_2^2.\end{aligned}$$

Now we prove (4.1). Let $\{(n_l, \varepsilon_l)\}$ be an arbitrary pair of sequences such that $n_l \nearrow \infty$ and $\varepsilon_l \searrow 0$. Similar to Lemma 3.1, we only need to verify

$$\lim_{l\to\infty} \int_{n_l}^{n_{l+m}} (D^{\mathcal{L}}_{t,k})^2 f \frac{dt}{t} = \lim_{l\to\infty} \int_{\varepsilon_{l+m}}^{\varepsilon_l} (D^{\mathcal{L}}_{t,k})^2 f \frac{dt}{t} = 0 \quad \forall m \ge 1.$$

In fact, we use functional calculus again such that

$$\Big\| \int_{n_l}^{n_{l+m}} (D^{\mathcal{L}}_{t,k})^2 f \frac{dt}{t}\Big\|^2 \le \int_0^\infty \Big| \int_{n_l}^{n_{l+m}} t^{2k}\lambda^k e^{-2t\sqrt{\lambda}} \frac{dt}{t}\Big|^2 dE_{f,f}(\lambda).$$

Computing the integral inside one is led to the estimate

$$\int_0^\infty \Big[\sum_{j=1}^{2k-1} \frac{1}{2^j} \frac{(2k-1)!}{(2k-j)!} (n_l\sqrt{\mathcal{L}})^{2k-j} + \frac{(2k-1)!}{2^{2k}}\Big] e^{-2n_l\sqrt{L}}\, dE_{f,f}(a), \quad \text{as } n_l \to \infty,$$

which tends to zero by the dominated convergence theorem. The rest of the proof is similar to that of Lemma 3.1. We omit the details. □

For $k \in \mathbb{Z}_+$, the area function associated with the Poisson semigroup is defined as

$$S^{\mathcal{L}}_{P,k}(f)(x) =: \Big(\int_0^\infty \int_{|x-y|<t} \big|D^{\mathcal{L}}_{t,k} f(y)\big|^2 \frac{dy\, dt}{t^{n+1}}\Big)^{1/2}, \quad x \in \mathbb{R}^n.$$

Lemma 4.2. *Let f be a finite linear combination of $H^1_{\mathcal{L}}$-atoms. There exists $c > 0$ such that $\|S^{\mathcal{L}}_P(f)\|_{L^1} \le c\|f\|_{H^1_{\mathcal{L}}}$.*

Proof. Fix an $H^1_{\mathcal{L}}$-atom a which is supported on $B=B(x_0,r)$. We have

$$\begin{aligned}\big\|S^{\mathcal{L}}_{P,k}(a)\big\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^{n+1}_+}\big|D^{\mathcal{L}}_{t,k}a(y)\big|^2\chi_{\Gamma(x)}(y,t)\frac{dy\,dt}{t^{n+1}}\Big]\,dx\\ &\lesssim \int_{\mathbb{R}^{n+1}_+}\big|D^{\mathcal{L}}_{t,k}a(y)\big|^2\frac{dy\,dt}{t}\simeq \big\|g^{\mathcal{L}}_{P,k}(a)\big\|^2_{L^2}\lesssim \frac{1}{8}\|a\|^2_{L^2},\end{aligned}$$

where in the last step we have used Lemma 4.1. Hölder's inequality indicates that

$$\begin{aligned}\int_{|x-x_0|\le 8r}S^{\mathcal{L}}_{P,k}(a)(x)\,dx &\lesssim |B_{8r}|^{1/2}\Big(\int_{|x-x_0|\le 8r}S^{\mathcal{L}}_{P}(a)(x)^2\,dx\Big)^{1/2}\\ &\lesssim |B|^{1/2}\|a\|_{L^2}\lesssim 1.\end{aligned}$$

Similar to Lemma 3.2, we will prove that the integral

$$I=:\int_{|x-x_0|>8r}S^{\mathcal{L}}_{P,k}(a)(x)\,dx$$

is bounded uniformly. For this purpose, we divide the proof into two cases.

