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COMPLEX INTERPOLATION OF PREDUAL SPACES OF GENERAL LOCAL MORREY-TYPE SPACES

DENNY IVANAL HAKIM

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Abstract. In this article, we investigate the complex interpolation of predual spaces of general local Morrey-type spaces. By showing that these spaces are equal to the associate space of general local Morrey-type spaces, we prove that predual spaces of general local Morrey-type spaces behave well under the first complex interpolation.

1. Introduction

In this article, we establish that the first complex interpolation functor behaves well for local block spaces. These spaces are known as the *predual of local Morrey spaces* (see [11]). Let us recall the definition of Morrey spaces introduced in [17]. Let $0 < p \le \infty$ and $0 \le \lambda \le n/p$. The Morrey space $\mathcal{M}_p^\lambda = \mathcal{M}_p^\lambda(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}_p^\lambda} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda}\|f\chi_{B(x,r)}\|_{L^p} < \infty,$$

where $B(x,r)$ denotes the open ball centered at x of radius r. We note that if $\lambda = 0$, then the Morrey space $\mathcal{M}_p^\lambda$ is equal to the Lebesgue space L^p. When $\lambda = n/p$, we have $\mathcal{M}_p^\lambda = L^\infty$.

Interpolation of Morrey spaces can be traced back to the work of Stampacchia [22], where a generalization of the Riesz–Thorin complex interpolation was investigated. Although it was known in [3] and [20] that Morrey spaces do not have an

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interpolation property in general, some recent results on real and complex interpolation of Morrey spaces have been established. The complex interpolation of Morrey spaces was discussed in [9] and later in [15] and [16], and their generalizations can be found in [12]–[14], and [24]. The real interpolation of local Morrey-type spaces was established in [6]. The local Morrey space $L\mathcal{M}_p^\lambda = L\mathcal{M}_p^\lambda(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L\mathcal{M}_p^\lambda} := \sup_{r>0} r^{-\lambda} \|f\chi_{B(r)}\|_{L^p} < \infty,$$

where $B(r)$ denotes the open ball centered at the origin of radius $r > 0$. Furthermore, the results in [6] were extended to general local Morrey-type spaces and more general spaces in [7] and [18]. The definitions of general local Morrey-type spaces and general global Morrey-type spaces are given as follows.

Definition 1.1. Let $1 < p, q \le \infty$, and let w be a nonnegative measurable function on $(0, \infty)$. The *general local Morrey-type space* $L\mathcal{M}_{pq,w} = L\mathcal{M}_{pq,w}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L\mathcal{M}_{pq,w}} := \big\| w(r) \|f\chi_{B(r)}\|_{L^p} \big\|_{L^q(0,\infty)} < \infty.$$

The *global Morrey-type space* $G\mathcal{M}_{pq,w} = G\mathcal{M}_{pq,w}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{G\mathcal{M}_{pq,w}} := \sup_{x\in\mathbb{R}^n} \big\| f(x+\cdot) \big\|_{L\mathcal{M}_{pq,w}} < \infty.$$

Note that if $w(r) = 1$, then $L\mathcal{M}_{p\infty,w} = G\mathcal{M}_{p\infty,w} = L^p$. When $w(r) = r^{-\lambda}$, we can recover $G\mathcal{M}_{p\infty,w} = \mathcal{M}_p^\lambda$ and $L\mathcal{M}_{p\infty,w} = L\mathcal{M}_p^\lambda$. Useful applications of these spaces can be found in [19], where some qualitative properties, as regularity results, were obtained for solutions of equations considered in a context where the natural geometry is not Euclidean but is given as a group structure that relies on Lie groups. In order to guarantee that $L\mathcal{M}_{pq,w}$ contains nonzero functions, we will assume the following condition on w.

Definition 1.2. Let $1 < p, q \le \infty$. We define

$$\Omega_q := \big\{ w : (0,\infty) \to (0,\infty) : \|w\|_{L^q(t_0,\infty)} < \infty \text{ for some } t_0 > 0 \big\}$$

and

$$\Omega_{pq} := \big\{ w : (0,\infty) \to (0,\infty) : \big\| r^{n/p} w(r) \big\|_{L^q(0,t)} < \infty \text{ and } \|w\|_{L^q(t,\infty)} < \infty, \forall t > 0 \big\}.$$

Remark 1.3. The authors in [5, Lemma 1] showed that $w \in \Omega_q$ if and only if $L\mathcal{M}_{pq,w}$ is not equal to the set of all functions equivalent to the zero function. In addition, $w \in \Omega_{pq}$ is a necessary and sufficient condition for $G\mathcal{M}_{pq,w}$ to be nontrivial.

In this article, we will give a description of complex interpolation spaces for a predual of general local Morrey-type spaces. It was proved in [11] (see Theorem 2.6) that the predual of $L\mathcal{M}_{pq,\tilde{w}}$ can be characterized as the local block space $LH_{p'q',w}$, where $1 < p < \infty$, $1 < q \le \infty$, $\frac{1}{p'} := 1 - \frac{1}{p}$, $\frac{1}{q'} := 1 - \frac{1}{q}$, $\tilde{w}(t) = t^{-1/q} w(t)$, and w satisfies the following doubling condition: there exists a constant $C > 1$ such that $C^{-1}w(r) \le w(s) \le Cw(r)$ for every r, s satisfying $\frac{r}{s} \in (\frac{1}{2}, 2)$. The definition of $LH_{p'q',w}$ is given as follows.

Definition 1.4. Let $1 < p < \infty$, $1 < q \le \infty$, $w \in \Omega_q$, and $r \in (0,\infty)$. Let p' and q' be defined by $\frac{1}{p'} := 1 - \frac{1}{p}$ and $\frac{1}{q'} := 1 - \frac{1}{q}$. A measurable function A is called a *(p', w, r)-block* if $\mathrm{supp}(A) \subseteq B(r)$ and $\|A\|_{L^{p'}} \le w(r)$. We define

$$\dot{\mathcal{A}}_w(L^{p'}) := \left\{ \left\{(A_j, 2^j)\right\}_{j=-\infty}^{\infty} : A_j \text{ is a } (p', w, 2^j)\text{-block} \right\}.$$

The *local block space* $LH_{p'q',w}$ is defined by

$$LH_{p'q',w} := \left\{ \sum_{j=-\infty}^{\infty} \lambda_j A_j : \{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'} \text{ and } \left\{(A_j, 2^j)\right\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'}) \right\}.$$

The space $LH_{p'q',w}$ is equipped with the norm

$$\|f\|_{LH_{p'q',w}} := \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{1/q'}, \tag{1.1}$$

where the infimum is taken over all decompositions $f = \sum_{j=-\infty}^{\infty} \lambda_j A_j$, $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$, and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$.

Let $[\cdot,\cdot]_\theta$ be Calderón's first complex interpolation space. Our main results are given in the following theorems.

