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WAVELET CHARACTERIZATIONS OF MUSIELAK–ORLICZ HARDY SPACES

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ABSTRACT. In this article, via establishing a new atomic characterization of the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ [which is essentially deduced from the known molecular characterization of $H^{\varphi}(\mathbb{R}^n)$] and some estimates on a new discrete Littlewood–Paley g-function and a Peetre-type maximal function, together with using the known intrinsic g-function characterization of $H^{\varphi}(\mathbb{R}^n)$, the authors obtain several equivalent characterizations of $H^{\varphi}(\mathbb{R}^n)$ in terms of wavelets, which extend the wavelet characterizations of both Orlicz–Hardy spaces and the weighted Hardy spaces, and are available to the typical and useful Musielak–Orlicz Hardy space $H^{\log}(\mathbb{R}^n)$. The novelty of this approach is that the new adapted atomic characterization of $H^{\varphi}(\mathbb{R}^n)$ compensates the inconvenience in applications of the supremum appearing in the original definition of atoms, which play crucial roles in the proof of the main theorem of this article and may have further potential applications.

1. Introduction

In recent decades, the real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ has played essential roles in the theory of harmonic analysis and partial differential equations (see [6], [25]–[27] for details). In the 1990s, the wavelet theory provided a new powerful tool for the real-variable theory of Hardy spaces. More precisely, Meyer [24] established several equivalent wavelet characterizations of $H^1(\mathbb{R}^n)$; Liu [22] established some equivalent wavelet

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characterizations of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$; Wu [31] obtained further a wavelet area integral characterization of the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ for any $p \in (0,1]$; and later, García-Cuerva and Martell [7] found a characterization of $H^p_\omega(\mathbb{R}^n)$ for any $p \in (0,1]$ in terms of wavelets without compact supports via the vector-valued Calderón–Zygmund theory.

From that point on, wavelet characterizations were established involving many more different Hardy-type spaces or other function spaces. More specifically, Hernández et al. [9] established the φ-transform and the wavelet characterizations of Herz-type spaces. Kopaliani [13] and Izuki [11] independently introduced the wavelet inequalities of Lebesgue spaces with variable exponents. Later, Wang and Liu [30] obtained the wavelet characterization of Herz-type Hardy spaces with variable exponents. Izuki, Nakai, and Sawano [12] further obtained the wavelet characterization for weighted Lebesgue spaces with variable exponents.

Recently, motivated by the wide application of function spaces of Musielak–Orlicz type in several branches of mathematics and physics (see, e.g., [2]–[4], [16], [32]), Ky [16] introduced the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which extends both the Orlicz–Hardy spaces and the weighted Hardy spaces. In particular, the typical Musielak–Orlicz Hardy space $H^{\log}(\mathbb{R}^n)$, related to the growth function

$$\theta(x,t) := \frac{t}{\ln(e+|x|)+\ln(e+t)}, \quad \forall x \in \mathbb{R}^n, \forall t \in [0,\infty), \tag{1.1}$$

plays a striking role in the bilinear decompositions of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. These bilinear decompositions have important applications in the optimal endpoint estimate for the *div-curl lemma*, involving the space $H^{\log}(\mathbb{R}^n)$, which has relations to an implicit conjecture from [4] (see also [1], [3]), and in the bilinear or the subbilinear decompositions, respectively, for the linear or the sublinear commutators of Calderón–Zygmund operators from [14] (see [15], [17], [32] for more applications of the above decompositions).

So far, a great deal of literature on the real-variable theory of $H^\varphi(\mathbb{R}^n)$ and its applications has appeared (see, e.g., [16], [18]–[21], [32], [33]). A natural question is whether or not we can establish a unified framework for the wavelet characterizations of all the Hardy-type spaces. Unfortunately, it was shown in [34, Remark 1.7(iv)] that the Musielak–Orlicz Hardy space and the variable exponent Hardy space cannot cover each other. However, in this article we establish several equivalent characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of wavelets, which extend the wavelet characterizations of both Orlicz–Hardy spaces and weighted Hardy spaces and are available to the typical and useful Musielak–Orlicz Hardy space $H^{\log}(\mathbb{R}^n)$.

To formulate the main result of this article, let us first recall some important details. Recall that a function $\phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ if $t \in (0,\infty)$, and $\lim_{t\to\infty} \phi(t) = \infty$. The following notions of uniformly lower-type and upper-type properties can be found in Ky [16] (see also [32]).

Definition 1.1. For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for each $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is Orlicz, φ is said to be of *uniformly lower-* (resp., *upper-*) *type* p if there exists a positive constant $C_{(p)}$, depending on p, such that, for any $x \in \mathbb{R}^n$,

$t \in [0,\infty)$ and $s \in (0,1)$ (resp., $s \in [1,\infty)$), $\varphi(x,st) \le C_{(p)} s^p \varphi(x,t)$. Moreover, let

$$i(\varphi) := \sup\big\{p \in (0,\infty) : \varphi \text{ is of uniformly lower type } p\big\}. \tag{1.2}$$

Note that $i(\varphi)$ may not be attainable (see, e.g., [32] for some examples). We now recall the notions of the uniformly Muckenhoupt condition and the uniformly reverse Hölder condition from [16] (see also [32]).

Definition 1.2. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is said to satisfy the *uniformly Muckenhoupt condition* for some $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$[\varphi]_{\mathbb{A}_q(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{\int_B \big[\varphi(y,t)\big]^{-1/(q-1)}\,dy\right\}^{q-1} < \infty$$

or, when $q = 1$,

$$[\varphi]_{\mathbb{A}_1(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \Big(\operatorname{ess\,sup}_{y\in B} \big[\varphi(y,t)\big]^{-1}\Big) < \infty,$$

where the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$. Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is said to satisfy the *uniformly reverse Hölder condition* for some $q \in (1,\infty]$, denoted by $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$[\varphi]_{\mathbb{RH}_q(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{\frac{1}{|B|}\int_B \big[\varphi(x,t)\big]^q\,dx\right\}^{1/q} \left\{\frac{1}{|B|}\int_B \varphi(y,t)\,dy\right\}^{-1} < \infty$$

or, when $q = \infty$,

$$[\varphi]_{\mathbb{RH}_\infty(\mathbb{R}^n)} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{\frac{1}{|B|}\int_B \varphi(y,t)\,dy\right\}^{-1} \Big\{\operatorname*{ess\,sup}_{y\in B} \varphi(y,t)\Big\} < \infty,$$

where the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

Now we recall the notion of growth functions.

Definition 1.3. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following conditions are satisfied.

- (i) φ is a *Musielak–Orlicz function*, namely:
  - $(\mathrm{i})_1$ the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for each $x \in \mathbb{R}^n$,
  - $(\mathrm{i})_2$ the function $\varphi(\cdot,t)$ is measurable for every $t \in [0,\infty)$.
- (ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.
- (iii) φ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Remark 1.4.

(i) Some nontrivial examples of growth functions can be found in [32, Example 1.1.5] or Remark 1.11 below.

(ii) From [32, Remark 1.7], it follows that, without loss of generality, we may assume that $\varphi(x,\cdot)$ is continuous and strictly increasing for any fixed $x \in \mathbb{R}^n$.

We then recall the notion of Musielak–Orlicz Hardy spaces. In what follows, we denote the *Schwartz class* and its *dual space*, respectively, by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$. For any $m \in \mathbb{N} := \{1, 2, \dots\}$, let $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and

$$\mathscr{S}_m(\mathbb{R}^n) := \Big\{h \in \mathscr{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \sup_{\{\beta\in\mathbb{Z}_+^n:|\beta|\le m+1\}} \big(1+|x|\big)^{(m+2)(n+1)} \big|\partial^\beta h(x)\big| \le 1\Big\},$$

where, for any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ and $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Then, for any $f \in \mathscr{S}'(\mathbb{R}^n)$, its *nontangential grand maximal function* f_m^* is defined by setting

$$f_m^*(x) := \sup_{h\in\mathscr{S}_m(\mathbb{R}^n)} \sup_{\{|y-x|<t,t\in(0,\infty)\}} \big|f * h_t(y)\big|, \quad \forall x \in \mathbb{R}^n,$$

where, for any $t \in (0,\infty)$, $h_t(\cdot) := t^{-n} h(\frac{\cdot}{t})$. In what follows, for any $s \in \mathbb{R}$, we use the *symbol* $\lfloor s \rfloor$ to denote the largest integer not greater than s. Let

$$m(\varphi) := \big\lfloor n\big[q(\varphi)/i(\varphi) - 1\big]\big\rfloor, \tag{1.3}$$

where $i(\varphi)$ is as in (1.2) and

$$q(\varphi) := \inf\big\{q \in [1,\infty) : \varphi \in \mathbb{A}_q(\mathbb{R}^n)\big\}. \tag{1.4}$$

We simply write $f^* := f^*_{m(\varphi)}$. Observe that $q(\varphi)$ may not be attainable (see, e.g., [32]). Recall that $L^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f satisfying that there exists $t \in (0,\infty)$ such that

$$\int_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{t}\Big)\,dx < \infty$$

equipped with the quasinorm

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\Big\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{\lambda}\Big)\,dx \le 1\Big\}.$$

Definition 1.5. Let φ be a growth function as in Definition 1.3. The *Musielak–Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$, equipped with the quasinorm

$$\|f\|_{H^\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}.$$

We also need to recall notions of Musielak–Orlicz Campanato spaces from [20, Definition 1.1]. In what follows, for any $s \in \mathbb{Z}_+$, we use the *symbol* $\mathcal{P}_s(\mathbb{R}^n)$ to denote the set of all polynomials on $\mathbb{R}^n$ with the order not greater than s.

Definition 1.6. Let φ be as in Definition 1.3, let $s \in \mathbb{Z}_+$, and let $q \in [1,\infty)$. The *Musielak–Orlicz Campanato space* $\mathrm{Lip}_{\varphi,q,s}(\mathbb{R}^n)$ is defined to be the set of all functions $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|g\|_{\mathrm{Lip}_{\varphi,q,s}(\mathbb{R}^n)} := \sup_B \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{ \int_B \left[\frac{|g(x) - P_B^s g(x)|}{\varphi(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})} \right]^q \right.$$
$$\left. \times \varphi\big(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big)\, dx \right\}^{1/q} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $P_B^s g$ denotes the unique polynomial $P \in \mathcal{P}_s(\mathbb{R}^n)$ such that, for any $h \in \mathcal{P}_s(\mathbb{R}^n)$ on B,

$$\int_B \big[g(x) - P(x)\big] h(x)\, dx = 0.$$

In what follows, for any quasi-Banach space E, let E^* denote its *dual space* and, for any $q \in [1,\infty)$, let q' denote its *conjugate index*, namely, $1/q + 1/q' = 1$.

Remark 1.7. Let φ be a growth function. By [20, Theorems 2.7 and 3.5], we know that the following hold.

(i) If $s \in \mathbb{Z}_+$ and $q \in [1, [q(\varphi)]')$, then $\mathrm{Lip}_{\varphi,q,s}(\mathbb{R}^n)$ is independent of the choice of q.

(ii) If $s \in [m(\varphi),\infty) \cap \mathbb{Z}_+$ with $m(\varphi)$ as in (1.3), then $(H^\varphi(\mathbb{R}^n))^* = \mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$.

(iii) Assume that $s \in [m(\varphi),\infty) \cap \mathbb{Z}_+$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$ and $f^* \in L^\varphi(\mathbb{R}^n)$. Indeed, if $f \in H^\varphi(\mathbb{R}^n)$, then, by (ii) of this remark, we know that $f \in (H^\varphi(\mathbb{R}^n))^{**} = (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$ and, by Definition 1.5, $f^* \in L^\varphi(\mathbb{R}^n)$. On the other hand, if $f \in (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$ and $f^* \in L^\varphi(\mathbb{R}^n)$, then, by [21, Remark 1.12(i)], we find that $\mathscr{S}(\mathbb{R}^n) \subset \mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$ and hence $f \in \mathscr{S}'(\mathbb{R}^n)$, which, together with Definition 1.5, shows $f \in H^\varphi(\mathbb{R}^n)$. This finishes the proof of the above claim.

In order to state the main result of this article, we also need to recall some known facts on wavelets. We begin with the following notion of the multiresolution analysis on $\mathbb{R}$ (see, e.g., [24], [29] for more details).