Case I: $r<m(x_0,\mu)$. By the cancelation property of a, we have

$$S^{\mathcal{L}}_{P,k}(a)(x)\le S_1(x)+S_2(x),$$

where

$$S_1(x)=:\Big[\int_0^{|x-x_0|/2}\int_{|x-y|<t}\Big(\int_B\big|D^{\mathcal{L}}_{t,k}(y,x')-D^{\mathcal{L}}_{t,k}(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}$$

and

$$S_2(x)=:\Big[\int_{|x-x_0|/2}^{\infty}\int_{|x-y|<t}\Big(\int_B\big|D^{\mathcal{L}}_{t,k}(y,x')-D^{\mathcal{L}}_{t,k}(y,x_0)\big|\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}.$$

For S_1, note that if $x'\in B$, then $|y-x'|\sim|y-x_0|\sim|x-x_0|$ and $|x'-x_0|<|y-x_0|/4$. Applying Proposition 2.19(ii), we have

$$\begin{aligned}S_1(x)&\lesssim\Big[\int_0^{|x-x_0|/2}\int_{|x-y|<t}\Big(\int_B\Big(\frac{|x'-x_0|}{t}\Big)^{\delta'}\frac{t|B|^{-1}\,dx'}{(t^2+|y-x_0|^2)^{(n+1)/2}}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim\Big[\int_0^{|x-x_0|/2}\Big(\frac{r}{t}\Big)^{2\delta'}t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)}\frac{dt}{t}\Big]^{1/2}\lesssim\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.\end{aligned}$$

For S_2, it is easy to see that $|x'-x_0|\le r<|x-x_0|/2\le t$. Proposition 2.19 gives

$$\begin{aligned}S_2(x)&\lesssim\Big[\int_{|x-x_0|/2}^{\infty}\int_{|x-y|<t}\Big(\int_B\Big(\frac{|x'-x_0|}{t}\Big)^{\delta'}t^{-n}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim\Big[\int_{|x-x_0|/2}^{\infty}\Big(\frac{r}{t}\Big)^{2\delta'}t^{-2n}\frac{dt}{t}\Big]^{1/2}\lesssim\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}.\end{aligned}$$

Finally, we obtain

$$\int_{|x-x_0|>8r}S^{\mathcal{L}}_{P}(a)(x)\,dx\lesssim\int_{|x-x_0|>8r}\frac{r^{\delta'}}{|x-x_0|^{n+\delta'}}\,dx\lesssim 1.$$

Case 2: $m(x_0,\mu)^{-1} < r < 4m(x_0,\mu)^{-1}$. For this case, we divide the integral defining $S^{\mathcal{L}}_{P,k}a$ into three parts: $S^{\mathcal{L}}_{P,k}(a)(x) \lesssim S_1'(x) + S_2'(x) + S_3'(x)$, where

$$S_1'(x) =: \Big[\int_0^{r/2}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2},$$
$$S_2'(x) =: \Big[\int_{r/2}^{|x-x_0|/4}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2},$$

and

$$S_3'(x) =: \Big[\int_{|x-x_0|/4}^{\infty}\int_{|x-y|<t}\Big(\int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,k}(y,x')g(x')\,dx'\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}.$$

For S_1', we have $|x'-y| \backsim |x-x_0|$. Using Proposition 2.19(i), we get

$$\begin{aligned} S_1'(x) &\lesssim \Big[\int_0^{r/2}\int_{|x-y|<t}\Big(\int_B \frac{t}{(t^2+|y-x'|^2)^{(n+1)/2}}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim \Big[\int_0^{r/2}\int_{|x-y|<t} t^{-2n}\Big(1+\frac{|x-x_0|}{t}\Big)^{-2(n+1)}\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim \Big[\int_0^{r/2} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+1)}\frac{dt}{t}\Big]^{1/2} \lesssim \frac{r}{|x-x_0|^{n+1}}. \end{aligned}$$

