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GENERALIZED FRAMES FOR OPERATORS ASSOCIATED WITH ATOMIC SYSTEMS

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ABSTRACT. In this paper, we investigate the g-frame and Bessel g-sequence related to a linear bounded operator K in Hilbert space, which we call a *K-g-frame* and a *K-dual Bessel g-sequence*, respectively. Since the frame operator for a K-g-frame may not be invertible, there is no classical canonical dual for a K-g-frame. So we characterize the concept of a canonical K-dual Bessel g-sequence of a K-g-frame that generalizes the classical dual of a g-frame. Moreover, we use a family of linear operators to characterize atomic systems. We also consider the construction of new atomic systems from given ones and bounded operators.

1. INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] in the context of nonharmonic Fourier series. In contrast to orthonormal bases, frames form redundant systems, thereby allowing nonunique but stable decompositions and expansions. Due to this property, frames have been widely applied in numerous applications, such as filter bank theory [11], sampling theory [10], signal and image processing [8], coding and communication [12], [13], and compressed sensing [2], [16]. For more details about the theory and applications of frames, we refer the reader to [3].

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Let $\mathcal{H}$ be a separable Hilbert space over the complex field. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a frame if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I}\left|\langle f, f_i\rangle\right|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The constants A and B are called *frame bounds*. Frames can be viewed as redundant bases that are generalizations of orthonormal bases. In coding theory, a signal vector f is encoded as $\{\langle f, f_i\rangle\}_{i\in I}$ against a sequence $\{f_i\}_{i\in I}$; then $\{\langle f, f_i\rangle\}_{i\in I}$ is sent to a receiver for decoding to reconstruct the signal f. If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then the last decoding process requires the canonical dual frame to do the job. However, in real-world applications, $\{f_i\}_{i\in I}$ may not be a frame or a base for $\mathcal{H}$ because of limited computation power and the fact that applications require real-time information without any delays. Then there is no classical canonical dual for $\{f_i\}_{i\in I}$.

In this case, Găvruţa [7] introduced a generalization of frames with a linear bounded operator K, called *K-frames*, when working in atomic systems for operators. K-frames are more general than frames in the sense that the lower frame bound condition holds only for the elements in the range of K and in the sense that they allow the reconstruction of the elements from the range of K in a stable way. Since K-frames are more general than frames, many properties for K-frames may not hold. As such, the frame operator of K-frames is not invertible and the synthesis operator for a K-frame is not surjective (see [7], [9], [15], [19]).

In [1], Asgari and Rahimi introduced g-frames for operators, which allow one to reconstruct elements from the range of bounded linear operators. These g-frames for K are what we call *K-g-frames*. Since there are few results on the K-dual Bessel g-sequences of K-g-frames, and since the frame operator for a K-g-frame may not be invertible, there is no classical canonical dual for a K-g-frame. Thus it is natural to extend the dual for g-frames to the case of K-g-frames and examine its properties. So we characterize the K-dual and canonical K-dual Bessel g-sequence of a given K-g-frame in this paper.

Christensen and Heil in [4] gave the concept of atomic decompositions in Banach spaces. Găvruţa in [7] gave both the definition of an atomic system for a bounded linear operator and a characterization of atomic systems by K-frames. We also give a characterization of atomic systems by a sequence of bounded operators in this paper. We consider the construction of new atomic systems from given ones and bounded operators.

Throughout the present paper, $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces and $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathcal{K}$, where I is a subset of $\mathbb{Z}$ and $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_i$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and kernel of T, respectively. And we denote by $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. Now we recall some definitions and results.

Definition 1.1 (see [17]). We call a sequence $\{\Lambda_i\}_{i\in I}$ a *generalized frame*, or simply a *g-frame*, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there are two positive constants A and

B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We call A and B the *lower* and *upper frame bounds*, respectively.

If we only have the upper bound, then we call $\{\Lambda_i\}_{i\in I}$ a *Bessel g-sequence*. Now define

$$\Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{\ell^2} := \Big\{\{f_i\}_{i\in I} \;\Big|\; f_i \in \mathcal{H}_i, \big\|\{f_i\}_{i\in I}\big\|_2^2 = \sum_{i\in I} \|f_i\|^2 < \infty\Big\}$$

with pointwise operators and inner product as

$$\big\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\big\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

Lemma 1.2 (see [14]). *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathcal{H}$ with bound B. Then for each sequence $\{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2}$, the series $\sum_{i\in I} \Lambda_i^*(f_i)$ converges unconditionally.*

In [17], Sun showed that every g-frame can be considered as a frame. More precisely, let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ and let $\{e_{i,j}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$. Then there exists a frame $\{u_{i,j}\}_{i\in I, j\in J_i}$ of $\mathcal{H}$ such that

$$u_{i,j} = \Lambda_i^* e_{i,j} \tag{1.1}$$

and

$$\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}, \quad \forall f \in \mathcal{H}$$

and

$$\Lambda_i^* g = \sum_{j\in J_i} \langle g, e_{i,j}\rangle u_{i,j}, \quad \forall g \in \mathcal{H}_i.$$

We call $\{u_{i,j}\}_{i\in I, j\in J_i}$ the *frame induced by $\{\Lambda_i\}_{i\in I}$ with respect to $\{e_{i,j}\}_{i\in I, j\in J_i}$*. The next lemma is a characterization of g-frame by a frame.

Lemma 1.3 (see [17]). *Let $\{\Lambda_i\}_{i\in I}$ be a family of linear operators, and let $u_{i,j}$ be defined as in (1.1). Then $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ if and only if $\{u_{i,j}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$.*

Definition 1.4. Let $K \in \mathcal{L}(\mathcal{H})$. We call a sequence $\{\Lambda_i\}_{i\in I}$ a *K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$* if there are two positive constants A and B such that

$$A\|K^* f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We call A and B the *lower and upper frame bounds*, respectively.

We call $\{\Lambda_i\}_{i\in I}$ a *tight K-g-frame* if $A\|K^* f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2$, and we call it a *Parseval K-g-frame* if $\|K^* f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2$. If we have only the second inequality, then we call it a *K-Bessel g-sequence*. In [1], the K-g-frame operator S_Λ is defined as

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in \mathcal{H},$$

which is a bounded, self-adjoint, and positive operator, where T_Λ is the synthesis operator of $\{\Lambda_i\}_{i\in I}$ and T_Λ^* is the analysis operator of $\{\Lambda_i\}_{i\in I}$.

The following result characterizes a K-g-frame in terms of a range inclusion property.

Lemma 1.5 (see [1]). *Let $\mathcal{H}$ be a separable Hilbert space, $K \in \mathcal{L}(\mathcal{H})$. Then a g-sequence $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i\in I}$ is a Bessel g-sequence for $\mathcal{H}$ and the range of synthesis operators $\mathcal{R}(T_\Lambda) \supset \mathcal{R}(K)$.*

The next result tells us that for any K-g-frame there always exists a Bessel g-sequence for $\mathcal{H}$ such that they give a decomposition formula for any element in the range of K.