Definition 1.8. A *multiresolution analysis* (for short, MRA) on $\mathbb{R}$ is defined to be an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces in $L^2(\mathbb{R})$ such that

(i) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{\theta\}$, where θ denotes the zero function;

(ii) for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f \in V_j$ if and only if $f(2^{-j}\cdot) \in V_0$;

(iii) for any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f \in V_0$ if and only if $f(\cdot - k) \in V_0$;

(iv) there exists a function $\phi \in L^2(\mathbb{R})$ (called *father wavelet*) such that

$$\big\{\phi_k(\cdot)\big\}_{k\in\mathbb{Z}} := \big\{\phi(\cdot - k)\big\}_{k\in\mathbb{Z}}$$

is an orthonormal basis of V_0.

For any fixed $s \in \mathbb{Z}_+$, according to [5, Section 4] (see also [29, Theorem 1.61(ii)]), we choose the *father* and the *mother wavelets* $\phi, \psi \in C_c^{s+1}(\mathbb{R})$ (the *set of all functions with compact supports having continuous derivatives up to order* $s+1$) such that $\widehat{\phi}(0)=(2\pi)^{-1/2}$ and, for any $l\in\{0,\dots,s+1\}$,

$$\int_{\mathbb{R}} x^l\psi(x)\,dx=0,$$

where $\widehat{\phi}$ denotes the *Fourier transform* of ϕ; namely, for any $\xi\in\mathbb{R}$,

$$\widehat{\phi}(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\phi(y)e^{-i\xi y}\,dy.$$

Following [3], throughout the whole article we *always assume* that

$$\operatorname{supp}\phi,\operatorname{supp}\psi\subset 1/2+m(-1/2,1/2), \tag{1.5}$$

where $1/2+m(-1/2,1/2)$ denotes the interval obtained from $(0,1)$ via a dilation by m centered at $1/2$; namely, $x\in 1/2+m(-1/2,1/2)$ if and only if $|x-1/2|<m/2$. Here $m\in[1,\infty)$ is a positive constant independent of the main parameters involved in the whole article.

The extension of the above considerations from 1-dimension to n-dimension can be realized by the standard procedure of tensor products. More precisely, let

$$\vec{\theta}_n:=(\overbrace{0,\dots,0}^{n\text{ times}})\qquad\text{and}\qquad E:=\{0,1\}^n\setminus\{\vec{\theta}_n\}.$$

Assume that $\mathcal{D}$ is the set of all *dyadic cubes* in $\mathbb{R}^n$, namely, for any $Q\in\mathcal{D}$, there exist $j\in\mathbb{Z}_+$ and $k:=\{k_1,\dots,k_n\}\in\mathbb{Z}^n$ such that

$$Q=Q_{j,k}:=\left\{x\in\mathbb{R}^n:\ k_i\le 2^jx_i<k_i+1\text{ for any }i\in\{1,\dots,n\}\right\}. \tag{1.6}$$

Let mQ be the m dilation of Q with the same center as Q and m as in (1.5). Based on the tensor product in [24, p. 108], we write, for any $\lambda:=(\lambda_1,\dots,\lambda_n)\in E$, $Q:=Q_{j,k}$ with $k:=(k_1,\dots,k_n)\in\mathbb{Z}^n$, $j\in\mathbb{Z}$, and $x=(x_1,\dots,x_n)$,

$$\begin{aligned}\psi_Q^\lambda(x)&:=2^{jn/2}\psi^{\lambda_1}(2^jx_1-k_1)\cdots\psi^{\lambda_n}(2^jx_n-k_n),\\ \phi_Q(x)&:=2^{jn/2}\phi(2^jx_1-k_1)\cdots\phi(2^jx_n-k_n),\end{aligned}$$

where $\psi^0:=\phi$ and $\psi^1:=\psi$.

A family $\{\psi_Q^\lambda\}_{Q\in\mathcal{D},\lambda\in E}\subset C^{s+1}(\mathbb{R}^n)$ (the *set of all functions having continuous derivatives up to order* $s+1$) is called an s*-order wavelet system* if $\{\psi_Q^\lambda\}_{Q\in\mathcal{D},\lambda\in E}$ satisfy

(i) $\{\psi_Q^\lambda\}_{Q\in\mathcal{D},\lambda\in E}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$;

(ii) ψ_Q^λ are compactly supported, namely,

$$\operatorname{supp}\psi_Q^\lambda\subset mQ; \tag{1.7}$$

(iii) there exists a positive constant C, depending on s, such that, for any $\beta:=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_+^n$ with $|\beta|:=\beta_1+\cdots+\beta_n\le s+1$,

$$\left|\partial^\beta\psi_Q^\lambda(x)\right|\le C2^{j|\beta|}2^{jn/2},\quad\forall x\in\mathbb{R}^n; \tag{1.8}$$

(iv) for any $\beta:=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_+^n$ with $|\beta|\le s$, $\int_{\mathbb{R}^n}x^\beta\psi_Q^\lambda(x)\,dx=0$, here and hereafter, for any $x:=(x_1,\dots,x_n)\in\mathbb{R}^n$, $x^\beta:=x_1^{\beta_1}\cdots x_n^{\beta_n}$.

The existence of the above s-order wavelet system is guaranteed by the classical results in [24, p. 108].

Thus, for any $f \in L^2(\mathbb{R}^n)$, we have

$$f = \sum_{\lambda\in E}\sum_{Q\in\mathcal{D}}(f,\psi_Q^\lambda)\psi_Q^\lambda = \sum_{\lambda\in E}\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^n}(f,\psi_{j,k}^\lambda)\psi_{j,k}^\lambda \quad \text{in } L^2(\mathbb{R}^n),$$

where $(\cdot,\cdot)$ denotes the *inner product* in $L^2(\mathbb{R}^n)$ and, for any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}^n$ with $Q=Q_{j,k}\in\mathcal{D}$ as in (1.6) and $\lambda\in E$,

$$\psi_{j,k}^\lambda := \psi_Q^\lambda.$$

As in [24, p. 142], we may assume that, for any $\lambda\in E$, there exists some set $W^\lambda\subset[0,1)^n$ satisfying $|W^\lambda|\ge\gamma>0$ and $|\psi^\lambda|\ge c_0\chi_{W^\lambda}$ for some fixed positive constants γ and c_0, where

$$\psi^\lambda := \psi^\lambda_{[0,1)^n}. \tag{1.9}$$

For every $\lambda\in E$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^n$, and $Q:=Q_{j,k}$, let

$$W_{j,k}^\lambda := \{x\in\mathbb{R}^n : 2^jx-k\in W^\lambda\} =: W_Q^\lambda. \tag{1.10}$$

Then we know that, for each $\lambda\in E$, $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^n$,

$$W_{j,k}^\lambda\subset Q_{j,k},\quad |W_{j,k}^\lambda|\ge\gamma|Q_{j,k}| \tag{1.11}$$

and

$$|\psi_{j,k}^\lambda|\ge c_0\frac{\chi_{W_{j,k}^\lambda}}{|Q_{j,k}|}. \tag{1.12}$$

In what follows, let

$$\Lambda := \big\{(\lambda,j,k):\lambda\in E,(j,k)\in\mathbb{Z}\times\mathbb{Z}^n\big\}. \tag{1.13}$$

Moreover, for any $j\in\mathbb{Z}$, let V_j be the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{\phi_Q\}_{|Q|=2^{-jn}}$. It is known that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA on $\mathbb{R}^n$, whose definition is an extension of MRA on $\mathbb{R}$ in Definition 1.8 (see [24, Chapter 2] or [29, Remark 1.52] for more details).

Let $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$. From Remark 1.7(iii), it follows that, if $f\in H^\varphi(\mathbb{R}^n)$, then $f\in(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$. On the other hand, by Corollary 2.2 below, we know that $\psi_{j,k}^\lambda\in\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$. Let $\langle\cdot,\cdot\rangle$ denote the *dual relation* between $\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$ and $(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$. Thus, following an idea used in [24, p. 177], we know that $\langle f,\psi_{j,k}^\lambda\rangle$ is well defined in the sense of the duality between $\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$ and $(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$. Now we are ready to state the main result of this article.

Theorem 1.9. *Suppose that φ is an growth function, suppose that $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$ with $m(\varphi)$ as in (1.3), $\{\psi_{j,k}^\lambda\}_{(\lambda,j,k)\in\Lambda}$ is an s-order wavelet system and, for any $f\in(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$, suppose that*

$$f=\sum_{(\lambda,j,k)\in\Lambda}\langle f,\psi_{j,k}^\lambda\rangle\psi_{j,k}^\lambda \quad \textit{in } \big(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)\big)^*. \tag{1.14}$$

Then the following statements are mutually equivalent:

(i) *$f \in H^{\varphi}(\mathbb{R}^n)$;*

(ii) *$\|f\|_{(\mathrm{i})} := \|[\sum_{(\lambda,j,k)\in\Lambda} |\langle f, \psi^{\lambda}_{j,k}\rangle|^2 |\psi^{\lambda}_{j,k}|^2]^{1/2}\|_{L^{\varphi}(\mathbb{R}^n)} < \infty$;*

(iii) *$\|f\|_{(\mathrm{ii})} := \|[\sum_{(\lambda,j,k)\in\Lambda} |\langle f, \psi^{\lambda}_{j,k}\rangle|^2 \frac{\chi_{Q_{j,k}}}{|Q_{j,k}|}]^{1/2}\|_{L^{\varphi}(\mathbb{R}^n)} < \infty$;*

(iv) *$\|f\|_{(\mathrm{iii})} := \|[\sum_{(\lambda,j,k)\in\Lambda} |\langle f, \psi^{\lambda}_{j,k}\rangle|^2 \frac{\chi_{W^{\lambda}_{j,k}}}{|Q_{j,k}|}]^{1/2}\|_{L^{\varphi}(\mathbb{R}^n)} < \infty$, where, for any $(\lambda, j, k) \in \Lambda$, $W^{\lambda}_{j,k} \subset Q_{j,k}$ is as in (1.10) and*

$$|W^{\lambda}_{j,k}| \sim |Q_{j,k}| \tag{1.15}$$

with the implicit positive constants independent of (λ, j, k).

Moreover, all the quasinorms $\|\cdot\|_{(\mathrm{i})}$, $\|\cdot\|_{(\mathrm{ii})}$ and $\|\cdot\|_{(\mathrm{iii})}$ are equivalent to $\|\cdot\|_{H^{\varphi}(\mathbb{R}^n)}$.

As a consequence of Theorem 1.9, we have the following conclusion.

Corollary 1.10. *Replacing the assumption (1.14) in Theorem 1.9 by $f \in L^2(\mathbb{R}^n)$, then all the conclusions in Theorem 1.9 still hold true.*

Remark 1.11. There exist the following interesting special cases of Theorem 1.9.

(i) For any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, let $\varphi(x, t) := t^p$, where $p \in (0, 1]$. Then Theorem 1.9 is just [24, p. 143, Theorem 1].

(ii) For any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, let $\varphi(x, t) := \omega(x)t^p$ with any given $p \in (0, 1]$. Then Theorem 1.9 covers [7, Theorem 4.2] and [31, Theorem 3.2].

(iii) For any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, let $\varphi(x, t) := \omega(x)\Phi(t)$, where ω is a nonnegative locally integrable function and Φ an Orlicz function. Then Theorem 1.9, even in such cases, seems new.

(iv) It was shown in [32, Lemma 1.1.3(v)] that another typical and useful growth function φ is defined by setting

$$\varphi(x, t) := \frac{t^{\alpha}}{[\mathrm{In}(e + |x|)]^{\beta} + [\mathrm{In}(e + t)]^{\gamma}}, \quad \forall x \in \mathbb{R}^n, \forall t \in [0, \infty)$$

with any giving $\alpha \in (0, 1]$, $\beta \in [0, \infty)$, and $\gamma \in [0, 2\alpha(1 + \mathrm{In}\, 2)]$. Then Theorem 1.9, even in such cases, is new. Recall that $H^{\log}(\mathbb{R}^n)$ with θ as in (1.1) plays an irreplaceable role in [3], [14].