For S_2', because $|x'-y| \backsim |x-x_0|$, it follows from Lemma 2.1 that $m(x',\mu)^{-1} \backsim m(x_0,\mu)^{-1} \backsim r$. Applying Proposition 2.19(i), we obtain

$$\begin{aligned} S_2'(x) &\lesssim \Big[\int_{r/2}^{|x-x_0|/4}\int_{|x-y|<t}\Big(\int_B \frac{t(1+tm(x_0,\mu))^{-M}}{(t^2+|y-x'|^2)^{(n+1)/2}}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim \Big[\int_{r/2}^{|x-x_0|} t^{-2n}\Big(\frac{t}{|x-x_0|}\Big)^{2(n+M+1)}\frac{1}{(tm(x_0,\mu))^{2M}}\frac{dt}{t}\Big]^{1/2} \lesssim \frac{r^M}{|x-x_0|^{n+M}}. \end{aligned}$$

For S_3', similarly, we have

$$\begin{aligned} S_3'(x) &\lesssim \Big[\int_{|x-x_0|/4}^{\infty}\int_{|x-y|<t}\Big(\int_B \frac{t[1+tm(x_0,\mu)]^{-M}}{(t^2+|y-x'|^2)^{(n+1)/2}}\frac{dx'}{|B|}\Big)^2\frac{dy\,dt}{t^{n+1}}\Big]^{1/2}\\ &\lesssim \Big[\int_{|x-x_0|/4}^{\infty}\frac{1}{t^{2n}(1+|y-x'|/t)^{2(n+1)}}\frac{1}{(tm(x_0,\mu))^{2M}}\frac{dt}{t}\Big]^{1/2} \lesssim \frac{r^M}{|x-x_0|^{n+M}}. \end{aligned}$$

Thus we integrate $S^{\mathcal{L}}_{P,k}(a)$ over $(B_{8r})^c$ to obtain

$$I \lesssim \int_{|x-x_0|>8r}\big[S_1'(x)+S_2'(x)+S_3'(x)\big]\,dx \lesssim \int_{|x-x_0|>8r}\frac{r^M}{|x-x_0|^{n+M}}\,dx \lesssim 1.$$

This completes the proof of Lemma 4.2. $\square$

4.2. Characterization associated with $e^{-t\sqrt{\mathcal{L}}}$.

Theorem 4.3. *Let $k \in \mathbb{Z}_+$. Suppose that μ satisfies (1.1) and (1.2) for all $x \in \mathbb{R}^n$, $0 < r < R$, where $B(x,r)$ denotes the (open) ball centered at x with radius r. For some $\delta > 0$, let $d\nu_{P,k}$ be the measure defined by (1.5).*

(1) *If $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, then $d\nu_{P,k}$ is a Carleson measure.*
(2) *Conversely, if $f \in L^1((1+|x|)^{-(n+1)}\,dx)$ and $d\nu_{P,k}$ is a Carleson measure, then $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$.*

Moreover, in either case there exists $C > 0$ such that

$$\frac{1}{C}\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}} \le \|d\nu_{P,k}\|_{\mathcal{C}} \le C\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.$$

Proof. We first prove (i). From Proposition 2.9 and the integrability of $(1+|y|)^{-n-1}|f(y)|$, we know that

$$D^{\mathcal{L}}_{t,2k}f(x) = \int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,2k}(x,y)f(y)\,dy$$

is a well-defined absolutely convergent integral for all $(x,t) \in \mathbb{R}^{n+1}_+$. Fix a ball $B = B(x_0, r)$. We wish to show that

$$\frac{1}{|B|}\int_0^r\int_B \left|D^{\mathcal{L}}_{t,2k}f(x)\right|^2\frac{dx\,dt}{t} \le c\|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}. \tag{4.2}$$

To do this, we split f into three parts:

$$f = (f - f_{B_{2r}})\chi_{B_{2r}} + (f - f_{B_{2r}})\chi_{(B_{2r})^c} + f_{B*} = f_1 + f_2 + f_{B_{2r}}.$$

For f_1, using Lemma 4.1 and Corollary 2.8, we have

$$\begin{aligned}\frac{1}{|B|}\int_0^r\int_B \left|D^{\mathcal{L}}_{t,2k}f_1(x)\right|^2\frac{dx\,dt}{t} &\lesssim \frac{1}{|B|}\int_B\int_0^\infty \left|D^{\mathcal{L}}_{t,2k}f_1(x)\right|^2\frac{dx\,dt}{t}\\ &\lesssim \frac{1}{|B|}\int_B \left|g^{\mathcal{L}}_{P,k}f_1(x)\right|^2 dx \lesssim \frac{1}{|B|}\|f_1\|_2^2 \lesssim \|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.\end{aligned}$$