Lemma 1.6 (see [1]). *Let $\mathcal{H}$ be a separable Hilbert space, and let $K \in \mathcal{L}(\mathcal{H})$. Let $\{\Lambda_i\}_{i\in I}$ be a family of linear operators. The following statements are equivalent:*

(1) *$\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.*
(2) *$\{\Lambda_i\}_{i\in I}$ is a Bessel g-sequence for $\mathcal{H}$ and there exists a Bessel g-sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ respect to $\{\mathcal{H}_i\}_{i\in I}$ such that*

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathcal{H}.$$

The following lemmas are key tools for the proofs of our main results.

Lemma 1.7 (see [5]). *Let $U, V \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

(1) *$\mathcal{R}(U) \subset \mathcal{R}(V)$.*
(2) *$UU^* \le \lambda VV^*$ for some $\lambda \ge 0$.*
(3) *There exists $Q \in \mathcal{L}(\mathcal{H})$ such that $U = VQ$.*

Moreover, if (i), (ii), and (iii) are valid, then there exists a unique operator Q such that

(1) *$\|Q\|^2 = \inf\{\mu : UU^* \le \mu VV^*\}$,*
(2) *$\mathcal{N}(U) = \mathcal{N}(C)$, and*
(3) *$\mathcal{R}(C) \subset \overline{\mathcal{R}(V^*)}$.*

Lemma 1.8 (see [3]). *Let $\mathcal{H}$ be a Hilbert space, and suppose that $T \in \mathcal{L}(\mathcal{H})$ has a closed range. Then there exists an operator $T^\dagger \in \mathcal{L}(\mathcal{H})$ for which*

$$\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp, \qquad \mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp, \qquad TT^\dagger f = f, \quad f \in \mathcal{R}(T).$$

We call the operator $T^\dagger$ the pseudo-inverse *of T. If T is invertible, then we have $T^{-1} = T^\dagger$.*

2. K-dual Bessel g-sequences for given K-g-frames

In this section, we mainly investigate the dual of a g-frame and a Bessel g-sequence related to a linear bounded operator K in Hilbert space.

Definition 2.1. Suppose that $K \in \mathcal{L}(\mathcal{H})$ and that $\{\Lambda_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$. A Bessel g-sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ is called a *K-dual Bessel g-sequence of* $\{\Lambda_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathcal{H}.$$

The following theorem characterizes a K-g-frame by operator decompositions and also gives a sufficient condition for a g-sequence to be a K-dual Bessel g-sequence of a K-g-frame. Note that $\{\delta_i\}_{i\in I}$ denotes the canonical basis of $\ell^2(I)$.

Theorem 2.2. *Suppose that $K \in \mathcal{L}(\mathcal{H})$ and that $\{\Lambda_i\}_{i\in I}$ is a Bessel g-sequence for $\mathcal{H}$ with the synthesis operator T_Λ. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if there exists a bounded operator $\Phi : (\sum_{i\in I} \oplus \mathcal{H}_i) \longrightarrow \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$. Moreover, if $\Gamma^* e_{i,j} = \Phi(e_{ij}\delta_i)$, $j \in J_i$, $i \in I$, then $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$.*

Proof. $\Longrightarrow$: Suppose that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$. By Lemma 1.5, we have $\mathcal{R}(K) \subset \mathcal{R}(T_\Lambda)$. Since $T_\Lambda : (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2} \longrightarrow \mathcal{H}$ and $K : \mathcal{H} \longrightarrow \mathcal{H}$, by Lemma 1.7 there exists a bounded operator $\Phi : (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2} \longrightarrow \mathcal{H}$ such that $K = T_\Lambda \Phi^*$. This implies that $K^* = \Phi T_\Lambda^*$.

$\Longleftarrow$: Suppose that there exists a bounded operator $\Phi : (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2} \longrightarrow \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$. Let $K = T_\Lambda \Phi^*$. Then $\mathcal{R}(K) \subset \mathcal{R}(T_\Lambda)$ by Lemma 1.7, and so $\{\Lambda_i\}_{i\in I}$ is a K-g-frame by Lemma 1.5.

Moreover, note that if $\{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2}$, then we have

$$\{f_i\}_{i\in I} = \sum_{i\in I} f_i \delta_i = \sum_{i\in I} \sum_{j\in J_i} \langle f_i, e_{ij}\rangle e_{ij}\delta_i.$$

Roughly speaking, $\{e_{i,j}\delta_i\}_{i\in I, j\in J_i}$ is an orthonormal basis of $(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell^2}$. In terms of the above proof, we know that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame. Let $u_{i,j}$ be defined as in (1.1). If $K^* = \Phi T_\Lambda^*$ and $\Gamma^* e_{i,j} = \Phi(e_{ij}\delta_i)$, $j \in J_i$, $i \in I$, then for all $f \in \mathcal{H}$ we have

$$\begin{aligned} K^* f = \Phi T_\Lambda^* f &= \Phi\Big(\sum_{i,j} \langle \Lambda_i f, e_{ij}\rangle e_{ij}\delta\Big) \\ &= \sum_{i\in I} \sum_{j\in J} \langle f, \Lambda_i^* e_{ij}\rangle \Phi(e_{ij}\delta) \\ &= \sum_{i\in I} \sum_{j\in J} \langle f, u_{ij}\rangle \Gamma_i^* e_{ij} \\ &= \sum_{i\in I} \Gamma_i^* \Big(\sum_{j\in J_i} \langle f, u_{ij}\rangle e_{ij}\Big) = \sum_{i\in I} \Gamma_i^* \Lambda_i f. \end{aligned}$$

Consequently, $Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f$, meaning that $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$. □

In Theorem 2.2, the sufficient condition for a K-dual Bessel g-sequence for a K-g-frame is also necessary.

