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REFLEXIVE SETS OF OPERATORS

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Abstract. For a set $\mathcal{M}$ of operators on a complex Banach space $\mathscr{X}$, the reflexive cover of $\mathcal{M}$ is the set $\mathrm{Ref}(\mathcal{M})$ of all those operators T satisfying $Tx \in \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$. Set $\mathcal{M}$ is reflexive if $\mathrm{Ref}(\mathcal{M}) = \mathcal{M}$. The notion is well known, especially for Banach algebras or closed spaces of operators, because it is related to the problem of invariant subspaces. We study reflexivity for general sets of operators. We are interested in how the reflexive cover behaves towards basic operations between sets of operators. It is easily seen that the intersection of an arbitrary family of reflexive sets is reflexive, as well. However this does not hold for unions, since the union of two reflexive sets of operators is not necessarily a reflexive set. We give some sufficient conditions under which the union of reflexive sets is reflexive. We explore how the reflexive cover of the sum (resp., the product) of two sets is related to the reflexive covers of summands (resp., factors). We also study the relation between reflexivity and convexity, with special interest in the question: under which conditions is the convex hull of a reflexive set reflexive?

1. Introduction

Let $\mathscr{X}$ be a complex Banach space and let $\mathcal{B}(\mathscr{X})$ be the Banach algebra of all bounded linear operators on $\mathscr{X}$. For a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$, let $\mathcal{M}x = \{Mx; M \in \mathcal{M}\}$ denote the orbit of $\mathcal{M}$ at vector $x \in \mathscr{X}$, and let $\overline{\mathcal{M}x}$ be its closure. The reflexive cover of $\mathcal{M}$ is set $\mathrm{Ref}(\mathcal{M}) = \{T \in \mathcal{B}(\mathscr{X}); Tx \in \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}\}$. Hence, an operator T is in $\mathrm{Ref}(\mathcal{M})$ if and only if, for every $x \in \mathscr{X}$ and every $\varepsilon > 0$, there exists an operator $M_{x,\varepsilon} \in \mathcal{M}$ such that $\|(T - M_{x,\varepsilon})x\| < \varepsilon$.

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The pure algebraic version of this concept is the algebraic reflexive cover of $\mathcal{M}$, the set which is given by $\mathrm{Ref_a}(\mathcal{M}) = \{T \in \mathcal{B}(\mathscr{X}); Tx \in \mathcal{M}x \text{ for every } x \in \mathscr{X}\}$. It is obvious that $\mathcal{M} \subseteq \mathrm{Ref_a}(\mathcal{M}) \subseteq \mathrm{Ref}(\mathcal{M}) \subseteq \mathcal{B}(\mathscr{X})$. In general, any inclusion in the previous chain can be proper for a suitable set $\mathcal{M}$; however, note that $\mathrm{Ref_a}(\mathcal{M}) = \mathrm{Ref}(\mathcal{M})$ whenever every orbit $\mathcal{M}x$ ($x \in \mathscr{X}$) is closed.

It is obvious that $\mathcal{M}_1 \subseteq \mathrm{Ref}(\mathcal{M}_1) \subseteq \mathrm{Ref}(\mathcal{M}_2)$ if $\emptyset \neq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{B}(\mathscr{X})$. Since, by the definition of the reflexive cover of a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$, we have $\overline{\mathrm{Ref}(\mathcal{M})x} = \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$, it follows that $\mathrm{Ref}(\mathrm{Ref}(\mathcal{M})) = \mathrm{Ref}(\mathcal{M})$. The algebraic reflexive cover has similar properties.

A basic question about a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is: how large is its reflexive cover? If $\mathrm{Ref}(\mathcal{M}) = \mathcal{M}$, then $\mathcal{M}$ is considered reflexive, and if $\mathrm{Ref}(\mathcal{M}) = \mathcal{B}(\mathscr{X})$, then $\mathcal{M}$ is considered topologically transitive. Similarly, if $\mathrm{Ref_a}(\mathcal{M}) = \mathcal{M}$, then $\mathcal{M}$ is considered algebraically reflexive, and it is considered transitive if $\mathrm{Ref_a}(\mathcal{M}) = \mathcal{B}(\mathscr{X})$. It is obvious that every reflexive set is algebraically reflexive and that every transitive set is topologically transitive. Due to technical reasons, we set $\mathrm{Ref}(\emptyset) = \emptyset$ (i.e., we consider the empty set to be reflexive).

The reflexivity of sets with some additional algebraic structure has been studied by many authors. Sometimes other terminology is used instead of *reflexivity*. The most extensive study has been devoted to algebras and linear spaces of operators because in that case reflexivity is intimately related to the problem of invariant subspaces (see [10, Section 9.2]). The theory had its beginnings in Sarason's article [11], where he proved that the weakly closed algebra of operators on a complex Hilbert space—which is generated by a normal operator, or an analytic Toeplitz operator, and the identity operator—is reflexive. Loginov and Shulman [8] extended the notion of reflexivity to linear spaces of operators. Hadwin et al. [6] studied reflexivity of $\mathscr{O}(S)$, the strongly closed semigroup generated by a nonzero operator S on a complex Hilbert space. An operator is considered orbit-reflexive if $\mathscr{O}(S)$ is reflexive (it has been proved in [6] that many operators are orbit-reflexive). Grivaux and Roginskaya [5] and, independently, Müller and Vršovský [9] have shown that there exist operators on a Hilbert space which are not orbit-reflexive.

The main goal of this paper is to study reflexivity of general sets of operators. We are interested in the question of how reflexivity is related to some basic operations between sets of operators. For instance, can the sum, the intersection, or the union of reflexive sets be considered reflexive sets, as well? Our article is organized as follows. In Section 2, we present a few examples of reflexive sets and we prove basic results related to the reflexivity of sums and products of sets of operators. Section 3 is devoted to the intersections and unions of sets of operators. It is easily seen that the intersection of an arbitrary collection of reflexive sets is reflexive, while on the other hand, the union of two reflexive sets is not reflexive in general. We give some sufficient conditions under which the SOT-closure (*strong operator topology-closure*) of the union of special collections of reflexive sets can be reflexive, as well. Finally, in Section 4, we consider reflexivity of convex sets of operators, showing that a convex set of operators is reflexive if its polar contains enough rank 1 operators.

At the end of this section we introduce our notation. For a complex Banach space $\mathscr{X}$, we denote by $\mathcal{B}(\mathscr{X})$ the Banach algebra of all bounded operators on $\mathscr{X}$. The identity operator is denoted by I. The unit ball and the unit sphere of $\mathscr{X}$ are denoted by $\mathscr{B}_{\mathscr{X}}$ and $\mathscr{S}_{\mathscr{X}}$, respectively. The topological dual of $\mathscr{X}$ is $\mathscr{X}^*$ and the pairing between these spaces is given by $(x,\xi) \mapsto \langle x,\xi\rangle$, where $x \in \mathscr{X}$ and $\xi \in \mathscr{X}^*$ are arbitrary and $\langle x,\xi\rangle$ denotes number $\xi(x)$. For an operator $T \in \mathcal{B}(\mathscr{X})$, the image is denoted by $\operatorname{im}(T)$ and the kernel by $\ker(T)$. More generally, for a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$, we denote $\operatorname{Im}(\mathcal{M}) = \vee_{M\in\mathcal{M}} \operatorname{im}(M)$, where $\vee$ means the closed linear span of a set of vectors, and $\operatorname{Ker}(\mathcal{M}) = \bigcap_{M\in\mathcal{M}} \ker(M)$. For $e \in \mathscr{X}$ and $\xi \in \mathscr{X}^*$, we denote by $e \otimes \xi$ the operator which is given by $x \mapsto \langle x,\xi\rangle e$ ($x \in \mathscr{X}$); if e and ξ are nonzero, then this is a bounded rank 1 operator. We denote the set of all bounded rank 1 operators and operator 0 by $\mathcal{F}_1(\mathscr{X})$, and we denote the linear space of all bounded finite rank operators by $\mathcal{F}(\mathscr{X})$. For nonempty sets $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ and $\Lambda \subseteq \mathbb{C}$, let $\Lambda\mathcal{M} = \{\lambda M; \lambda \in \Lambda, M \in \mathcal{M}\}$. Similarly, for nonempty sets $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{B}(\mathscr{X})$, we denote $\mathcal{M}_1\mathcal{M}_2 = \{M_1M_2; M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}$ and $\mathcal{M}_1 + \mathcal{M}_2 = \{M_1 + M_2; M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}$. If one of the sets is a singleton, then by the above notation we write that single element instead of the singleton; for instance, $\Lambda M = \{\lambda M; \lambda \in \Lambda\}$, $\mathcal{M} + N = \{M + N; M \in \mathcal{M}\}$ and so forth.

2. Examples and basic properties

We begin this section with some simple examples of reflexive sets. The first example has been presented in [3]. For the sake of completeness, we include here a simple proof which relies on the following known lemma (see [2]) that will be used several times throughout this article.

Lemma 2.1. *Let $\mathscr{X}$ be a complex vector space and let $\mathscr{Y}_1, \dots, \mathscr{Y}_n \subseteq \mathscr{X}$ be linear subspaces. If $\mathscr{X} = \mathscr{Y}_1 \cup \cdots \cup \mathscr{Y}_n$, then $\mathscr{X} = \mathscr{Y}_k$ for at least one index $k \in \{1, \dots, n\}$.*

Proof. Towards a contradiction, assume that each $\mathscr{Y}_i$ ($1 \le i \le n$) is a proper subspace of $\mathscr{X}$. Then for every $i \in \{1, \dots, n\}$ there exists a vector $e_i \in \mathscr{X} \setminus \mathscr{Y}_i$. Let $(\lambda_j)_{j=1}^\infty$ be a sequence of pairwise distinct complex numbers. For each $j \in \mathbb{N}$, let $x_j = e_1 + \lambda_j e_2 + \cdots + \lambda_j^{n-1} e_n$. Because of $\mathscr{X} = \mathscr{Y}_1 \cup \cdots \cup \mathscr{Y}_n$, there exists an index k such that $\mathscr{Y}_k$ contains infinitely many vectors from this sequence. Assume that $j_1, \dots, j_n$ are distinct indices such that $x_{j_1}, \dots, x_{j_n} \in \mathscr{Y}_k$. Of course, any linear combination of these vectors is in $\mathscr{Y}_k$, as well. For $1 \le l \le n$, let $\gamma_l = 1$ if $l = k$ and let $\gamma_l = 0$ otherwise. The matrix of the linear system

$$\lambda_{j_1}^m z_1 + \cdots + \lambda_{j_n}^m z_n = \gamma_{m+1} \quad (m = 0, \dots, n-1)$$

is a Vandermonde matrix with nonzero determinant. Hence, the system has a unique solution $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. It follows that

$$e_k = \sum_{m=0}^{n-1} \Big(\sum_{t=1}^{n} \alpha_t \lambda_{j_t}^m \Big) e_{m+1} = \sum_{t=1}^{n} \alpha_t \Big(\sum_{m=0}^{n-1} \lambda_{j_t}^m e_{m+1} \Big) = \alpha_1 x_{j_1} + \cdots + \alpha_n x_{j_n},$$

which gives the contradiction $e_k \in \mathscr{Y}_k$. □

Proposition 2.2. *Every nonempty finite set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is reflexive.*

Proof. Note first that for a finite set $\mathcal{M}$, every orbit $\mathcal{M}x$ ($x \in \mathscr{X}$) is closed. Hence $\mathrm{Ref}(\mathcal{M}) = \mathrm{Ref}_{\mathrm{a}}(\mathcal{M})$. Suppose that $\mathcal{M} = \{M_1, \dots, M_n\}$, and let $T \in \mathrm{Ref}(\mathcal{M})$. Then $Tx \in \{M_1x, \dots, M_nx\}$ for every $x \in \mathscr{X}$. Since $Tx = M_ix$ if and only if $x \in \ker(T - M_i)$, we have $\mathscr{X} = \ker(T - M_1) \cup \dots \cup \ker(T - M_n)$. Now we apply Lemma 2.1 to conclude that $T = M_k$ for some index $k \in \{1, \dots, n\}$. □

For nonempty sets $\mathscr{U}, \mathscr{V} \subseteq \mathscr{X}$, let $\mathcal{M}_{\mathscr{U},\mathscr{V}} = \{M \in \mathcal{B}(\mathscr{X}); M\mathscr{U} \subseteq \mathscr{V}\}$. If $\mathscr{U} = \mathscr{V}$, then we will use the shorter notation $\mathcal{M}_{\mathscr{U}}$, and in the case when a set is singleton we will write that vector instead of the set; for instance $\mathcal{M}_{u,v}$ denotes the set of all operators M such that $Mu = v$. It is easily seen that $\mathcal{M}_{\mathscr{U},\mathscr{V}} = \bigcap_{u\in\mathscr{U}} \bigcup_{v\in\mathscr{V}} \mathcal{M}_{u,v}$. Note that $\mathcal{M}_{\mathscr{U},\mathscr{V}}$ is a semigroup if $\mathscr{V} \subseteq \mathscr{U}$. In particular, every $\mathcal{M}_{\mathscr{U}}$ is a semigroup containing the identity operator.

Proposition 2.3. *Let $\mathscr{U}, \mathscr{V} \subseteq \mathscr{X}$ be nonempty sets. Then $\mathrm{Ref}(\mathcal{M}_{\mathscr{U},\mathscr{V}}) \subseteq \mathcal{M}_{\mathscr{U},\overline{\mathscr{V}}}$. If $\mathscr{V}$ is closed, then $\mathcal{M}_{\mathscr{U},\mathscr{V}}$ is reflexive.*

Proof. Let $T \in \mathrm{Ref}(\mathcal{M}_{\mathscr{U},\mathscr{V}})$. For every $u \in \mathscr{U}$ and $\varepsilon > 0$ there exists $M_{u,\varepsilon} \in \mathcal{M}_{\mathscr{U},\mathscr{V}}$ such that $\|Tu - M_{u,\varepsilon}u\| < \varepsilon$. Since $M_{u,\varepsilon}u \in \mathscr{V}$ for all u and ε, we have $Tu \in \overline{\mathscr{V}}$. We conclude that $\mathrm{Ref}(\mathcal{M}_{\mathscr{U},\mathscr{V}}) \subseteq \mathcal{M}_{\mathscr{U},\overline{\mathscr{V}}}$. If $\mathscr{V}$ is closed, then the opposite inclusion holds as well. □

Remark 2.4. Note that $\mathcal{M}_{\mathscr{B}_{\mathscr{X}}}$ is the semigroup of all contractions and that $\mathcal{M}_{\mathscr{S}_{\mathscr{X}}}$ is the semigroup of all isometries. Hence, by Proposition 2.3, these sets are reflexive.