This article is organized as follows. In Section 2, we first recall some known results on the atomic and the molecular characterizations of $H^{\varphi}(\mathbb{R}^n)$. We then introduce a new adapted atomic Musielak–Orlicz Hardy space $H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$ and show that $H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$ coincides with $H^{\varphi}(\mathbb{R}^n)$ via the known molecular characterization of $H^{\varphi}(\mathbb{R}^n)$ (see Theorem 2.12 below). The approach used in the proof of Theorem 2.12 is essentially an elaborate modification of the well-known argument given by Taibleson and Weiss in [28]. We also introduce a wavelet variant of atoms and show that they are positive constant multiples of the original atoms defined in Definition 2.4 (see Lemma 2.17 below), which are vital to the proof of Theorem 1.9. Then, via introducing a discrete Littlewood–Paley g-functions and applying the boundedness of the intrinsic Littlewood–Paley g-function from [21] and some ideas from [7], we establish some estimates on a discrete Peetre-type maximal function (see Proposition 2.21 below). Observe that the new adapted atomic characterization of $H^{\varphi}(\mathbb{R}^n)$ in Theorem 2.12 compensates the inconvenience in applications of the supremum appearing in the original definition of atoms (see Definition 2.4), which may have further potential applications. Section 3 is devoted to the proofs of Theorem 1.9 and Corollary 1.10 on several wavelet characterizations of the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$. By the estimate on the Peetre-type maximal function in Section 2, the wavelet characterizations of Lebesgue spaces from [24], two new variants of atomic characterizations established in Section 2 and some standard arguments on the wavelet characterizations of the classical Hardy spaces, we then finish the proof of Theorem 1.9 in this section. The proof of Corollary 1.10 is just a slight modification of the proof of Theorem 1.9.

Finally, we note some conventions on notation. Throughout the whole article, $\mathbb{N}$ stands for the *set of all positive integers*, $\mathbb{Z}_+$ the *set of all nonnegative integers* and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$; C stands for a *positive constant* which is independent of the main parameters, but it may vary from line to line. Moreover, we use $C_{(\rho,\beta,\dots)}$ to denote a positive constant depending on the indicated parameters $\rho, \beta, \dots$. If, for two real functions f and g, $f \le Cg$, then we write $f \lesssim g$; if $f \lesssim g \lesssim f$, the we write $f \sim g$. For any subset E of $\mathbb{R}^n$, we use χ_E to denote its *characteristic function*. Furthermore, $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$ represent the duality relation and the $L^2(\mathbb{R}^n)$ inner product, respectively. To simplify the notation, if there exists no ambiguity, we denote $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$ simply by $(\cdot,\cdot)$.

2. Preliminaries

In this section, we first recall the known atomic and the known molecular characterizations of $H^{\varphi}(\mathbb{R}^n)$ and then we establish a new atomic characterization of $H^{\varphi}(\mathbb{R}^n)$. We also introduce a wavelet variant of atoms and establish an estimate on a Peetre-type maximal function. We begin with showing that the wavelets belong to Musielak–Orlicz Campanato spaces.

For any $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$, let $\widetilde{\mathcal{C}}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ be the class of all functions $\eta \in C^s(\mathbb{R}^n)$ (the *set of all functions having continuous derivatives up to order* s) such that, for any $\nu \in \mathbb{Z}_+^n$, with $|\nu| \le s$, and for any $x \in \mathbb{R}^n$,

$$\left|\partial^{\nu}\eta(x)\right| \le \left(1+|x|\right)^{-n-\epsilon} \tag{2.1}$$

and, for any $\nu \in \mathbb{Z}_+^n$, with $|\nu| = s$, and for any $x_1, x_2 \in \mathbb{R}^n$,

$$\left|\partial^{\nu}\eta(x_1) - \partial^{\nu}\eta(x_2)\right| \le |x_1-x_2|^{\alpha}\left[\left(1+|x_1|\right)^{-n-\epsilon} + \left(1+|x_2|\right)^{-n-\epsilon}\right]. \tag{2.2}$$

In what follows, we write $\widetilde{\mathcal{C}}_{\epsilon,s}(\mathbb{R}^n) := \widetilde{\mathcal{C}}_{(1,\epsilon),s}(\mathbb{R}^n)$.

Proposition 2.1. *Let φ be a growth function. Then*

(i) *if $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+1),1]$, and $\varphi \in \mathbb{A}_{p(1+[\alpha+s]/n)}(\mathbb{R}^n)$, then $\widetilde{\mathcal{C}}_{(\alpha,\epsilon),s}(\mathbb{R}^n) \subset \mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$;*

(ii) *if $s \in [m(\varphi),\infty) \cap \mathbb{N}$, with $m(\varphi)$ as in (1.3), and $\epsilon \in (1+s,\infty)$, then $\widetilde{\mathcal{C}}_{\epsilon,s}(\mathbb{R}^n) \subset \mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$.*

Proof. By a careful check on the proof of [21, Proposition 2.3] (see also [32, Proposition 6.2.1]), we know that (i) holds true. Indeed, in the proof of [21, Proposition 2.3] (or [32, Propostion 6.2.1]), no vanishing moment of the considered functions was used.

Now, let $s \in [m(\varphi), \infty) \cap \mathbb{N}$ and $\epsilon \in (1+s, \infty)$. To prove (ii), based on (i), it suffices to show that $\varphi \in \mathbb{A}_{1+(1+s)/n}(\mathbb{R}^n)$. Indeed, we first claim that $i(\varphi) \le 1$ with $i(\varphi)$ as in (1.2). Otherwise, if $i(\varphi) > 1$, then there exists $p_0 \in (1, i(\varphi))$ such that φ is of uniformly lower-type p_0, which, combined with the uniformly upper-type 1 of φ, further implies that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$, and $s \in (0,1)$, $s\varphi(x,t) \lesssim \varphi(x, st) \lesssim s^{p_0}\varphi(x,t)$ and hence that $s^{1-p_0} \lesssim 1$, where the implicit positive constants are independent of s, t, and x. Letting $s \to 0$, we then have $\infty = \lim_{s\to 0} s^{1-p_0} \lesssim 1$, which is a contradiction. Thus, $i(\varphi) \le 1$, which completes the proof of the preceding claim.

Moreover, by this claim and $s \ge m(\varphi)$, we obtain

$$s > n\big[q(\varphi)/i(\varphi) - 1\big] - 1 \ge n\big[q(\varphi) - 1\big] - 1,$$

and hence $q(\varphi) < 1 + (1+s)/n$. From this, the definition of $q(\varphi)$ in (1.4), and [32, Lemma 1.1.3(i)], we deduce that there exists $q_0 \in (q(\varphi), 1 + (1+s)/n)$ such that $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n) \subset \mathbb{A}_{1+(1+s)/n}(\mathbb{R}^n)$, which completes the proof of (ii) and hence of Proposition 2.1. $\square$

As a consequence of Proposition 2.1, we have the following conclusion.

Corollary 2.2. *Let φ be a growth function and let $s \in [m(\varphi), \infty) \cap \mathbb{Z}_+$ with $m(\varphi)$ as in (1.3). Then, for any $(\lambda, j, k) \in \Lambda$ with Λ as in (1.13), $\psi^\lambda_{j,k} \in \mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)$.*

Proof. Let φ be a growth function, let $s \in [m(\varphi), \infty) \cap \mathbb{Z}_+$ with $m(\varphi)$ as in (1.3), and let $\epsilon \in (1+s, \infty)$. By Proposition 2.1(ii), it suffices to show that, for any $(\lambda, j, k) \in \Lambda$, there exists a positive constant $\widetilde{C}$, depending on j and k, such that $\widetilde{C}\psi^\lambda_{j,k} \in \widetilde{\mathcal{C}}_{\epsilon,s}(\mathbb{R}^n)$. Indeed, by the fact that $\psi^\lambda_{j,k} \in C^{s+1}(\mathbb{R}^n)$ with compact support, we easily know that there exists a positive constant $\widetilde{C}$, depending on j and k, such that $\widetilde{C}\psi^\lambda_{j,k}$ satisfies (2.1). We now show that $\widetilde{C}\psi^\lambda_{j,k}$ also satisfies (2.2). To this end, let $\nu \in \mathbb{Z}^n_+$, with $|\nu| = s$, and let $x_1, x_2 \in \mathbb{R}^n$. We consider the following two cases.

Case (1): $|x_1 - x_2| \ge 1$. In this case, since $\psi^\lambda_{j,k} \in C^{s+1}(\mathbb{R}^n)$ has compact support, it follows that

$$\begin{aligned}
&\big|\partial^\nu \psi^\lambda_{j,k}(x_1) - \partial^\nu \psi^\lambda_{j,k}(x_2)\big| \\
&\quad \le \big|\partial^\nu \psi^\lambda_{j,k}(x_1)\big| + \big|\partial^\nu \psi^\lambda_{j,k}(x_2)\big| \lesssim \big(1 + |x_1|\big)^{-n-\epsilon} + \big(1 + |x_2|\big)^{-n-\epsilon} \\
&\quad \lesssim |x_1 - x_2|\Big[\big(1 + |x_1|\big)^{-n-\epsilon} + \big(1 + |x_2|\big)^{-n-\epsilon}\Big],
\end{aligned}$$

which is the desired estimate.

Case (2): $|x_1 - x_2| \le 1$. In this case, by (1.7) and (1.8), we conclude that, if $x_1, x_2 \notin mQ_{j,k}$, then

$$\big|\partial^\nu \psi^\lambda_{j,k}(x_1) - \partial^\nu \psi^\lambda_{j,k}(x_2)\big| = 0;$$

if $x_1 \in mQ_{j,k}$ and $x_2 \notin mQ_{j,k}$, then, by the mean value theorem for derivatives, we have

$$\begin{aligned}\left|\partial^\nu \psi^\lambda_{j,k}(x_1) - \partial^\nu \psi^\lambda_{j,k}(x_2)\right| &\lesssim |x_1 - x_2| \lesssim |x_1 - x_2|\left(1 + |x_1|\right)^{-n-\epsilon} \\ &\lesssim |x_1 - x_2|\left[\left(1 + |x_1|\right)^{-n-\epsilon} + \left(1 + |x_2|\right)^{-n-\epsilon}\right]\end{aligned}$$

and, by symmetry, a similar estimate also holds true if $x_1 \notin mQ_{j,k}$ and $x_2 \in mQ_{j,k}$; finally, if $x_1, x_2 \in mQ_{j,k}$, then, by the mean value theorem for derivatives again, we obtain

$$\begin{aligned}\left|\partial^\nu \psi^\lambda_{j,k}(x_1) - \partial^\nu \psi^\lambda_{j,k}(x_2)\right| &\lesssim |x_1 - x_2| \\ &\lesssim |x_1 - x_2|\left[\left(1 + |x_1|\right)^{-n-\epsilon} + \left(1 + |x_2|\right)^{-n-\epsilon}\right].\end{aligned}$$

This finishes the proof of (2.2) and hence of Corollary 2.2. □

Now we recall the notion of atomic Musielak–Orlicz Hardy spaces from [16] (see also [32]).

Definition 2.3. For any measurable set S in $\mathbb{R}^n$, the *space* $L^q_\varphi(S)$ for any $q \in [1,\infty]$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$, supported on S, such that

$$\|f\|_{L^q_\varphi(S)} := \begin{cases} \sup_{t\in(0,\infty)}\left[\frac{1}{\varphi(S,t)}\int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx\right]^{1/q} < \infty & \text{when } q \in [1,\infty), \\ \|f\|_{L^\infty(\mathbb{R}^n)} < \infty & \text{when } q = \infty. \end{cases}$$

Here and hereafter, $\varphi(S,t) := \int_S \varphi(x,t)\,dx$ for any $t \in [0,\infty)$.