Theorem 2.3. *Suppose that $K \in \mathcal{L}(\mathcal{H})$ and that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ with the synthesis operator T_Λ. Then $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$ if and only if there exists a bounded operator $\Phi : (\sum_{i\in I} \oplus \mathcal{H}_i) \longrightarrow \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$ and $\Gamma^* e_{i,j} = \Phi(e_{ij}\delta_i)$, $j \in J_i$, $i \in I$.*

Proof. The sufficient condition has been proved in Theorem 2.2. Now we show that the necessary condition holds. Suppose that $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$. Then the synthesis operator for $\{\Gamma_i\}_{i\in I}$ satisfies the conditions. In fact, $\{\Gamma_i\}_{i\in I}$ is a Bessel g-sequence, and for all $f \in \mathcal{H}$ we have

$$K^*f = \sum_{i\in I} \Gamma_i^* \Lambda_i f.$$

Let Φ be the synthesis operator of $\{\Gamma_i\}_{i\in I}$. Then

$$\Phi(e_{ij}\delta_i) = \sum_{i\in I} \Gamma_i^* e_{ij}\delta_i = \sum_{i\in I} u_{ij}\delta_i = \sum_{j\in J_i} u_{ij} = \sum_{i\in J_i} \langle e_{ij}, e_{ij}\rangle u_{ij} = \Gamma_i^* e_{ij}.$$

So a calculation as above shows that

$$\begin{aligned} K^*f &= \sum_{i\in I} \Gamma_i^* \Lambda_i f = \sum_{i\in I} \Gamma_i^* \Big(\sum_{j\in J_i} \langle f, u_{ij}\rangle e_{ij} \Big) \\ &= \Phi\Big(\sum_{i,j} \langle \Lambda_i f, e_{ij}\rangle e_{ij}\delta \Big) \\ &= \Phi T_\Lambda^* f. \end{aligned}$$

So $K^* = \Phi T_\Lambda^*$. $\square$

In frame theory, we know that the synthesis operator of a canonical dual frame obtains the minimal norm of the set of the norms of synthesis operators of all dual frames. Now, we prove this result for a K-g-frame.

Theorem 2.4. *Suppose that $K \in \mathcal{L}(\mathcal{H})$ and that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ with optimal lower frame bound A. If $\Gamma = \{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$, then $\|T_\Gamma\|^2 \geq A$, where T_Γ denotes the synthesis operator of Γ. Moreover, there exists a unique K-dual Bessel g-sequence $\Theta = \{\Theta_i\}_{i\in I}$ of $\{\Lambda_i\}_{i\in I}$ such that $\|T_\Theta\|^2 = A$, where T_Θ denotes the synthesis operator of Θ.*

Proof. Suppose that $C > 0$ is a lower K-g-frame bound of $\{\Lambda_i\}_{i\in I}$. Then for any $f \in \mathcal{H}$ we have

$$\sum_{i\in I} \|\Lambda_i f\|^2 \geq C\|K^*f\|^2, \quad \forall f \in \mathcal{H}.$$

So $C\|K^*f\|^2 \leq \|T_\Lambda f\|^2$ for all $f \in \mathcal{H}$. This implies that $\|K^*f\|^2 \leq \frac{1}{C}\|T_\Lambda f\|^2$ for all $f \in \mathcal{H}$. So

$$\begin{aligned} A &= \max\big\{\lambda > 0 : \lambda\|K^*f\|^2 \leq \|T_\Lambda f\|^2, \forall f \in \mathcal{H}\big\} \\ &= \inf\big\{\mu : \|K^*f\|^2 \leq \mu\|T_\Lambda f\|^2, \forall f \in \mathcal{H}\big\}. \end{aligned}$$

Since $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$, for any $f \in \mathcal{H}$ we have

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f = T_\Lambda T_\Gamma^* f.$$

So $K = T_\Lambda T_\Gamma^*$. Thus $KK^* = T_\Lambda T_\Gamma^* T_\Gamma T_\Lambda^* \leq \|T_\Gamma\|^2 T_\Lambda T_\Lambda^*$. So for any $f \in \mathcal{H}$, we have

$$\|K^*f\|^2 = \langle K^*f, K^*f\rangle = \langle KK^*f, f\rangle \leq \|T_\Gamma\|^2 \langle T_\Lambda T_\Lambda^* f, f\rangle = \|T_\Gamma\|^2 \|T_\Lambda f\|^2.$$

So $\|T_\Gamma\|^2 \geq A$. Since $\{\Lambda_i\}_{i\in I}$ is a K-g-frame, we have that $\mathcal{R}(K) \subset \mathcal{R}(T_\Lambda)$. By Lemma 1.7, there exists a unique bounded operator $\Phi : (\sum_{i\in I} \oplus \mathcal{H}_i) \longrightarrow \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$ and

$$\|\Phi\|^2 = \inf\{\mu : \|K^* f\|^2 \leq \mu \|T_\Lambda f\|^2, \forall f \in \mathcal{H}\} = A.$$

Let $\Theta_i^* e_{ij} = \Phi(e_{ij}\delta_i)$. Then it is easy to check that $\Theta = \{\Theta_i\}_{i\in I}$ is a Bessel g-sequence, since for any $f \in \mathcal{H}$ we have

$$K^* f = \Phi T_\Lambda^* f = \sum_{i\in I} \Gamma_i^* \Lambda_i f.$$

So $Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f$ and Θ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$, since for all $f \in \mathcal{H}$,

$$\begin{aligned} T_\Theta^* f &= \sum_{i\in I}\sum_{j\in J_i} \langle \Theta_i f, e_{ij}\rangle e_{ij}\delta_i = \sum_{i\in I}\sum_{j\in J_i} \langle f, \Theta_i^* e_{ij}\rangle e_{ij}\delta_i \\ &= \sum_{i\in I}\sum_{j\in J_i} \langle f, \Phi e_{ij}\delta_i\rangle e_{ij}\delta_i = \sum_{i\in I}\sum_{j\in J_i} \langle \Phi^* f, e_{ij}\delta_i\rangle e_{ij}\delta_i = \Phi^* f. \end{aligned}$$

So $T_\Theta = \Phi$, and hence $\|T_\Theta\|^2 = \|\Phi\|^2 = A$. □

Remark 2.5. Let $K \in \mathcal{L}(\mathcal{H})$, and let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame with an optimal lower K-g-frame bound A.

(1) Let $\{\Gamma_i\}_{i\in I}$ be the K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$ satisfying $\|T_\Gamma\|^2 = A$; we call $\{\Gamma_i\}_{i\in I}$ the *canonical K-dual Bessel g-sequence of* $\{\Lambda_i\}_{i\in I}$.

(2) When $K = I_\mathcal{H}$, the K-g-frame is exactly a g-frame; in this case, the canonical I-dual Bessel g-sequence is exactly the canonical dual g-frame.

In [19], Xiao et al. show that a K-frame for $\mathcal{H}$ has a dual frame on the closed subspace $\mathcal{R}(K)$, which is derived from a dual frame. Our next result is a generalization of [18, Theorem 8], which provides a characterization of a K-g-frame by a dual g-frame on $\mathcal{R}(K)$.