The following proposition is a simple generalization of a well-known fact that every 1-dimensional space of operators is reflexive.

Proposition 2.5. *Let $M \in \mathcal{B}(\mathscr{X})$ and let $\Lambda \subseteq \mathbb{C}$ be a nonempty set. Then $\mathrm{Ref}(\Lambda M) = \overline{\Lambda}M$. In particular, if $M \neq 0$, then ΛM is a reflexive set if and only if Λ is closed.*

Proof. If $M = 0$, then there is nothing to prove. Assume therefore that $M \neq 0$. Let $T \in \mathrm{Ref}(\Lambda M)$. Since $M \neq 0$ there exists $x \in \mathscr{X}$ such that $Mx \neq 0$. From $Tx \in \overline{\Lambda Mx} = \overline{\Lambda}Mx$ it follows that there exists $\lambda_x \in \overline{\Lambda}$ such that $Tx = \lambda_x Mx$. Let $y \in \mathscr{X}$ be arbitrary. If Mx and My are linearly dependent and $\gamma \in \mathbb{C}$ is such that $My = \gamma Mx$, then it follows from $T(y - \gamma x) \in \overline{\Lambda}M(y - \gamma x) = \{0\}$ that $Ty = \gamma Tx = \gamma\lambda_x Mx = \lambda_x My$. Suppose now that Mx and My are linearly independent and let $\lambda_y, \lambda_{x+y} \in \overline{\Lambda}$ be such that $Ty = \lambda_y My$ and $T(x + y) = \lambda_{x+y}M(x+y)$. It follows from $T(x+y) = \lambda_{x+y}M(x+y) = \lambda_{x+y}Mx + \lambda_{x+y}My$ and $T(x+y) = Tx + Ty = \lambda_x Mx + \lambda_y My$ that $(\lambda_{x+y} - \lambda_x)Mx + (\lambda_{x+y} - \lambda_y)My = 0$, which gives $\lambda_y = \lambda_{x+y} = \lambda_x$. Hence we have $T = \lambda M$ for some $\lambda \in \overline{\Lambda}$. The opposite inclusion is obvious. □

Proposition 2.5 does not hold if we replace a single operator M by two or more operators. For instance, if M_1, M_2 are distinct operators, then $\mathbb{C}\{M_1, M_2\}$ is the union of 1-dimensional spaces $\mathbb{C}M_1$ and $\mathbb{C}M_2$ which is not necessarily reflexive, as we see later in Example 3.2.

There exist nonreflexive linear spaces of dimension 2 or more. For later use we include an example which is based on the well-known fact that the reflexive cover of $\mathcal{M} = \{\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right]; \alpha, \beta \in \mathbb{C}\}$ is $\mathrm{Ref}(\mathcal{M}) = \{\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right]; \alpha, \beta, \gamma \in \mathbb{C}\}$. In the example we will need the following lemma.

Lemma 2.6. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set. If $T \in \mathrm{Ref}(\mathcal{M})$, then $\mathrm{im}(T) \subseteq \mathrm{Im}(\mathcal{M})$ and $\mathrm{Ker}(\mathcal{M}) \subseteq \ker(T)$. Hence, $\mathrm{Im}(\mathrm{Ref}(\mathcal{M})) = \mathrm{Im}(\mathcal{M})$ and $\mathrm{Ker}(\mathrm{Ref}(\mathcal{M})) = \mathrm{Ker}(\mathcal{M})$.*

We omit a simple proof of the lemma.

Example 2.7. Assume that $\dim(\mathscr{X}) \geq 2$. Let $e_1, e_2 \in \mathscr{S}_{\mathscr{X}}$ be linearly independent, and let $\xi_1, \xi_2 \in \mathscr{X}^*$ be such that $\langle e_j, \xi_k \rangle = \delta_{jk}$ for $1 \leq j, k \leq 2$ (the Kronecker's delta). Let $\Lambda \subseteq \mathbb{C}$ be a closed set such that $0 \in \Lambda$. We claim that the reflexive cover of $\mathcal{M} = \{\alpha(e_1 \otimes \xi_1 + e_2 \otimes \xi_2) + \beta e_1 \otimes \xi_2; \alpha \in \Lambda, \beta \in \mathbb{C}\}$ is $\mathrm{Ref}(\mathcal{M}) = \{\alpha e_1 \otimes \xi_1 + \beta e_1 \otimes \xi_2 + \gamma e_2 \otimes \xi_2; \alpha, \gamma \in \Lambda, \beta \in \mathbb{C}\}$. If $\Lambda = \{0\}$, then our claim follows, by Proposition 2.5. Assume therefore that $\{0\} \subsetneq \Lambda$. Since Λ is a closed set, every orbit $\mathcal{M}x = \Lambda(\langle x, \xi_1 \rangle e_1 + \langle x, \xi_2 \rangle e_2) + \mathbb{C}\langle x, \xi_2 \rangle e_1$ is a closed subset of $\mathscr{X}$. Hence $\mathrm{Ref}(\mathcal{M}) = \mathrm{Ref}_{\mathrm{a}}(\mathcal{M})$. It is clear that $\mathrm{Im}(\mathcal{M}) = \vee\{e_1, e_2\}$, $\mathrm{Ker}(\mathcal{M}) = \ker(\xi_1) \cap \ker(\xi_2)$ and $\mathscr{X} = \mathrm{Im}(\mathcal{M}) \oplus \mathrm{Ker}(\mathcal{M})$. It follows, by Lemma 2.6, that every operator $T \in \mathrm{Ref}(\mathcal{M})$ is a linear combination of rank 1 operators $e_j \otimes \xi_k$ $(1 \leq j, k \leq 2)$. Let $T = \tau_{11} e_1 \otimes \xi_1 + \tau_{12} e_1 \otimes \xi_2 + \tau_{21} e_2 \otimes \xi_1 + \tau_{22} e_2 \otimes \xi_2$, where $\tau_{jk} \in \mathbb{C}$. Since $Te_1 \in \mathcal{M}e_1$, there exists $M_{e_1} = \alpha_{e_1}(e_1 \otimes \xi_1 + e_2 \otimes \xi_2) + \beta_{e_1} e_1 \otimes \xi_2$ such that $Te_1 = \tau_{11} e_1 + \tau_{21} e_2$ is equal to $M_{e_1} e_1 = \alpha_{e_1} e_1$. It follows that $\tau_{11} \in \Lambda$ and $\tau_{21} = 0$. Similarly, it follows from $Te_2 \in \mathcal{M}e_2$ that $\tau_{22} \in \Lambda$. Hence, every operator in $\mathrm{Ref}(\mathcal{M})$ is of the form as claimed above. On the other hand, let $T = \tau_{11} e_1 \otimes \xi_1 + \tau_{12} e_1 \otimes \xi_2 + \tau_{22} e_2 \otimes \xi_2$ with $\tau_{11}, \tau_{22} \in \Lambda$, $\tau_{12} \in \mathbb{C}$. Since every $x \in \mathscr{X}$ can be written as $x = \langle x, \xi_1 \rangle e_1 + \langle x, \xi_2 \rangle e_2 + x_0$, where $x_0 \in \mathrm{Ker}(\mathcal{M})$, we can show that $Tx \in \mathcal{M}x$. Indeed, if $\langle x, \xi_2 \rangle = 0$, let $M_x = \tau_{11}(e_1 \otimes \xi_1 + e_2 \otimes \xi_2) \in \mathcal{M}$, and if $\langle x, \xi_2 \rangle \neq 0$, let $M_x = \tau_{22}(e_1 \otimes \xi_1 + e_2 \otimes \xi_2) + ((\tau_{11} - \tau_{22})\frac{\langle x, \xi_1 \rangle}{\langle x, \xi_2 \rangle} + \tau_{12}) e_1 \otimes \xi_2 \in \mathcal{M}$. It is easily seen that $Tx = M_x x$. This proves our claim. Hence, set $\mathcal{M}$ is not reflexive. If $\Lambda = \mathbb{C}$, then $\mathcal{M}$ is a nonreflexive 2-dimensional linear space. This example also shows that the linear span of a reflexive set of operators is not necessarily reflexive: if $M_1, M_2 \in \mathcal{B}(\mathscr{X})$ span a nonreflexive 2-dimensional space, then $\{M_1, M_2\}$ is a reflexive set, by Proposition 2.2, but its linear span is not reflexive.

We have already mentioned some basic properties of the reflexive cover. Now we list a few more. We omit the proof of the following proposition since it relies on a simple fact that $\overline{\mathrm{Ref}(\mathcal{M})x} = \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$. We use the following notation: the closure of a set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ in (SOT) is denoted by $\overline{\mathcal{M}}^{\mathrm{SOT}}$.

Proposition 2.8. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set. Then*

(i) *$\mathrm{Ref}(\mathcal{M})$ is SOT-closed, which means that $\overline{\mathcal{M}}^{\mathrm{SOT}} \subseteq \mathrm{Ref}(\mathcal{M})$ and $\mathrm{Ref}(\mathcal{M}) = \mathrm{Ref}(\overline{\mathcal{M}}^{\mathrm{SOT}})$;*

(ii) *if $\mathcal{M}$ is a bounded set, then $\mathrm{Ref}(\mathcal{M})$ is bounded, as well, and $\sup_{M \in \mathcal{M}} \|M\| = \sup_{T \in \mathrm{Ref}(\mathcal{M})} \|T\|$.*

In general, $\operatorname{Ref}(\mathcal{M})$ is not closed in the *weak operator topology* (WOT). For instance, let S be the unilateral shift on ℓ^2; that is, let $S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$. Let $\mathcal{M} = \{S^n; n \geq 0\}$. It is well known and easy to see that 0 is in the WOT-closure of $\mathcal{M}$, and hence it is in the WOT-closure of $\operatorname{Ref}(\mathcal{M})$. On the other hand, since S^n is an isometry for every $n \geq 0$ and the set of all isometries is reflexive (see Remark 2.4), each operator $T \in \operatorname{Ref}(\mathcal{M})$ is an isometry, which means that $0 \notin \operatorname{Ref}(\mathcal{M})$.

Proposition 2.9. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be nonempty sets. Then the following hold.*

(i) *If $\mathcal{M}_1 + \mathcal{M}_2 \subseteq \mathcal{M}$, then $\overline{\operatorname{Ref}(\mathcal{M}_1) + \operatorname{Ref}(\mathcal{M}_2)}^{\mathrm{SOT}} \subseteq \operatorname{Ref}(\mathcal{M})$.*

(ii) *If $\mathcal{M}_1\mathcal{M}_2 \subseteq \mathcal{M}$, then $\overline{\operatorname{Ref}(\mathcal{M}_1)\operatorname{Ref}(\mathcal{M}_2)}^{\mathrm{SOT}} \subseteq \operatorname{Ref}(\mathcal{M})$.*

Proof. (i) Assume that $T_1 \in \operatorname{Ref}(\mathcal{M}_1)$ and that $T_2 \in \operatorname{Ref}(\mathcal{M}_2)$. Let $x \in \mathscr{X}$ and $\varepsilon > 0$ be arbitrary. Then there exist $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ such that $\|T_1x - M_1x\| < \frac{\varepsilon}{2}$ and $\|T_2x - M_2x\| < \frac{\varepsilon}{2}$. It follows that $\|(T_1 + T_2)x - (M_1 + M_2)x\| < \varepsilon$. Hence $\operatorname{Ref}(\mathcal{M}_1) + \operatorname{Ref}(\mathcal{M}_2) \subseteq \operatorname{Ref}(\mathcal{M})$. Since, by Proposition 2.8(i), reflexive cover is SOT-closed, the assertion follows.

(ii) Let $M_1 \in \mathcal{M}_1$ and $T_2 \in \operatorname{Ref}(\mathcal{M}_2)$ be arbitrary. For $x \in \mathscr{X}$ and $\varepsilon > 0$, there exists $M_2 \in \mathcal{M}_2$ such that $\|T_2x - M_2x\| < \varepsilon$. It follows that $\|M_1T_2x - M_1M_2x\| < \|M_1\|\varepsilon$. Since x and ε are arbitrary we conclude that $M_1T_2 \in \operatorname{Ref}(\mathcal{M}_1\mathcal{M}_2) \subseteq \operatorname{Ref}(\mathcal{M})$. Now, let $T_1 \in \operatorname{Ref}(\mathcal{M}_1)$ and $T_2 \in \operatorname{Ref}(\mathcal{M}_2)$ be arbitrary. Again, for every $x \in \mathscr{X}$ and every $\varepsilon > 0$, there exists $M_1 \in \mathcal{M}_1$ such that $\|T_1(T_2x) - M_1(T_2x)\| < \varepsilon$. Since $M_1T_2 \in \operatorname{Ref}(\mathcal{M})$ we may conclude that $T_1T_2 \in \operatorname{Ref}(\operatorname{Ref}(\mathcal{M})) = \operatorname{Ref}(\mathcal{M})$. ☐

Corollary 2.10. *If $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is a semigroup (resp., a linear space, or an algebra), then $\operatorname{Ref}(\mathcal{M})$ is a semigroup (resp., a linear space, or an algebra).*

Proposition 2.11. *If $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is a nonempty set such that $\dim(\vee\mathcal{M}) < \infty$, then $\dim(\vee\operatorname{Ref}(\mathcal{M})) < \infty$.*

Proof. Observe first that $\mathcal{M} \subseteq \vee\mathcal{M}$ implies that $\operatorname{Ref}(\mathcal{M}) \subseteq \operatorname{Ref}(\vee\mathcal{M})$ and consequently that $\vee\operatorname{Ref}(\mathcal{M}) \subseteq \operatorname{Ref}(\vee\mathcal{M})$ since $\operatorname{Ref}(\vee\mathcal{M})$ is SOT-closed linear space. Denote $\mathcal{S} = \vee\mathcal{M}$ and assume that it is a finite-dimensional space. Then each orbit $\mathcal{S}x$ $(x \in \mathscr{X})$ is a finite-dimensional linear subspace of $\mathscr{X}$ which means that it is closed. It follows that $\operatorname{Ref}(\mathcal{S}) = \operatorname{Ref}_{\mathrm{a}}(\mathcal{S})$. By [7, Corollary 2.9], we conclude that $\operatorname{Ref}_{\mathrm{a}}(\mathcal{S})$ is finite-dimensional, which gives, by the inclusion above, that $\dim(\vee\operatorname{Ref}(\mathcal{M})) < \infty$. ☐

In general the sum of two reflexive sets is not reflexive. Again a 2-dimensional nonreflexive space can serve as an example: it is the sum of any two distinct 1-dimensional subspaces, which are reflexive, by Proposition 2.5.