Recall that, for any given $q \in (0,\infty]$, the *space* $L^q(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that, when $q \in (0,\infty)$,

$$\|f\|_{L^q(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} \left|f(x)\right|^q dx\right]^{1/q} < \infty$$

and, when $q = \infty$,

$$\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left|f(x)\right| < \infty.$$

Definition 2.4. A triplet (φ, q, s) is said to be *admissible* if $q \in (q(\varphi),\infty]$ and $s \in \mathbb{Z}_+$ satisfies $s \geq m(\varphi)$, where $q(\varphi)$ and $m(\varphi)$ are defined, respectively, as in (1.4) and (1.3). Let (φ, q, s) be admissible. A measurable function a is called a (φ, q, s)-*atom* if the following three conditions hold true:

(i) $a \in L^q_\varphi(Q)$ for some cube Q;
(ii) $\|a\|_{L^q_\varphi(Q)} \leq \|\chi_Q\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;
(iii) for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leq s$,

$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0.$$

The *atomic Musielak–Orlicz Hardy space* $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying that there exists a sequence $\{b_j\}_{j\in\mathbb{N}}$ of constant multiples of some (φ, q, s)-atoms, respectively, related to some cubes $\{Q_j\}_{j\in\mathbb{N}}$, with the property

$$\sum_{j=1}^{\infty} \varphi\big(Q_j, \|b_j\|_{L^q_\varphi(Q_j)}\big) < \infty,$$

such that $f = \sum_{j=1}^{\infty} b_j$ in $\mathscr{S}'(\mathbb{R}^n)$. Moreover, let

$$\Lambda\big(\{b_j\}_{j=1}^{\infty}\big) := \inf\left\{\lambda \in (0,\infty) : \sum_{j=1}^{\infty} \varphi\left(Q_j, \frac{\|b_j\|_{L^q_\varphi(Q_j)}}{\lambda}\right) \le 1\right\}$$

and then define

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)} := \inf\left\{\Lambda\big(\{b_j\}_{j=1}^{\infty}\big) : f = \sum_{j=1}^{\infty} b_j \text{ in } \mathscr{S}'(\mathbb{R}^n)\right\},$$

where the infimum is taken over all decompositions of f as above.

The following atomic characterization of $H^\varphi(\mathbb{R}^n)$ is taken from [16, Theorem 3.1] (see also [32, Theorem 1.3.17]).

Theorem 2.5. *Let (φ, q, s) be admissible. Then $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n) = H^\varphi(\mathbb{R}^n)$ with equivalent quasinorms.*

Remark 2.6. Let (φ, q, s) be admissible. Then, by Theorem 2.5, $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ is independent of the choices of q and s in the sense of equivalent quasinorms.

We also need to recall the molecular characterization of $H^\varphi(\mathbb{R}^n)$. To this end, we first review the notion of molecular Musielak–Orlicz Hardy spaces from [10] (see also [32]).

Definition 2.7. Let φ be a growth function, $q \in (1,\infty)$, and $s \in \mathbb{Z}_+$ and $\epsilon \in (0,\infty)$. A measurable function α is called a $(\varphi, q, s, \epsilon)$-*molecule*, related to the ball B, if the following two conditions hold true:

(i) for any $j \in \mathbb{Z}_+$,

$$\|\alpha\|_{L^q(U_j(B))} \le 2^{-j\epsilon} |2^j B|^{1/q} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)},$$

where $U_0(B) := B$ and $U_j(B) := 2^j B \setminus 2^{j-1} B$ for any $j \in \mathbb{N}$;

(ii) for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \le s$,

$$\int_{\mathbb{R}^n} x^\beta \alpha(x)\, dx = 0.$$

The *molecular Musielak–Orlicz Hardy space* $H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(\varphi, q, s, \epsilon)$-molecules, respectively, related to some balls $\{B_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, with the property

$$\sum_{j=1}^{\infty} \varphi\left(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\right) < \infty,$$

such that $f=\sum_{j=1}^{\infty}\lambda_j\alpha_j$ in $\mathscr{S}'(\mathbb{R}^n)$. Moreover, let

$$\Lambda\big(\{\lambda_j\alpha_j\}_{j=1}^{\infty}\big):=\inf\Big\{\lambda\in(0,\infty):\sum_{j=1}^{\infty}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^{\varphi}(\mathbb{R}^n)}}\Big)\le 1\Big\}$$

and then define

$$\|f\|_{H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)}:=\inf\Big\{\Lambda\big(\{\lambda_j\alpha_j\}_{j=1}^{\infty}\big):f=\sum_{j=1}^{\infty}\lambda_j\alpha_j\ \text{in}\ \mathscr{S}'(\mathbb{R}^n)\Big\},$$

where the infimum is taken over all decompositions of f as above.

The following molecular characterization of $H^{\varphi}(\mathbb{R}^n)$ is taken from [10, Theorem 4.13] (see also [32, Theorem 3.2.10]).

Theorem 2.8. *Let φ be a growth function, let $\epsilon\in(\max\{n+s,nq(\varphi)/i(\varphi)\},\infty)$, let $s\in\mathbb{Z}_+$, and let $q\in(q(\varphi)[r(\varphi)]',\infty)$, where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (1.4) and (1.2), and*

$$r(\varphi):=\sup\big\{q\in(1,\infty]:\varphi\in\mathbb{RH}_q(\mathbb{R}^n)\big\}. \tag{2.3}$$

Then $H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)=H^{\varphi}(\mathbb{R}^n)$ with equivalent quasinorms.

Recall that $r(\varphi)$ may not be attainable (see, e.g., [32]).

Remark 2.9. Let φ,q,s, and ϵ be as in Theorem 2.8. Then, by Theorem 2.8, we know that $H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ is independent of the choices of q,s, and ϵ in the sense of equivalent quasinorms.

In order to obtain Theorem 1.9, we also need to establish a new atomic characterization for $H^{\varphi}(\mathbb{R}^n)$. We now introduce the notion of adapted atomic Musielak–Orlicz Hardy spaces.

Definition 2.10. Let (φ,q,s) be admissible. A function $\alpha\in L^q(\mathbb{R}^n)$ is called an *adapted* $(\varphi,q,s)_A$*-atom* related to some ball B if part (iii) of Definition 2.4 holds true and, instead of (i) and (ii) of Definition 2.4, a satisfies

(i)$'$ $\operatorname{supp} a\subset B$,

(ii)$'$ $\|a\|_{L^q(\mathbb{R}^n)}\le\frac{|B|^{1/q}}{\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}}$.

The *adapted atomic Musielak–Orlicz Hardy space* $H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$ is defined to be the space of all $f\in\mathscr{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{a_j\}_{j\in\mathbb{N}}$ of some (φ,q,s)-atoms, respectively, related to some balls $\{B_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$, with the property

$$\sum_{j=1}^{\infty}\varphi\Big(B_j,\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{\varphi}(\mathbb{R}^n)}}\Big)<\infty,$$

such that $f=\sum_{j=1}^{\infty}\lambda_j a_j$ in $\mathscr{S}'(\mathbb{R}^n)$. Moreover, let

$$\Lambda\big(\{b_j\}_{j=1}^{\infty}\big):=\inf\Big\{\lambda\in(0,\infty):\sum_{j=1}^{\infty}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^{\varphi}(\mathbb{R}^n)}}\Big)\le 1\Big\}$$

and then define

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)} := \inf\Big\{\Lambda\big(\{b_j\}_{j=1}^{\infty}\big) : f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in } \mathscr{S}'(\mathbb{R}^n)\Big\},$$

where the infimum is taken over all decompositions of f as above.

Remark 2.11. It is obvious that an adapted $(\varphi,\infty,s)_A$-atom is just a (φ,∞,s)-atom.

Then we establish a new atomic characterization of $H^{\varphi}(\mathbb{R}^n)$ as follows.

Theorem 2.12. *Let φ be a growth function, let $s \in \mathbb{Z}_+ \cap [m(\varphi),\infty)$, and let $q \in (q(\varphi)[r(\varphi)]',\infty)$, where $m(\varphi)$, $q(\varphi)$ and $r(\varphi)$ are, respectively, as in (1.3), (1.4) and (2.3). Then $H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n) = H^{\varphi}(\mathbb{R}^n)$ with equivalent quasinorms.*

Proof. Let φ be a growth function, let $\epsilon \in (\max\{n+s, nq(\varphi)/i(\varphi)\},\infty)$, let $s \in \mathbb{Z}_+ \cap [m(\varphi),\infty)$, and let $q \in (q(\varphi)[r(\varphi)]',\infty)$. Then (φ,q,s) is admissible.

First, we observe that any adapted $(\varphi,q,s)_A$-atom is a (φ,q,s,ϵ)-molecule. Thus, by Theorem 2.8, we have

$$H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n) \subset H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n) = H^{\varphi}(\mathbb{R}^n)$$

and, for any $f \in H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$,

$$\|f\|_{H^{\varphi}(\mathbb{R}^n)} \sim \|f\|_{H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)} \lesssim \|f\|_{H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)}.$$

Now we show that $H^{\varphi}(\mathbb{R}^n) = H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n) \subset H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$. Indeed, let α be any fixed (φ,q,s,ϵ)-molecule related to some ball $B := B(x_B,r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$. It suffices to show that α is a countable linear combination of adapted $(\varphi,q,s)_A$-atoms and adapted $(\varphi,\infty,s)_A$-atoms. To do this, for any $k \in \mathbb{Z}_+$, we let $\alpha_k := \alpha\chi_{U_k(B)}$, where $U_k(B)$ is as in Definition 2.7(i), and we also use the *symbol* $\mathcal{P}_k(\mathbb{R}^n)$ to denote the set of all polynomials with order not greater than s restricted on $U_k(B)$, namely,

$$\mathcal{P}_k(\mathbb{R}^n) := \operatorname{span}\big\{x^{\beta}\chi_{U_k(B)} : \beta \in \mathbb{Z}^n_+, |\beta| \le s\big\}.$$

By the proof of [23, p. 41, (3.5)], we conclude that, for any $k \in \mathbb{Z}_+$, there exists $P_k \in \mathcal{P}_k(\mathbb{R}^n)$ such that, for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \le s$,

$$\int_{U_k(B)} x^{\beta}\big[\alpha_k(x) - P_k(x)\big]\,dx = 0. \tag{2.4}$$

Let us now prove that, for any $k \in \mathbb{Z}_+$, $\alpha_k - P_k$ is a constant multiple of an adapted $(\varphi,q,s)_A$-atom and that $\sum_{k=0}^{\infty} P_k$ also has a decomposition in terms of adapted $(\varphi,\infty,s)_A$-atoms. If these facts hold true, then

$$\alpha = \sum_{k=0}^{\infty}\alpha_k = \sum_{k=0}^{\infty}(\alpha_k - P_k) + \sum_{k=0}^{\infty} P_k, \tag{2.5}$$

and we can easily finish the proof of Theorem 2.12.

As in the proof of [23, Theorem 7.2], for any $k \in \mathbb{Z}_+$ and s as above, let $\{Q_\ell^k : \ell \in \mathbb{Z}_+^n, |\ell| \le s\}$ be the orthogonal polynomials with weight $1/|U_k(B)|$ by means of the Gram–Schmidt orthogonalization process from

$$\left\{x^\beta \chi_{U_k(B)} : \beta \in \mathbb{Z}_+^n, |\beta| \le s\right\}.$$

Thus, for any $k \in \mathbb{Z}_+$ and $\ell, \beta \in \mathbb{Z}_+^n$ with $|\ell|, |\beta| \le s$, we have

$$\langle Q_\ell^k, Q_\beta^k \rangle_w := \frac{1}{|U_k(B)|} \int_{U_k(B)} Q_\ell^k(x) Q_\beta^k(x)\, dx = \delta_{\ell,\beta},$$

where $\delta_{\ell,\beta} := 1$ if $\ell = \beta$ and $\delta_{\ell,\beta} := 0$ if $\ell \neq \beta$. From [23, (7.1)], it follows that

$$P_k = \sum_{\ell \in \mathbb{Z}_+^n, |\ell| \le s} \langle \alpha_k, Q_\ell^k \rangle_w Q_\ell^k \chi_{U_k(B)}.$$

By [23, Lemma 4.1], we know that

$$\sup_{x \in U_k(B)} \left|P_k(x)\right| \lesssim \frac{1}{|U_k(B)|} \|\alpha_k\|_{L^1(\mathbb{R}^n)},$$

which, together with the Minkowski inequality and the Hölder inequality, implies that

$$\begin{aligned} \|\alpha_k - P_k\|_{L^q(\mathbb{R}^n)} &\le \|\alpha_k\|_{L^q(U_k(B))} + \|P_k\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|\alpha_k\|_{L^q(U_k(B))} + \|P_k\|_{L^\infty(\mathbb{R}^n)} \left|U_k(B)\right|^{1/q} \\ &\lesssim \|\alpha_k\|_{L^q(U_k(B))} + \|\alpha_k\|_{L^1(U_k(B))} \frac{1}{|U_k(B)|^{1/q'}} \\ &\lesssim \|\alpha_k\|_{L^q(U_k(B))} \le C_0 2^{-k\epsilon} |2^k B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \end{aligned} \tag{2.6}$$

where C_0 is a positive constant independent of k and B.