For f_2, similar to (3.4), we can get

$$\left|D^{\mathcal{L}}_{t,2k}(f)_2(x)\right| \lesssim \frac{t}{r}\sum_{k=1}^\infty \frac{k+1}{2^k}\|f\|_{\mathrm{BMO}} \lesssim \frac{t}{r}\|f\|_{\mathrm{BMO}},$$

which gives

$$\frac{1}{|B|}\int_0^r\int_0^B \left|D^{\mathcal{L}}_{t,2k}(f_2)(x)\right|^2\frac{dx\,dt}{t} \lesssim \int_0^r\left(\frac{t}{r}\|f\|_{\mathrm{BMO}}\right)^2\frac{dt}{t} \lesssim \|f\|^2_{\mathrm{BMO}_{\mathcal{L}}}.$$

Now we deal with the term $f_{B_{2r}}$. At first, we assume that $r < m(x_0,\mu)^{-1}$. It follows from Proposition 2.1 that $m(x,\mu)^{-1} \sim m(x_0,\mu)^{-1}$ for $x \in B$. We can make use of Lemma 2.11 and Proposition 2.19(iii) to get

$$\begin{aligned}\frac{1}{|B|}\int_0^r\int_B \left|D^{\mathcal{L}}_{t,2k}(f_{B_{2r}})(x)\right|^2\frac{dx\,dt}{t} &\lesssim \frac{|f_{B_{2r}}|^2}{|B|}\int_0^r\int_B \big(tm(x,\mu)\big)^{2\delta}\frac{dx\,dt}{t}\\ &\lesssim |f_{B_{2r}}|^2\big(rm(x_0,\mu)\big)^{2\delta}\end{aligned}$$

$$\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2 \big(1+\log\big(rm(x_0,\mu)\big)^{-1}\big)^2 \big(rm(x_0,\mu)\big)^{2\delta}$$
$$\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2.$$

Finally, suppose that $r \geq m(x_0,\mu)^{-1}$. We choose from Proposition 2.9 a finite family of critical balls $\{\mathcal{B}_k\}$ such that $B \subset \bigcup \mathcal{B}_k$ and $\sum |\mathcal{B}_k| \lesssim |B|$. By Proposition 2.19(iii) and the fact that $|f_{B_{2r}}| \leq \|f\|_{\mathrm{BMO}_{\mathcal{L}}}$, we know that

$$\begin{aligned}\frac{1}{|B|}\int_0^r \int_B \big|D_{t,2k}^{\mathcal{L}}(f_{B_{2r}})(x)\big|^2 \frac{dx\,dt}{t} &= \frac{|f_{B_{2r}}|^2}{|B|}\int_0^r \int_B \big|D_{t,2k}^{\mathcal{L}}(x,y)\,dy\big|^2 \frac{dx\,dt}{t}\\ &= \frac{\|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2}{|B|}\sum_k (C_k + D_k),\end{aligned}$$

where

$$C_k =: \int_0^{1/m(x_k,u)} \int_{\mathcal{B}_k} \big(tm(x_k,\mu)\big)^{2\delta} \frac{dx\,dt}{t}$$

and

$$D_k =: \int_{1/m(x_k,\mu)}^{\infty} \int_{\mathcal{B}_k} \frac{dx}{1+rm(x_k,\mu)^{2M-2\delta}} \frac{dt}{t}.$$

It is easy to get

$$C_k \lesssim |\mathcal{B}_k| \int_0^{1/m(x_k,\mu)} \frac{t^{2\delta-1}}{m(x_k,\mu)^{2\delta}} \frac{dx\,dt}{t} \lesssim |\mathcal{B}_k|$$

and

$$D_k \lesssim \int_{1/m(x_k,\mu)}^{\infty} \int_{\mathcal{B}_k} \frac{1}{\big(tm(x_k,\mu)^{2M-2\delta}\big)} \frac{dx\,dt}{t} \lesssim |\mathcal{B}_k|.$$

Thus we have

$$\frac{1}{|B|}\int_0^r \int_B \big|D_{t,2k}^{\mathcal{L}}(f_{B_{2r}})(x)\big|^2 \frac{dx\,dt}{t} \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2 \frac{1}{|B|}\sum_k |\mathcal{B}_k| \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}}^2.$$

According to the arguments above, (4.2) holds. Thus we have $\|\nu_{P,k}\|_{\mathcal{C}} < \infty$. This establishes Theorem 4.3(i).