Proposition 2.12. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set and let $N \in \mathcal{B}(\mathscr{X})$. Then $\operatorname{Ref}(\mathcal{M} + N) = \operatorname{Ref}(\mathcal{M}) + N$; in particular, $\mathcal{M}$ is reflexive if and only if $\mathcal{M} + N$ is reflexive.*

Proof. For each $x \in \mathscr{X}$, one has $(\mathcal{M}+N)x = \mathcal{M}x+Nx$, which gives $\overline{(\mathcal{M} + N)x} = \overline{\mathcal{M}x}+Nx$. Let $T \in \operatorname{Ref}(\mathcal{M})$. Then $(T+N)x = Tx+Nx \in \overline{\mathcal{M}x}+Nx = \overline{(\mathcal{M} + N)x}$

for every $x \in \mathscr{X}$. Hence $T + N \in \operatorname{Ref}(\mathcal{M} + N)$. On the other hand, assume that $T \in \operatorname{Ref}(\mathcal{M} + N)$. Then $Tx \in \overline{(\mathcal{M} + N)x} = \overline{\mathcal{M}x} + Nx$ for every $x \in \mathscr{X}$. Hence $(T - N)x \in \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$. This proves that $T - N \in \operatorname{Ref}(\mathcal{M})$ and therefore that $T \in \operatorname{Ref}(\mathcal{M}) + N$. $\square$

Next proposition, which can be seen as an extension of Proposition 2.5, shows that many subsets of a reflexive 2-dimensional linear subspace $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ are reflexive.

Proposition 2.13. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be a reflexive 2-dimensional linear subspace. For arbitrary $M_1, M_2 \in \mathcal{S}$ and nonempty closed sets $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$, the set $\mathcal{M} = \Lambda_1 M_1 + \Lambda_2 M_2$ is reflexive.*

Proof. If M_1, M_2 are linearly dependent, say $M_2 = \mu M_1$ ($\mu \in \mathbb{C}$), then $\mathcal{M} = (\Lambda_1 + \mu\Lambda_2)M_1$ is reflexive, by Proposition 2.5. Assume therefore that M_1, M_2 are linearly independent. Let $T \in \operatorname{Ref}(\mathcal{M})$. Since $\mathcal{M} \subseteq \mathcal{S}$ and $\mathcal{S}$ is reflexive, we have $T = \alpha_1 M_1 + \alpha_2 M_2$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. We have to show that $\alpha_j \in \Lambda_j$ ($j = 1, 2$). If there exists $x \in \mathscr{X}$ such that $M_1 x$, $M_2 x$ are linearly independent, then it follows from $Tx \in \mathcal{M}x$ that $\alpha_1 M_1 x + \alpha_2 M_2 x = \lambda_1 M_1 x + \lambda_2 M_2 x$, for some $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, which gives $\alpha_j = \lambda_j \in \Lambda_j$ ($j = 1, 2$). Suppose now that $M_1 x$, $M_2 x$ are linearly dependent for every $x \in \mathscr{X}$. By [4, Theorem 2.3], there exist a nonzero $e \in \mathscr{X}$ and linearly independent $\xi_1, \xi_2 \in \mathscr{X}^*$ such that $M_1 = e \otimes \xi_1$ and $M_2 = e \otimes \xi_2$. Hence $T = e \otimes (\alpha_1 \xi_1 + \alpha_2 \xi_2)$. Let $x_1, x_2 \in \mathscr{X}$ be such that $\langle x_i, \xi_j \rangle = \delta_{ij}$ ($1 \leq i, j \leq 2$). Since $Tx_j \in \mathcal{M}x_j$, for $j = 1, 2$, there exist $\lambda_{ij} \in \Lambda_i$ ($1 \leq i, j \leq 2$) such that $Tx_j = (\lambda_{1j} M_1 + \lambda_{2j} M_2)x_j$ ($j = 1, 2$) which gives $\alpha_1 = \lambda_{11} \in \Lambda_1$ and $\alpha_2 = \lambda_{22} \in \Lambda_2$. Hence $T \in \mathcal{M}$. $\square$

In the proof of Proposition 2.13 we used Theorem 2.3 from [4] which gives the complete description of a pair of locally linearly dependent operators. A similar characterization for a triple of locally linearly dependent operators is given by [4, Theorem 2.4]. As mentioned on page 1258 of [4], the problem of finding a similar characterization for a collection of $n \geq 4$ locally linearly dependent operators seems to become extremely difficult. Because of this it seems that a result similar to Proposition 2.13, but for $\mathcal{S}$ of dimension $n \geq 4$, would be hard to prove.

Proposition 2.14. *Let $A \in \mathcal{B}(\mathscr{X})$ be left invertible with a left inverse $A^l \in \mathcal{B}(\mathscr{X})$, and let $B \in \mathcal{B}(\mathscr{X})$ be right invertible with a right inverse $B^r \in \mathcal{B}(\mathscr{X})$. If $\mathcal{N}$ is a nonempty subset of $\mathcal{B}(\mathscr{X})$, then $\operatorname{Ref}(\mathcal{N}) = A^l \operatorname{Ref}(A\mathcal{N}B)B^r$. In particular, if $A\mathcal{N}B$ is reflexive, then $\mathcal{N}$ is reflexive.*

Proof. Since $\mathcal{M} = A\mathcal{N}B$ and $\mathcal{N} = A^l\mathcal{M}B^r$, we have $A\operatorname{Ref}(\mathcal{N})B \subseteq \operatorname{Ref}(\mathcal{M})$ and $A^l \operatorname{Ref}(\mathcal{M})B^r \subseteq \operatorname{Ref}(\mathcal{N})$, by Proposition 2.9 (note that singletons are reflexive sets, by Proposition 2.2). It follows that

$$\operatorname{Ref}(\mathcal{N}) = A^l\big(A\operatorname{Ref}(\mathcal{N})B\big)B^r \subseteq A^l \operatorname{Ref}(\mathcal{M})B^r \subseteq \operatorname{Ref}(\mathcal{N})$$

which gives $\operatorname{Ref}(\mathcal{N}) = A^l \operatorname{Ref}(\mathcal{M})B^r$. If $\mathcal{M}$ is reflexive, then $\operatorname{Ref}(\mathcal{N}) = A^l\mathcal{M}B^r = \mathcal{N}$; that is, $\mathcal{N}$ is reflexive. $\square$

Corollary 2.15. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set and let $A, B \in \mathcal{B}(\mathscr{X})$ be invertible. Then $\operatorname{Ref}(A\mathcal{M}B) = A\operatorname{Ref}(\mathcal{M})B$. In particular, $\mathcal{M}$ is reflexive if and only if $A\mathcal{M}B$ is reflexive.*

For a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$, let $\operatorname{Sgr}_l(\mathcal{M}) = \{A \in \mathcal{B}(\mathscr{X}); A\mathcal{M} \subseteq \mathcal{M}\}$ and $\operatorname{Sgr}_r(\mathcal{M}) = \{B \in \mathcal{B}(\mathscr{X}); \mathcal{M}B \subseteq \mathcal{M}\}$. It is easily seen that $\operatorname{Sgr}_l(\mathcal{M})$ and $\operatorname{Sgr}_r(\mathcal{M})$ are semigroups containing I. By Corollary 2.10, $\operatorname{Ref}(\operatorname{Sgr}_l(\mathcal{M}))$ and $\operatorname{Ref}(\operatorname{Sgr}_r(\mathcal{M}))$ are semigroups as well. If $I \in \mathcal{M}$, then $\operatorname{Sgr}_l(\mathcal{M}) \subseteq \mathcal{M}$ and $\operatorname{Sgr}_r(\mathcal{M}) \subseteq \mathcal{M}$. On the other hand, $\mathcal{M} \subseteq \operatorname{Sgr}_l(\mathcal{M})$ if and only if $\mathcal{M}$ is a semigroup if and only if $\mathcal{M} \subseteq \operatorname{Sgr}_r(\mathcal{M})$. Hence, if $\mathcal{M}$ is a semigroup containing I, then $\operatorname{Sgr}_l(\mathcal{M}) = \mathcal{M} = \operatorname{Sgr}_r(\mathcal{M})$. However, in general, these semigroups are not necessarily related with $\mathcal{M}$ through inclusions.

Proposition 2.16. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set.*

(i) *If $\mathcal{M}$ is reflexive, then semigroups $\operatorname{Sgr}_l(\mathcal{M})$ and $\operatorname{Sgr}_r(\mathcal{M})$ are reflexive as well.*

(ii) *Operator $A \in \mathcal{B}(\mathscr{X})$ is in $\operatorname{Sgr}_l(\operatorname{Ref}(\mathcal{M}))$ if and only if $A\overline{\mathcal{M}x} \subseteq \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$. Similarly, $B \in \operatorname{Sgr}_r(\operatorname{Ref}(\mathcal{M}))$ if and only if $\overline{\mathcal{M}Bx} \subseteq \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$.*

Proof. (i) It follows from the definition of $\operatorname{Sgr}_l(\mathcal{M})$ that $\operatorname{Sgr}_l(\mathcal{M})\mathcal{M} \subseteq \mathcal{M}$,. By Proposition 2.9, we have $\operatorname{Ref}(\operatorname{Sgr}_l(\mathcal{M}))\operatorname{Ref}(\mathcal{M}) \subseteq \operatorname{Ref}(\mathcal{M})$. Hence, if $\mathcal{M}$ is reflexive, then $\operatorname{Ref}(\operatorname{Sgr}_l(\mathcal{M}))\mathcal{M} \subseteq \mathcal{M}$ which gives, by the definition of $\operatorname{Sgr}_l(\mathcal{M})$, that $\operatorname{Ref}(\operatorname{Sgr}_l(\mathcal{M})) \subseteq \operatorname{Sgr}_l(\mathcal{M})$. This proves the reflexivity of $\operatorname{Sgr}_l(\mathcal{M})$. The reflexivity of $\operatorname{Sgr}_r(\mathcal{M})$ is proved similarly.

(ii) If $A \in \operatorname{Sgr}_l(\operatorname{Ref}(\mathcal{M}))$, then $A\operatorname{Ref}(\mathcal{M})x \subseteq \operatorname{Ref}(\mathcal{M})x$ for every $x \in \mathscr{X}$. By the continuity of A, $A\overline{\operatorname{Ref}(\mathcal{M})x} \subseteq \overline{\operatorname{Ref}(\mathcal{M})x}$, and therefore $A\overline{\mathcal{M}x} \subseteq \overline{\mathcal{M}x}$ as $\overline{\operatorname{Ref}(\mathcal{M})x} = \overline{\mathcal{M}x}$ for every x. To prove the opposite implication, assume that $A\overline{\mathcal{M}x} \subseteq \overline{\mathcal{M}x}$ for every $x \in \mathscr{X}$. Let $T \in \operatorname{Ref}(\mathcal{M})$ be arbitrary. For every $x \in \mathscr{X}$ and $\varepsilon > 0$ there exists $M_{x,\varepsilon} \in \mathcal{M}$ such that $\|Tx - M_{x,\varepsilon}x\| < \varepsilon$. By assumption, $AM_{x,\varepsilon}x \in \overline{\mathcal{M}x}$, which means that there exists $N_{x,\varepsilon} \in \mathcal{M}$ such that $\|AM_{x,\varepsilon}x - N_{x,\varepsilon}x\| < \varepsilon$. It follows that $\|ATx - N_{x,\varepsilon}x\| < (\|A\| + 1)\varepsilon$. Hence $AT \in \operatorname{Ref}(\mathcal{M})$. The second part of (ii) is proved similarly. □

Example 2.17. Let $M \in \mathcal{B}(\mathscr{X})$ and $\Lambda \subseteq \mathbb{C}$ be a nonempty closed set. Let $\mathcal{M} = \Lambda M$. We claim that

$$\operatorname{Sgr}_l(\mathcal{M}) = \big\{A \in \mathcal{B}(\mathscr{X}); \exists\lambda_A \in \Lambda : \lambda_A\Lambda \subseteq \Lambda \text{ and } \operatorname{im}(M) \subseteq \ker(\lambda_A I - A)\big\}$$

and

$$\operatorname{Sgr}_r(\mathcal{M}) = \big\{B \in \mathcal{B}(\mathscr{X}); \exists\lambda_B \in \Lambda : \lambda_B\Lambda \subseteq \Lambda \text{ and } \operatorname{im}(\lambda_B I - B) \subseteq \ker(M)\big\}.$$

If $M = 0$ or $\Lambda = \{0\}$, then $\mathcal{M} = \{0\}$, which gives $\operatorname{Sgr}_l(\mathcal{M}) = \operatorname{Sgr}_r(\mathcal{M}) = \mathcal{B}(\mathscr{X})$. Assume therefore that $M \neq 0$ and that Λ contains a nonzero number. Since, for a nonzero number $\lambda \in \Lambda$, we have $\mathcal{M} = (\frac{1}{\lambda}\Lambda)(\lambda M)$, we may assume without loss of generality that $1 \in \Lambda$, which implies that $M \in \mathcal{M}$.

We will check only the first equality. If $A \in \operatorname{Sgr}_l(\mathcal{M})$, then $AM \in \mathcal{M}$ and therefore there exists $\lambda_A \in \Lambda$ such that $(\lambda_A I - A)M = 0$, which means that $\operatorname{im}(M) \subseteq \ker(\Lambda I - A)$. For arbitrary $\lambda \in \Lambda$, we have $A(\lambda M) \in \mathcal{M}$ (i.e., $\lambda_A\lambda M \in \mathcal{M}$). Since

$M \neq 0$, we conclude that $\Lambda_A \lambda \in \Lambda$. On the other hand, if $\mathrm{im}(M) \subseteq \ker(\lambda_A I - A)$ for $\lambda_A \in \Lambda$ satisfying $\lambda_A \Lambda \subseteq \Lambda$, then $(\lambda_A I - A)(\lambda M)x = 0$ for every $x \in \mathscr{X}$, and therefore $A(\lambda M) = \lambda_A \lambda M \in \mathrm{Sgr}_l(\mathcal{M})$.