Theorem 1.5. *Let $\theta \in (0,1)$, $1 < p_0, p_1 < \infty$, $w_0 \in \Omega_{p_0\infty}$, and $w_1 \in \Omega_{p_1\infty}$. Assume that w_0 and w_1 satisfy the doubling condition, and assume that $\lim_{t\to\infty} w_0(t) = 0$ and $\lim_{t\to\infty} w_1(t) = 0$. In addition, assume that $w_0(t)^{p_0} = w_1(t)^{p_1}$. Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad w := w_0^{1-\theta} w_1^{\theta}.$$

Define $\frac{1}{p_0'} := 1 - \frac{1}{p_0}$, $\frac{1}{p_1'} := 1 - \frac{1}{p_1}$, and $\frac{1}{p'} := 1 - \frac{1}{p}$. Then

$$[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta = LH_{p'1,w}.$$

Theorem 1.6. *Let $\theta \in (0,1)$, $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 < \infty$, $w_0 \in \Omega_{p_0q_0}$, and $w_1 \in \Omega_{p_1q_1}$. Assume that w_0 and w_1 satisfy the doubling condition. In addition, assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$ and $w_0(t)^{q_0} = w_1(t)^{q_1}$. Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \text{and} \qquad w := w_0^{1-\theta} w_1^{\theta}.$$

Define $\frac{1}{p_0'} := 1 - \frac{1}{p_0}$, $\frac{1}{p_1'} := 1 - \frac{1}{p_1}$, $\frac{1}{p'} := 1 - \frac{1}{p}$, $\frac{1}{q_0'} := 1 - \frac{1}{q_0}$, $\frac{1}{q_1'} := 1 - \frac{1}{q_1}$, and $\frac{1}{q'} := 1 - \frac{1}{q}$. Then

$$[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta = LH_{p'q',w}.$$

We note that the complex interpolation of the predual of Morrey spaces can be seen in [4] and [23].

Our article is organized as follows. We recall the definition of complex interpolation spaces and some properties of $LH_{p'q',w}$ in Section 2. In Section 3, we prove the Fatou property of $LH_{p'q',w}$ following the idea in [21]. We also give a characterization of the associate space of general local Morrey type spaces. The proofs of Theorems 1.5 and 1.6 will be given in Section 4. In Section 5, we prove several lemmas used in Section 4. Finally, throughout the paper, by C_α we mean a positive constant that depends only on α. The inequality $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$ and C is independent of appropriate quantities. We write $A \sim B$ if and only if $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

2.1. Complex interpolation method. In this section, we recall the definition of the first complex interpolation method given in [2] and [8]. Let $\overline{X} = (X_0, X_1)$ be a compatible couple of Banach spaces. Let $\overline{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$, and let S be its interior.

Definition 2.1 (Calderón's first complex interpolation space [8, p. 114]). A function $F : \overline{S} \to X_0 + X_1$ is called a *first complex interpolation functor* if F satisfies the following properties:

(1) F is continuous on $\overline{S}$, F is holomorphic in S, and $\sup_{z\in\overline{S}} \|F(z)\|_{X_0+X_1} < \infty$,

(2) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

Denote by $\mathcal{F}(X_0, X_1)$ the set of all first complex interpolation functors. The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\big\{\sup_{t\in\mathbb{R}} \big\|F(it)\big\|_{X_0},\ \sup_{t\in\mathbb{R}} \big\|F(1 + it)\big\|_{X_1}\big\}.$$

Definition 2.2. Let $\theta \in (0, 1)$. The first complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) is defined by

$$[X_0, X_1]_\theta := \big\{f \in X_0 + X_1 :\ f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\big\}.$$

The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\big\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\big\}.$$

The fact that $[X_0, X_1]_\theta$ is a Banach space can be seen in [8] and [2, Theorem 4.1.2].

2.2. Some properties of $LH_{p'q',w}$. A nontrivial member of the space $LH_{p'q',w}$ is an $L^{p'}$-function supported on the ball B of dyadic radius.

Lemma 2.3. *Let $1 < p < \infty$, $1 < q \le \infty$, and $w \in \Omega_q$. If $A \in L^{p'}$ and $\mathrm{supp}(A) \subseteq B(2^j)$ for some $j \in \mathbb{Z}$, then $A \in LH_{p'q',w}$ and*

$$\|A\|_{LH_{p'q',w}} \le \frac{\|A\|_{L^{p'}}}{w(2^j)}. \tag{2.1}$$

Proof. If $\|A\|_{L^{p'}} = 0$, then $A = 0$, so (2.1) is trivial. Hence, we may assume that $\|A\|_{L^{p'}} \neq 0$. Define $\tilde{A} := \frac{w(2^j)A}{\|A\|_{L^{p'}}}$. Since $\mathrm{supp}(\tilde{A}) \subseteq B(2^j)$ and $\|\tilde{A}\|_{L^{p'}} = w(2^j)$, we know that $\tilde{A}$ is a $(p', w, 2^j)$-block. Moreover, (2.1) follows from $A = \frac{\|A\|_{L^{p'}}}{w(2^j)}\tilde{A}$. $\square$

We prove the following basic properties of $LH_{p'q',w}$. The first one is the lattice property of $LH_{p'q',w}$.

Lemma 2.4. *Let $1 < p < \infty$, $1 < q \leq \infty$, and $w \in \Omega_{pq}$. If $0 \leq f(x) \leq g(x)$ and $g \in LH_{p'q',w}$, then $f \in LH_{p'q',w}$ and*

$$\|f\|_{LH_{p'q',w}} \leq \|g\|_{LH_{p'q',w}}. \tag{2.2}$$

Proof. Given $\varepsilon > 0$, let $g = \sum_{j=-\infty}^{\infty} \lambda_j A_j$, where $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ satisfy

$$\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\varepsilon)\|g\|_{LH_{p'q',w}}. \tag{2.3}$$

For each $j \in \mathbb{Z}$, define $B_j := \chi_{\{g\neq 0\}}\frac{f}{g}A_j$. Then, $\{(B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ and

$$f = \chi_{\{g\neq 0\}}\frac{f}{g}\sum_{j=-\infty}^{\infty} \lambda_j A_j = \sum_{j=-\infty}^{\infty} \lambda_j B_j.$$

Consequently, $f \in LH_{p'q',w}$. From (2.3), it follows that

$$\|f\|_{LH_{p'q',w}} \leq (1+\varepsilon)\|g\|_{LH_{p'q',w}}. \tag{2.4}$$

By taking $\varepsilon \to 0$, we get (2.2). $\square$

Lemma 2.5. *Let $1 < p < \infty$, $1 < q \leq \infty$, and $w \in \Omega_{pq}$. If $f \in LH_{p'q',w}$, then $|f| \in LH_{p'q',w}$ with*

$$\big\||f|\big\|_{LH_{p'q',w}} = \|f\|_{LH_{p'q',w}}. \tag{2.5}$$

Proof. Let $\delta > 0$. Then there exist $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ such that

$$f = \sum_{j=-\infty}^{\infty} \lambda_j A_j, \qquad \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\delta)\|f\|_{LH_{p'q',w}}.$$

Since $|f| \leq \sum_{j=-\infty}^{\infty} |\lambda_j||A_j|$, $\{|\lambda_j|\}_{j=-\infty}^{\infty} \in \ell^{q'}$, and $\{(|A_j|, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$, by Lemma 2.4, we have $|f| \in LH_{p'q',w}$ and

$$\big\||f|\big\|_{LH_{p'q',w}} \leq \Big\|\sum_{j=-\infty}^{\infty} |\lambda_j||A_j|\Big\|_{LH_{p'q',w}} \leq \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\delta)\|f\|_{LH_{p'q',w}}.$$

By letting $\delta \to 0$, we get

$$\big\||f|\big\|_{LH_{p'q',w}} \leq \|f\|_{LH_{p'q',w}}. \tag{2.6}$$

Since $|f| \in LH_{p'q',w}$, for any $\varepsilon > 0$, we can find $\{\alpha_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and

$$\{(B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$$

such that

$$|f| = \sum_{j=\infty}^{\infty} \alpha_j B_j, \qquad \Big(\sum_{j=-\infty}^{\infty} |\alpha_j|^{q'}\Big)^{1/q'} \le (1+\varepsilon)\big\||f|\big\|_{LH_{p'q',w}}.$$

From $f = \sum_{j=-\infty}^{\infty} \alpha_j(\operatorname{sgn}(f)B_j)$ and $\{(\operatorname{sgn}(f)B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$, it follows that

$$\|f\|_{LH_{p'q',w}} \le \Big(\sum_{j=-\infty}^{\infty} |\alpha_j|^{q'}\Big)^{1/q'} \le (1+\varepsilon)\big\||f|\big\|_{LH_{p'q',w}}.$$

Since ε is arbitrary, we have

$$\|f\|_{LH_{p'q',w}} \le \big\||f|\big\|_{LH_{p'q',w}}. \tag{2.7}$$

Thus, (2.5) follows from (2.6) and (2.7). □

Now, we state the result in [11] about a characterization of a predual of general local Morrey-type spaces as certain local block spaces.