Let $\mu_k := C_0 2^{-k\epsilon} \|\chi_{2^k B}\|_{L^\varphi(\mathbb{R}^n)} / \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}$ and let $a_k := \frac{\alpha_k - P_k}{\mu_k}$. This condition, combined with (2.4) and (2.6) and the fact that $\mathrm{supp}(\alpha_k - P_k) \subset 2^k B$, implies that, for any $k \in \mathbb{Z}_+$, a_k is an adapted $(\varphi, q, s)_A$-atom and $\alpha_k - P_k = \mu_k a_k$. Moreover, by the Minkowski inequality, (2.6) and $\epsilon > n\frac{q(\varphi)}{i(\varphi)} \ge n > \frac{n}{q}$, we conclude that

$$\begin{aligned} \left\|\sum_{k=0}^\infty (\alpha_k - P_k)\right\|_{L^q(\mathbb{R}^n)} &\le \sum_{k=0}^\infty \|\alpha_k - P_k\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{k=0}^\infty 2^{-k(\epsilon - \frac{n}{q})} |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\ &\lesssim |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \end{aligned}$$

which further implies that

$$\sum_{k=0}^\infty (\alpha_k - P_k) = \sum_{k=0}^\infty \mu_k a_k \quad \text{in } L^q(\mathbb{R}^n). \tag{2.7}$$

Now we turn to the atomic decomposition of $\sum_{k=0}^\infty P_k$. For any $k \in \mathbb{Z}_+$ and s as above, let $\{P_\ell^k : \ell \in \mathbb{Z}_+^n, |\ell| \le s\}$ be a dual basis of $\{x^\alpha \chi_{U_k(B)} : \alpha \in \mathbb{Z}_+^n, |\alpha| \le s\}$

with respect to the weight $1/|U_k(B)|$; namely, for any $k \in \mathbb{Z}_+$ and $\alpha, \ell \in \mathbb{Z}_+^n$ with $|\ell|, |\alpha| \le s$, $P_\ell^k \in \mathcal{P}_k(\mathbb{R}^n)$ and

$$\langle P_\ell^k, x^\alpha \rangle_w := \frac{1}{|U_k(B)|} \int_{U_k(B)} x^\alpha P_\ell^k(x)\, dx = \delta_{\alpha,\ell}. \tag{2.8}$$

Thus, for any $x \in \mathbb{R}^n$, we have

$$Q_\ell^k(x) = \sum_{\nu \in \mathbb{Z}_+^n, |\nu| \le s} \langle Q_\ell^k, x^\nu \rangle_w x^\nu \qquad \text{and} \qquad P_\ell^k(x) = \sum_{\nu \in \mathbb{Z}_+^n, |\nu| \le s} \langle x^\ell, Q_\nu^k \rangle_w Q_\nu^k(x),$$

and that property leads to

$$P_k(x) = \sum_{\ell \in \mathbb{Z}_+^n, |\ell| \le s} \langle \alpha_k, x^\ell \rangle_w P_\ell^k(x) \chi_{U_k(B)}(x). \tag{2.9}$$

Furthermore, from [23, (7.2)], we deduce that, for any $k \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}_+^n$ with $|\ell| \le s$, and $x \in \mathbb{R}^n$,

$$\left|P_\ell^k(x)\right| \lesssim (2^{k-1} r_B)^{-|\ell|}. \tag{2.10}$$

For any $k \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}_+^n$ with $|\ell| \le s$, let

$$N_\ell^k := \sum_{j=k}^\infty \left|U_j(B)\right| \langle \alpha_j, x^\ell \rangle_w.$$

Then, by the Lebesgue dominated convergence theorem, we have, for any $\ell \in \mathbb{Z}_+^n$ with $|\ell| \le s$,

$$N_\ell^0 := \sum_{j=0}^\infty \left|U_j(B)\right| \langle \alpha_j, x^\ell \rangle_w = \sum_{j=0}^\infty \int_{\mathbb{R}^n} \alpha_j(x) x^\ell \, dx = \int_{\mathbb{R}^n} \alpha(x) x^\ell \, dx = 0 \tag{2.11}$$

and, by the Hölder inequality and $\epsilon > s + n \ge |\ell| + n$, we obtain, for any $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}_+^n$ with $|\ell| \le s$,

$$\begin{aligned} |N_\ell^k| &\le \sum_{j=k}^\infty \int_{U_j(B)} \left|\alpha_j(x) x^\ell\right| dx \le \sum_{j=k}^\infty \|\alpha_j\|_{L^q(U_j(B))} (2^j r_B)^{|\ell| + \frac{n}{q'}} \\ &\lesssim \sum_{j=k}^\infty 2^{-j\epsilon} |2^j B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} (2^j r_B)^{|\ell| + \frac{n}{q'}} \sim \sum_{j=k}^\infty 2^{-j(\epsilon - |\ell| - n)} |B|^{1 + \frac{|\ell|}{n}} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\ &\sim 2^{-k(\epsilon - |\ell| - n)} |B|^{1 + \frac{|\ell|}{n}} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \end{aligned} \tag{2.12}$$

which, together with (2.10), implies that, for any $k \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}_+^n$ with $|\ell| \le s$,

$$\begin{aligned} \left|U_k(B)\right|^{-1} |N_\ell^k| |P_\ell^k \chi_{U_k(B)}| &\lesssim 2^{-kn} |B|^{-1} 2^{-k(\epsilon - n - |\ell|)} 2^{-k|\ell|} r_B^{-|\ell|} |B|^{1 + \frac{|\ell|}{n}} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\ &\sim 2^{-k\epsilon} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \end{aligned} \tag{2.13}$$

Moreover, by (2.9), (2.11) and the Abel transform, we write

$$\begin{aligned}\sum_{k=0}^{\infty} P_k &= \sum_{k=0}^{\infty} \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \langle \alpha_k, x^\ell\rangle_w P_\ell^k \chi_{U_k(B)} \\ &= \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \sum_{k=0}^{\infty} \frac{N_\ell^k - N_\ell^{k+1}}{|U_k(B)|} P_\ell^k \chi_{U_k(B)} \\ &= \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \sum_{k=0}^{\infty} N_\ell^{k+1}\left[P_\ell^{k+1}\frac{\chi_{U_{k+1}(B)}}{|U_{k+1}(B)|} - P_\ell^k \frac{\chi_{U_k(B)}}{|U_k(B)|}\right] \\ &=: \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \sum_{k=0}^{\infty} b_\ell^k,\end{aligned}$$

where, for any $k\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$,

$$b_\ell^k := N_\ell^{k+1}\left[P_\ell^{k+1}\frac{\chi_{U_{k+1}(B)}}{|U_{k+1}(B)|} - P_\ell^k \frac{\chi_{U_k(B)}}{|U_k(B)|}\right].$$

From (2.13), (2.10), and (2.12), we deduce that

$$\|b_\ell^k\|_{L^\infty(\mathbb{R}^n)} \le C_1 2^{-k\epsilon} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \tag{2.14}$$

where C_1 is a positive constant independent of k, ℓ, and B. For any $k\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_+^n$ with $|\ell|\le s$, let $\mu_\ell^k := C_1 2^{-k\epsilon}\|\chi_{2^{k+1}B}\|_{L^\varphi(\mathbb{R}^n)}/\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}$ and $a_\ell^k := b_\ell^k/\mu_\ell^k$. Then

$$\|a_\ell^k\|_{L^\infty(\mathbb{R}^n)} \le \|\chi_{2^{k+1}B}\|_{L^\varphi(\mathbb{R}^n)}^{-1}.$$

By (2.8) and the definition of a_ℓ^k, we know that, for any $k\in\mathbb{Z}_+$ and $\ell,\beta\in\mathbb{Z}_+^n$ with $|\ell|,|\beta|\le s$,

$$\int_{\mathbb{R}^n} a_\ell^k(x) x^\beta\,dx = 0.$$

Furthermore, observe that $\operatorname{supp} a_\ell^k \subset 2^{k+1}B$ for any $k\in\mathbb{N}$. Thus, a_ℓ^k is an adapted $(\varphi,\infty,s)_A$-atom and $b_\ell^k = \mu_\ell^k a_\ell^k$. Similarly to (2.7), we find that

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \mu_\ell^k a_\ell^k \quad \text{in } L^q(\mathbb{R}^n).$$

By this, (2.5) and (2.7), we conclude that

$$\alpha = \sum_{k=0}^{\infty}(\alpha_k - P_k) + \sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \mu_k a_k + \sum_{k=0}^{\infty} \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \mu_\ell^k a_\ell^k \tag{2.15}$$

hold true in $L^q(\mathbb{R}^n)$ and hence in $\mathscr{S}'(\mathbb{R}^n)$.

By $\epsilon > n\frac{q(\varphi)}{i(\varphi)}$, we know that there exists $p_0 \in [1,\infty)$ such that $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\epsilon > \frac{p_0}{i(\varphi)} \geq \frac{p_0}{p}$, where φ is of uniformly lower-type $p \in (0,1]$. From this, (2.6), (2.14), and [32, Lemma 1.1.3(iv)], we further deduce that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \varphi\Big(2^k B, \frac{\lambda|\mu_k|}{\|\chi_{2^k B}\|_{L^\varphi(\mathbb{R}^n)}}\Big) \\
&\qquad + \sum_{k=0}^{\infty} \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \varphi\Big(2^{k+1} B, \frac{\lambda|\mu_\ell^k|}{\|\chi_{2^{k+1} B}\|_{L^\varphi(\mathbb{R}^n)}}\Big) \\
&\quad \lesssim \sum_{k=0}^{\infty} \varphi\Big(2^k B, 2^{-k\epsilon}\frac{\lambda}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big) \\
&\qquad + \sum_{k=0}^{\infty} \sum_{\{\ell\in\mathbb{Z}_+^n:|\ell|\le s\}} \varphi\Big(2^{k+1} B, 2^{-(k+1)\epsilon}\frac{\lambda}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big) \\
&\quad \lesssim \sum_{k=0}^{\infty} 2^{-kp\epsilon}\varphi\Big(2^k B, \frac{\lambda}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big) \lesssim \sum_{k=0}^{\infty} 2^{-kp(\epsilon-\frac{p_0}{p})}\varphi\Big(B, \frac{\lambda}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big) \\
&\quad \lesssim \varphi\Big(B, \frac{\lambda}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big). \qquad (2.16)
\end{aligned}$$

Let $f \in H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$. We know that there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_{j=1}^\infty$ of $(\varphi, q, s, \epsilon)$-molecules such that

$$f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \quad \text{in } \mathscr{S}'(\mathbb{R}^n)$$

and

$$\|f\|_{H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)} \sim \Lambda\big(\{\lambda_j\alpha_j\}_{j=1}^\infty\big). \qquad (2.17)$$

Then, by (2.15), we find that, for any $j \in \mathbb{N}$, there exist a sequence $\{a_{j,k}\}_{k\in\mathbb{N}}$ of adapted $(\varphi, q, s)_A$-atoms and $\{\mu_{j,k}\}_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $\alpha_j = \sum_{k=1}^\infty \mu_{j,k} a_{j,k}$ in $\mathscr{S}'(\mathbb{R}^n)$. Thus, we have

$$f = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \lambda_j \mu_{j,k} a_{j,k} \quad \text{in } \mathscr{S}'(\mathbb{R}^n),$$

which, combined with Theorem 2.8, (2.16), and (2.17), further implies that

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)} \lesssim \Lambda\big(\{\lambda_j\mu_{j,k}a_{j,k}\}_{j,k=1}^\infty\big) \lesssim \Lambda\big(\{\lambda_j\alpha_j\}_{j=1}^\infty\big) \sim \|f\|_{H^{q,s,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)} \sim \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.12. □

Remark 2.13. Let φ, q, s be as in Theorem 2.12. Then, by this theorem, we know that $H^{\varphi,q,s}_{\mathrm{at,A}}(\mathbb{R}^n)$ is independent of the choices of q and s in the sense of equivalent quasinorms.