Note that $\mathrm{Sgr}_l(\mathcal{M})$ and $\mathrm{Sgr}_r(\mathcal{M})$ are reflexive sets since $\mathcal{M}$ is reflexive, by Proposition 2.5. We mention the following special case. Let $\mathscr{Y}$ be a complemented closed subspace of $\mathscr{X}$, with complement $\mathscr{Z}$, and let $\Lambda \subseteq \mathbb{C}$ be a multiplicative semigroup containing 1. Then sets $\{A \in \mathcal{B}(\mathscr{X}); \exists \lambda_A \in \Lambda \text{ such that } \mathscr{Y} \subseteq \ker(\lambda_A I - A)\}$ and $\{B \in \mathcal{B}(\mathscr{X}); \exists \lambda_B \in \Lambda \text{ such that } \mathrm{im}(\lambda_B I - B) \subseteq \mathscr{Z}\}$ are reflexive. Indeed, since $\mathscr{Y}$ is complemented there exists an idempotent $P \in \mathcal{B}(\mathscr{X})$ such that $\mathrm{im}(P) = \mathscr{Y}$ and $\ker(P) = \mathscr{Z}$. We take $\mathcal{M} = \Lambda P$ and therefore the above sets are precisely $\mathrm{Sgr}_l(\mathcal{M})$ (resp., $\mathrm{Sgr}_r(\mathcal{M})$).

3. Reflexivity of intersections and unions

In this section, we are concerned with the question: are the intersection and the SOT-closure of the union of reflexive sets reflexive?

Proposition 3.1. *Let $\{\mathcal{M}_i; i \in \mathbb{I}\}$ be an arbitrary family of nonempty subsets of $\mathcal{B}(\mathscr{X})$. Then*

$$\mathrm{Ref}\Big(\bigcap_{i\in\mathbb{I}} \mathcal{M}_i\Big) \subseteq \bigcap_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i) \qquad \text{and} \qquad \mathrm{Ref}\Big(\bigcup_{i\in\mathbb{I}} \mathcal{M}_i\Big) \supseteq \overline{\bigcup_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i)}^{\mathrm{SOT}}.$$

(i) *If every $\mathcal{M}_i$ $(i \in \mathbb{I})$ is reflexive, then the intersection $\bigcap_{i\in\mathbb{I}} \mathcal{M}_i$ is reflexive.*

(ii) *If $\overline{\bigcup_{i\in\mathbb{I}} \mathcal{M}_i}^{\mathrm{SOT}}$ is reflexive, then $\overline{\bigcup_{i\in\mathbb{I}} \mathcal{M}_i}^{\mathrm{SOT}} = \overline{\bigcup_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i)}^{\mathrm{SOT}}$.*

Proof. Since $\bigcap_{i\in\mathbb{I}} \mathcal{M}_i \subseteq \mathcal{M}_j$ we have $\mathrm{Ref}(\bigcap_{i\in\mathbb{I}} \mathcal{M}_i) \subseteq \mathrm{Ref}(\mathcal{M}_j)$ for every $j \in \mathbb{I}$. Hence $\mathrm{Ref}(\bigcap_{i\in\mathbb{I}} \mathcal{M}_i) \subseteq \bigcap_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i)$. To prove (i), assume that every $\mathcal{M}_i$ $(i \in \mathbb{I})$ is reflexive. Then $\bigcap_{i\in\mathbb{I}} \mathcal{M}_i \subseteq \mathrm{Ref}(\bigcap_{i\in\mathbb{I}} \mathcal{M}_i) \subseteq \bigcap_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i) = \bigcap_{i\in\mathbb{I}} \mathcal{M}_i$.

We have $\bigcup_{i\in\mathbb{I}} \mathcal{M}_i \supseteq \mathcal{M}_j$ and therefore $\mathrm{Ref}(\bigcup_{i\in\mathbb{I}} \mathcal{M}_i) \supseteq \mathrm{Ref}(\mathcal{M}_j)$, for every $j \in \mathbb{I}$. It follows that $\mathrm{Ref}(\bigcup_{i\in\mathbb{I}} \mathcal{M}_i) \supseteq \overline{\bigcup_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i)}^{\mathrm{SOT}}$, by Proposition 2.8(i). Suppose that $\overline{\bigcup_{i\in\mathbb{I}} \mathcal{M}_i}^{\mathrm{SOT}}$ is reflexive. Then, by Proposition 2.8(i),

$$\overline{\bigcup_{i\in\mathbb{I}} \mathcal{M}_i}^{\mathrm{SOT}} = \mathrm{Ref}\Big(\bigcup_{i\in\mathbb{I}} \mathcal{M}_i\Big) \supseteq \overline{\bigcup_{i\in\mathbb{I}} \mathrm{Ref}(\mathcal{M}_i)}^{\mathrm{SOT}} \supseteq \overline{\bigcup_{i\in\mathbb{I}} \mathcal{M}_i}^{\mathrm{SOT}}. \tag{3.1}$$

□

Although the intersection of an arbitrary family of reflexive sets is reflexive, the next example shows that the union of two reflexive sets is not necessarily reflexive.

Example 3.2. Let $e_1, e_2 \in \mathscr{S}_{\mathscr{X}}$ be linearly independent vectors, and let $\xi_1, \xi_2 \in \mathscr{S}_{\mathscr{X}^*}$ be such that $\langle e_j, \xi_k \rangle = \delta_{jk}$ $(1 \le j, k \le 2)$. Let $\mathcal{M} = \{\mu e_1 \otimes \xi_1; \mu \in \mathbb{C}\}$ and, for a number α, let $A = \alpha e_1 \otimes \xi_2$. Set $\mathcal{M}$ is a 1-dimensional linear space, and hence it is reflexive and SOT-closed. Singleton $\{A\}$ is reflexive and SOT-closed, as well. We claim that $\mathrm{Ref}(\mathcal{M} \cup \{A\}) = \mathcal{M} \cup (\mathcal{M} + A)$. It is obvious that $\mathcal{M} \subseteq \mathrm{Ref}(\mathcal{M} \cup \{A\})$. Let us show that every $T = M + A$, where $M \in \mathcal{M}$, is in $\mathrm{Ref}(\mathcal{M} \cup \{A\})$. We have $T = e_1 \otimes (\mu\xi_1 + \alpha\xi_2)$ for some $\mu \in \mathbb{C}$. Let $x \in \mathscr{X}$ be an arbitrary vector. We distinguish two cases. If $\langle x, \xi_1 \rangle = 0$, then it follows that $Tx = Ax$. On the other hand, if $\langle x, \xi_1 \rangle \neq 0$, then let $M_x = (\mu + \alpha \frac{\langle x, \xi_2 \rangle}{\langle x, \xi_1 \rangle}) e_1 \otimes \xi_1 \in \mathcal{M}$. It is not hard to check that $Tx = M_x x$. This proves that $Tx \in (\mathcal{M} \cup \{A\})x$ for every $x \in \mathscr{X}$. Hence $T \in \mathrm{Ref}(\mathcal{M} \cup \{A\})$.

Now we prove the inclusion $\mathrm{Ref}(\mathcal{M} \cup \{A\}) \subseteq \mathcal{M} \cup (\mathcal{M} + A)$. It holds, by Proposition 2.5, if $\alpha = 0$ (i.e., $A = 0$). Suppose therefore that $\alpha \neq 0$. Note that for every $T \in \mathrm{Ref}(\mathcal{M} \cup \{A\})$, we have $\mathrm{Ker}(\mathcal{M} \cup \{A\}) \subseteq \ker(T)$ and $\mathrm{im}(T) \subseteq \mathrm{Im}(\mathcal{M} \cup \{A\})$. From the definition of $\mathcal{M}$ and A it follows that $\mathrm{Im}(\mathcal{M} \cup \{A\}) = \mathbb{C}e_1$. Hence, either $T = 0$ or T is rank 1 operator with image $\mathbb{C}e_1$. In any case, $T = e_1 \otimes \zeta$ for some $\zeta \in \mathscr{X}^*$. If we write $\zeta = \langle e_1, \zeta \rangle \xi_1 + \langle e_2, \zeta \rangle \xi_2 + \rho$, where $\rho \in \mathscr{X}^*$ is such that $\langle e_j, \rho \rangle = 0$ $(j = 1, 2)$, then it is not hard to check that $\rho = 0$, which means that $T = e_1 \otimes (\langle e_1, \zeta \rangle \xi_1 + \langle e_2, \zeta \rangle \xi_2)$. If $\langle e_2, \zeta \rangle = 0$, then $T = \langle e_1, \zeta \rangle e_1 \otimes \xi_1 \in \mathcal{M}$. Assume therefore that $\langle e_2, \zeta \rangle \neq 0$. Since $Me_2 = 0$ for every $M \in \mathcal{M}$ and $Ae_2 = \alpha e_1$, we have $(\mathcal{M} \cup \{A\})e_2 = \{0, \alpha e_1\}$. It follows from $Te_2 = \langle e_2, \zeta \rangle e_1 \in \{0, \alpha e_1\}$ and $\langle e_2, \zeta \rangle \neq 0$ that $T = \langle e_1, \zeta \rangle e_1 \otimes \xi_1 + \alpha e_1 \otimes \xi_2 \in \mathcal{M} + A$.

Let $\mathcal{M}$ be as before and let $\mathcal{N} = \mathbb{C}e_1 \otimes \xi_2$. It is clear that $\mathcal{M}, \mathcal{N}$ are 1-dimensional linear spaces, which implies that they are reflexive. For every $\alpha \in \mathbb{C}$, it follows from $\mathcal{M} \cup \{\alpha e_1 \otimes \xi_2\} \subseteq \mathcal{M} \cup \mathcal{N}$ that $\mathrm{Ref}(\mathcal{M} \cup \{\alpha e_1 \otimes \xi_2\}) \subseteq \mathrm{Ref}(\mathcal{M} \cup \mathcal{N})$. We already know that $\mathrm{Ref}(\mathcal{M} \cup \{\alpha e_1 \otimes \xi_2\}) = \mathcal{M} \cup (\mathcal{M} + \alpha e_1 \otimes \xi_2)$. Hence, $\mathcal{M} + \mathcal{N} = \bigcup_{\alpha \in \mathbb{C}} (\mathcal{M} \cup (\mathcal{M} + \alpha e_1 \otimes \xi_2)) \subseteq \mathrm{Ref}(\mathcal{M} \cup \mathcal{N})$. On the other hand, if $T \in \mathrm{Ref}(\mathcal{M} \cup \mathcal{N})$, then it follows from $\mathrm{im}(T) \subseteq \mathrm{Im}(\mathcal{M} \cup \mathcal{N}) = \mathbb{C}e_1$ and $\ker(\xi_1) \cap \ker(\xi_2) = \mathrm{Ker}(\mathcal{M} \cup \mathcal{N}) \subseteq \ker(T)$ that $T \in \mathcal{M} + \mathcal{N}$. Hence, $\mathrm{Ref}(\mathcal{M} \cup \mathcal{N}) = \mathcal{M} + \mathcal{N}$.

Inclusion $\mathrm{Ref}(\mathcal{M} \cup \mathcal{N}) \supseteq \mathcal{M} + \mathcal{N}$ in the last equality of Example 3.2 follows from the following more general result.

Proposition 3.3. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_n \subseteq \mathcal{B}(\mathscr{X})$ $(n \geq 2)$ be linear spaces. If, for every $x \in \mathscr{X}$, there exists an index $j_x \in \{1, \ldots, n\}$ such that $\bigcup_{i=1}^n \mathcal{M}_i x \subseteq \mathcal{M}_{j_x} x$, then $\overline{\mathcal{M}_1 + \cdots + \mathcal{M}_n}^{\mathrm{SOT}} \subseteq \mathrm{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$.*

Proof. Let $T = M_1 + \cdots + M_n$, where $M_j \in \mathcal{M}_j$ $(1 \leq j \leq n)$. For every $x \in \mathscr{X}$, we have $Tx = M_1 x + \cdots + M_n x \in \mathcal{M}_1 x + \cdots + \mathcal{M}_n x$. Since $\mathcal{M}_j x$ are linear sets and $\bigcup_{i=1}^n \mathcal{M}_i x \subseteq \mathcal{M}_{j_x} x$, we have $\mathcal{M}_1 x + \cdots + \mathcal{M}_n x \subseteq (\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)x$. It follows that $T \in \mathrm{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$, and consequently $\mathcal{M}_1 + \cdots + \mathcal{M}_n \subseteq \mathrm{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$. Now we apply Proposition 2.8(i). $\square$

Recall that operators $M_1, \ldots, M_n \in \mathcal{B}(\mathscr{X})$ are locally linearly dependent if, for every $x \in \mathscr{X}$, vectors $M_1 x, \ldots, M_n x$ are linearly dependent.

Corollary 3.4. *Let $M_1, M_2 \in \mathcal{B}(\mathscr{X})$ be arbitrary operators and let $\mathcal{M}_j = \mathbb{C}M_j$ $(j = 1, 2)$. If M_1, M_2 are locally linearly dependent, then $\mathcal{M}_1 + \mathcal{M}_2 \subseteq \mathrm{Ref}(\mathcal{M}_1 \cup \mathcal{M}_2)$. On the other hand, if M_1, M_2 are not locally linearly dependent, then $T \in \mathcal{M}_1 + \mathcal{M}_2$ is in* $\mathrm{Ref}(\mathcal{M}_1 \cup \mathcal{M}_2)$ *if and only if $T \in \mathcal{M}_1 \cup \mathcal{M}_2$.*

Proof. If M_1, M_2 are locally linearly dependent, then, for every $x \in \mathscr{X}$, either $\mathcal{M}_1 x \subseteq \mathcal{M}_2 x$ or $\mathcal{M}_1 x \supseteq \mathcal{M}_2 x$. It follows, by Proposition 3.3, that $\mathcal{M}_1 + \mathcal{M}_2 \subseteq \mathrm{Ref}(\mathcal{M}_1 \cup \mathcal{M}_2)$. Assume now that M_1, M_2 are not locally linearly dependent.