Theorem 2.6 ([11, Theorem 4.1]). *Let $1 < p < \infty$, $1 < q \le \infty$, and $w \in \Omega_{pq}$. Suppose that w satisfies the doubling condition. Define*

$$\tilde{w}(t) := \begin{cases} t^{-1/q} w(t), & q < \infty, \\ w(t), & q = \infty. \end{cases}$$

Then $(LH_{p'q',w})^* \simeq L\mathcal{M}_{pq,\tilde{w}}$ in the following sense.

(1) *For every $g \in L\mathcal{M}_{pq,\tilde{w}}$, the mapping L_g defined by*

$$L_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\,dx \quad (f \in LH_{p'q',w})$$

is a bounded linear functional on $LH_{p'q',w}$.

(2) *For every $L \in (LH_{p'q',w})^*$, there exists $g \in L\mathcal{M}_{pq,\tilde{w}}$ such that*

$$L = L_g \qquad \text{and} \qquad \|L\|_{(LH_{p'q',w})^*} \sim \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}.$$

3. A characterization of the associate space of $L\mathcal{M}_{pq,w}$

In this section, we prove that the associate space of general local Morrey-type spaces can be realized as certain block spaces. First, we recall the definition of associate space (see [1, Chapter 1]).

Definition 3.1. Let X be a Banach space of measurable functions on $\mathbb{R}^n$. The *associate space of* X, denoted by X', is defined to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{X'} := \sup_{\|g\|_X \le 1} \int_{\mathbb{R}^n} \big|f(x)g(x)\big|\,dx$$

is finite.

Next, we prove the Fatou property of $LH_{p'q',w}$.

Proposition 3.2. *Let $1 < p < \infty$, $1 < q \le \infty$, and $w \in \Omega_{pq}$. In addition, for $q = \infty$ only, assume that $\lim_{t\to\infty} w(t) = 0$. Suppose that $\{f_k\}_{k=1}^{\infty} \subseteq LH_{p'q',w}$ satisfies $0 \le f_k(x) \le f_{k+1}(x)$ for almost everywhere $x \in \mathbb{R}^n$ and for every $k \in \mathbb{N}$. If*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}} < \infty,$$

then $f := \lim_{k\to\infty} f_k \in LH_{p'q',w}$ and

$$\|f\|_{LH_{p'q',w}} = \sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}}. \tag{3.1}$$

Proof. Let $M := \sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}}$ and let $\varepsilon > 0$. For every $k \in \mathbb{N}$, we can choose $\{\lambda_{j,k}\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and $\{(A_{j,k}, 2^j)\}_{j=\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ such that $f_k = \sum_{j=-\infty}^{\infty} \lambda_{j,k} A_{j,k}$ and

$$\Big(\sum_{j=-\infty}^{\infty} |\lambda_{j,k}|^{q'}\Big)^{1/q'} \le (1+\varepsilon)\|f_k\|_{LH_{p'q',w}}.$$

Therefore, for each $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have

$$|\lambda_{j,k}| \le (1+\varepsilon)M$$

and

$$\|A_{j,k}\|_{L^{p'}} \le w(2^j). \tag{3.2}$$

Consequently, there exist $\{\lambda_{j,k_\ell}\}_{\ell=1}^{\infty} \subseteq \{\lambda_{j,k}\}_{k=1}^{\infty}$, $\{A_{j,k_\ell}\}_{\ell=1}^{\infty} \subseteq \{A_{j,k}\}_{k=1}^{\infty}$, $\lambda_j \in \mathbb{C}$, and $A_j \in L^{p'}$ such that $\{\lambda_{j,k_\ell}\}_{\ell=1}^{\infty}$ converges to λ_j and

$$\lim_{\ell\to\infty} \int_{\mathbb{R}^n} A_{j,k_\ell}(x)h(x)\,dx = \int_{\mathbb{R}^n} A_j(x)h(x)\,dx \tag{3.3}$$

for every $h \in L^p$. Moreover,

$$\begin{aligned}\big\|\{\lambda_j\}_{j\in\mathbb{Z}}\big\|_{\ell^{q'}} &= \Big(\sum_{j=-\infty}^{\infty} \lim_{\ell\to\infty} |\lambda_{j,k_\ell}|^{q'}\Big)^{1/q'} \\ &\le \liminf_{\ell\to\infty}\Big(\sum_{j=-\infty}^{\infty} |\lambda_{j,k_\ell}|^{q'}\Big)^{1/q'} \le (1+\varepsilon)M, \end{aligned}\tag{3.4}$$

$\mathrm{supp}(A_j) \subseteq B(2^j)$, and

$$\|A_j\|_{L^{p'}} = \sup_{\|\tilde{g}\|_{L^p}\le 1} \lim_{\ell\to\infty} \int_{\mathbb{R}^n} A_{j,k_\ell}(y)\tilde{g}(y)\,dy \le \limsup_{\ell\to\infty} \|A_{j,k_\ell}\|_{L^{p'}} \le w(2^j). \tag{3.5}$$

Define $g := \sum_{j=-\infty}^{\infty} \lambda_j A_j$. From (3.4) and (3.5), it follows that $g \in LH_{p'q',w}$. Therefore, if we can prove that

$$f(x) = g(x) \quad \text{a.e. } x \in \mathbb{R}^n, \tag{3.6}$$

then $f \in LH_{p'q',w}$. The proof of (3.6) goes as follows. Let $x \in \mathbb{R}^n \setminus \{0\}$. Once we can show that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} g(y)\,dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \tag{3.7}$$

for all balls $B(x,r)$ which satisfy $0 \notin B(x,2r)$, we have (3.6), by virtue of the Lebesgue differentiation theorem. By substituting $h(y) := \chi_{B(x,r)}(y)$ in (3.3) and using the fact that $B(x,r) \cap B(2^j) = \emptyset$ for every $j < \log_2 r$, we have

$$\begin{aligned}\int_{B(x,r)} g(y)\,dy &= \int_{B(x,r)} \sum_{j=-\infty}^{\infty} \lambda_j A_j(y)\,dy\\ &= \int_{B(x,r)} \sum_{j\in\mathbb{Z}, j\ge \log_2 r} \lambda_j A_j(y)\,dy.\end{aligned}$$

As a consequence of (3.4) and (3.5), we have

$$\begin{aligned}&\int_{B(x,r)} \sum_{j\in\mathbb{Z}, j\ge \log_2 r} \big|\lambda_j A_j(y)\big|\,dy\\ &\quad\le \big|B(x,r)\big|^{\frac{1}{p}} \sum_{j\in\mathbb{Z}, j\ge \log_2 r} \|\lambda_j A_j\|_{L^{p'}}\\ &\quad\le \big|B(x,r)\big|^{\frac{1}{p}} \Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\Big)^{\frac{1}{q'}} \Big(\sum_{j\in\mathbb{Z}, j\ge \log_2 r} \|A_j\|_{L^{p'}}^q\Big)^{\frac{1}{q}}\\ &\quad\le \big|B(x,r)\big|^{\frac{1}{p}} (1+\varepsilon) M \Big(\sum_{j\in\mathbb{Z}, j\ge \log_2 r} w(2^j)^q\Big)^{\frac{1}{q}},\end{aligned}$$

so, by using the doubling condition and $w \in \Omega_{pq}$, we get

$$\begin{aligned}\int_{B(x,r)} \sum_{j\in\mathbb{Z}, j\ge \log_2 r} \big|\lambda_j A_j(y)\big|\,dy &\lesssim \big|B(x,r)\big|^{\frac{1}{p}} (1+\varepsilon) M \Big(\sum_{j\in\mathbb{Z}, j\ge \log_2 r} \int_{2^j}^{2^{j+1}} \frac{w(t)^q}{t}\,dt\Big)^{\frac{1}{q}}\\ &\le \frac{|B(x,r)|^{\frac{1}{p}}(1+\varepsilon)M}{r^{\frac{1}{q}}} \Big(\int_r^\infty w(t)^q\,dt\Big)^{\frac{1}{q}} < \infty.\end{aligned}$$