Theorem 2.6. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathcal{H}$ with a frame operator S_Λ. If $\{\Lambda_i\}_{i\in I}$ has a dual g-frame on $\mathcal{R}(K)$ and $S_\Lambda(\mathcal{R}(K)) \subset \mathcal{R}(K)$, then it is a K-g-frame for $\mathcal{H}$.*

Proof. Assume that $\{\Gamma\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ on $\mathcal{R}(K)$. Then each $f \in \mathcal{H}$ can be expressed as $f = f_1 + f_2$, where $f_1 \in \mathcal{R}(K)$ and $f_2 \in (\mathcal{R}(K))^\perp$. Then

$$\begin{aligned} \sum_{i\in I} \|\Lambda_i f\|^2 &= \sum_{i\in I} \big\|\Lambda(f_1 + f_2)\big\|^2 \\ &= \sum_{i\in I} \|\Lambda_i f_1\|^2 + \sum_{i\in I} \|\Lambda_i f_2\|^2 + 2\,\mathrm{Re} \sum_{i\in I} \langle \Lambda_i^* \Lambda_i f_1, f_2\rangle. \end{aligned}$$

Note that $\sum_{i\in I} \langle \Lambda_i^* \Lambda_i f_1, f_1\rangle = S_\Lambda f_1 \in S_\Lambda(\mathcal{R}(K)) \subset \mathcal{R}(K)$, and so we have

$$\sum_{i\in I} \langle \Lambda_i^* \Lambda_i f_1, f_2\rangle = 0.$$

Hence

$$\sum_{i\in I}\|\Lambda_i f\|^2=\sum_{i\in I}\|\Lambda_i f_1\|^2+\sum_{i\in I}\|\Lambda_i f_2\|^2.$$

By Lemma 1.2, $\sum_{i\in I}\Gamma_i^*\Lambda_i f_1$ converges and so does $\sum_{i\in I}\pi_{\mathcal{R}(K)}\Gamma_i^*\Lambda_i f_1$, where $\pi_{\mathcal{R}(K)}$ is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(K)$. Then for each $g\in\mathcal{R}(K)$ we have

$$\langle g,f_1\rangle=\Big\langle\sum_{i\in I}\Lambda^*\Gamma_i g,f_1\Big\rangle=\sum_{i\in I}\langle g,\Gamma_i^*\Lambda_i f_1\rangle=\Big\langle g,\sum_{i\in I}\pi_{\mathcal{R}(K)}\Gamma_i^*\Lambda_i f_1\Big\rangle.$$

It follows that

$$f_1=\sum_{i\in I}\pi_{\mathcal{R}(K)}\Gamma_i^*\Lambda_i f_1.$$

Thus

$$\begin{aligned}\|K^*f\|^4&=\big\|K^*(f_1+f_2)\big\|^4=\|K^*f_1\|^4=\big\|\langle K^*f_1,K^*f_1\rangle\big\|^2=\big\|\langle f_1,KK^*f_1\rangle\big\|^2\\&=\Big\|\Big\langle\sum_{i\in I}\pi_{\mathcal{R}(K)}\Gamma_i^*\Lambda_i f_1,KK^*f_1\Big\rangle\Big\|^2=\Big\|\sum_{i\in I}\langle\Lambda_i f_1,\Gamma_i\pi_{\mathcal{R}(K)}KK^*f_1\rangle\Big\|^2\\&\le\sum_{i\in I}\|\Lambda_i f_1\|^2\sum_{i\in I}\|\Gamma_i\pi_{\mathcal{R}(K)}KK^*f_1\|^2\\&\le D\|K\|^2\|K^*f\|^2\sum_{i\in I}\|\Lambda_i f_1\|^2,\end{aligned}$$

where D is the Bessel bound of $\{\Gamma_i\}_{i\in I}$. Then we have

$$\sum_{i\in I}\|\Lambda_i f_1\|^2\ge D^{-1}\|K\|^{-1}\|K^*f\|^2.$$

Hence

$$\sum_{i\in I}\|\Lambda_i f\|^2=\sum_{i\in I}\|\Lambda_i f_1\|^2+\sum_{i\in I}\|\Lambda_i f_2\|^2\ge\sum_{i\in I}\|\Lambda_i f_1\|^2\ge D^{-1}\|K\|^{-1}\|K^*f\|^2.\qquad\square$$

3. Atomic systems

In this section, we study atomic systems for K by a sequence of bounded operators. In order to understand the main results in this section, let us introduce some basic knowledge on atomic systems, as follows.

Definition 3.1 (see [7]). Let $K\in\mathcal{L}(\mathcal{H})$. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called an *atomic system for K*, if the following conditions are satisfied:

(1) $\{f_i\}_{i\in I}$ is a Bessel sequence;
(2) there exists $c>0$ such that for every $f\in\mathcal{H}$ there exists $a=\{a_i\}_{i\in I}\in\ell^2(I)$ such that $\|a\|_{\ell^2}\le c\|f\|$ and $Kf=\sum_{i\in I}a_i f_i$.

The following lemma characterizes an atomic system in terms of a K-frame.

Lemma 3.2 (see [7]). *Let $\{f_i\}_{i\in I}$ be a sequence in $\mathcal{H}$, and let $K\in\mathcal{L}(\mathcal{H})$. Then $\{f_i\}_{i\in I}$ is an atomic system for K if and only if $\{f_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$.*

We now give a characterization of an atomic system with a sequence of linear operators.

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in I}$ be a family of linear operators. Then the following statements are equivalent:*

(1) *$\{\Lambda_i\}_{i\in I}$ is an atomic system for K;*
(2) *$\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$;*
(3) *there exists a g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ such that $Kf = \sum_{i\in I}\Lambda_i^*\Gamma_i f$.*

Proof. The proof can be easily obtained by Lemmas 1.3, 1.6, and 3.2. □

We can find that each atomic system is associated with a bounded operator K on $\mathcal{H}$. We study a class of operators in $\mathcal{L}(\mathcal{H})$ associated with given atomic systems.

Theorem 3.4. *Let $K_1, K_2 \in \mathcal{L}(\mathcal{H})$. If $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1 and K_2, and α, β are scalars, then $\{\Lambda_i\}_{i\in I}$ is an atomic system for $\alpha K_1 + \beta K_2$ and K_1K_2.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1 and K_2, there are positive constants $A_n, B_n > 0$ $(n = 1, 2)$ such that

$$A_n\|K_n^*f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le B_n\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{3.1}$$

Since

$$\begin{aligned}\|K_1^*f\|^2 = \frac{1}{|\alpha|^2}\|\alpha K_1^*f\|^2 &= \frac{1}{|\alpha|^2}\big\|(\alpha K_1^* + \beta K_2^*)f - \beta K_2^*f\big\|^2 \\ &\ge \frac{1}{|\alpha|^2}\big\|(\alpha K_1^* + \beta K_2^*)f\big\|^2 - \frac{1}{|\alpha|^2}\|\beta K_2^*f\|^2,\end{aligned}$$

we have

$$\begin{aligned}\big\|(\alpha K_1^* + \beta K_2^*)f\big\|^2 &\le |\alpha|^2\|K_1^*f\|^2 + |\beta|^2\|K_2^*f\|^2 \\ &\le \frac{1}{2}\Big(|\alpha|^2\|K_1^*f\|^2 + |\beta|^2\|K_2^*f\|^2 \\ &\quad + \frac{A_1}{A_2}|\beta|^2\|K_1^*f\|^2 + \frac{A_2}{A_1}|\alpha|^2\|K_2^*f\|^2\Big) \\ &= \frac{A_2|\alpha|^2 + A_1|\beta|^2}{2A_1A_2}\big(A_1\|K_1^*f\|^2 + A_2\|K_2^*f\|^2\big).\end{aligned}$$