Then there exists $x \in \mathscr{X}$ such that M_1x, M_2x are linearly independent vectors. If $T = \alpha_1 M_1 + \alpha_2 M_2 \in \mathcal{M}_1 + \mathcal{M}_2$ is in $\operatorname{Ref}(\mathcal{M}_1 \cup \mathcal{M}_2)$, then it follows from $Tx \in \overline{(\mathcal{M}_1 \cup \mathcal{M}_2)x} = \mathcal{M}_1 x \cup \mathcal{M}_2 x$ that $Tx \in \mathcal{M}_j x$ for an index $j \in \{1,2\}$. Hence there is $\lambda \in \mathbb{C}$ such that $\alpha_1 M_1 x + \alpha_2 M_2 x = \lambda M_j x$. Since M_1x, M_2x are linearly independent, it follows that $\alpha_i = \lambda$ if $i = j$ and $\alpha_i = 0$ if $i \neq j$; that is, $T \in \mathcal{M}_1 \cup \mathcal{M}_2$. □

In the rest of this section we are interested in conditions which imply reflexivity of the SOT-closure of the union of reflexive sets. For some families of reflexive sets it is not hard to show that (the SOT-closure of) the union is reflexive. For instance, let $u \in \mathscr{X}$ and let $\mathscr{V} \subseteq \mathscr{X}$ be a nonempty closed set. Then, by Proposition 2.3, $\bigcup_{v\in\mathscr{V}} \mathcal{M}_{u,v} = \mathcal{M}_{u,\mathscr{V}}$ is reflexive.

Proposition 3.5. *Let $\mathscr{U}, \mathscr{V} \subseteq \mathscr{X}$ be nonempty sets, $\mathscr{V}$ closed. The set $\mathcal{M} = \bigcup_{u\in\mathscr{U}} \mathcal{M}_{u,\mathscr{V}}$ is reflexive if and only if $\mathscr{U} = \Lambda u_0$ for some $u_0 \in \mathscr{X}$ and $\emptyset \neq \Lambda \subseteq \mathbb{C}$. If $\mathcal{M}$ is not reflexive, then it is transitive.*

Proof. Assume that $\mathscr{U} = \Lambda u_0$, where $u_0 \in \mathscr{X}$ and $\emptyset \neq \Lambda \subseteq \mathbb{C}$. If $\mathscr{U} = \{0\}$, then $\mathcal{M} = \mathcal{M}_{0,\mathscr{V}}$, which is reflexive. Let $\mathscr{U} \neq \{0\}$. We may suppose that $\mathscr{U} = \Lambda u_0$, where $u_0 \neq 0$ and $1 \in \Lambda$. Let $T \in \operatorname{Ref}(\mathcal{M})$ be arbitrary. For every $\varepsilon > 0$, there exists $M_\varepsilon \in \mathcal{M}$ such that $\|Tu_0 - M_\varepsilon u_0\| < \varepsilon$. It follows that $\|T(\lambda u_0) - M_\varepsilon(\lambda u_0)\| < |\lambda|\varepsilon$ for every $\lambda \in \Lambda$. Fix λ and let $\varepsilon \to 0$. Since $M_\varepsilon(\lambda u_0) \in \mathscr{V}$ and $\mathscr{V}$ is closed, we have $T(\lambda u_0) \in \mathscr{V}$. Hence $T \in \mathcal{M}$.

Assume now that $\mathscr{U}$ is not of the form Λu_0, for some $u_0 \in \mathscr{X}$ and that $\emptyset \neq \Lambda \subseteq \mathbb{C}$. Then there exist linearly independent vectors $u_1, u_2 \in \mathscr{U}$. Let $v_1, v_2 \in \mathscr{V}$ be arbitrary and let $\xi_1, \xi_2 \in \mathscr{X}^*$ be such that $\langle u_j, \xi_k\rangle = \delta_{jk}$ $(1 \le j,k \le 2)$. We claim that, for every $T \in \mathcal{B}(\mathscr{X})$, one has $Tx \in (\mathcal{M}_{u_1,v_1} \cup \mathcal{M}_{u_2,v_2})x$ $(x \in \mathscr{X})$. Indeed, if $x \in \mathscr{X}$ is such that $\langle x, \xi_1\rangle = 0$, then let $M_x = T + (v_1 - Tu_1) \otimes \xi_1$. Since $M_x u_1 = v_1$, we see that $M_x \in \mathcal{M}_{u_1,v_1}$. We also have $M_x x = Tx$. Now let $x \in \mathscr{X}$ be such that $\langle x, \xi_1\rangle \neq 0$. For $M_x = T + \frac{\langle x,\xi_2\rangle}{\langle x,\xi_1\rangle}(Tu_2 - v_2) \otimes \xi_1 + (v_2 - Tu_2) \otimes \xi_2$, we have $M_x u_2 = v_2$, which means that $M_x \in \mathcal{M}_{u_2,v_2}$ and that $M_x x = Tx$. This proves that $\mathcal{M}_{u_1,v_1} \cup \mathcal{M}_{u_2,v_2}$ is transitive. Since $\mathcal{M}_{u_1,v_1} \cup \mathcal{M}_{u_2,v_2} \subseteq \mathcal{M}$, we conclude that $\mathcal{M}$ is transitive, as well. □

Corollary 3.4 shows that local linear dependence can be an obstacle for reflexivity of the union of reflexive sets. In the following theorem, we avoid this by a condition on the images of the involved sets of operators.

Theorem 3.6. *Let $n \ge 2$ and let $\mathcal{M}_j \subseteq \mathcal{B}(\mathscr{X})$ $(1 \le j \le n)$ be such that $0 \in \mathcal{M}_j$ for every j and $\operatorname{Im}(\mathcal{M}_j) \cap \operatorname{Im}(\mathcal{M}_k) = \{0\}$ if $j \neq k$. Then $\operatorname{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n) = \operatorname{Ref}(\mathcal{M}_1) \cup \cdots \cup \operatorname{Ref}(\mathcal{M}_n)$; in particular, if every $\mathcal{M}_j$ $(1 \le j \le n)$ is reflexive, then the union $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n$ is reflexive.*

Proof. It is obvious that $\operatorname{Ref}(\mathcal{M}_1) \cup \cdots \cup \operatorname{Ref}(\mathcal{M}_n) \subseteq \operatorname{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$. Assume that $T \in \operatorname{Ref}(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$. For every $x \in \mathscr{X}$, we have $Tx \in \overline{(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)x} \subseteq \overline{\mathcal{M}_1 x} \cup \cdots \cup \overline{\mathcal{M}_n x}$. Hence, if we define $\mathscr{Y}_j = \{x \in \mathscr{X}; Tx \in \overline{\mathcal{M}_j x}\}$ $(1 \le j \le n)$, then $\mathscr{X} = \mathscr{Y}_1 \cup \cdots \cup \mathscr{Y}_n$. We claim that each $\mathscr{Y}_j$ is a linear subspace of $\mathscr{X}$. Indeed, let $x_1, x_2 \in \mathscr{Y}_j$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ be arbitrary. Since $0 \in \mathcal{M}_j$, we have $\alpha_1 x_1 + \alpha_2 x_2 \in \mathscr{Y}_j$ if $T(\alpha_1 x_1 + \alpha_2 x_2) = 0$. Suppose that $T(\alpha_1 x_1 + \alpha_2 x_2) \neq 0$. We have $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T x_1 + \alpha_2 T x_2 \in \alpha_1 \overline{\mathcal{M}_j x_1} + \alpha_2 \overline{\mathcal{M}_j x_2} \subseteq \operatorname{Im}(\mathcal{M}_j)$ and $T(\alpha_1 x_1 + \alpha_2 x_2) \in \overline{\mathcal{M}_1(\alpha_1 x_1 + \alpha_2 x_2)} \cup \cdots \cup \overline{\mathcal{M}_n(\alpha_1 x_1 + \alpha_2 x_2)}$. Because of $\operatorname{Im}(\mathcal{M}_j) \cap \operatorname{Im}(\mathcal{M}_k) = \{0\}$ when $j \neq k$, we may conclude that $T(\alpha_1 x_1 + \alpha_2 x_2) \in \overline{\mathcal{M}_j(\alpha_1 x_1 + \alpha_2 x_2)}$; that is, $\alpha_1 x_1 + \alpha_2 x_2 \in \mathscr{Y}_j$. Now it follows from $\mathscr{X} = \mathscr{Y}_1 \cup \cdots \cup \mathscr{Y}_n$ that $\mathscr{X} = \mathscr{Y}_j$ for at least one index j, by Lemma 2.1. Hence, $T \in \operatorname{Ref}(\mathcal{M}_j)$. □

Corollary 3.7. *Let $\mathcal{M}$ be a set of operators which contains operator 0, let A be a nonzero operator such that $\operatorname{Im}(\mathcal{M}) \cap \operatorname{im}(A) = \{0\}$, and let $\Lambda \subseteq \mathbb{C}$ be a nonempty closed set. Then $\operatorname{Ref}(\mathcal{M} \cup \Lambda A) = \operatorname{Ref}(\mathcal{M}) \cup \Lambda A$.*

Proof. Denote $\Lambda_0 = \Lambda \cup \{0\}$. Since $0 \in \mathcal{M}$, we have $\mathcal{M} \cup \Lambda A = \mathcal{M} \cup \Lambda_0 A$. Hence, by Theorem 3.6 and Proposition 2.5, $\operatorname{Ref}(\mathcal{M} \cup \Lambda A) = \operatorname{Ref}(\mathcal{M}) \cup \Lambda_0 A = \operatorname{Ref}(\mathcal{M}) \cup \Lambda A$. □

In Corollary 3.7, we can replace set ΛA by any reflexive set $\mathcal{N} \subseteq \mathcal{B}(\mathscr{X})$ such that $\operatorname{Im}(\mathcal{M}) \cap \operatorname{Im}(\mathcal{N}) = \{0\}$ and $\mathcal{N} \cup \{0\}$ is reflexive. In the following example we describe a family of reflexive sets $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ such that $\mathcal{M} \cup \{0\}$ is reflexive, as well.

Example 3.8. Let $A, B \in \mathcal{B}(\mathscr{X})$ be linearly independent, A injective, and let $\mathcal{M} = \mathbb{C}A + B$. By Propositions 2.5 and 2.12, $\mathcal{M}$ is reflexive. It is obvious that $0 \notin \mathcal{M}$. Denote by $\mathscr{K}$ the closed linear span of all kernels $\ker(\alpha A + B)$ $(\alpha \in \mathbb{C})$.

Claim: *If $\mathscr{K}$ is a proper subspace of $\mathscr{X}$, then $\mathcal{M} \cup \{0\}$ is reflexive.*

To prove the claim, assume that $T \in \operatorname{Ref}(\mathcal{M} \cup \{0\})$ and $T \neq 0$. Since $\ker(T) \neq \mathscr{X}$ we have that $\mathscr{W}_T = \mathscr{X} \setminus (\mathscr{K} \cup \ker(T))$ is an open dense subset of $\mathscr{X}$. For $x \in \mathscr{W}_T$, we have $Tx \neq 0$ and therefore it follows from $Tx \in \overline{(\mathcal{M} \cup \{0\})x} = \mathcal{M}x \cup \{0\}$ that there exists $\lambda_x \in \mathbb{C}$ such that $Tx = (\lambda_x A + B)x$. Let $x \in \mathscr{W}_T$ and $\alpha \neq 0$ be arbitrary. Then $\alpha x \in \mathscr{W}_T$ and therefore $T(\alpha x) = (\lambda_{\alpha x} A + B)(\alpha x)$, which gives $Tx = (\lambda_{\alpha x} A + B)x$. Since $Ax \neq 0$, we conclude that $\lambda_{\alpha x} = \lambda_x$. Assume now that $x, y \in \mathscr{W}_T$ are linearly independent. By injectivity of A, vectors Ax and Ay are linearly independent, too. It is obvious that there exists number $\omega \neq 0$ such that $x + \omega y \in \mathscr{W}_T$. It follows from $Tx = (\lambda_x A + B)x$, $Ty = (\lambda_y A + B)y$, and $T(x + \omega y) = (\lambda_{x+\omega y} A + B)(x + \omega y)$ that $\lambda_{x+\omega y} - \lambda_x)Ax + \omega(\lambda_{x+\omega y} - \lambda_y)Ay = 0$ which gives $\lambda_x = \lambda_{x+\omega y} = \lambda_y$. We have seen that there exists $\lambda \in \mathbb{C}$ such that $Tx = (\lambda A + B)x$ for every $x \in \mathscr{W}_T$. Now let $x \in \mathscr{K} \cup \ker(T)$. Since $\mathscr{W}_T$ is dense in $\mathscr{X}$ there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathscr{K} \cup \ker(T)$ which converges to x. It follows from $\|Tx - (\lambda A + B)x\| \leq \|T(x - x_n)\| + \|(\lambda A + B)(x_n - x)\|$ that $Tx = (\lambda A + B)x$. We conclude that $T = \lambda A + B$. Thus, $\mathcal{M} \cup \{0\}$ is reflexive.

For the next two results we need the following lemma.