Consequently, by virtue of the dominated convergence theorem, we have

$$\int_{B(x,r)} g(y)\,dy = \sum_{j\in\mathbb{Z}, j\ge \log_2 r} \int_{B(x,r)} \lambda_j A_j(y)\,dy.$$

Therefore, once we prove that

$$\sum_{j\in\mathbb{Z}: j\ge \log_2 r} \lambda_j \int_{B(x,r)} A_j(y)\,dy = \lim_{\ell\to\infty} \sum_{j\in\mathbb{Z}: j\ge \log_2 r} \lambda_{j,k_\ell} \int_{B(x,r)} A_{j,k_\ell}(y)\,dy, \tag{3.8}$$

we have

$$\begin{aligned}\int_{B(x,r)} g(y)\,dy &= \lim_{\ell\to\infty} \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\,dy \\ &= \lim_{\ell\to\infty} \int_{B(x,r)} \sum_{j=-\infty}^{\infty} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\,dy \\ &= \lim_{\ell\to\infty} \int_{B(x,r)} f_{k_\ell}(y)\,dy = \int_{B(x,r)} f(y)\,dy,\end{aligned}$$

so we arrive at (3.7). Hence, we only need to verify (3.8).

Now, assume that $q < \infty$. Since $\int_1^\infty \frac{w(t)^q}{t}\,dt \leq \int_1^\infty w(t)^q\,dt < \infty$, for every $\delta > 0$, there exists $J \in \mathbb{N} \cap (\log_2 r, \infty)$ such that

$$\Big(\int_{2^J-1}^{\infty} \frac{w(t)^q}{t}\,dt\Big)^{1/q} < \delta. \tag{3.9}$$

Therefore, for every $\ell \in \mathbb{N}$, we have

$$\begin{aligned}&\Big|\sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_j A_j(y)\,dy - \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\,dy\Big| \\ &\quad\leq \sum_{j\in\mathbb{Z}:j\geq J} \int_{B(x,r)} \big|\lambda_j A_j(y)\big|\,dy + \sum_{j\in\mathbb{Z}:j\geq J} \int_{B(x,r)} \big|\lambda_{j,k_\ell} A_{j,k_\ell}(y)\big|\,dy \\ &\qquad + \sum_{j\in Z:\log_2 r\leq j\leq J} \Big|\lambda_j \int_{B(x,r)} A_j(y)\,dy - \lambda_{j,k_\ell} \int_{B(x,r)} A_{j,k_\ell}(y)\,dy\Big|. \end{aligned}\tag{3.10}$$

Let I_1, I_2, and I_3 be the first, second, and third term in the right-hand side of (3.10), respectively. By using Hölder's inequality and (3.9), we have

$$\begin{aligned}I_1 &\leq \Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\Big)^{1/q'} \Big(\sum_{j=J}^{\infty}\Big(\int_{B(x,r)} \big|A_j(y)\big|\,dy\Big)^q\Big)^{1/q} \\ &\leq (1+\varepsilon)M\Big(\sum_{j=J}^{\infty}\big(\|A_j\|_{L^{p'}}\big|B(x,r)\big|^{1/p}\big)^q\Big)^{1/q} \\ &\leq (1+\varepsilon)M\big|B(x,r)\big|^{1/p}\Big(\sum_{j=J}^{\infty} w(2^j)^q\Big)^{1/q} \\ &\leq (1+\varepsilon)M\big|B(x,r)\big|^{1/p}\Big(\int_{2^J-1}^{\infty} \frac{w(t)^q}{t}\,dt\Big)^{1/q} \\ &\leq (1+\varepsilon)M\big|B(x,r)\big|^{1/p}\delta.\end{aligned}\tag{3.11}$$

Likewise,

$$I_2 \leq (1+\varepsilon)M\big|B(x,r)\big|^{1/p}\delta. \tag{3.12}$$

Meanwhile, from $\lim_{\ell\to\infty}\lambda_{j,k_\ell}=\lambda_j$ and (3.3), it follows that

$$\lim_{\ell\to\infty} I_3 = \sum_{j\in\mathbb{Z}:\log_2 r\le j\le J} \lim_{\ell\to\infty}\left|\lambda_j \int_{B(x,r)} A_j(y)\,dy - \lambda_{j,k_\ell}\int_{B(x,r)} A_{j,k_\ell}(y)\,dy\right| = 0. \tag{3.13}$$

By combining (3.10)–(3.13), we obtain

$$\begin{aligned}&\limsup_{\ell\to\infty}\left|\sum_{j\in\mathbb{Z}:j\ge\log_2 r}\int_{B(x,r)}\lambda_j A_j(y)\,dy - \sum_{j\in\mathbb{Z}:j\ge\log_2 r}\int_{B(x,r)}\lambda_{j,k_\ell}A_{j,k_\ell}(y)\,dy\right|\\ &\quad\le 2(1+\varepsilon)M\left|B(x,r)\right|^{1/p}\delta,\end{aligned} \tag{3.14}$$

so by taking $\delta\to 0^+$, we get (3.8) when $q<\infty$.

Now, we consider the case $q=\infty$. Since $\lim_{t\to\infty} w(t)=0$, for each $\delta>0$, there exists $K\in\mathbb{N}\cap(\log_2 r,\infty)$ such that

$$w(2^K)<\delta.$$

For every $\ell\in\mathbb{N}$, we have

$$\begin{aligned}&\left|\sum_{j\in\mathbb{Z}:j\ge\log_2 r}\int_{B(x,r)}\lambda_j A_j(y)\,dy - \sum_{j\in\mathbb{Z}:j\ge\log_2 r}\int_{B(x,r)}\lambda_{j,k_\ell}A_{j,k_\ell}(y)\,dy\right|\\ &\quad\le I_4+I_5+I_6,\end{aligned} \tag{3.15}$$

where

$$I_4 := \sum_{j=K}^{\infty}|\lambda_j|\int_{B(x,r)}\left|A_j(y)\right|dy,\qquad I_5 := \sum_{j=K}^{\infty}|\lambda_{j,k_\ell}|\int_{B(x,r)}\left|A_{j,k_\ell}(y)\right|dy,$$

and

$$I_6 := \left|\sum_{j\in\mathbb{Z}:\log_2 r\le j<K}\lambda_j\int_{B(x,r)}A_j(y)\,dy - \sum_{j\in\mathbb{Z}:\log_2 r\le j<K}\lambda_{j,k_\ell}\int_{B(x,r)}A_{j,k_\ell}(y)\,dy\right|.$$

By Hölder's inequality, we get

$$\begin{aligned}I_4 &\le \sum_{j=K}^{\infty}|\lambda_j|\|A_j\|_{L^{p'}}\left|B(x,r)\right|^{1/p}\\ &\le \left|B(x,r)\right|^{1/p}\sum_{j=K}^{\infty}|\lambda_j|w(2^j)\\ &\le \left|B(x,r)\right|^{1/p}w(2^K)\sum_{j=-\infty}^{\infty}|\lambda_j| \le \left|B(x,r)\right|^{1/p}(1+\varepsilon)M\delta.\end{aligned} \tag{3.16}$$

Similarly,

$$I_5 \le \left|B(x,r)\right|^{1/p}(1+\varepsilon)M\delta. \tag{3.17}$$

From $\lim_{\ell\to\infty} \lambda_{j,k_\ell} = \lambda_j$ and (3.3), it follows that $\lim_{\ell\to\infty} I_6 = 0$. We combine this and (3.15)–(3.17) to obtain

$$\begin{aligned}\limsup_{\ell\to\infty}&\Big|\sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_j A_j(y)\,dy - \sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_{j,k_\ell}A_{j,k_\ell}(y)\,dy\Big| \\ &\leq 2\big|B(x,r)\big|^{1/p}(1+\varepsilon)M\delta.\end{aligned}$$

Therefore, by taking $\delta\to 0$, we have (3.8) for $q=\infty$.