Hence

$$\sum_{i\in I}\|\Lambda_i f\|^2 \ge \frac{1}{2}\big(A_1\|K_1^*f\|^2 + A_2\|K_2^*f\|^2\big) \ge \frac{A_1A_2}{A_2|\alpha|^2 + A_1|\beta|^2}\big\|(\alpha K_1^* + \beta K_2^*)f\big\|^2,$$

and from inequalities (3.1), we get

$$\sum_{i\in I}\|\Lambda_i f\|^2 \le \frac{B_1 + B_2}{2}\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is an $(\alpha K_1 + \beta K_2)$-g-frame. By Theorem 3.3, $\{\Lambda_i\}_{i\in I}$ is an atomic system for $\alpha K_1 + \beta K_2$.

Now for each $f \in \mathcal{H}$, we have

$$\big\|(K_1K_2)^* f\big\|^2 = \|K_2^* K_1^* f\|^2 \le \|K_2^*\|^2 \|K_1^* f\|^2.$$

Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1, we have

$$\frac{A_1}{\|K_2^*\|^2}\big\|(K_1K_2)^* f\big\|^2 \le A_1\|K_1^* f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le B_1\|f\|^2, \quad \forall f \in \mathcal{H}.$$

By Theorem 3.3, $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1K_2. □

The following results provide the construction of a new atomic system from given ones.

Theorem 3.5. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two atomic systems for K, and let the corresponding synthesis operators be T_Λ and T_Γ, respectively. If $T_\Lambda T_\Gamma^* = 0$ and U or V is surjective satisfying $UK^* = K^*U$ or $VK^* = K^*V$, then $\{\Lambda_i U + \Gamma_i V\}_{i\in I}$ is an atomic system for K.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are two atomic systems for K, by Theorem 3.3, $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are two K-g-frames for $\mathcal{H}$, and so there exist $B_1 \ge A_1 > 0$ and $B_2 \ge A_2 > 0$ such that

$$A_1\|K^* f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le B_1\|f\|^2, \qquad A_2\|K^* f\|^2 \le \sum_{i\in I}\|\Gamma_i f\|^2 \le B_2\|f\|^2.$$

Since $T_\Lambda T_\Gamma^* = 0$, for any $f \in \mathcal{H}$, we have

$$\sum_{i\in I}\Lambda_i^*\Gamma_i f = \sum_{i\in I}\Gamma_i^*\Lambda_i f = 0.$$

Therefore, for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in I}\big\|(\Lambda_i U + \Gamma_i V) f\big\|^2 &= \sum_{i\in I}\langle \Lambda_i U + \Gamma_i V f, \Lambda_i U + \Gamma_i V f\rangle \\
&= \sum_{i\in I}\|\Lambda_i U f\|^2 + \sum_{i\in I}\|\Gamma_i V f\|^2 + 2\,\mathrm{Re}\sum_{i\in I}\langle \Lambda_i^*\Gamma_i V f, U f\rangle \\
&= \sum_{i\in I}\|\Lambda_i U f\|^2 + \sum_{i\in I}\|\Gamma_i V f\|^2 \\
&\le B_1\|Uf\|^2 + B_2\|Vf\|^2 \le \big(B_1\|U\|^2 + B_2\|V\|^2\big)\|f\|^2.
\end{aligned}$$

Without loss of generality, assume that U is surjective; then there exists $C > 0$ such that $\|Uf\|^2 \ge C\|f\|^2$ for any $f \in \mathcal{H}$. Since $UK^* = K^*U$, we have

$$\begin{aligned}
\sum_{i\in I}\big\|(\Lambda_i U + \Gamma_i V) f\big\|^2 &= \sum_{i\in I}\|\Lambda_i U f\|^2 + \sum_{i\in I}\|\Gamma_i V f\|^2 \\
&\ge \sum_{i\in I}\|\Lambda_i U f\|^2 \ge A_1\|K^* U f\|^2 \\
&= A_1\|UK^* f\|^2 \ge A_1 C\|K^* f\|^2.
\end{aligned}$$

So $\{\Lambda_i U + \Gamma_i V\}_{i\in I}$ is a K-g-frame and thus an atomic system for K by Theorem 3.3. □

Let $B = 0$, and we get the following corollary.

Corollary 3.6. *Suppose that $K \in \mathcal{L}(\mathcal{H})$ and that $\{\Lambda_i\}_{i\in I}$ is an atomic system for K. If U is surjective and $UK^* = K^*U$, then $\{\Lambda_i U\}_{i\in I}$ is an atomic system for K.*

Let $U = V = I_{\mathcal{H}}$; then we obtain the following corollary for a K-g-frame.

Corollary 3.7. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two Parseval K-g-frames for $\mathcal{H}$, with synthesis operators T_Λ and T_Γ, respectively. If $T_\Lambda T_\Gamma^* = 0$, then $\{\Lambda_i + \Gamma\}_{i\in I}$ is a 2-tight K-g-frame for $\mathcal{H}$.*

Theorem 3.8. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two atomic systems for K, and let the corresponding synthesis operators be T_1 and T_2, respectively. If $T_\Lambda T_\Gamma^* = 0$ and $U_i \in \mathcal{L}(\mathcal{H})$ satisfies $\mathcal{R}(T_i) \subset \mathcal{R}(U_i^* T_i)$, for $i = 1, 2$, then $\{\Lambda_i U_1 + \Gamma_i U_2\}_{i\in I}$ is an atomic system for K.*

Proof. Since $T_1 T_2^* = 0$, we have

$$\begin{aligned}\sum_{i\in I}\big\|(\Lambda_i U_1 + \Gamma_i U_2)f\big\|^2 &= \sum_{i\in I}\|\Lambda_i U_1 f\|^2 + \sum_{i\in I}\|\Gamma_i U_2 f\|^2\\ &= \|T_1^* U_1 f\|^2 + \|T_2^* U_2 f\|^2\\ &= \big\|(U_1^* T_1)^* f\big\|^2 + \big\|(U_2^* T_2)^* f\big\|^2. \end{aligned}\tag{3.2}$$

Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are atomic systems, they are K-g-frames by Theorem 3.3. Thus from Lemma 1.5, we have that $\mathcal{R}(K) \subset \mathcal{R}(T_i) \subset \mathcal{R}(U_i^* T_i)$. So by Lemma 1.7, for each $i = 1, 2$, there exists $\lambda_i > 0$ such that

$$KK^* \le \lambda_i (U_i^* T_i)(U_i^* T_i)^*.$$

By (3.2), for each $f \in \mathcal{H}$, we have

$$\sum_{i\in I}\big\|(\Lambda_i U_1 + \Gamma_i U_2)f\big\|^2 = \big\|(U_1^* T_1)^* f\big\|^2 + \big\|(U_2^* T_2)^* f\big\|^2 \ge \Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\|K^* f\|^2.$$

Hence $\{\Lambda_i U + \Gamma_i V\}_{i\in I}$ is a K-g-frame and thus an atomic system for K by Theorem 3.3. □

Before the following result, we need a simple lemma, which is a generalization of [19, Theorem 3.5].