Now we prove (ii). Let $f \in L^1((1+|x|)^{-(n+1)}\,dx)$ such that

$$d\nu_{P,k}(x,t) =: \big|D_{t,2k}^{\mathcal{L}} f(x)\big|^2 \frac{dx\,dt}{t}$$

is a Carleson measure. We want to prove that $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$. By Lemma 2.10, it suffices to show that the linear functional

$$H_{\mathcal{L}}^1 \ni a \to \Phi_f(a) =: \int_{\mathbb{R}^n} f(x)a(x)\,dx,$$

which is defined at least over finite linear combinations of $H_{\mathcal{L}}^1$-atoms, satisfies the estimate

$$\big|\Phi_f(a)\big| \leq c\|\nu_{P,k}\|_{\mathcal{C}}^{1/2}\|a\|_{H_{\mathcal{L}}^1}.$$

For this purpose, let

$$\begin{cases} F(x,t) =: D^{\mathcal{L}}_{t,2k} f(x), & (x,t) \in \mathbb{R}^{n+1}_+, \\ G(x,t) =: D^{\mathcal{L}}_{t,2k} a(x), & (x,t) \in \mathbb{R}^{n+1}_+. \end{cases}$$

We only need to prove the following identity:

$$\frac{1}{8} \int_{\mathbb{R}^n} f(x) \overline{a(x)}\, dx = \int_{\mathbb{R}^{n+1}_+} F(x,t) \overline{G(x,t)} \frac{dx\, dt}{t}. \tag{4.3}$$

Note that (4.3) is clearly valid when $f, a \in L^2(\mathbb{R}^n)$. Hence we should justify the convergence of the integrals in the case when $f \in L^1((1+|x|)^{-(n+1)}\, dx)$ and a is an $H^1_{\mathcal{L}}$-atom.

If (4.3) holds, then, noting that $\|\nu_{P,k}\|_{\mathcal{C}} = \|\mathcal{I}(F)\|^2_{L^\infty}$, we can deduce from Lemma 2.12 that

$$\left| \frac{1}{8} \int_{\mathbb{R}^n} f(x) \overline{a(x)}\, dx \right| \le \big\|\mathcal{I}(F)\big\|_{L^\infty} \big\|\mathcal{G}(G)\big\|_{L^1} \le \|\nu_{P,k}\|^{1/2}_{\mathcal{C}} \big\|\mathcal{G}(G)\big\|_{L^1}.$$

On the other hand, it is easy to see that $\mathcal{G}(G)(x) = S^{\mathcal{L}}_{P,2k}(a)(x)$. It follows from Lemma 4.2 that $\|\mathcal{G}(G)\|_{L^1} \le C\|a\|_{H^1_{\mathcal{L}}}$ and

$$\left| \frac{1}{8} \int_{\mathbb{R}^n} f(x) \overline{a(x)}\, dx \right| \le C \|\nu_{P,k}\|^{1/2}_{\mathcal{C}} \|a\|_{H^1_{\mathcal{L}}},$$

which implies that f is a bounded linear functional on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Now we begin to prove (4.3). By Lemmas 2.12, 3.2, and the dominated convergence theorem, we obtain that the following integral is absolutely convergent and satisfies

$$V = \int_{\mathbb{R}^{n+1}_+} F(x,t) \overline{G(x,t)} \frac{dx\, dt}{t} = \lim_{\varepsilon \to 0, N \to \infty} \int_\varepsilon^N \int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,2k} f(x) \overline{D^{\mathcal{L}}_{t,2k} a(x)} \frac{dx\, dt}{t}.$$