Finally, we prove (3.1). From $f=\sum_{j=-\infty}^{\infty}\lambda_j A_j$ and (3.4), it follows that

$$\|f\|_{LH_{p'q',w}} \leq (1+\varepsilon)M.$$

Since ε is arbitrary, we have

$$\|f\|_{LH_{p'q',w}} \leq M. \tag{3.18}$$

On the other hand, because of Lemma 2.4 and $0\leq f_k(x)\leq f(x)$ for all $k\in\mathbb{N}$, we have

$$\|f_k\|_{LH_{p'q',w}} \leq \|f\|_{LH_{p'q',w}},$$

and hence

$$M \leq \|f\|_{LH_{p'q',w}}. \tag{3.19}$$

Thus, (3.1) follows from (3.18) and (3.19). $\square$

Now, we arrive at the following characterization of the associate space of general local Morrey-type spaces. We note that a different characterization of the associate space of local Morrey-type spaces can be seen in [10, Theorem 4.3].

Proposition 3.3. *Let $1<p<\infty$, $1<q\leq\infty$, and $w\in\Omega_{pq}$. Assume that w satisfies the doubling condition. If $q=\infty$, assume also that $\lim_{t\to\infty} w(t)=0$. Define $\tilde{w}(t) := \begin{cases} t^{-1/q}w(t), & q<\infty, \\ w(t), & q=\infty.\end{cases}$ Then $(L\mathcal{M}_{pq,\tilde{w}})' = LH_{p'q',w}$.*

Proof. Let $f\in LH_{p'q',w}$. By Theorem 2.6, for every $g\in L\mathcal{M}_{pq,\tilde{w}}$ with $\|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\leq 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\big|f(x)g(x)\big|\,dx &= \int_{\mathbb{R}^n}\frac{f(x)}{\operatorname{sgn}(f(x))}\big|g(x)\big|\,dx \\ &\lesssim \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\|f\|_{LH_{p'q',w}} \leq \|f\|_{LH_{p'q',w}} < \infty.\end{aligned}$$

Therefore, $f\in(L\mathcal{M}_{pq,\tilde{w}})'$. Hence, $LH_{p'q',w}\subseteq(L\mathcal{M}_{pq,\tilde{w}})'$.

Let $f\in(L\mathcal{M}_{pq,\tilde{w}})'$ of norm 1. Then

$$\sup_{\|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\leq 1}\int_{\mathbb{R}^n}\big|f(x)g(x)\big|\,dx = 1. \tag{3.20}$$

Without loss of generality, we may assume that $f\geq 0$. For each $k\in\mathbb{N}$, define

$$g_k(x) := 2^k\chi_{B(2^k)}(x) \qquad \text{and} \qquad f_k(x) := \min\big(f(x), g_k(x)\big).$$

Since $\operatorname{supp}(g_k)=B(2^k)$ and $g_k\in L^{p'}$, by Lemma 2.3, we have $g_k\in LH_{p'q',w}$. From $0\leq f_k(x)\leq g_k(x)$ and Lemma 2.4, it follows that $f_k\in LH_{p'q',w}$. Thanks to the

Hahn–Banach theorem, there exists $L \in (LH_{p'q',w})^*$ such that $\|L\|_{(LH_{p'q',w})^*} = 1$ and

$$\|f_k\|_{LH_{p'q',w}} = \big|L(f_k)\big|. \tag{3.21}$$

According to Theorem 2.6, there exists $h \in L\mathcal{M}_{pq,\tilde{w}}$ such that $\|h\|_{L\mathcal{M}_{pq,\tilde{w}}} \sim 1$ and $L = L_h$. Consequently, by combining (3.20) and (3.21), we get

$$\|f_k\|_{LH_{p'q',w}} = \big|L_h(f_k)\big| \le \int_{\mathbb{R}^n} f_k(x)\big|h(x)\big|\,dx \le \|h\|_{L\mathcal{M}_{pq,\tilde{w}}} \sim 1,$$

so, $\sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}} \lesssim 1$. Since $\{f_k\}_{k=1}^\infty$ is increasing and $\lim_{k\to\infty} f_k(x) = f(x)$, by virtue of Proposition 3.2, we have $f \in LH_{p'q',w}$ and

$$\|f\|_{LH_{p'q',w}} \lesssim 1.$$

Hence, $(L\mathcal{M}_{pq,w})' \subseteq LH_{p'q',w}$. This completes the proof. □

We will use the following corollary.

Corollary 3.4. *Let $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 \le \infty$, $w_0 \in \Omega_{p_0q_0}$, and $w_1 \in \Omega_{p_1q_1}$. Assume that w_0 and w_1 satisfy the doubling condition. If $q_0 = \infty$ and $q_1 = \infty$, assume also that $\lim_{t\to\infty} w_0(t) = 0$ and $\lim_{t\to\infty} w_1(t) = 0$. Define*

$$\tilde{w}_0(t) := \begin{cases} t^{-1/q_0} w_0(t), & q_0 < \infty, \\ w_0(t), & q_0 = \infty, \end{cases} \qquad \tilde{w}_1(t) := \begin{cases} t^{-1/q_1} w_1(t), & q_1 < \infty, \\ w_1(t), & q_1 = \infty. \end{cases}$$

If $f \in LH_{p_0'q_0',w_0} + LH_{p_1'q_1',w_1}$ and $g \in L\mathcal{M}_{p_0q_0,\tilde{w}_0} \cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}$, then $fg \in L^1$ and

$$\int_{\mathbb{R}^n} \big|f(x)g(x)\big|\,dx \lesssim \|f\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} \|g\|_{L\mathcal{M}_{p_0q_0,\tilde{w}_0}\cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}}. \tag{3.22}$$

Proof. Let $f_0 \in LH_{p_0'q_0',w_0}$ and $f_1 \in LH_{p_1'q_1',w_1}$ be such that $f = f_0 + f_1$ and

$$\|f_0\|_{LH_{p_0'q_0',w_0}} + \|f_1\|_{LH_{p_1'q_1',w_1}} \le \|f\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}}. \tag{3.23}$$

Then, by Proposition 3.3, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \big|f(x)g(x)\big|\,dx &\le \int_{\mathbb{R}^n} \big|f_0(x)g(x)\big|\,dx + \int_{\mathbb{R}^n} \big|f_1(x)g(x)\big|\,dx \\ &\le \|f_0\|_{(L\mathcal{M}^{p_0q_0,\tilde{w}_0})'} \|g\|_{L\mathcal{M}^{p_0q_0,\tilde{w}_0}} + \|f_1\|_{(L\mathcal{M}^{p_1q_1,\tilde{w}_1})'} \|g\|_{L\mathcal{M}^{p_1q_1,\tilde{w}_1}} \\ &\lesssim \big(\|f_0\|_{LH_{p_0'q_0',w_0}} + \|f_1\|_{LH_{p_1'q_1',w_1}}\big) \|g\|_{L\mathcal{M}_{p_0q_0,\tilde{w}_0}\cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}}. \end{aligned} \tag{3.24}$$

Thus, (3.22) follows from (3.23) and (3.24). □

4. Proof of main theorems

4.1. Proof of Theorem 1.5. Without loss of generality, we may assume that $p_0 > p_1$. Since $\lim_{t\to\infty} w_0(t) = \lim_{t\to\infty} w_1(t) = 0$, we have $\lim_{t\to\infty} w(t) = 0$.