We further introduce a variant of atoms adapted from [7, Definition 4.17]. To this end, we first introduce some notation. Let φ be a growth function, let $s \in [m(\varphi), \infty) \cap \mathbb{Z}_+$ with $m(\varphi)$ as in (1.3), and let $\{\psi_{j,k}^{\lambda}\}_{(\lambda,j,k)\in\Lambda}$ be an s-order wavelet system. In what follows, for any $f \in (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$, let

$$\mathcal{W}_{\psi,(\mathrm{i})} f := \Big[\sum_{(\lambda,j,k)\in\Lambda} \big|\langle f, \psi_{j,k}^{\lambda}\rangle\big|^2 |\psi_{j,k}^{\lambda}|^2 \Big]^{1/2}, \tag{2.18}$$

$$\mathcal{W}_{\psi,(\mathrm{ii})} f := \Big[\sum_{(\lambda,j,k)\in\Lambda} \big|\langle f, \psi_{j,k}^{\lambda}\rangle\big|^2 \frac{\chi_{Q_{j,k}}}{|Q_{j,k}|} \Big]^{1/2} \tag{2.19}$$

and

$$\mathcal{W}_{\psi,(\mathrm{iii})} f := \Big[\sum_{(\lambda,j,k)\in\Lambda} \big|\langle f, \psi_{j,k}^{\lambda}\rangle\big|^2 \frac{\chi_{W_{j,k}^{\lambda}}}{|Q_{j,k}|} \Big]^{1/2}, \tag{2.20}$$

where Λ is as in (1.13). By Corollary 2.2, we know that $\mathcal{W}_{\psi,(\mathrm{i})} f$, $\mathcal{W}_{\psi,(\mathrm{ii})} f$, and $\mathcal{W}_{\psi,(\mathrm{iii})} f$ are well defined.

Definition 2.14. Let φ be a growth function, let s be as in Definition 2.4, and let ψ be the mother wavelet of an s-order wavelet system. For any given $q \in (1, \infty)$, a function $a \in L^2(\mathbb{R}^n)$ is called an *adapted* $(\varphi, q, s, \psi)_A$*-atom* if there exists a dyadic cube R such that $\operatorname{supp} a \subset mR$ with m as in (1.7),

$$a = \sum_{\lambda\in E} \sum_{\substack{Q\subset R\\ Q\in\mathcal{D}}} (a, \psi_Q^{\lambda}) \psi_Q^{\lambda}$$

and

$$\|\mathcal{W}_{\psi,(\mathrm{ii})} a\|_{L^q(\mathbb{R}^n)} = \Big\| \Big[\sum_{\lambda\in E} \sum_{\substack{Q\subset R\\ Q\in\mathcal{D}}} \big|(a, \psi_Q^{\lambda})\big|^2 \frac{\chi_Q}{|Q|} \Big]^{1/2} \Big\|_{L^q(\mathbb{R}^n)} \le \frac{|mR|^{1/q}}{\|\chi_{mR}\|_{L^{\varphi}(\mathbb{R}^n)}},$$

where $\mathcal{W}_{\psi,(\mathrm{ii})}$ is as in (2.19).

Remark 2.15. Observe that, if φ is a growth function, $s \in [m(\varphi), \infty) \cap \mathbb{Z}_+$, with $m(\varphi)$ as in (1.3), and $q \in (1, \infty)$, then $L^q(\mathbb{R}^n) \subset (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$ and hence, for any $f \in L^2(\mathbb{R}^n)$ and $(\lambda, j, k) \in \Lambda$ with Λ as in (1.13), $\langle f, \psi_{j,k}^{\lambda}\rangle = (f, \psi_{j,k}^{\lambda})$. Moreover, if there exists a dyadic cube R such that $\operatorname{supp} f \subset mR$ with m as in (1.7), in this case, by the orthogonality of $\{\psi_{j,k}^{\lambda}\}_{(lz,j,k)\in\Lambda}$, we have

$$f = \sum_{\lambda\in E} \sum_{\substack{Q\subset R\\ Q\in\mathcal{D}}} (f, \psi_Q^{\lambda}) \psi_Q^{\lambda}.$$

Thus, Definition 2.14 is well defined.

Then we recall the wavelet characterizations of Lebesgue spaces from [24].

Theorem 2.16 ([24, p. 165, Theorem 1]). *Let $q \in (1,\infty)$. Then there exists a positive constant $C := C_{(q,n)}$, depending only on q and n, such that, for any $f \in L^q(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{L^q(\mathbb{R}^n)} \le \|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)},$$

where $\mathcal{W}_{\psi,(\mathrm{ii})}$ is as in (2.19).

The following conclusion is needed in the proof of Theorem 1.9, whose proof borrows some ideas from the proof of [7, Lemma 4.18].

Lemma 2.17. *Let φ, ψ and q be as in Definition 2.14. If a is an adapted $(\varphi,q,0,\psi)_A$-atom related to a cube R, then there exists a positive harmless constant c, independent of a, such that a/c is an adapted $(\varphi,q,0)_A$-atom.*

Proof. Observe that $\operatorname{supp} a \subset mR$ with m as in (1.7). From the Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^n} a(x)\,dx = \sum_{\lambda\in E}\sum_{\substack{Q\subset R\\ Q\in\mathcal{D}}}(a,\psi_Q^\lambda)\int_{\mathbb{R}^n}\psi_Q^\lambda(x)\,dx = 0.$$

Furthermore, by Theorem 2.16, we know that

$$\|a\|_{L^q(\mathbb{R}^n)} \lesssim \|\mathcal{W}_{\psi,(\mathrm{ii})}a\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|mR|^{1/q}}{\|\chi_{mR}\|_{L^\varphi(\mathbb{R}^n)}}.$$

This shows that a is a positive constant multiple of an adapted $(\varphi,q,0)_A$-atom (see Definition 2.10), which completes the proof of Lemma 2.17. $\square$

We now recall the definition of intrinsic g-functions from [21]. For any $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the class of all functions $\eta \in C^s(\mathbb{R}^n)$ such that $\operatorname{supp}\eta \subset \{x\in\mathbb{R}^n : |x|\le 1\}$,

$$\int_{\mathbb{R}^n}\eta(x)x^\gamma\,dx = 0 \quad \text{for any } \gamma\in\mathbb{Z}_+^n \text{ with } |\gamma|\le s,$$

and such that there exists a positive constant C, depending on s, such that, for any $\nu \in \mathbb{Z}_+^n$, with $|\nu| = s$, and any $x_1, x_2 \in \mathbb{R}^n$,

$$\left|\partial^\nu\eta(x_1) - \partial^\nu\eta(x_2)\right| \le C|x_1-x_2|^\alpha.$$

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\infty)$, we write

$$A_{\alpha,s}(f)(y,t) := \sup_{\eta\in\mathcal{C}_{\alpha,s}(\mathbb{R}^n)}\left|f*\eta_t(y)\right|,$$

where $\eta_t(\cdot) := t^{-n}\eta(\frac{\cdot}{t})$ for any $t\in(0,\infty)$. Then the *intrinsic g-function* from [21] is defined by setting, for any $x\in\mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) := \left\{\int_0^\infty \left[A_{\alpha,s}(f)(x,t)\right]^2\frac{dt}{t}\right\}^{1/2}.$$

The following conclusion is from [21] (see also [32, Theorem 6.3.3]). Recall that $f \in \mathscr{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity if, for every $\psi \in \mathscr{S}(\mathbb{R}^n)$, $f*\psi_t \to 0$ in $\mathscr{S}'(\mathbb{R}^n)$ as $t\to\infty$.

Proposition 2.18 ([21, Theorem 1.6]). *Let φ be a growth function, let $\alpha \in (0,1]$, let $s \in \mathbb{Z}_+$, let $\epsilon \in (\alpha+s,\infty)$, let $p_0 \in (n/(n+\alpha+s),1]$, and let $\varphi \in \mathbb{A}_{p_0[1+(\alpha+s)/n]}(\mathbb{R}^n)$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in (\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in L^\varphi(\mathbb{R}^n)$; moreover, there exists a positive constant C such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{H^\varphi(\mathbb{R}^n)} \le \left\|g_{\alpha,s}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \le C\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

In order to prove Theorem 1.9, we need to introduce a discrete variant of the Littlewood–Paley g-function $g^\lambda(f)$. For any $\lambda \in E$, $f \in L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, we write

$$\widetilde{g}^\lambda(f)(x) := \left[\sum_{j\in\mathbb{Z}}\left|f * \psi^\lambda_{2^{-j}}(x)\right|^2\right]^{1/2},$$

where ψ^λ is as (1.9). Then we formulate a useful conclusion from [21] (see also [32, Theorem 6.2.4]).

Proposition 2.19 ([21, Theorem 2.6]). *Let $\lambda \in E$, let $\alpha \in (0,1]$, let $s \in \mathbb{Z}_+$, and let $\epsilon \in (\max\{s,\alpha\},\infty)$. Then there exists a positive constant C such that, for any f satisfying*

$$\left|f(\cdot)\right|\left(1+|\cdot|\right)^{-n-\epsilon} \in L^1(\mathbb{R}^n) \tag{2.21}$$

and for any $x \in \mathbb{R}^n$, it holds true that

$$\widetilde{g}^\lambda(f)(x) \le C g_{\alpha,s}(f)(x).$$

Observe that, for any $\lambda \in E$, $\psi^\lambda \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ and, if $f \in L^2(\mathbb{R}^n)$, then f satisfies (2.21) by the Hölder inequality. From these, we easily deduce the conclusion of Proposition 2.19, the details of which we omit. Propositions 2.18 and 2.19 imply the following conclusion, whose details we also omit.

Corollary 2.20. *Let φ be a growth function and let $\lambda \in E$. If it is the case that $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widetilde{g}^\lambda(f) \in L^\varphi(\mathbb{R}^n)$; moreover, there exists a positive constant $C_{(\lambda)}$, depending on λ, such that, for any $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,*

$$\left\|\widetilde{g}^\lambda(f)\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(\lambda)}\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

For any $\lambda \in E$, $j \in \mathbb{Z}$, $\nu \in (0,\infty)$, $f \in L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, we introduce a variant of the *Peetre-type maximal functions* defined by setting

$$\psi^{\lambda,**}_{j,\nu}(f)(x) := \sup_{y\in\mathbb{R}^n}\frac{|f * \psi^\lambda_{2^{-j}}(x-y)|}{[1+2^j|y|]^\nu}.$$

Similarly to [7], we obtain the following corresponding result.

Proposition 2.21 ([7, Proposition 4.8]). *Let φ be a growth function and let $\nu \in (q(\varphi)/p,\infty)$. Then there exists a positive constant $C_{(\lambda,\nu)}$, depending on λ and ν, such that, for any $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,*

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\left|\psi^{\lambda,**}_{j,\nu}(f)\right|^2\right\}^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(\lambda,\nu)}\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

Proof. Let M be the *Hardy–Littlewood maximal function* defined by setting

$$Mf(x) := \sup_{B\ni x: B \text{ ball}} \frac{1}{|B|}\int_B \left|f(y)\right| dy, \quad \forall f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ that contain x.

By some arguments analogous to those used in the proof of [8, p. 271] in 1-dimensional case, we conclude that, for any $j \in \mathbb{Z}$, $\nu \in (0,\infty)$, $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$\psi_{j,\nu}^{\lambda,**} f(x) \lesssim \left[M\left(|f * \psi^\lambda_{2^{-j}}|^{1/\nu}\right)(x)\right]^\nu,$$

where the implicit positive constant depends only on λ, ν and n. From this, the Fefferman–Stein vector-valued inequality on Musielak–Orlicz spaces (see [32, Theorem 2.1.4]), and Propositions 2.18 and 2.19, we deduce that, for any $\nu \in (q(\varphi)/p, \infty)$ and $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\left\|\left[\sum_{j\in\mathbb{Z}}\left|\psi_{j,\nu}^{\lambda,**}(f)\right|^2\right]^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} &\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}}\left[M\left(|\psi^\lambda_{2^{-j}} * f|^{1/\nu}\right)\right]^{2\nu}\right\}^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} \\
&\sim \left\|\left\{\sum_{j\in\mathbb{Z}}\left[M\left(|\psi^\lambda_{2^{-j}} * f|^{1/\nu}\right)\right]^{2\nu}\right\}^{1/(2\nu)}\right\|^\nu_{L^{\widetilde{\varphi}}(\mathbb{R}^n)} \\
&\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}}|\psi^\lambda_{2^{-j}} * f|^2\right\}^{1/(2\nu)}\right\|^\nu_{L^{\widetilde{\varphi}}(\mathbb{R}^n)} \sim \left\|\widetilde{g}^\lambda(f)\right\|_{L^\varphi(\mathbb{R}^n)} \\
&\lesssim \left\|g_{\alpha,s}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)},
\end{aligned}$$

where $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and, for any $(x,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty)$, $\widetilde{\varphi}(x,t) := \varphi(x,t^\nu)$, and $\widetilde{\varphi}$ is of lower type $p\nu \in (q(\varphi),\infty)$. This finishes the proof of Proposition 2.21. □

3. Proofs of Theorem 1.9 and Corollary 1.10

This section is devoted to the proofs of Theorem 1.9 and Corollary 1.10. We first show Theorem 1.9.