Lemma 3.9. *Let $S, T \in \mathcal{B}(\mathscr{X})$. Assume that $x_1, \ldots, x_n \in \mathscr{X}$ $(n \geq 2)$ are such that $Sx_1, \ldots, Sx_n$ are linearly independent. Let $\Lambda \subseteq \mathbb{C}$ be a nonempty set, and let $\Omega = \{\omega \in \mathbb{C}; \exists \lambda_\omega \in \Lambda : T(x_1 + \omega x_2 + \cdots + \omega^{n-1} x_n) = \lambda_\omega S(x_1 + \omega x_2 + \cdots + \omega^{n-1} x_n)\}$. If the cardinality of Ω is at least $n+1$, then there exists $\lambda \in \Lambda$ such that $T(x_1 + \omega x_2 + \cdots + \omega^{n-1} x_n) = \lambda S(x_1 + \omega x_2 + \cdots + \omega^{n-1} x_n)$ for all $\omega \in \Omega$.*

Proof. Assume that $\omega_0, \omega_1, \ldots, \omega_n$ are distinct numbers in Ω. By the assumption,

$$\sum_{k=1}^{n} \omega_j^{k-1} T x_k = \lambda_\omega \sum_{k=1}^{n} \omega_j^{k-1} S x_k \quad \text{for some } \lambda_\omega \in \Lambda (j = 0, 1, \ldots, n). \tag{3.2}$$

Denote

$$V_0 = \begin{bmatrix} 1 & \omega_0 & \cdots & \omega_0^{n-1} \\ 1 & \omega_2 & \cdots & \omega_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n & \cdots & \omega_n^{n-1} \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} 1 & \omega_1 & \cdots & \omega_1^{n-1} \\ 1 & \omega_2 & \cdots & \omega_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n & \cdots & \omega_n^{n-1} \end{bmatrix}.$$

Let $D_0 = \operatorname{diag}[\lambda_0, \lambda_2, \ldots, \lambda_n]$ and $D_1 = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ be diagonal matrices. Note that the Vandermonde matrices V_0 and V_1 are invertible since $\omega_0, \omega_1, \ldots, \omega_n$ are distinct numbers. For an $n \times n$ matrix A and an operator $B \in \mathcal{B}(\mathscr{X})$, we have linear operator $A \otimes B$ on $\mathbb{C}^n \otimes \mathscr{X}$ which is given by

$$(A \otimes B)(v_1 \otimes z_1 + \cdots + v_k \otimes z_k) = Av_1 \otimes Bz_1 + \cdots + Av_k \otimes Bz_k,$$

where $v_1, \ldots, v_k \in \mathbb{C}^n$ and $z_1, \ldots, z_n \in \mathscr{X}$ are arbitrary. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. Using (3.2) we can write

$$(V_i \otimes I)(e_1 \otimes Tx_1 + \cdots + e_n \otimes Tx_n) = (D_i V_i \otimes I)(e_1 \otimes Sx_1 + \cdots + e_n \otimes Sx_n),$$

for $i = 0, 1$, which gives

$$\big((V_0^{-1} D_0 V_0 - V_1^{-1} D_1 V_1) \otimes I\big)(e_1 \otimes Sx_1 + \cdots + e_n \otimes Sx_n) = 0,$$

that is,

$$(V_0^{-1} D_0 V_0 - V_1^{-1} D_1 V_1) e_1 \otimes Sx_1 + \cdots + (V_0^{-1} D_0 V_0 - V_1^{-1} D_1 V_1) e_n \otimes Sx_n = 0.$$

Since $Sx_1, \ldots, Sx_n$ are linearly independent, we have

$$(V_0^{-1} D_0 V_0 - V_1^{-1} D_1 V_1) e_k = 0 \quad \text{for all } k = 1, \ldots, n.$$

Hence $V_0^{-1} D_0 V_0 - V_1^{-1} D_1 V_1 = 0$ (i.e., D_0 and D_1 are similar). It follows that these diagonal matrices have the same eigenvalues. We conclude that $\lambda_1 = \lambda_0$ and therefore that all λ_ω's are equal. $\square$

Theorem 3.10. *Let $M_1, \ldots, M_n \in \mathcal{B}(\mathscr{X})$ $(n \geq 2)$ be such that $\overline{\operatorname{im}(M_j)} \cap \overline{\operatorname{im}(M_k)} = \{0\}$ if $j \neq k$. Let $\Lambda_j \subseteq \mathbb{C}$ $(j = 1, \ldots, n)$ be nonempty closed sets. Then $\mathcal{M} = \Lambda_1 M_1 \cup \cdots \cup \Lambda_n M_n$ is a reflexive set of operators.*

Proof. Let $T \in \operatorname{Ref}(\mathcal{M})$. Since, for every $x \in \mathscr{X}$, the orbit $\mathcal{M}x$ is a closed subset of $\mathscr{X}$, we have

$$Tx \in \Lambda_1 M_1 x \cup \cdots \cup \Lambda_n M_n x \quad (x \in \mathscr{X}). \tag{3.3}$$

Assume that $T = 0$. If $0 \in \Lambda_j$ for some $j \in \{1, \ldots, n\}$, then $T = 0 = 0 \cdot M_j \in \mathcal{M}$. Suppose that $0 \notin \Lambda_j$ for every $j \in \{1, \ldots, n\}$. Then (3.3) gives $0 \in \Lambda_1 M_1 x \cup \cdots \cup \Lambda_n M_n x$ for every $x \in \mathscr{X}$, and we conclude that for each $x \in \mathscr{X}$ there is $j_x \in \{1, \ldots, n\}$ such that $M_{j_x} x = 0$. Hence $\ker(M_1) \cup \cdots \cup \ker(M_n) = \mathscr{X}$. By Lemma 2.1, there exists $k \in \{1, \ldots, n\}$ such that $\ker(M_k) = \mathscr{X}$ (i.e., $M_k = 0$). We conclude that $T = 0 = M_k \in \mathcal{M}$.

Assume now that $T \neq 0$, which means that $\overline{\mathrm{im}(T)} \neq \{0\}$. It follows from (3.3) that $\overline{\mathrm{im}(T)} \subseteq \overline{\mathrm{im}(M_1)} \cup \cdots \cup \overline{\mathrm{im}(M_n)}$. Let $\mathscr{Y}_j = \overline{\mathrm{im}(T)} \cap \overline{\mathrm{im}(M_j)}$ for $j = 1, \ldots, n$. Each $\mathscr{Y}_j$ is a closed subspace of $\overline{\mathrm{im}(T)}$. It is clear that $\overline{\mathrm{im}(T)} = \mathscr{Y}_1 \cup \cdots \cup \mathscr{Y}_n$ and that $\mathscr{Y}_j \cap \mathscr{Y}_k = \{0\}$ if $j \neq k$. Again we use Lemma 2.1 to conclude that there exists $k \in \{1, \ldots, n\}$ such that $\overline{\mathrm{im}(T)} = \mathscr{Y}_k \subseteq \overline{\mathrm{im}(M_k)}$, which also gives $\mathscr{Y}_j = \{0\}$ if $j \neq k$.

First we consider the case $\dim(\mathrm{im}(T)) = 1$ (i.e., T is a rank 1 operator). Let $e \in \mathscr{X}$ be such that $Te \neq 0$ and let $\xi \in \mathscr{X}^*$ be such that $\langle e, \xi\rangle = 1$. Then $T = Te \otimes \xi$. Since $Te \neq 0$, there exists a nonzero $\lambda_e \in \Lambda_k$ such that $Te = \lambda_e M_k e$; note that $M_k e$ is also nonzero. An arbitrary vector $x \in \mathscr{X}$ can be written as $x = \alpha e + y$ where $\alpha \in \mathbb{C}$ and $y \in \ker(T)(= \ker(\xi))$. Assume that $\alpha \neq 0$. Then $Tx = \alpha Te \neq 0$, which means that there exists a nonzero $\lambda_x \in \Lambda_k$ such that $Tx = \lambda_x M_k x$. This and $Te = \lambda_e M_k e$ together give $M_k y = \alpha \frac{\lambda_e - \lambda_x}{\lambda_x} M_k e$, which implies that $M_k x = \alpha \frac{\lambda_e}{\lambda_x} M_k e = \frac{\alpha}{\lambda_x} Te$. Since x is an arbitrary vector, we conclude that M_k is a rank 1 operator whose image is spanned by Te. Let $0 \neq \eta \in \mathscr{X}^*$ be such that $M_k = Te \otimes \eta$. If ξ and η were linearly independent, then there would exist $f \in \mathscr{X}$ such that $\langle f, \xi\rangle \neq 0$ and $\langle f, \eta\rangle = 0$. It would then follow that $Tf = \langle f, \xi\rangle Te \neq 0$ and therefore $Tf = \lambda_f M_k f$ for some $\lambda_f \in \Lambda_k$. However, this is impossible since $M_k f = 0$. We conclude that ξ and η are linearly dependent. Let $\mu \in \mathbb{C}$ be such that $\xi = \mu\eta$. It follows from $Te = \lambda_e M_k e$ that $\mu = \lambda_e \in \Lambda_k$. Hence $T \in \Lambda_k M_k$.

Now we consider the case $\dim(\mathrm{im}(T)) \geq 2$. Let $e_1, e_2 \in \mathscr{X}$ be such that Te_1 and Te_2 are linearly independent. Then there exist nonzero $\lambda_1, \lambda_2 \in \Lambda_k$ such that $Te_1 = \lambda_1 M_k e_1$ and $Te_2 = \lambda_2 M_k e_2$. It follows that $M_k e_1$ and $M_k e_2$ are linearly independent, as well. Since Te_1 and Te_2 are linearly independent, $T(e_1 + \omega e_2) \neq 0$ for every $\omega \in \mathbb{C}$, which means that there exists $\lambda_\omega \in \Lambda_k$ such that $T(e_1 + \omega e_2) = \lambda_\omega M_k(e_1 + \omega e_2)$. By Lemma 3.9, there exists $\lambda \in \Lambda_k$ such that $\lambda_\omega = \lambda$ for every $\omega \in \mathbb{C}$. In particular, $\lambda = \lambda_1 = \lambda_2 \neq 0$. Let $y \in \ker(T)$ be arbitrary. Then e_1 and $e_2 + y$ are such that Te_1 and $T(e_2 + y)$ are linearly independent. As before, we see that there exists $\mu \in \Lambda_k$ such that $T(e_1 + \omega(e_2 + y)) = \mu M_k(e_1 + \omega(e_2 + y))$ for any $\omega \in \mathbb{C}$. Since we already have $Te_1 = \lambda M_k e_1$, it follows that $\mu = \lambda$. If $\omega = 1$ in $T(e_1 + \omega(e_2 + y)) = \lambda M_k(e_1 + \omega(e_2 + y))$, then we get $0 = Ty = \lambda M_k y$. Hence $\ker(T) \subseteq \ker(M_k)$. Note that the opposite inclusion holds as well because $Tx \neq 0$, for a vector $x \in \mathscr{X}$, implies that $Tx = \lambda_x M_k x$ for some $\lambda_x \in \Lambda_k$. Thus, $\ker(T) = \ker(M_k)$. Now let $x \in \mathscr{X}$ be arbitrary. If it is a linear combination of Te_1 and Te_2 (say, $Tx = \alpha_1 Te_1 + \alpha_2 Te_2$), then $T(x - \alpha_1 e_1 - \alpha_2 e_2) = 0$ and therefore $M_k(x - \alpha_1 e_2 - \alpha_2 e_2) = 0$. It follows that $Tx = \lambda\alpha_1 M_k e_1 + \lambda\alpha_2 M_k e_2 + \lambda M_k(x - \alpha_1 e_1 - \alpha_2 e_2) = \lambda M_k x$. In the case when Te_1, Te_2 and Tx are linearly independent, vectors $M_k e_1$, $M_k e_2$ and $M_k x$ are linearly independent as well. Since $T(e_1 + \omega e_2 + \omega^2 x) \neq 0$ for every $\omega \in \mathbb{C}$, we thus have $T(e_1 + \omega e_2 + \omega^2 x) = \lambda_\omega M_k(e_1 + \omega e_2 + \omega^2 x)$ for some $\lambda_\omega \in \Lambda_k$. By Lemma 3.9, there exists $\mu \in \Lambda_k$ such that $T(e_1 + \omega e_2 + \omega^2 x) = \mu M_k(e_1 + \omega e_2 + \omega^2 x)$ for any $\omega \in \mathbb{C}$. Using $Te_1 = \lambda M_k e_1$, we conclude that $\mu = \lambda$. We have shown that $Tx = \lambda M_k x$ for any $x \in \mathscr{X}$. Hence $T \in \Lambda_k M_k$. $\square$

If $n=2$, then Theorem 3.10 is a particular case of Corollary 3.7. However, for $n\geq 3$, the assertion of Theorem 3.10 cannot be deduced from Theorem 3.6. In the next theorem we replace conditions on images by injectivity of the involved operators to prove the reflexivity of the union.

Theorem 3.11. *Let $M_1,\dots,M_n\in\mathcal{B}(\mathscr{X})$ $(n\geq 2)$ be linearly independent injective operators, and let $\Lambda_1,\dots,\Lambda_n\subseteq\mathbb{C}$ be nonempty closed sets. Then $\mathcal{M}=\Lambda_1M_1\cup\dots\cup\Lambda_nM_n$ is a reflexive set.*

Proof. Note first that, for every $x\in\mathscr{X}$, the orbit $\mathcal{M}x$ is closed, and it is the union of closed sets Λ_jM_jx $(1\leq j\leq n)$. Hence, if $T\in\mathrm{Ref}(\mathcal{M})$, then for every $x\in\mathscr{X}$ there exists an index j_x such that $Tx\in\Lambda_{j_x}M_{j_x}x$. If there exists an index $j\in\{1,\dots,n\}$ such that $Tx\in\Lambda_jM_jx$ for every $x\in\mathscr{X}$, then $T\in\Lambda_jM_j$, by Proposition 2.5. Towards a contradiction, assume that this is not the case. Suppose that there exist indices $j_1,\dots,j_m\in\{1,\dots,n\}$ $(2\leq m\leq n)$ such that $Tx\in\Lambda_{j_1}M_{j_1}x\cup\dots\cup\Lambda_{j_m}M_{j_m}x$ for every $x\in\mathscr{X}$, and suppose that for every j_i there exists a vector $e_{j_i}\in\mathscr{X}$ such that $Te_{j_i}\in\Lambda_{j_i}M_{j_i}e_{j_i}$ and $Te_{j_i}\notin\Lambda_{j_l}M_{j_l}e_{j_i}$ if $i\neq l$. Without loss of generality, we may assume that $j_i=i$.

It is obvious that $e_1,\dots,e_m$ are nonzero vectors. Let d be the dimension of $\vee\{e_1,\dots,e_m\}$. Without loss of generality, we may assume that this linear space is spanned by $e_1,\dots,e_d$. For every index $j\in\{1,\dots,n\}$ the vectors $M_je_1,\dots,M_je_d$ are linearly independent because M_j is an injective operator. If $d=1$, then $e_i=\alpha_ie_1$ for every $2\leq i\leq m$, where $\alpha_i\in\mathbb{C}$. This would give a contradiction $Te_i\in\Lambda_1M_1e_i$. Hence $2\leq d\leq m$.

For every $\omega\in\mathbb{C}$, let $x_\omega=e_1+\omega e_2+\dots+\omega^{d-1}e_d$ and let $\Omega_j=\{\omega\in\mathbb{C};Tx_\omega\in\Lambda_jM_jx_\omega\}$ for $1\leq j\leq m$. It is clear that $\Omega_1\cup\dots\cup\Omega_m=\mathbb{C}$. Hence, there exists an index $k\in\{1,\dots,m\}$ such that Ω_k is an infinite set. By Lemma 3.9, there exists $\lambda\in\Lambda_k$ such that $Tx_\omega=\lambda M_kx_\omega$ for every $\omega\in\Omega_k$. Let $\omega_1,\dots,\omega_d$ be distinct nonzero numbers from Ω_k. It follows from

$$\begin{aligned}&Te_1+\omega_iTe_2+\dots+\omega_i^{d-1}Te_d\\&\quad=\lambda M_ke_1+\lambda\omega_iM_ke_2+\dots+\lambda\omega_i^{d-1}M_ke_d\quad(1\leq i\leq d)\end{aligned}$$

that $Te_i=\lambda M_ke_i$ for every $i\in\{1,\dots,d\}$. Since $d\geq 2$, there exists $l\in\{1,\dots,d\}$ such that $l\neq k$, and we have a contradiction $Te_l\in\Lambda_kM_ke_l$. □

We close this section with an example in which we apply Theorems 3.10 and 3.11.