Let $f \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. We will show that $f \in LH_{p'1,w}$. According to Proposition 3.3, it suffices to show that

$$\sup_{\|g\|_{L\mathcal{M}_{p\infty,w}}\le 1} \int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}. \tag{4.1}$$

Choose $F \in \mathcal{F}(LH_{p_0'1,w_0}, LH_{p_1'1,w_1})$ such that $f = F(\theta,\cdot)$ and

$$\|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})} \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}. \tag{4.2}$$

Let $g \in L\mathcal{M}_{p\infty,w}$ with $\|g\|_{L\mathcal{M}_{p\infty,w}} \le 1$ and $M := \sup_{z\in\overline{S}} \|F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}$. For $k \in \mathbb{N}$ and $z \in \overline{S}$, define

$$H_k(z,x) := \chi_{\{f\neq 0\}}(x)\frac{|f(x)|}{f(x)}\left|g(x)\right|^{p(\frac{1-z}{p_0}+\frac{z}{p_1})}\chi_{\{\frac{1}{k}\le|g|\le k\}}(x) \quad (x\in\mathbb{R}^n)$$

and

$$\phi_k(z) := \int_{\mathbb{R}^n} F(z,x)H_k(z,x)\,dx. \tag{4.3}$$

Since

$$\phi_k(\theta) = \int_{\mathbb{R}^n} f(x)H_k(\theta,x)\,dx = \int_{\mathbb{R}^n} \left|f(x)g(x)\right|\chi_{\{\frac{1}{k}\le|g|\le k\}}(x)\,dx,$$

we have

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx = \lim_{k\to\infty} \phi_k(\theta).$$

Hence, in order to obtain (4.1), we only need to prove that

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}, \tag{4.4}$$

for all $k \in \mathbb{N}$.

Now, let us prove (4.4). For every $z \in \overline{S}$, we have

$$\left|H_k(z,x)\right| \le \left|g(x)\right|^{\frac{p}{p_0}}\left|g(x)\right|^{(\frac{p}{p_1}-\frac{p}{p_0})\operatorname{Re}(z)}\chi_{\{\frac{1}{k}\le|g|\le k\}}(x) \le k^{\frac{p}{p_1}-\frac{p}{p_0}}\left|g(x)\right|^{\frac{p}{p_0}} \tag{4.5}$$

and

$$\begin{aligned}\left|H_k(z,x)\right| &\le \left|g(x)\right|^{\frac{p}{p_1}}\left|g(x)\right|^{-(\frac{p}{p_1}-\frac{p}{p_0})(1-\operatorname{Re}(z))}\chi_{\{\frac{1}{k}\le|g|\le k\}}(x)\\ &\le k^{\frac{p}{p_1}-\frac{p}{p_0}}\left|g(x)\right|^{\frac{p}{p_1}}.\end{aligned} \tag{4.6}$$

Note that our assumptions imply that $w_0(r)^{p_0} = w_1(r)^{p_1} = w(r)^p$. Therefore,

$$\left\||g|^{\frac{p}{p_0}}\right\|_{L\mathcal{M}_{p_0\infty,w_0}} = \sup_{r>0} w(r)^{\frac{p}{p_0}}\|g\|_{L^p(B(r))}^{\frac{p}{p_0}} = \|g\|_{L\mathcal{M}_{p\infty,w}}^{\frac{p}{p_0}} \le 1 \tag{4.7}$$

and

$$\left\||g|^{\frac{p}{p_1}}\right\|_{L\mathcal{M}_{p_1\infty,w_1}} = \sup_{r>0} w(r)^{\frac{p}{p_1}}\|g\|_{L^p(B(r))}^{\frac{p}{p_1}} = \|g\|_{L\mathcal{M}_{p\infty,w}}^{\frac{p}{p_1}} \le 1. \tag{4.8}$$

By combining (4.5)–(4.8), we have $H_k(z) \in L\mathcal{M}_{p_0\infty,w_0} \cap L\mathcal{M}_{p_1\infty,w_1}$ with

$$\big\|H_k(z,\cdot)\big\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}} \le k^{\frac{p}{p_1}-\frac{p}{p_0}}. \tag{4.9}$$

From the last inequality and Corollary 3.4, it follows that

$$\big|\phi_k(z)\big| \lesssim k^{\frac{p}{p_1}-\frac{p}{p_0}}\big\|F(z,\cdot)\big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}. \tag{4.10}$$

Therefore,

$$\sup_{z\in\overline{S}}\big|\phi_k(z)\big| \lesssim k^{\frac{p}{p_1}-\frac{p}{p_0}}M < \infty. \tag{4.11}$$

Next, we estimate $|\phi_k(z)|$ on the boundary of $\overline{S}$. Let $t \in \mathbb{R}$. As a consequence of (4.7) and $|H_k(it,x)| \le |g(x)|^{\frac{p}{p_0}}$, we have

$$\big\|H_k(it,\cdot)\big\|_{L\mathcal{M}_{p_0\infty,w_0}} \le 1.$$

From the last inequality and Proposition 3.3, it follows that

$$\begin{aligned}\big|\phi_k(it)\big| &\le \int_{\mathbb{R}^n}\big|F(it,x)H_k(it,x)\big|\,dx\\ &\le \big\|F(it,\cdot)\big\|_{(L\mathcal{M}_{p_0\infty,w_0})'}\\ &\lesssim \big\|F(it,\cdot)\big\|_{LH_{p_0'1,w_0}}\\ &\le \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})}.\end{aligned} \tag{4.12}$$

By a similar argument, we also have

$$\big|\phi_k(1+it)\big| \lesssim \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})}. \tag{4.13}$$

In view of Lemma 5.1 and (4.11), we may use the three-lines lemma together with (4.2), (4.12), and (4.13) to obtain

$$\begin{aligned}\phi_k(\theta) &\le \big(\sup_{t\in\mathbb{R}}\big|\phi_k(it)\big|\big)^{1-\theta}\big(\sup_{t\in\mathbb{R}}\big|\phi_k(1+it)\big|\big)^{\theta}\\ &\le \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})} \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}.\end{aligned}$$

Thus, the proof of $[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \subseteq LH_{p'1,w}$ is complete.