Proof of Theorem 1.9. We first observe that (1.15) follows from (1.11). Then we only need to show that (i) through (iv) of Theorem 1.9 are mutually equivalent. Indeed, we prove (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) in sequence.

(i) $\Longrightarrow$ (ii). We first notice that $H^\varphi(\mathbb{R}^n)$ is a quasi-Banach space and that $H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H^\varphi(\mathbb{R}^n)$. Thus, it suffices to show that, for any $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\|\mathcal{W}_{\psi,(\mathrm{i})} f\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)},$$

where $\mathcal{W}_{\psi,(\mathrm{i})} f$ is as in (2.18). Indeed, for any $\lambda \in E$ and $f \in H^\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left|(f, \psi^\lambda_{j,k})\right| &= 2^{jn/2}\left|\int_{\mathbb{R}^n} f(x)\overline{\psi^\lambda(2^j x - k)}\, dx\right| = 2^{-jn/2}\left|\widetilde{\psi}^\lambda_{2^{-j}} * f(2^{-j}k)\right| \\
&\lesssim 2^{-jn/2} \sup_{y\in Q^\lambda_{j,k}} \left|\widetilde{\psi}^\lambda_{2^{-j}} * f(y)\right|,
\end{aligned}$$

where $\widetilde{\psi}(x) := \overline{\psi(-x)}$ for any $x \in \mathbb{R}^n$. Fix $\lambda \in E$ and $j \in \mathbb{Z}$. For any given $\nu \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$, by (1.8) and the finitely overlapped property of $\{mQ_{j,k}^\lambda\}_{k\in\mathbb{Z}^n}$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}^\lambda\rangle\right|^2 \left|\psi_{j,k}^\lambda(x)\right|^2 \\
&\quad\lesssim \sum_{k\in\mathbb{Z}^n} \Big[\sup_{y\in Q_{j,k}^\lambda} \left|\widetilde{\psi}_{2^{-j}}^\lambda * f(y)\right|\Big]^2 \chi_{mQ_{j,k}}(x) \\
&\quad\lesssim \sum_{k\in\mathbb{Z}^n} \Big[\sup_{|z|\le\sqrt{n}2^{-j}} \left|\widetilde{\psi}_{2^{-j}}^\lambda * f(x-z)\right|\Big]^2 \chi_{mQ_{j,k}}(x) \\
&\quad\lesssim \Big[\sup_{|z|\le\sqrt{n}2^{-j}} \frac{|\widetilde{\psi}_{2^{-j}}^\lambda * f(x-z)|}{(1+2^j|z|)^\nu} \left(1+2^j|z|\right)^\nu\Big]^2 \\
&\quad\lesssim \left[\psi_{j,\nu}^{\lambda,**} f(x)\right]^2,
\end{aligned}$$

where the implicit positive constants only depend on ν, m and n. By this and Proposition 2.21, choosing $\nu \in (q(\varphi)/p, \infty)$, we conclude that

$$\begin{aligned}
\|\mathcal{W}_{\psi,(\mathrm{i})} f\|_{L^\varphi(\mathbb{R}^n)} &= \left\|\Big\{\sum_{\lambda\in E}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}^\lambda\rangle\right|^2 |\psi_{j,k}^\lambda|^2\Big\}^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} \\
&\lesssim \sum_{\lambda\in E} \left\|\Big\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}^\lambda\rangle\right|^2 |\psi_{j,k}^\lambda|^2\Big\}^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} \\
&\lesssim \sum_{\lambda\in E} \left\|\Big\{\sum_{j\in\mathbb{Z}} \left|\psi_{j,\nu}^{\lambda,**}(f)\right|^2\Big\}^{1/2}\right\|_{L^\varphi(\mathbb{R}^n)} \\
&\lesssim \|f\|_{H^\varphi(\mathbb{R}^n)},
\end{aligned} \tag{3.1}$$

which completes the proof for (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iv). This is an easy consequence of the fact (1.12). Moreover, we have

$$\|\mathcal{W}_{\psi,(\mathrm{iii})} f\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|\mathcal{W}_{\psi,(\mathrm{i})} f\|_{L^\varphi(\mathbb{R}^n)}, \tag{3.2}$$

where $\mathcal{W}_{\psi,(\mathrm{iii})} f$ is as in (2.20).

(iv) $\Longrightarrow$ (iii). We first notice that, for any $s \in (0, \infty)$ and $(\lambda, j, k) \in \Lambda$,

$$\chi_{Q_{j,k}} \lesssim \left[M(\chi_{W_{j,k}^\lambda})\right]^{1/s}. \tag{3.3}$$

Indeed, we know that, for every $x \in Q_{j,k}$,

$$1 \sim \left[\frac{|W_{j,k}^\lambda|}{|Q_{j,k}|}\right]^{1/s} \sim \left\{\frac{1}{|Q_{j,k}|}\int_{Q_{j,k}} \chi_{W_{j,k}^\lambda}(y)\,dy\right\}^{1/s} \lesssim \left[M(\chi_{W_{\alpha,\beta}^k})(x)\right]^{1/s},$$

which shows (3.3).

Moreover, by (3.3), with $s:=r/2$ and $r\in(0,\min\{1,p/q(\varphi)\})$, the Fefferman–Stein vector-valued inequality on Musielak–Orlicz spaces (see, e.g., [32, Theorem 2.1.4] with r replaced by $2/r$) and the increasing property of φ, we obtain

$$\begin{aligned}
\|f\|_{(\mathrm{ii})} &= \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left|\langle f,\psi_{j,k}^{\lambda}\rangle\right|^2\frac{\chi_{Q_{j,k}}}{|Q_{j,k}|}\right\}^{1/2}\right\|_{L^{\varphi}(\mathbb{R}^n)}\\
&\lesssim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\frac{|\langle f,\psi_{j,k}^{\lambda}\rangle|^2}{|Q_{j,k}|}\left[M(\chi_{W_{j,k}^{\lambda}})\right]^{2/r}\right\}^{1/2}\right\|_{L^{\varphi}(\mathbb{R}^n)}\\
&\sim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left[M\left(\left[\frac{|\langle f,\psi_{j,k}^{\lambda}\rangle|}{|Q_{j,k}|^{1/2}}\chi_{W_{j,k}^{\lambda}}\right]^r\right)\right]^{2/r}\right\}^{r/2}\right\|_{L^{\widetilde{\varphi}}(\mathbb{R}^n)}^{1/r}\\
&\lesssim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left[\frac{|\langle f,\psi_{j,k}^{\lambda}\rangle|}{|Q_{j,k}|^{1/2}}\chi_{W_{j,k}^{\lambda}}\right]^2\right\}^{r/2}\right\|_{L^{\widetilde{\varphi}}(\mathbb{R}^n)}^{1/r}\\
&\sim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left|\langle f,\psi_{j,k}^{\lambda}\rangle\right|^2\frac{\chi_{W_{j,k}^{\lambda}}}{|Q_{j,k}|}\right\}^{1/2}\right\|_{L^{\varphi}(\mathbb{R}^n)}\sim\|f\|_{(\mathrm{iii})}, \qquad (3.4)
\end{aligned}$$

where, for any $x\in\mathbb{R}^n$ and $t\in(0,\infty)$, $\widetilde{\varphi}(x,t):=\varphi(x,t^{1/r})$ is of uniformly lower-type $p/r\in(q(\varphi),\infty)$. This shows that (iv) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). We first reduce the proof to showing that, for any $f\in L^2(\mathbb{R}^n)$ with $\mathcal{W}_{\psi,(\mathrm{ii})}f\in L^{\varphi}(\mathbb{R}^n)$, $f\in H^{\varphi}(\mathbb{R}^n)$ and

$$\|f\|_{H^{\varphi}(\mathbb{R}^n)}\lesssim\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^{\varphi}(\mathbb{R}^n)}, \qquad (3.5)$$

where $\mathcal{W}_{\psi,(\mathrm{ii})}f$ is as in (2.19). Indeed, we assume that (3.5) holds true. Then, for any

$$f=\sum_{(\lambda,j,k)\in\Lambda}\langle f,\psi_{j,k}^{\lambda}\rangle\psi_{j,k}^{\lambda}\in\left(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n)\right)^* \qquad (3.6)$$

with $\mathcal{W}_{\psi,(\mathrm{ii})}f\in L^{\varphi}(\mathbb{R}^n)$ and Λ as in (1.13), let $\{\Lambda_N\}_{N\in\mathbb{N}}$ be an increasing sequence of finite subsets of Λ satisfying that $\bigcup_{N\in\mathbb{N}}\Lambda_N=\Lambda$, and for any $N\in\mathbb{N}$,

$$f_N:=\sum_{(\lambda,k,j)\in\Lambda_N}\langle f,\psi_{j,k}^{\lambda}\rangle\psi_{j,k}^{\lambda}.$$

For any $N,M\in\mathbb{N}$ with $M>N$, by (3.5), Remark 1.4(ii) and [32, Lemma 1.10(ii)], we obtain

$$\|f_N-f_M\|_{H^{\varphi}(\mathbb{R}^n)}\lesssim\left\|\left[\sum_{(\lambda,j,k)\in\Lambda_M\setminus\Lambda_N}\left|\langle f,\psi_{j,k}^{\lambda}\rangle\right|^2\frac{\chi_{Q_{j,k}}}{|Q_{j,k}|}\right]^{1/2}\right\|_{L^{\varphi}(\mathbb{R}^n)}\to0,$$

as $N\to\infty$. By the completion of $H^{\varphi}(\mathbb{R}^n)$, we know that there exists $\widetilde{f}\in H^{\varphi}(\mathbb{R}^n)$ such that $\widetilde{f}:=\lim_{N\to\infty}f_N$ in $H^{\varphi}(\mathbb{R}^n)$ and hence in $(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$. Thus, by this and (3.6), we conclude that $f=\widetilde{f}$ in $(\mathrm{Lip}_{\varphi,1,s}(\mathbb{R}^n))^*$, which, combined with $\widetilde{f}\in H^{\varphi}(\mathbb{R}^n)$, implies that $f\in H^{\varphi}(\mathbb{R}^n)$. This is the desired conclusion.

Now we start to prove (3.5). For any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{W}_{\psi,(\mathrm{ii})} f \in L^\varphi(\mathbb{R}^n)$, we aim to show that

$$f = \sum_{\lambda \in E} \sum_{Q \in \mathcal{D}} (f, \psi_Q^\lambda) \psi_Q^\lambda = \sum_{k \in \mathbb{Z}} \sum_{i \in \Delta_k} b(k,i), \tag{3.7}$$

where $\{b(k,i) : k \in \mathbb{Z}, i \in \Delta_k\}$ are some adapted $(\varphi, 2, 0, \psi)_A$-atoms which are determined later. For any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : \mathcal{W}_{\psi,(\mathrm{ii})} f(x) > 2^k\}$,

$$\mathcal{D}_k := \left\{ Q \in \mathcal{D} : |Q \cap \Omega_k| \geq \frac{1}{2}|Q|, |Q \cap \Omega_{k+1}| < \frac{1}{2}|Q| \right\}$$

and $\widetilde{\mathcal{D}} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$.

Then we claim that, for any $Q \in \widetilde{\mathcal{D}}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{D}_k$. Indeed, if there exist $k, \widetilde{k} \in \mathbb{Z}$ such that $Q \in \mathcal{D}_k \cap \mathcal{D}_{\widetilde{k}}$, then we have

$$\left|Q \cap (\Omega_k \setminus \Omega_{\widetilde{k}+1})\right| \geq |Q \cap \Omega_k| - |Q \cap \Omega_{\widetilde{k}+1}| > \frac{1}{2}|Q| - \frac{1}{2}|Q| = 0.$$

This shows that there exists $z \in Q \cap (\Omega_k \setminus \Omega_{\widetilde{k}+1}) \neq \emptyset$, which implies that

$$2^k < \mathcal{W}_{\psi,(\mathrm{ii})} f(z) \leq 2^{\widetilde{k}+1}$$

and hence that $k < \widetilde{k} + 1$. From this, it further follows that $k \leq \widetilde{k}$. Similarly, by $|Q \cap (\Omega_{\widetilde{k}} \setminus \Omega_{k+1})| > 0$ and the symmetry, we obtain $\widetilde{k} \leq k$. This shows that $k = \widetilde{k}$, which completes the proof of the preceding claim.