For each $t > 0$, using Fubini's theorem, we obtain

$$\int_{\mathbb{R}^n} D^{\mathcal{L}}_{t,2k} f(x) \overline{D^{\mathcal{L}}_{t,2k} a(x)}\, dx = \int_{\mathbb{R}^n} f(y) \overline{(D^{\mathcal{L}}_{t,2k})^2 a(y)}\, dy.$$

Then we get

$$\begin{aligned} V &= \lim_{\varepsilon \to 0, N \to \infty} \int_\varepsilon^N \Big[\int_{\mathbb{R}^n} f(y) \overline{(D^{\mathcal{L}}_{t,2k})^2 a(y)}\, dy \Big] \frac{dt}{t} \\ &= \lim_{\varepsilon \to 0, N \to \infty} \int_{\mathbb{R}^n} f(y) \Big[\int_\varepsilon^N \overline{(D^{\mathcal{L}}_{t,2k})^2 a(y) \frac{dt}{t}} \Big]\, dy. \end{aligned} \tag{4.4}$$

Because $f \in L^1((1+|x|)^{-(n+1)}\, dx)$, it is easy to check the absolute integrability in these steps by Lemma 2.13 and the fact that $|(D^{\mathcal{L}}_{t,2k})^2(x,y)| \lesssim t^{-n} \times (1+|x-y|/t)^{-n+1}$.

Finally, we also need to prove the following estimate:

$$\sup_{\varepsilon, N > 0} \left| \int_\varepsilon^N (D^{\mathcal{L}}_{t,2k})^2 a(y) \frac{dt}{t} \right| \le C_{y_0,r} \big(1+|y|\big)^{-(n+1)}, \quad y \in \mathbb{R}^n. \tag{4.5}$$

We denote by $H_{\varepsilon,k}(\cdot,\cdot)$ the integral kernel of the operator $\int_\varepsilon^\infty (D^{\mathcal{L}}_{t,2k})^2 \frac{dt}{t}$. Similar to Theorem 3.3(ii), we can use a direct calculus to get

$$\Big|\int_\varepsilon^\infty (D^{\mathcal{L}}_{t,k})^2 \frac{dt}{t}\Big| = \frac{1}{2^{4k}}\Big[\sum_{j=1}^{4k-1} \frac{(4k-1)!}{(4k-j)!} D^{\mathcal{L}}_{2\varepsilon,4k-j} + (4k-1)! P^{\mathcal{L}}_{2\varepsilon}\Big],$$

which implies that $H_{\varepsilon,k}(\cdot,\cdot)$ has the same properties for the kernels $P^{\mathcal{L}}_{t,k}(\cdot,\cdot)$ and $D^{\mathcal{L}}_{t,k}(\cdot,\cdot)$; that is, $H_{\varepsilon,k}(\cdot,\cdot)$ satisfies the assumption of Lemma 2.13. Note that

$$\begin{aligned}\Big|\int_\varepsilon^N (D^{\mathcal{L}}_{t,2k})^2 a(y)\frac{dt}{t}\Big| &= \Big|\int_\varepsilon^\infty (D^{\mathcal{L}}_{t,2k})^2 a(y)\frac{dt}{t} - \int_N^\infty (D^{\mathcal{L}}_{t,2k})^2 a(y)\frac{dt}{t}\Big| \\ &\le \sup_{\varepsilon>0}\Big|\int_{\mathbb{R}^n} H_{\varepsilon,k}(x,y)a(y)\,dy\Big| + \sup_{N>0}\Big|\int_{\mathbb{R}^n} H_{N,k}(x,y)a(y)\,dy\Big|.\end{aligned}$$

Thus (4.5) holds. Indeed, (4.5) allows passing to the limit inside the integral in (4.4). Combining Lemma 4.1, we have $V = \frac{1}{8}\int_{\mathbb{R}^n} f(y)\overline{a(y)}\,dy$. This completes the proof of Theorem 4.3. □

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School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, People's Republic of China.

E-mail address: 1193976187@qq.com; ptli@qdu.edu.cn; zhkzhc@aliyun.com