Conversely, we will show that $LH_{p'1,w} \subseteq [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. Let $f \in LH_{p'1,w}$. Then, there exist $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^1$ and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ such that

$$f = \sum_{j=-\infty}^{\infty}\lambda_j A_j \qquad\text{and}\qquad \sum_{j=-\infty}^{\infty}|\lambda_j| \lesssim \|f\|_{LH_{p'1,w}}. \tag{4.14}$$

For every $J \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_J(z,\cdot) := \sum_{j=-J}^{J}\lambda_j\frac{w_0(2^j)^{1-z}w_1(2^j)^z}{w(2^j)^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}}\operatorname{sgn}\big(A_j(\cdot)\big)\big|A_j(\cdot)\big|^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}. \tag{4.15}$$

We claim that

$$F_J(\theta,\cdot) \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \tag{4.16}$$

and

$$\lim_{\substack{J,K\to\infty\\ J>K}} \left\| F_J(\theta,\cdot) - F_K(\theta,\cdot)\right\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta} = 0. \tag{4.17}$$

The proofs of (4.16) and (4.17) will be given in Section 5.1. As a consequence of (4.17), there exists $g \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$ such that

$$\lim_{J\to\infty} \left\| F_J(\theta,\cdot) - g\right\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta} = 0. \tag{4.18}$$

From (4.18) and $[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \subseteq LH_{p'1,w}$, it follows that

$$\lim_{J\to\infty} \left\| F_J(\theta,\cdot) - g\right\|_{LH_{p'1,w}} = 0. \tag{4.19}$$

Since $f - F_J(\theta,\cdot) = \sum_{j\in\mathbb{Z},|j|>J} \lambda_j A_j(\cdot)$, we have

$$\left\| f - F_J(\theta,\cdot)\right\|_{LH_{p'1,w}} \le \sum_{j\in\mathbb{Z},|j|>J} |\lambda_j| \to 0.$$

Therefore,

$$\lim_{J\to\infty} \left\| f - F_J(\theta,\cdot)\right\|_{LH_{p'1,w}} = 0. \tag{4.20}$$

By combining (4.19) and (4.20), we have $f = g$. Hence, $f \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$.

4.2. Proof of Theorem 1.6. Without loss of generality, assume that $p_0 > p_1$. First, we prove that

$$[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta \subseteq LH_{p'q',w}. \tag{4.21}$$

Let $f \in [LH_{p_0'q_0,w_0}, LH_{p_1'q_1',w_1}]_\theta$. We will show that $f \in LH_{p'q',w}$. Let $\tilde{w}(t) := t^{-\frac{1}{q}} w(t)$. By Proposition 3.3, we only need to show that

$$\sup_{\|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\le 1} \int_{\mathbb{R}^n} \left| f(x)g(x)\right| dx \lesssim \|f\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta}. \tag{4.22}$$

Let $g \in L\mathcal{M}_{pq,\tilde{w}}$ with $\|g\|_{L\mathcal{M}_{pq,\tilde{w}}} \le 1$. By the definition of $[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta$, there exists $F \in \mathcal{F}(LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1})$ such that $f = F(\theta,\cdot)$ and

$$\|F\|_{\mathcal{F}(LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1})} \lesssim \|f\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta}. \tag{4.23}$$

For $k \in \mathbb{N}$ and $z \in \overline{S}$, define

$$H_k(z,x) := \chi_{\{f\neq 0\}}(x) \frac{|f(x)|}{f(x)} \left| g(x)\right|^{p(\frac{1-z}{p_0}+\frac{z}{p_1})} \chi_{\{\frac{1}{k}\le |g|\le k\}}(x)$$

and

$$\phi_k(z) := \int_{\mathbb{R}^n} F(z,x) H_k(z,x)\, dx.$$

Since

$$\begin{aligned}\lim_{k\to\infty}\phi_k(\theta) &= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(x)H_k(\theta,x)\,dx\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n}\left|f(x)g(x)\right|\chi_{\{\frac{1}{k}\le|g|\le k\}}(x)\,dx\\ &= \int_{\mathbb{R}^n}\left|f(x)g(x)\right|dx,\end{aligned}$$

the inequality (4.22) can be obtained if we can show that

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta} \tag{4.24}$$

for all $k\in\mathbb{N}$.

The proof of (4.24) goes as follows. Let $z\in\overline{S}$. Let $\tilde{w_0}(t) := t^{-\frac{1}{q_0}}w_0(t)$ and $\tilde{w_1}(t) := t^{-\frac{1}{q_1}}w_1(t)$. From $w_0(t)^{q_0} = w_1(t)^{q_1}$, $w(t) = w_0(t)^{1-\theta}w_1(t)^\theta$, and $\frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_0}$, it follows that

$$w_0(t)^{q_0} = w_1(t)^{q_1} = w(t)^q,$$

which yields

$$\tilde{w}_0(t)^{q_0} = \tilde{w}_1(t)^{q_1} = \tilde{w}(t)^q.$$

By using this identity, for every $j\in\{0,1\}$, we have

$$\begin{aligned}\left\||g|^{\frac{p}{p_j}}\right\|_{L\mathcal{M}_{p_jq_j,\tilde{w}_j}} &= \left\|\tilde{w}(r)^{\frac{q}{q_j}}\left\||g|^{\frac{p}{p_j}}\chi_{B(r)}\right\|_{L^{p_j}(B(r))}\right\|_{L^{q_j}(0,\infty)}\\ &= \left\|\left(\tilde{w}(r)\|g\chi_{B(r)}\|_{L^p(B(r))}\right)^{\frac{p}{p_j}}\right\|_{L^{q_j}(0,\infty)}\\ &= \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}^{\frac{p}{p_j}} \le 1.\end{aligned} \tag{4.25}$$

From the inequalities (4.5), (4.6), and (4.25), it follows that $H_k(z,\cdot)\in L\mathcal{M}_{p_0q_0,w_0}\cap L\mathcal{M}_{p_1q_1,w_1}$ and

$$\left\|H_k(z,\cdot)\right\|_{L\mathcal{M}_{p_0q_0,w_0}\cap L\mathcal{M}_{p_1q_1,w_1}} \le k^{\frac{p}{p_0}-\frac{p}{p_1}}. \tag{4.26}$$

Therefore, by combining (3.22) and (4.26), we have

$$\sup_{z\in\overline{S}}\left|\phi_k(z)\right| \lesssim k^{\frac{p}{p_1}-\frac{p}{p_0}}\sup_{z\in\overline{S}}\left\|F(z)\right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} < \infty. \tag{4.27}$$

Next, we estimate $|\phi_k(j+it)|$ for every $j\in\{0,1\}$ and $t\in\mathbb{R}$. Since $|H_k(j+it,\cdot)|\le|g|^{p/p_j}$, by virtue of (4.25), we have

$$\left\|H_k(j+it,\cdot)\right\|_{L\mathcal{M}_{p_0q_0,\tilde{w}_0}} \le 1. \tag{4.28}$$

Consequently, by combining (4.23), (4.28), and Proposition 3.3, we see that

$$\begin{aligned}\left|\phi_k(j+it)\right| &\le \int_{\mathbb{R}^n}\left|F(j+it,x)H_k(j+it,x)\right|dx\\ &\le \left\|F(j+it,\cdot)\right\|_{(L\mathcal{M}_{p_0q_0,\tilde{w}_0})'}\end{aligned}$$

$$\begin{aligned}&\sim \big\|F(j+it,\cdot)\big\|_{LH_{p_0'q_0',w_0}}\\&\le \|F\|_{\mathcal{F}(LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1})} \lesssim \|f\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta}.\end{aligned} \tag{4.29}$$

By using an argument analogous to that of the proof of Lemma 5.1, we have that $\phi_k(z)$ is continuous on $\overline{S}$ and holomorphic in S. Since (4.27) holds. we may use the three-lines lemma and the estimate (4.29) to obtain

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta},$$

as desired. Thus, the proof of (4.21) is complete.