We now claim that, for any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{W}_{\psi,(\mathrm{ii})} f \in L^\varphi(\mathbb{R}^n)$, $\lambda \in E$, and $Q \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$,

$$(f, \psi_Q^\lambda) = 0. \tag{3.8}$$

If $(f, \psi_Q^\lambda) \neq 0$, then there exists $k_0 \in \mathbb{Z}$ such that $|(f, \psi_Q^\lambda)||Q|^{-1/2} \geq 2^{k_0}$, which shows that, for any $x \in Q$, $\mathcal{W}_{\psi,(\mathrm{ii})} f(x) > 2^{k_0}$ and hence $Q \subset \Omega_{k_0}$. Therefore,

$$|Q \cap \Omega_{k_0}| = |Q| \geq \frac{1}{2}|Q| > 0.$$

On the other hand, by $\mathcal{W}_{\psi,(\mathrm{ii})} f \in L^\varphi(\mathbb{R}^n)$, we know that $\lim_{k \to \infty} |\Omega_k| = 0$. From these facts and the decreasing property of Ω_k with respect to k, we deduce that there exists $k \in \mathbb{Z}$ such that

$$|Q \cap \Omega_k| \geq \frac{1}{2}|Q|, \qquad |Q \cap \Omega_{k+1}| < \frac{1}{2}|Q|, \tag{3.9}$$

which further implies that $Q \in \mathcal{D}_k \subset \widetilde{\mathcal{D}}$. This finishes the proof of the claim (3.8).

Observe that, due to the nesting property of dyadic cubes, for any $Q \in \mathcal{D}_k$, there exists a unique maximal dyadic cube $\widetilde{Q} \in \mathcal{D}_k$ such that $Q \subset \widetilde{Q}$. Let $\{\widetilde{Q}_k^i \in \mathcal{D}_k : i \in \Delta_k\}$ be the collection of all such maximal dyadic cubes in $\mathcal{D}_k$. Then

$$\widetilde{\mathcal{D}} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k = \bigcup_{k \in \mathbb{Z}} \bigcup_{i \in \Delta_k} \{Q \in \mathcal{D}_k : Q \subset \widetilde{Q}_k^i\}.$$

From this, (3.7), and (3.8), we deduce that, for any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{W}_{\psi,(\mathrm{ii})} f \in L^\varphi(\mathbb{R}^n)$,

$$\begin{aligned} f &= \sum_{\lambda\in E}\sum_{Q\in\mathcal{D}} (f, \psi_Q^\lambda)\psi_Q^\lambda \\ &= \sum_{\lambda\in E}\sum_{Q\in\widetilde{\mathcal{D}}} (f, \psi_Q^\lambda)\psi_Q^\lambda \\ &= \sum_{\lambda\in E}\sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\Big\{\sum_{Q\subset\widetilde{Q}_k^i, Q\in\mathcal{D}_k} (f, \psi_Q^\lambda)\psi_Q^\lambda\Big\} \\ &=: \sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k} b(k,i), \end{aligned} \tag{3.10}$$

where, for any $k \in \mathbb{Z}$ and $i \in \Lambda$,

$$b(k,i) := \sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i, Q\in\mathcal{D}_k} (f, \psi_Q^\lambda)\psi_Q^\lambda.$$

We now prove that $b(k,i)$ is a multiple of an adapted $(\varphi, 2, 0, \psi)_A$-atom. By the orthogonality of the wavelets, for every $k \in \mathbb{Z}$ and $i \in \Lambda$, we have

$$\mathcal{W}_{\psi,(\mathrm{ii})} b(k,i) := \Big[\sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i, Q\in\mathcal{D}_k} \big|(f, \psi_Q^\lambda)\big|^2 \frac{\chi_Q}{|Q|}\Big]^{1/2}.$$

Moreover, for every $k \in \mathbb{Z}$ and $i \in \Lambda$, let

$$\lambda(k,i) := \big\|\mathcal{W}_{\psi,(\mathrm{ii})} b(k,i)\big\|_{L^2(\mathbb{R}^n)} \frac{\|\chi_{m\widetilde{Q}_k^i}\|_{L^\varphi(\mathbb{R}^n)}}{|m\widetilde{Q}_k^i|^{1/2}}$$

and

$$a(k,i) := \begin{cases} 0 & \text{when } \lambda(k,i) = 0, \\ \frac{1}{\lambda(k,i)} \sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i, Q\in\mathcal{D}_k} (f, \psi_Q^\lambda)\psi_Q^\lambda & \text{when } \lambda(k,i) \neq 0. \end{cases}$$

Then

$$\big\|\mathcal{W}_{\psi,(\mathrm{ii})} a(k,i)\big\|_{L^2(\mathbb{R}^n)} \le \frac{1}{|\lambda(k,i)|}\big\|\mathcal{W}_{\psi,(\mathrm{ii})} b(k,i)\big\|_{L^2(\mathbb{R}^n)} = \frac{|m\widetilde{Q}_k^i|^{1/2}}{\|\chi_{m\widetilde{Q}_k^i}\|_{L^\varphi(\mathbb{R}^n)}},$$

which implies that $a(k,i)$ is an adapted $(\varphi, 2, 0, \psi)_A$-atom supported on the cube $m\widetilde{Q}_k^i$, where m is as in (1.5). From this and Lemma 2.17, we deduce that $a(k,i)$ is a positive harmless constant multiple of an adapted $(\varphi, 2, 0)_A$-atom supported on the cube $m\widetilde{Q}_k^i$.

Using some ideas from [32, Lemma 1.3.15], we conclude that, for any $t \in (0, \infty)$, there exists a positive constant C, independent of f and t, such that

$$\sum_{k\in\mathbb{Z}} \varphi\Big(\Omega_k, \frac{2^k}{t}\Big) \le C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{\mathcal{W}_{\psi,(\mathrm{ii})} f(x)}{t}\Big)\, dx. \tag{3.11}$$

Furthermore, by (3.9), we find that, for any $t \in (0,\infty)$,

$$
\begin{aligned}
&\frac{1}{|m\widetilde{Q}_k^i|}\int_{\widetilde{Q}_k^i}\left[\mathcal{W}_{\psi,(\mathrm{ii})}b(k,i)(x)\right]^2\,dx\\
&\quad\sim \frac{1}{|m\widetilde{Q}_k^i|}\int_{\widetilde{Q}_k^i}\sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\left|(f,\psi_Q^\lambda)\right|^2\frac{\chi_Q(x)}{|Q|}\,dx\\
&\quad\sim \sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\frac{|(f,\psi_Q^\lambda)|^2}{|Q|}\frac{|Q|}{|m\widetilde{Q}_k^i|}\lesssim \sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\frac{|(f,\psi_Q^\lambda)|^2}{|Q|}\frac{|Q|}{|m\widetilde{Q}_k^i|}\\
&\quad\lesssim \sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\frac{|(f,\psi_Q^\lambda)|^2}{|Q|}\frac{|Q\setminus\Omega_{k+1}|}{|m\widetilde{Q}_k^i|}\\
&\quad\sim \frac{1}{|m\widetilde{Q}_k^i|}\sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\int_{Q\setminus\Omega_{k+1}}\frac{|(f,\psi_Q^\lambda)|^2}{|Q|}\,dx\\
&\quad\sim \frac{1}{|m\widetilde{Q}_k^i|}\int_{\widetilde{Q}_k^i\setminus\Omega_{k+1}}\sum_{\lambda\in E}\sum_{Q\subset\widetilde{Q}_k^i,Q\in\mathcal{D}_k}\frac{|(f,\psi_Q^\lambda)|^2\chi_Q(x)}{|Q|}\,dx\\
&\quad\lesssim \frac{1}{|m\widetilde{Q}_k^i|}\int_{\widetilde{Q}_k^i\setminus\Omega_{k+1}}\left[\mathcal{W}_{\psi,(\mathrm{ii})}f(x)\right]^2\,dx\lesssim 2^{2(k+1)}\frac{|\widetilde{Q}_k^i\setminus\Omega_{k+1}|}{|m\widetilde{Q}_k^i|}\lesssim 2^{2k},
\end{aligned}
$$

which further implies that

$$
\frac{1}{|m\widetilde{Q}_k^i|^{1/2}}\left\|\mathcal{W}_{\psi,(\mathrm{ii})}b(k,i)\right\|_{L^2(\mathbb{R}^n)}\lesssim 2^k.
$$

From this, (3.10), [32, parts (iv) and (vi) of Lemma 1.1.3], and (3.11), we deduce that there exists a fixed $p_0\in[1,\infty)$ such that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\varphi\Big(m\widetilde{Q}_k^i,\frac{|\lambda(k,i)|}{\|\chi_{\widetilde{mQ_k^i}}\|_{L^\varphi(\mathbb{R}^n)}\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\
&\quad\sim \sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\varphi\Big(m\widetilde{Q}_k^i,\frac{\|\mathcal{W}_{\psi,(\mathrm{ii})}b(k,i)\|_{L^2(\mathbb{R}^n)}}{|m\widetilde{Q}_k^i|^{1/2}\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\varphi\Big(m\widetilde{Q}_k^i,\frac{2^k}{\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\varphi\Big(\widetilde{Q}_k^i\cap\Omega_k,\frac{2^k}{\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\Big[\frac{|m\widetilde{Q}_k^i|}{|\widetilde{Q}_k^i\cap\Omega_k|}\Big]^{p_0}\\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{i\in\Delta_k}\varphi\Big(\widetilde{Q}_k^i\cap\Omega_k,\frac{2^k}{\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\lesssim \sum_{k\in\mathbb{Z}}\varphi\Big(\Omega_k,\frac{2^k}{\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\
&\quad\lesssim \int_{\mathbb{R}^n}\varphi\Big(x,\frac{\mathcal{W}_{\psi,(\mathrm{ii})}f}{\|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}}\Big)\,dx\sim 1,
\end{aligned}
$$

which, together with Theorem 2.12, implies that

$$\|f\|_{H^\varphi(\mathbb{R}^n)} \sim \|f\|_{H^{\varphi,2,0}_{\mathrm{at},A}(\mathbb{R}^n)} \lesssim \Lambda\left(\left\{b(k,i)\right\}_{k,i}\right) \lesssim \|\mathcal{W}_{\psi,(\mathrm{ii})}f\|_{L^\varphi(\mathbb{R}^n)}.$$

This finishes the proof that (iii) $\Longrightarrow$ (i).

Thus, (i) through (iv) are mutually equivalent.

Furthermore, from (3.1), (3.2), (3.4), and (3.5), we deduce that $\|\cdot\|_{(\mathrm{i})}$, $\|\cdot\|_{(\mathrm{ii})}$ and $\|\cdot\|_{(\mathrm{iii})}$ are equivalent to $\|\cdot\|_{H^\varphi(\mathbb{R}^n)}$, which completes the proof of Theorem 1.9. □

Finally, we prove Corollary 1.10.

Proof of Corollary 1.10. Let $f \in L^2(\mathbb{R}^n)$. Then, by the proof of Theorem 1.9, we know that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii). Observe that the assumption (1.14) is only used in the proof of "(iii) $\Longrightarrow$ (i)" in Theorem 1.9, where we indeed prove that, for any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{W}_{\psi,(\mathrm{ii})}f \in L^\varphi(\mathbb{R}^n)$, $f \in H^\varphi(\mathbb{R}^n)$ as an intermediate result. This shows (iii) $\Longrightarrow$ (i) and all the quasinorms $\|\cdot\|_{(\mathrm{i})}$, $\|\cdot\|_{(\mathrm{ii})}$ and $\|\cdot\|_{(\mathrm{iii})}$ are equivalent to $\|\cdot\|_{H^\varphi(\mathbb{R}^n)}$, which completes the proof of Corollary 1.10. □

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