Lemma 3.9. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathcal{H}$ with a frame operator S_Λ. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if there exists $\lambda > 0$ such that $S_\Lambda \ge \lambda KK^*$.*

Proof. $\{\Lambda_i\}_{i\in I}$ is K-g-frame with frame bounds A, B and a frame operator S_Λ if and only if

$$A\|K^* f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 = \langle S_\Lambda f, f\rangle \le B\|f\|^2, \quad \forall f \in \mathcal{H};$$

that is,

$$\langle AKK^*f, f\rangle \le \langle S_\Lambda f, f\rangle \le \langle Bf, f\rangle, \quad \forall f \in \mathcal{H}.$$

So the conclusion holds. $\square$

Theorem 3.10. *Let $\{\Lambda_i\}_{i\in I}$ be an atomic system for K, and let S_Λ be the frame operator of $\{\Lambda_i\}_{i\in I}$. Let U be a positive operator; then $\{\Lambda_i+\Lambda_i U\}_{i\in I}$ is an atomic system for K. Moreover, for any natural number n, $\{\Lambda_i+\Lambda_i U^n\}_{i\in I}$ is an atomic system for K.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K, by Lemma 1.6, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$. Then by Lemma 3.9 there exists $\lambda > 0$ such that $S_\Lambda \ge \lambda KK^*$. The frame operator for $\{\Lambda_i + \Lambda_i U\}_{i\in I}$ is $(I_{\mathcal{H}}+U)^*S_\Lambda(I_{\mathcal{H}}+U)$. In fact, for each $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I}(\Lambda_i+\Lambda_i U)^*(\Lambda_i+\Lambda_i U)f &= (I_{\mathcal{H}}+U)^*\sum_{i\in I}\Lambda_i^*\Lambda_i(I_{\mathcal{H}}+U)f\\ &= (I_{\mathcal{H}}+U)^*S_\Lambda(I_{\mathcal{H}}+U)f.\end{aligned}$$

Since $U, S > 0$, $(I_{\mathcal{H}}+U)^*S_\Lambda(I_{\mathcal{H}}+U) \ge S_\Lambda \ge \lambda KK^*$, and again by Lemma 3.9, we can conclude that $\{\Lambda_i+\Lambda_i U\}_{i\in I}$ is a K-g-frame and an atomic system for K by Theorem 3.3. For any natural number n, the frame operator for $\{\Lambda_i+\Lambda_i U^n\}_{i\in I}$ is $(I_{\mathcal{H}}+U^n)^*S_\Lambda(I_{\mathcal{H}}+U^n)$. Similarly, $\{\Lambda_i+\Lambda_i U^n\}_{i\in I}$ is an atomic system for K. $\square$

Corollary 3.11. *Let $\{\Lambda_i\}_{i\in I}$ be an atomic system for K, and let S_Λ be the frame operator of $\{\Lambda_i\}_{i\in I}$. Let $\{I_1, I_2\}$ be a partition of I, and let S_j be the frame operator for the Bessel g-sequence $\{\Lambda_i\}_{i\in I_j}$, $j = 1, 2$. Then $\{\Lambda_i+\Lambda_i S_1^a\}_{i\in I_1}\cup\{\Lambda_i+\Lambda_i S_2^b\}_{i\in I_2}$ is an atomic system for K and for any natural numbers a and b.*

Proof. For any natural number a, we can define S^a by

$$S^a f = \sum_{I\in I}(\Lambda_i S^{\frac{a-1}{2}})^*\Lambda_i S^{\frac{a-1}{2}} f.$$

Note that $(S^a)^* = S^a$. For each $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I_1}(\Lambda_i+\Lambda_i S_1^a)^*(\Lambda_i+\Lambda_i S_1^a)f &= (I_{\mathcal{H}}+S_1^a)^*\sum_{i\in I}\Lambda_i^*\Lambda_i(I_{\mathcal{H}}+S_1^a)f\\ &= (I_{\mathcal{H}}+S_1^a)^*S_1(I_{\mathcal{H}}+S_1^a)f\\ &= (S_1+2S_1^{1+a}+S_1^{1+2a})f.\end{aligned}$$

Thus the frame operators for $\{\Lambda_i+\Lambda_i S_1^a\}_{i\in I_1}$ and $\{\Lambda_i+\Lambda_i S_2^b\}_{i\in I_2}$ are $S_1+2S_1^{1+a}+S_1^{1+2a}$ and $S_1+2S_2^{1+b}+S_2^{1+2b}$, respectively. Let S_0 be the frame operator for $\{\Lambda_i+\Lambda_i S_1^a\}_{i\in I_1}\cup\{\Lambda_i+\Lambda_i S_2^a\}_{i\in I_2}$. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K, from Theorem 3.3 we know that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$. Then there exists $\lambda > 0$ such that $S \ge \lambda KK^*$ and $S_0 \ge S_1 + S_2 = S \ge \lambda KK^*$. Hence $\{\Lambda_i+\Lambda_i S_1^a\}_{i\in I_1}\cup\{\Lambda_i+\Lambda_i S_2^a\}_{i\in I_2}$ is a K-g-frame for $\mathcal{H}$ by Lemma 3.9. Again by Theorem 3.3, $\{\Lambda_i+\Lambda_i S_1^a\}_{i\in I_1}\cup\{\Lambda_i+\Lambda_i S_2^a\}_{i\in I_2}$ is an atomic system for K. $\square$

Corollary 3.12. *Let $\{\Lambda_i\}_{i\in I}$ be an atomic system for K. Let $\{I_j\}_{j\in J}$ be a partition of I, and let S_j be the frame operator for the Bessel g-sequence $\{\Lambda_i\}_{i\in I_j}$, $j \in J \subset \mathbb{Z}$. Then*

$$\bigcup_{\substack{i\in I_j\\ j\in J}} \{\Lambda_i + \Lambda_i S_j^{(j-1)}\}_{i\in I_j}$$

is an atomic system for K.

Atomic decomposition has potential applications in encoding and decoding problems. The following result characterizes the atomic decomposition and reconstruction by a family of bounded operators.