Example 3.12. Let $\mathcal{A}$ be a complex Banach algebra. For $a\in\mathcal{A}$, let $L_a:\mathcal{A}\to\mathcal{A}$ be the left multiplication by a on $\mathcal{A}$; that is, L_a is a bounded linear operator on $\mathcal{A}$ given by $L_ax=ax$ $(x\in\mathcal{A})$. It is clear that the closure of the image of L_a is $\mathcal{R}_a=\overline{a\mathcal{A}}$, the closed right ideal in $\mathcal{A}$ generated by a. Operator L_a is injective if and only if a is not a left zero-divisor (i.e., $ax\neq 0$ for every $0\neq x\in\mathcal{A}$).

Assume that $a_1,\dots,a_n\in\mathcal{A}$ is an n-tuple of elements such that either every a_j $(1\leq j\leq n)$ is not a left zero-divisor or $\mathcal{R}_{a_i}\cap\mathcal{R}_{a_j}=\{0\}$ if $i\neq j$. If $T\in\mathcal{B}(\mathcal{A})$ is such that for every $x\in\mathcal{A}$ there exist $\lambda_x\in\mathbb{C}$ and an index j_x satisfying $Tx=\lambda_xa_{j_x}x$, then, by Theorems 3.10 and 3.11, there exist $\lambda\in\mathbb{C}$ and an index j such that $T=\lambda L_{a_j}$.

4. Reflexivity of convex sets

In this section we study reflexivity of convex sets of operators. For a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$, we denote by $\operatorname{conv}(\mathcal{M})$ its convex hull.

Proposition 4.1. *Let* $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ *be nonempty. Then* $\operatorname{conv}(\operatorname{Ref}(\mathcal{M})) \subseteq \operatorname{Ref}(\operatorname{conv}(\mathcal{M}))$. *If* $\mathcal{M}$ *is convex, then* $\operatorname{Ref}(\mathcal{M})$ *is convex.*

Proof. We prove the second part of the proposition first. Assume that $\mathcal{M}$ is convex. Then $\overline{\mathcal{M}x}$ is convex for every $x \in \mathscr{X}$. Let $T_1, T_2 \in \operatorname{Ref}(\mathcal{M})$ be arbitrary and let $t \in [0,1]$. Then $(tT_1 + (1-t)T_2)x = t(T_1x) + (1-t)(T_2x) \in \overline{\mathcal{M}x}$, which means that $tT_1 + (1-t)T_2 \in \operatorname{Ref}(\mathcal{M})$. Now let $\mathcal{M}$ be arbitrary. Since $\mathcal{M} \subseteq \operatorname{conv}(\mathcal{M})$, we have $\operatorname{Ref}(\mathcal{M}) \subseteq \operatorname{Ref}(\operatorname{conv}(\mathcal{M}))$. By the first part of the proof, $\operatorname{Ref}(\operatorname{conv}(\mathcal{M}))$ is convex and therefore $\operatorname{conv}(\operatorname{Ref}(\mathcal{M})) \subseteq \operatorname{Ref}(\operatorname{conv}(\mathcal{M}))$. □

The next corollary gives a necessary condition for the convex hull of $\mathcal{M}$ to be reflexive.

Corollary 4.2. *Let* $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ *be a nonempty set. If* $\operatorname{conv}(\mathcal{M})$ *is reflexive, then* $\operatorname{conv}(\operatorname{Ref}(\mathcal{M})) = \operatorname{conv}(\mathcal{M})$.

Proof. Since $\operatorname{conv}(\mathcal{M})$ is reflexive we have $\operatorname{conv}(\operatorname{Ref}(\mathcal{M})) \subseteq \operatorname{conv}(\mathcal{M})$, by Proposition 4.1. On the other hand, the inclusion $\mathcal{M} \subseteq \operatorname{Ref}(\mathcal{M})$ implies $\operatorname{conv}(\mathcal{M}) \subseteq \operatorname{conv}(\operatorname{Ref}(\mathcal{M}))$. □

It follows from Corollary 4.2 that $\operatorname{conv}(\mathcal{M})$ is not reflexive if $\operatorname{Ref}(\mathcal{M}) \not\subseteq \operatorname{conv}(\mathcal{M})$. The next example shows that condition $\operatorname{Ref}(\mathcal{M}) \subseteq \operatorname{conv}(\mathcal{M})$ is not sufficient for reflexivity of $\operatorname{conv}(\mathcal{M})$ even if $\mathcal{M}$ is a reflexive set.

Example 4.3. Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be a nonreflexive 2-dimensional linear space spanned by injective operators M_1 and M_2. For instance, space $\mathcal{S}$ in Example 2.7 has this property if $\dim(\mathscr{X}) = 2$. By Theorem 3.11, $\mathcal{M} = \{\lambda M_1; \lambda \in \mathbb{C}\} \cup \{\mu M_2; \mu \in \mathbb{C}\}$ is a reflexive set. On the other hand, its convex hull is $\operatorname{conv}(\mathcal{M}) = \mathcal{S}$, a nonreflexive set.

Let Φ be a bounded linear functional on $\mathcal{B}(\mathscr{X})$. Then $\mathrm{P}(T) = \frac{1}{2}(\Phi(T) + \overline{\Phi(T)})$, where $T \in \mathcal{B}(\mathscr{X})$ is arbitrary, defines a bounded $\mathbb{R}$-linear functional, and one has $\Phi(T) = \mathrm{P}(T) - i\mathrm{P}(iT)$ for every $T \in \mathcal{B}(\mathscr{X})$. Hence, P is the real part of Φ, and $\mathrm{Q}(T) = -\mathrm{P}(iT)$ $(T \in \mathcal{B}(\mathscr{X}))$ is the imaginary part of Φ. On the other hand, if P is a bounded $\mathbb{R}$-linear functional on $\mathcal{B}(\mathscr{X})$, then the above equality defines a bounded ($\mathbb{C}$-)linear functional. Every bounded finite-rank operator $F = e_1 \otimes \xi_1 + \cdots + e_k \otimes \xi_k$ on $\mathscr{X}$ defines by

$$\Phi_F(T) = \langle Te_1, \xi_1\rangle + \cdots + \langle Te_k, \xi_k\rangle \quad \big(T \in \mathcal{B}(\mathscr{X})\big) \tag{4.1}$$

a bounded linear functional on $\mathcal{B}(\mathscr{X})$. Its real part is $\mathrm{P}_F(T) = \operatorname{Re}(\Phi_F(T))$ $(T \in \mathcal{B}(\mathscr{X}))$.

For a nonempty closed set $G \subseteq \mathbb{C}$ and $\Phi \in \mathcal{B}(\mathscr{X})^*$, let $\mathcal{M}_\Phi(G) = \{M \in \mathcal{B}(\mathscr{X}); \Phi(M) \in G\}$.

Proposition 4.4. *Let* $e \in \mathscr{X}$ *and* $\xi \in \mathscr{X}^*$ *be nonzero and let* $G \subseteq \mathbb{C}$ *be a nonempty closed set. Then* $\mathcal{M}_{\Phi_{e\otimes\xi}}(G)$ *is a reflexive set. In particular,* $\ker(\Phi_{e\otimes\xi})$ *and* $\ker(\mathrm{P}_{e\otimes\xi})$ *are reflexive.*

Proof. Let $T \in \mathrm{Ref}(\mathcal{M}_{\Phi_{e\otimes\xi}}(G))$. For every $\varepsilon > 0$, there exists $M_\varepsilon \in \mathcal{M}_{\Phi_{e\otimes\xi}}(G)$ such that $\|Te - M_\varepsilon e\| < \varepsilon$. Hence $|\Phi_{e\otimes\xi}(T) - \Phi_{e\otimes\xi}(M_\varepsilon)| = |\langle Te - M_\varepsilon e, \xi\rangle| \leq \|Te - M_\varepsilon e\|\|\xi\| < \varepsilon\|\xi\|$. Since $\Phi_{e\otimes\xi}(M_\varepsilon) \in G$ and G is closed, we conclude that $\Phi_{e\otimes\xi}(T) \in G$; that is, $T \in \mathcal{M}_{\Phi_{e\otimes\xi}}(G)$. For the last part, observe that $\ker(\Phi_{e\otimes\xi}) = \mathcal{M}_{\Phi_{e\otimes\xi}}(\{0\})$ and that $\ker(\mathrm{P}_{e\otimes\xi}) = \mathcal{M}_{\Phi_{e\otimes\xi}}(i\mathbb{R})$. □

Recall that a closed subspace of $\mathcal{B}(\mathscr{X})$ has (complex) codimension 1 if and only if it is the kernel of a functional in $\mathcal{B}(\mathscr{X})^*$. Similarly, a closed subspace of $\mathcal{B}(\mathscr{X})$ has real codimension 1 if and only if it is the kernel of the real part of a functional in $\mathcal{B}(\mathscr{X})^*$.

Proposition 4.5. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a closed subspace of complex codimension 1 and let $\Phi \in \mathcal{B}(\mathscr{X})^*$. Then $\mathcal{M}$ is reflexive if and only if there exist $0 \neq e \in \mathscr{X}$ and $0 \neq \xi \in \mathscr{X}^*$ such that $\mathcal{M} = \ker(\Phi_{e\otimes\xi})$. If Φ is not of the form $\Phi_{e\otimes\xi}$ for some rank 1 operator $e \otimes \xi$, then $\ker(\Phi)$ is topologically transitive.*

Proof. Let $\Phi \in \mathcal{B}(\mathscr{X})^*$ be such that $\mathcal{M} = \ker(\Phi)$. If $\Phi = \Phi_{e\otimes\xi}$ for some nonzero $e \in \mathscr{X}$ and $\xi \in \mathscr{X}^*$, then $\mathcal{M}$ is reflexive, by Proposition 4.4. To prove the opposite implication, assume that $\mathcal{M}$ is reflexive. For every $x \in \mathscr{X}$, the orbit $\overline{\mathcal{M}x}$ is a closed subspace of $\mathscr{X}$. If for every $x \neq 0$ the orbit $\overline{\mathcal{M}x}$ were equal to $\mathscr{X}$, then we would have $Tx \in \overline{\mathcal{M}x}$ for every $T \in \mathcal{B}(\mathscr{X})$ and every $x \in \mathscr{X}$ and therefore $\mathcal{M}$ would not be reflexive but topologically transitive. Hence, there exists $0 \neq e \in \mathscr{X}$ such that $\overline{\mathcal{M}e} \neq \mathscr{X}$. It follows that there exists $0 \neq \xi \in \mathscr{X}^*$ such that $\overline{\mathcal{M}e} \subseteq \ker(\xi)$. This gives that $\Phi_{e\otimes\xi}$ is nonzero and $\Phi_{e\otimes\xi}(M) = \langle Me, \xi\rangle = 0$ for every $M \in \mathcal{M}$, that is, $\mathcal{M} \subseteq \ker(\Phi_{e\otimes\xi})$. Since $\mathcal{M}$ is of complex codimension 1 the last inclusion has to be an equality.

If Φ is not of the form $\Phi_{e\otimes\xi}$ for some rank 1 operator $e \otimes \xi$, then $\ker(\Phi)$ is not reflexive, by the first part of this proposition. Hence, $\mathrm{Ref}(\ker(\Phi))$ is a closed subspace of $\mathcal{B}(\mathscr{X})$ which contains $\ker(\Phi)$ as a proper subspace. This is possible only if $\mathrm{Ref}(\ker(\Phi)) = \mathcal{B}(\mathscr{X})$. □

Corollary 4.6. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a closed real subspace of (real) codimension 1. Then $\mathcal{M}$ is reflexive if and only if there exist $0 \neq e \in \mathscr{X}$ and $0 \neq \xi \in \mathscr{X}^*$ such that $\mathcal{M} = \ker(\mathrm{P}_{e\otimes\xi})$. If P is not of the form $\mathrm{P}_{e\otimes\xi}$ for some rank 1 operator $e \otimes \xi$, then $\ker(\mathrm{P})$ is topologically transitive.*

Proof. Let $\mathcal{M} = \ker(\mathrm{P})$, where P is a bounded $\mathbb{R}$-linear functional on $\mathcal{B}(\mathscr{X})$. Let Q be defined by $\mathrm{Q}(T) = -\mathrm{P}(iT)$ $(T \in \mathcal{B}(\mathscr{X}))$ and let $\Phi = \mathrm{P} + i\mathrm{Q}$. Hence, Q is $\mathbb{R}$-linear functional and $\Phi \in \mathcal{B}(\mathscr{X})^*$. If $\mathrm{P} = \mathrm{P}_{e\otimes\xi}$, then $\mathcal{M}$ is reflexive, by Proposition 4.4. Assume now that $\mathcal{M} = \ker(\mathrm{P})$ is reflexive. Since $T \in \ker(\mathrm{P})$ if and only if $iT \in \ker(\mathrm{Q})$, it is not hard to see that $\ker(\mathrm{Q})$ is reflexive as well. It follows that $\ker(\Phi) = \ker(\mathrm{P}) \cap \ker(\mathrm{Q})$ is reflexive. Hence, by Proposition 4.5, there exist $0 \neq e \in \mathscr{X}$ and $0 \neq \xi \in \mathscr{X}^*$ such that $\Phi = \Phi_{e\otimes\xi}$ and therefore $\mathrm{P} = \mathrm{P}_{e\otimes\xi}$.

If P is not of the form $\mathrm{P}_{e\otimes\xi}$, then Φ is not of the form $\Phi_{e\otimes\xi}$, which means that $\ker(\Phi)$ is topologically transitive. Since $\ker(\Phi) \subseteq \ker(\mathrm{P})$, we conclude that $\ker(\mathrm{P})$ is topologically transitive, as well. □

Let $\mathrm{P} \neq 0$ be a bounded $\mathbb{R}$-linear functional on $\mathcal{B}(\mathscr{X})$, and let $c \in \mathbb{R}$. We designate $\mathcal{H}_{\mathrm{P}}(c) = \{T \in \mathcal{B}(\mathscr{X}); \mathrm{P}(T) \geq c\}$ a closed half-space of $\mathcal{B}(\mathscr{X})$. Since

$\mathrm{P} \neq 0$, there exists $E_{\mathrm{P}} \in \mathcal{B}(\mathscr{X})$ such that $\mathrm{P}(E_{\mathrm{P}}) = 1$. It is easily seen that $\mathcal{H}_{\mathrm{P}}(c) = \mathcal{H}_{\mathrm{P}}(0) + cE_{\mathrm{P}}$.