Theorem 3.13. *Let $\{\Lambda_i\}_{i\in I}$ be an atomic system for K which has closed range. Then there is a g-frame $\Theta = \{\Theta_i = \Gamma_i K^\dagger|_{\mathcal{R}(K)}\}_{i\in I}$ such that*

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f = \sum_{i\in I} \Theta_i^* \Lambda_i f, \quad \forall f \in \mathcal{R}(K),$$

where $\{\Gamma_i\}_{i\in I}$ is defined in (iii) of Theorem 3.3.

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K, by (iii) of Theorem 3.3, there exists a Bessel g-sequence $\{\Gamma_i\}_{i\in I}$ such that

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f.$$

Since $\{\Gamma_i\}_{i\in I}$ is Bessel g-sequence, there exists $B > 0$ such that

$$\sum_{i\in I} \|\Gamma_i f\|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Hence

$$\sum_{i\in I} \|\Gamma_i K^\dagger f\|^2 \le B\|K^\dagger f\|^2 \le B\|K^\dagger\|^2\|f\|^2.$$

Since $\mathcal{R}(K)$ is closed, by Lemma 1.8 there exists $K^\dagger$ of K such that $f = KK^\dagger f$, $\forall f \in \mathcal{R}(K)$. And then

$$f = KK^\dagger f = \sum_{i\in I} \Lambda_i^* \Gamma_i K^\dagger f = \sum_{i\in I} \Lambda_i^* \Gamma_i K^\dagger|_{\mathcal{R}(K)} f.$$

Also,

$$\begin{aligned}
\|f\|^4 &= \left|\langle f, f\rangle\right|^2 = \left|\left\langle f, \sum_{i\in I} \Lambda_i^* \Gamma_i K^\dagger|_{\mathcal{R}(K)} f\right\rangle\right|^2 = \left|\sum_{i\in I} \langle f, \Lambda_i^* \Gamma_i K^\dagger|_{\mathcal{R}(K)} f\rangle\right|^2 \\
&= \left|\sum_{i\in I} \langle \Lambda_i f, \Gamma_i K^\dagger|_{\mathcal{R}(K)} f\rangle\right|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \sum_{i\in I} \|\Gamma_i K^\dagger|_{\mathcal{R}(K)} f\|^2 \\
&\le \sum_{i\in I} \|\Gamma_i K^\dagger|_{\mathcal{R}(K)} f\|^2 C\|f\|^2,
\end{aligned}$$

where C is a g-Bessel bound of $\{\Lambda_i\}_{i\in I}$. Therefore,

$$\frac{1}{C} \leq \sum_{i\in I} \|\Gamma_i K^\dagger|_{\mathcal{R}(K)} f\|^2, \quad \forall f \in \mathcal{R}(K).$$

Thus $\{\Gamma_i K^\dagger|_{\mathcal{R}(K)}\}_{i\in I}$ is a g-frame for $\mathcal{R}(K)$. Next we prove that $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are interchangeable on $\mathcal{R}(K)$. In fact, for any $f, g \in \mathcal{R}(K) \subset \mathcal{H}$,

$$\langle f, g\rangle = \Big\langle \sum_{i\in I} \Lambda_i^* \Theta_i f, g \Big\rangle = \sum_{i\in I} \langle \Lambda_i^* \Theta_i f, g\rangle = \sum_{i\in I} \langle f, \Theta_i^* \Lambda_i g\rangle = \Big\langle f, \sum_{i\in I} \Theta_i^* \Lambda_i g \Big\rangle. \qquad \square$$

Example 3.14. Let $\mathcal{H} = \mathbb{C}^3$, and let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$. Let $\mathcal{H}_i = \operatorname{span}\{e_i\}$, and let $\Gamma_i = e_i^*$ for $i = 1, 2, 3$. Obviously, $\{\Gamma_i\}_{i=1}^3$ is a Parseval g-frame for $\mathcal{H}$. Now define $K \in \mathcal{L}(\mathcal{H})$ as follows:

$$K : \mathcal{H} \longrightarrow \mathcal{H}, \qquad Ke_1 = e_1, \qquad Ke_2 = e_2, \qquad Ke_3 = e_1.$$

For any $f \in \mathcal{H}$, we have $f = \sum_{i=1}^3 \Gamma_i^* \Gamma_i f$, and it follows that

$$Kf = \sum_{i=1}^3 K\Gamma_i^* \Gamma_i f = \sum_{i=1}^3 (\Gamma_i K^*)^* \Gamma_i f = \sum_{i=1}^3 \Lambda^* \Gamma_i f,$$

where $\Lambda_i = \Gamma_i K^*$ for $i = 1, 2, 3$. By (iii) of Theorem 3.3, we know that $\{\Lambda_i\}_{i=1}^3$ is an atomic system for K. In other words, for any $f \in \mathcal{H}$, we can reconstruct f by an atomic system $\{\Lambda_i\}_{i=1}^3$, although $\operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i=1}^3 = \operatorname{span}\{e_1, e_2\} \subset \mathcal{H}$.

On the other hand, for all $f \in \mathcal{H}$, we may not get $Kf = \sum_{i\in I} \Gamma_i^* \Lambda_i f$. If we take $f = e_3$, then we have

$$Ke_3 = e_1 \neq \sum_{i=1}^3 \Gamma_i^* \Lambda_i f = \langle e_3, e_1\rangle e_1 + \langle e_3, e_2\rangle e_2 + \langle e_3, e_1\rangle e_3 = 0.$$

Since $\mathcal{R}(K) = \operatorname{span}\{e_1, e_2\}$ is closed, by Lemma 1.8 there exists $K^\dagger$ of K such that $KK^\dagger f = f$ for all $f \in \mathcal{R}(K)$. Hence we have $KK^\dagger e_1 = e_1$, $KK^\dagger e_2 = e_2$. Now we can take $K^\dagger$ as follows:

$$K^\dagger e_1 = \frac{e_1 + e_3}{2}, \qquad K^\dagger e_2 = e_2, \qquad K^\dagger e_3 = 0.$$

For any $f \in \mathcal{H}$, there exist c_i, for $i = 1, 2, 3$ such that $f = c_1 e_1 + c_2 e_2 + c_3 e_3$. Then we have

$$\begin{aligned}
\big\langle (K^\dagger|_{\mathcal{R}(K)})^* e_1, f \big\rangle &= \big\langle e_1, (K^\dagger|_{\mathcal{R}(K)}) f \big\rangle = \langle e_1, c_1 K^\dagger e_1 + c_2 K^\dagger e_2\rangle \\
&= \Big\langle e_1, c_1 \frac{e_1 + e_3}{2} + c_2 e_2 \Big\rangle = \Big\langle e_1, c_1 \frac{e_1 + e_3}{2} \Big\rangle \\
&= \Big\langle \frac{e_1}{2}, c_1 e_1 \Big\rangle = \frac{c_2}{2} = \Big\langle \frac{e_1}{2}, c_1 e_1 + c_2 e_2 + c_3 e_3 \Big\rangle \\
&= \Big\langle \frac{e_1}{2}, f \Big\rangle.
\end{aligned}$$