Corollary 4.7. *Let* $\mathrm{P} \neq 0$ *be a bounded* $\mathbb{R}$*-linear functional on* $\mathcal{B}(\mathscr{X})$ *and let* $c \in \mathbb{R}$. *Half-space* $\mathcal{H}_{\mathrm{P}}(c)$ *is reflexive if and only if* $\mathrm{P} = \mathrm{P}_{e\otimes\xi}$ *for some* $0 \neq e \in \mathscr{X}$, $0 \neq \xi \in \mathscr{X}^*$. *If it is not reflexive, then it is topologically transitive.*

Proof. We may assume that $c = 0$. Let $\Phi \in \mathcal{B}(\mathscr{X})^*$ be such that P is its real part. Then $\mathscr{H}_{\mathrm{P}}(0) = \mathcal{M}_{\Phi}(\mathbb{C}_+)$, where $\mathbb{C}_+ = \{z \in \mathbb{C}; \mathrm{Re}(z) \geq 0\}$. If $\mathrm{P} = \mathrm{P}_{e\otimes\xi}$, for some $0 \neq e \in \mathscr{X}$, $0 \neq \xi \in \mathscr{X}^*$, then $\Phi = \Phi_{e\otimes\xi}$ and, by Proposition 4.4, $\mathcal{M}_{\Phi}(\mathbb{C}_+)$ is reflexive. On the other hand, if P is not of the form $\mathrm{P}_{e\otimes\xi}$, then $\ker(\mathrm{P})$ is topologically transitive, by Corollary 4.6. Since $\ker(\mathrm{P}) \subseteq \mathcal{H}_{\mathrm{P}}(0)$, we conclude that $\mathcal{H}_{\mathrm{P}}(0)$ is topologically transitive also. ☐

Recall that every closed convex set is the intersection of a family of closed half spaces.

Corollary 4.8. *If* $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ *is a closed convex set such that there exist an index set* $\mathbb{I}$ *and* $e_i \in \mathscr{X}$, $\xi_i \in \mathscr{X}^*$, $c_i \in \mathbb{R}$ $(i \in \mathbb{I})$ *such that* $\mathcal{M} = \bigcap_{i\in\mathbb{I}} \mathcal{H}_{\mathrm{P}_{e_i\otimes\xi_i}}(c_i)$, *then* $\mathcal{M}$ *is reflexive.*

Proof. By Corollary 4.7, every half space $\mathcal{H}_{\mathrm{P}_{e_i\otimes\xi_i}}(c_i)$ $(i \in \mathbb{I})$ is reflexive. Hence, by Proposition 3.1(i), $\mathcal{M}$ is reflexive. ☐

It is well known that $(\mathcal{B}(\mathscr{X}), \mathcal{F}(\mathscr{X}))$ is a dual pair in the sense of [1, Definition 5.90] if the pairing is given by $(T, F) \mapsto \mathrm{P}_F(T)$, where $T \in \mathcal{B}(\mathscr{X})$ and $F \in \mathcal{F}(\mathscr{X})$ are arbitrary. Recall that $F \in \mathcal{F}(\mathscr{X})$ defines $\Phi_F \in \mathcal{B}(\mathscr{X})^*$ (see (4.1)) and that P_F is the real part of Φ_F. The one-sided polar of a nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is

$$\mathcal{M}^\circ = \big\{F \in \mathcal{F}(\mathscr{X}); \mathrm{P}_F(T) \leq 1 \text{ for all } T \in \mathcal{M}\big\}.$$

Similarly, the one-sided polar of a nonempty set $\mathcal{N} \subseteq \mathcal{F}(\mathscr{X})$ is defined by

$$\mathcal{N}^\circ = \big\{T \in \mathcal{B}(\mathscr{X}); \mathrm{P}_F(T) \leq 1 \text{ for all } F \in \mathcal{N}\big\}.$$

Denote $G_1 = \{z \in \mathbb{C}; \mathrm{Re}(z) \leq 1\}$. It is obvious that,

$$\mathcal{N}^\circ = \bigcap_{F\in\mathcal{N}} \mathcal{M}_{\Phi_F}(G_1).$$

Hence, if $\mathcal{N} \subseteq \mathcal{F}_1(\mathscr{X})$, then, by Propositions 4.4 and 3.1(i), $\mathcal{N}^\circ$ is a reflexive subset of $\mathcal{B}(\mathscr{X})$.

Theorem 4.9. *If* $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ *is a convex set such that* $0 \in \mathcal{M}$, *then*

$$\mathrm{Ref}(\mathcal{M}) = \big(\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X})\big)^\circ.$$

Proof. Since $\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}) \subseteq \mathcal{M}^\circ$, we have that $\mathcal{M} \subseteq \mathcal{M}^{\circ\circ} \subseteq (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$. As we mentioned before this theorem, $(\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$ is reflexive and therefore $\mathrm{Ref}(\mathcal{M}) \subseteq (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$. To prove the opposite inclusion, assume that $S \in \mathcal{B}(\mathscr{X})$ is not in $\mathrm{Ref}(\mathcal{M})$. Then there exist $e \in \mathscr{X}$ and $\varepsilon > 0$ such that $\|Se - Me\| > \varepsilon$ for every $M \in \mathcal{M}$. It follows that $Se \notin \overline{\mathcal{M}e}$. Note that $Se \neq 0$ since $0 \in \mathcal{M}e$. By [1, Corollary 5.80], there exists a bounded $\mathbb{R}$-functional ρ on $\mathscr{X}$ and $a \in \mathbb{R}$

such that $\rho(x) \leq a$ for every $x \in \overline{\mathcal{M}e}$ and $\rho(Se) > a$. Without loss of generality, we may assume that $a = 1$. Let $\xi \in \mathscr{X}^*$ be given by $\langle x, \xi \rangle = \rho(x) - i\rho(ix)$. Then $\mathrm{P}_{e\otimes\xi}(T) = \mathrm{Re}(\langle Te, \xi \rangle) = \rho(Te)$ for every $T \in \mathcal{B}(\mathscr{X})$. Hence $\mathrm{P}_{e\otimes\xi}(M) \leq 1$ for every $M \in \mathcal{M}$, which means that $e \otimes \xi \in \mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X})$. On the other hand, $\mathrm{P}_{e\otimes\xi}(S) > 1$ and therefore $S \notin (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$. □

We omit a simple proof of the following corollary.

Corollary 4.10. *A nonempty convex set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is reflexive if and only if $\mathcal{M} - M = ((\mathcal{M} - M)^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$, for some (and therefore for every) $M \in \mathcal{M}$.*

Corollary 4.11. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be such that $0 \in \mathrm{conv}(\mathcal{M})$. Then $\mathrm{conv}(\mathcal{M})$ is reflexive if and only if*

$$\mathrm{conv}(\mathcal{M}) = \big(\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X})\big)^\circ.$$

Proof. Since $\mathcal{M} \subseteq \mathrm{conv}(\mathcal{M})$, we have $\mathcal{M}^\circ \supseteq \mathrm{conv}(\mathcal{M})^\circ$. On the other hand, if $F \in \mathcal{M}^\circ$ then $\mathrm{P}_F(t_1M_1 + \cdots + t_kM_k) \leq 1$ for arbitrary $M_1, \ldots, M_k \in \mathcal{M}$ and $t_1, \ldots, t_k \in [0, 1]$ such that $t_1 + \cdots + t_k = 1$. Hence $F \in \mathrm{conv}(\mathcal{M})^\circ$, and therefore $\mathcal{M}^\circ = \mathrm{conv}(\mathcal{M})^\circ$. It follows, by Theorem 4.9, that $\mathrm{Ref}(\mathrm{conv}(\mathcal{M})) = (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$. If $\mathrm{conv}(\mathcal{M})$ is reflexive, then the last equality implies that $\mathrm{conv}(\mathcal{M}) = (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$. On the other hand, if $\mathrm{conv}(\mathcal{M}) = (\mathcal{M}^\circ \cap \mathcal{F}_1(\mathscr{X}))^\circ$ holds, then $\mathrm{conv}(\mathcal{M})$ is reflexive because the polar of every subset of $\mathcal{F}_1(\mathscr{X})$ is reflexive. □

By Proposition 2.2, every finite set of operators is reflexive. We already know that the linear span of a reflexive set of operators is not necessarily reflexive. What about the convex hull of a finite set of operators? The answer to this question is trivial in the case of a single operator. For two operators $M_1, M_2 \in \mathcal{B}(\mathscr{X})$, we have $\mathrm{conv}(\{M_1, M_2\}) = \{tM_1 + (1-t)M_2; t \in [0, 1]\} = [0, 1](M_1 - M_2) + M_2$. Hence, by Propositions 2.5 and 2.12(i), $\mathrm{conv}(\{M_1, M_2\})$ is reflexive. Note that when we consider reflexivity of the convex hull of operators $M_0, M_1, \ldots, M_n \in \mathcal{B}(\mathscr{X})$, we may assume, by Proposition 2.12(i), that $M_0 = 0$. Let $S_n = \{(s_1, \ldots, s_n) \in \mathbb{R}^n; 0 \leq s_1, \ldots, s_n, s_1 + \cdots + s_n \leq 1\}$. Then $\mathrm{conv}(\{0, M_1, \ldots, M_n\}) = \{s_1M_1 + \cdots + s_nM_n; (s_1, \ldots, s_n) \in S_n\}$.

Recall that operators $M_1, \ldots, M_n \in \mathcal{B}(\mathscr{X})$ are *locally linearly independent* (*LLI*) if for every nonzero vector $x \in \mathscr{X}$ vectors $M_1x, \ldots, M_2x$ are linearly independent. For the last result in this paper, we need a slightly stronger assumption on the involved operators. We say that $M_1, \ldots, M_n$ are 2-locally linearly independent over $\mathbb{R}$ (briefly, $\mathbb{R}$-2-LLI) if, for every pair of vectors $x_1, x_2 \in \mathscr{X}$ which are linearly independent over $\mathbb{R}$, the set of $2n$ vectors M_ix_j $(1 \leq i \leq n, 1 \leq j \leq 2)$ are linearly independent over $\mathbb{R}$. It is obvious that for every nonzero $x \in \mathscr{X}$ vectors $M_1x, \ldots, M_nx$ are linearly independent over $\mathbb{R}$ if $M_1, \ldots, M_n$ are $\mathbb{R}$-2-LLI.

Theorem 4.12. *If $M_1, \ldots, M_n \in \mathcal{B}(\mathscr{X})$ are $\mathbb{R}$-2-LLI, then*

$$\mathcal{M} = \mathrm{conv}\big(\{0, M_1, \ldots, M_n\}\big) = \big\{s_1M_1 + \cdots + s_nM_n; (s_1, \ldots, s_n) \in S_n\big\}$$

is a reflexive set.

Proof. Note first that, for every $x \in \mathscr{X}$, the orbit $\mathcal{M}x$ is a closed subset of $\mathscr{X}$ because it is the convex hull of finite set $\{0, M_1x, \ldots, M_nx\}$. Assume that

$T \in \operatorname{Ref}(\mathcal{M})$. Then $Tx \in \mathcal{M}x$ for every $x \in \mathscr{X}$, which means that there exists $s(x) = (s_1(x), \ldots, s_n(x)) \in S_n$ such that $Tx = s_1(x)M_1x + \cdots + s_n(x)M_nx$. We claim that $s(x)$ is uniquely determined by x if $x \neq 0$. Indeed, assume that $x \neq 0$ and that there are $s(x), s'(x) \in S_n$ such that $Tx = s_1(x)M_1x + \cdots + s_n(x)M_nx$ and $Tx = s'_1(x)M_1x + \cdots + s'_n(x)M_nx$. It follows that

$$\big(s_1(x) - s'_1(x)\big)M_1x + \cdots + \big(s_n(x) - s'_n(x)\big)M_nx = 0.$$

Since $M_1, \ldots, M_n$ are $\mathbb{R}$-2-LLI and therefore $\mathbb{R}$-1-LLI, we have $s_j(x) = s'_j(x)$ for $j = 1, \ldots, n$. Thus, $s : x \mapsto s(x)$ is given a well-defined mapping from $\mathscr{X} \setminus \{0\}$ to S_n.

If $x \in \mathscr{X} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then it follows from $Tx = s_1(x)M_1x + \cdots + s_n(x)M_nx$ that $T(\lambda x) = s_1(x)M_1(\lambda x) + \cdots + s_n(x)M_n(\lambda x)$. On the other hand, we have $T(\lambda x) = s_1(\lambda x)M_1(\lambda x) + \cdots + s_n(\lambda x)M_n(\lambda x)$, as well. Hence, $s_j(\lambda x) = s_j(x)$ for $j = 1, \ldots, n$.

Let $x, y \in \mathscr{X}$ be linearly independent. Then

$$T(x+y) = \sum_{j=1}^{n} s_j(x+y)M_j(x+y) = \sum_{j=1}^{n} s_j(x+y)M_jx + \sum_{j=1}^{n} s_j(x+y)M_jy$$

and

$$T(x+y) = Tx + Ty = \sum_{j=1}^{n} s_j(x)M_jx + \sum_{j=1}^{n} s_j(y)M_jy,$$

which gives

$$\sum_{j=1}^{n}\big(s_j(x) - s_j(x+y)\big)M_jx + \sum_{j=1}^{n}\big(s_j(y) - s_j(x+y)\big)M_jy = 0.$$

It follows, because of $\mathbb{R}$-2-LLI, that $s_j(x+y) = s_j(x) = s_j(y)$ for $j = 1, \ldots, n$. We have shown that $s : \mathscr{X} \setminus \{0\} \to S_n$ is a constant mapping; that is, there exists $s = (s_1, \ldots, s_n) \in S_n$ such that $Tx = s_1M_1x + \cdots + s_nM_nx$ for every $x \in \mathscr{X} \setminus \{0\}$. Since the last equality holds trivially for $x = 0$, we conclude that $T = s_1M_1 + \cdots + s_nM_n \in \mathcal{M}$. □

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