Similarly, we have

$$\big\langle (K^\dagger|_{\mathcal{R}(K)})^* e_2, f \big\rangle = c_2 = \langle e_2, f\rangle, \qquad \big\langle (K^\dagger|_{\mathcal{R}(K)})^* e_3, f \big\rangle = \frac{c_1}{2} = \Big\langle \frac{e_1}{2}, f \Big\rangle.$$

It follows that

$$\Theta_1^* = (\Gamma_1 K^\dagger|_{\mathcal{R}(K)})^* = (K^\dagger|_{\mathcal{R}(K)})^*\Gamma_i^* = (K^\dagger|_{\mathcal{R}(K)})^* e_1 = \frac{e_1}{2},$$

$$\Theta_2^* = (\Gamma_2 K^\dagger|_{\mathcal{R}(K)})^* = (K^\dagger|_{\mathcal{R}(K)})^*\Gamma_2^* = (K^\dagger|_{\mathcal{R}(K)})^* e_2 = e_2,$$

$$\Theta_3^* = (\Gamma_3 K^\dagger|_{\mathcal{R}(K)})^* = (K^\dagger|_{\mathcal{R}(K)})^*\Gamma_3^* = (K^\dagger|_{\mathcal{R}(K)})^* e_3 = \frac{e_1}{2}.$$

Let $g \in \mathcal{H}$ and $g = c_1e_1 + c_2e_2$. Then

$$\begin{aligned}\sum_{i\in I} \Lambda_i^*\Theta_i g &= \langle e_1/2, g\rangle e_1 + \langle e_2, g\rangle e_2 + \langle e_1/2, g\rangle e_1 \\ &= \langle e_1/2, c_1e_1 + c_2e_2\rangle + \langle e_2, c_1e_1 + c_2e_2\rangle e_2 + \langle e_1/2, c_1e_1 + c_2e_2\rangle e_1 \\ &= \frac{1}{2}c_1e_1 + c_2e_2 + \frac{1}{2}c_1e_1 = c_1e_1 + c_2e_2 = g\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in I} \Theta_i^*\Lambda_i g &= \langle e_1, g\rangle e_1/2 + \langle e_2, g\rangle e_2 + \langle e_1, g\rangle e_1/2 \\ &= \langle e_1, c_1e_1 + c_2e_2\rangle e_1/2 + \langle e_2, c_1e_1 + c_2e_2\rangle e_2 + \langle e_1, c_1e_1 + c_2e_2\rangle e_1/2 \\ &= \frac{1}{2}c_1e_1 + c_2e_2 + \frac{1}{2}c_1e_1 = c_1e_1 + c_2e_2 = g.\end{aligned}$$

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References

1. M. S. Asgari and H. Rahimi, *Generalized frames for operators in Hilbert spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17** (2014), no. 2, art. ID 1450013. Zbl 1294.41023. MR3212683. DOI 10.1142/S0219025714500131. 207, 208, 209
2. W. Chen, M. R. Rodrigues, and I. J. Wassell, *Projection design for statistical compressive sensing: A tight frame based approach*, IEEE Trans. Signal Process. **61** (2013), no. 8, 2016–2029. DOI 10.1109/TSP.2013.2245661. 206
3. O. Christensen, *An Introduction to Frames and Riesz Bases*, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 2003. Zbl 1017.42022. MR1946982. DOI 10.1007/978-0-8176-8224-8. 206, 209
4. O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. **185** (1997), no. 1, 33–47. Zbl 0868.42013. MR1452474. DOI 10.1002/mana.3211850104. 207
5. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415. Zbl 0146.12503. MR0203464. DOI 10.2307/2035178. 209
6. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. Zbl 0049.32401. MR0047179. DOI 10.2307/1990760. 206
7. L. Găvruţa, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), no. 1, 139–144. Zbl 1230.42038. MR2854166. DOI 10.1016/j.acha.2011.07.006. 207, 213
8. V. K. Goyal, J. Kovačević, and A. K. Jonathan, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), no. 3, 203–233. Zbl 0992.94009. MR1829801. DOI 10.1006/acha.2000.0340. 206

9. X. X. Guo, *Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. **444** (2016), no. 1, 598–609. Zbl 06607789. MR3523393. DOI 10.1016/j.jmaa.2016.06.055. 207
10. T. Košir and M. Omladič, *Normalized tight vs. general frames in sampling problems*, Adv. Oper. Theory. **2** (2017), no. 2, 114–125. Zbl 1370.42025. DOI 10.22034/aot.1611-1063. 206
11. J. Kovačević, P. L. Dragotti, and V. K. Goyal, *Filter bank frame expansions with erasures*, IEEE Trans. Inform. Theory. **48** (2002), no. 6, 1439–1450. Zbl 1061.94541. MR1909466. DOI 10.1109/TIT.2002.1003832. 206
12. J. S. Leng and D. G. Han, *Optimal dual frames for erasures, II*, Linear Algebra Appl. **435** (2011), no. 6, 1464–1472. Zbl 1235.42028. MR2807163. DOI 10.1016/j.laa.2011.03.043. 206
13. J. S. Leng, D. G. Han, and T. Z. Huang, *Optimal dual frames for communication coding with probabilistic erasures*, IEEE Trans. Signal Process. **59** (2011), no. 11, 5380–5389. MR2883607. DOI 10.1109/TSP.2011.2162955. 206
14. A. Najati, M. H. Faroughi, and A. Rahimi, *g-Frames and stability of g-frames in Hilbert spaces*, Methods Funct. Anal. Topology **14** (2008), no. 3, 271–286. Zbl 1164.42026. MR2458491. 208
15. G. Ramu and P. S. Johnson, *Frame operators of K-frames*, SeMA J. **73** (2016), no. 2, 171–181. Zbl 1349.42065. MR3502147. DOI 10.1007/s40324-016-0062-4. 207
16. Q. Sun and W.-S. Tang, *Nonlinear frames and sparse reconstructions in Banach spaces*, J. Fourier Anal. Appl. **23** (2015), no. 5, 1–35. MR3704759. DOI 10.1007/s00041-016-9501-y. 206
17. W. C. Sun, *g-Frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), no. 1, 437–452. Zbl 1129.42017. MR2239250. DOI 10.1016/j.jmaa.2005.09.039. 207, 208
18. Z.-Q. Xiang and Y.-M. Li, *Frame sequences and dual frames for operators*, Sci. Asia **42** (2016), no. 3, 222–230. DOI 10.2306/scienceasia1513-1874.2016.42.222. 212
19. X. C. Xiao, Y. C. Zhu, and L. Găvrua, *Some properties of K-frames in Hilbert spaces*, Results Math. **63** (2013), no. 3–4, 1243–1255. Zbl 1268.42067. MR3057367. DOI 10.1007/s00025-012-0266-6. 207, 212, 216

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