Now, suppose that $f \in LH_{p'q',w}$. We will show that $f \in [LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta$. Write

$$f = \sum_{j=-\infty}^{\infty} \lambda_j A_j \tag{4.30}$$

for some $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ satisfying

$$\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\Big)^{\frac{1}{q'}} \lesssim \|f\|_{LH_{p'q',w}}. \tag{4.31}$$

For $J \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_J(z,\cdot) := \sum_{j=-J}^{J} \mathrm{sgn}(\lambda_j)|\lambda_j|^{q'(\frac{1-z}{q_0'}+\frac{z}{q_1'})} \frac{w_0(2^j)^{1-z}w_1(2^j)^z}{w(2^j)^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}} \mathrm{sgn}\big(A_j(\cdot)\big)\big|A_j(\cdot)\big|^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}. \tag{4.32}$$

We claim that

$$F_J(\theta,\cdot) \in [LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta \tag{4.33}$$

and

$$\lim_{\substack{J,K\to\infty\\ J>K}} \big\|F_J(\theta,\cdot)-F_K(\theta,\cdot)\big\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta} = 0. \tag{4.34}$$

We postpone the proofs of (4.33) and (4.34) to Section 5.2. As a consequence of (4.33) and (4.34), there exists $g \in [LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta$ such that

$$\lim_{J\to\infty} \big\|F_J(\theta,\cdot)-g\big\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta} = 0. \tag{4.35}$$

By combining (4.35) and $[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta \subseteq LH_{p'q',w}$, we have

$$\lim_{J\to\infty} \big\|F_J(\theta,\cdot)-g\big\|_{LH_{p'q',w}} = 0. \tag{4.36}$$

From $f - F_J(\theta,\cdot) = \sum_{j\in\mathbb{Z},|j|>J} \lambda_j A_j(\cdot)$, (4.36), and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$, it follows that

$$\begin{aligned}\|f-g\|_{LH_{p'q',w}} &\le \big\|f-F_J(\theta,\cdot)\big\|_{LH_{p'q',w}} + \big\|F_J(\theta,\cdot)-g\big\|_{LH_{p'q',w}}\\&\le \Big(\sum_{j\in\mathbb{Z},|j|>J} |\lambda_j|^{q'}\Big)^{\frac{1}{q'}} + \big\|F_J(\theta,\cdot)-g\big\|_{LH_{p'q',w}} \to 0\end{aligned}$$

as $J \to \infty$. Consequently, $f = g$. Hence, $f \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. Thus, the proof of Theorem 1.6 is complete. $\square$

5. Auxiliary lemmas

In this section, we provide the proof of continuity and holomorphicity of the function $\phi_k(z)$ in Section 4.1. We also prove (4.16), (4.17), (4.33), and (4.34).

Lemma 5.1. *Let $\phi_k(z)$ be defined by (4.3). Then $\phi_k(z)$ is continuous on $\overline{S}$ and holomorphic in S.*

Proof. Let $z_1, z_2 \in \overline{S}$, $P := \frac{p}{p_1} - \frac{p}{p_0}$, and $M := \sup_{z\in\overline{S}} \|F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}$. By combining (3.22), (4.9), and

$$\begin{aligned}\big|H_k(z_2,x) - H_k(z_1,x)\big| &\le \big|H_k(z_1,x)\big|\big(\big|g(x)\big|^{P|z_2-z_1|} - 1\big)\\ &\le \big|H_k(z_1,x)\big|(k^{P|z_2-z_1|} - 1),\end{aligned}$$

we obtain

$$\begin{aligned}&\big|\phi_k(z_2) - \phi_k(z_1)\big|\\ &\quad\le \int_{\mathbb{R}^n} \big|F(z_2,x) - F(z_1,x)\big|\big|H_k(z_2,x)\big|\,dx\\ &\qquad+ \int_{\mathbb{R}^n} \big|F(z_1,x)\big(H_k(z_2,x) - H_k(z_1,x)\big)\big|\,dx\\ &\quad\le k^P\big\|F(z_2,\cdot) - F(z_1,\cdot)\big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} + k^P M(k^{P|z_2-z_1|} - 1). \qquad (5.1)\end{aligned}$$

Hence, the continuity of $\phi_k(z)$ on $\overline{S}$ follows from (5.1) and the continuity of $F : \overline{S} \to LH_{p_0'1,w_0} + L\mathcal{M}_{p_1'1,w_1}$.

Now, we prove the holomorphicity of $\phi_k(z)$. Let $z \in S$. Then there exists $F'(z,\cdot) \in LH_{p_0'1,w_0} + LH_{p_1'1,w_1}$ such that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}} \Big\|\frac{F(z,\cdot) - F(z,\cdot)}{h} - F'(z,\cdot)\Big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} = 0. \qquad (5.2)$$

Define $H_k'(z,\cdot) = P \cdot H_k(z,\cdot)\log|g|$ and

$$\phi_k'(z) := \int_{\mathbb{R}^n} F'(z,x)H_k(z,x) + F(z,x)H_k'(z,x)\,dx.$$

Let $h \in \mathbb{C}$ be such that $z + h \in \overline{S}$. From (3.22), it follows that

$$\Big|\frac{\phi_k(z+h) - \phi_k(z)}{h} - \phi_k'(z)\Big| \le I_1 + I_2 + I_3, \qquad (5.3)$$

where

$$\begin{aligned}I_1 :=\ &\big\|H_k(z,\cdot)\big\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_0\infty,w_0}}\\ &\times \Big\|\frac{F(z,\cdot) - F(z,\cdot)}{h} - F'(z,\cdot)\Big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}, \qquad (5.4)\end{aligned}$$

$$I_2 := \big\|F(z+h,\cdot)\big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} \times \Big\|\frac{H_k(z+h,\cdot)-H_k(z,\cdot)}{h} - H_k'(z,\cdot)\Big\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}}, \tag{5.5}$$

and

$$I_3 := \big\|H_k'(z,\cdot)\big\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}} \big\|F(z+h,\cdot)-F(z,\cdot)\big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}. \tag{5.6}$$

Combining (4.9), (5.3)–(5.6), and

$$\Big|\frac{H_k(z+h,x)-H_k(z,x)}{h} - H_k'(z,x)\Big| \le \big|H_k(z,x)\big|(k^{P|h|}-1)P\log k,$$

we get

$$\begin{aligned}
&\Big|\frac{\phi_k(z+h)-\phi_k(z)}{h} - \phi_k'(z)\Big| \\
&\quad\le k^P\Big\|\frac{F(z,\cdot)-F(z,\cdot)}{h} - F'(z,\cdot)\Big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} + MC_{k,P}(k^{P|h|}-1) \\
&\qquad + Pk^P(\log k)\big\|F(z+h,\cdot)-F(z,\cdot)\big\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}.
\end{aligned} \tag{5.7}$$

As a result of the continuity of $F(z,\cdot)$ and (5.2), the right-hand side of (5.7) tends to zero as $h \to 0$, so $\phi_k'(z)$ is the derivative of $\phi_k(z)$. Thus, $\phi_k(z)$ is holomorphic. □

5.1. Proofs of (4.16) and (4.17). In this section, f is any function in $LH_{p'1,w}$ such that (4.14) holds, and $F_J(z)$ is defined by (4.15). For each $j \in [-J,J]\cap\mathbb{Z}$, we define

$$u_j := \frac{w_0(2^j)^{-1}w_1(2^j)}{w(2^j)^{\frac{p'}{p_1'}-\frac{p'}{p_0'}}}|A_j|^{\frac{p'}{p_1'}-\frac{p'}{p_0'}}. \tag{5.8}$$

For each $j \in [-J,J]\cap\mathbb{Z}$ and $z \in \overline{S}$, define

$$G_j(z,\cdot) := \frac{w_0(2^j)^{1-z}w_1(2^j)^z}{w(2^j)^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}}\operatorname{sgn}\big(A_j(\cdot)\big)\big|A_j(\cdot)\big|^{p'(\frac{1-z}{p_0'}+\frac{z}{p_1'})}, \tag{5.9}$$

$$G_{j,0}(z,\cdot) := G_j(z,\cdot)\chi_{\{u_j\le 1\}}, \qquad G_{j,1}(z,\cdot) := G_j(z,\cdot) - G_{j,0}(z,\cdot). \tag{5.10}$$

We prove (4.16) by checking the conditions given in Definition 2.1. We will use the following calculation of the norm of some blocks.

Lemma 5.2. *Let $k \in \{0,1\}$. Then, for each $j \in \mathbb{Z}\cap[-J,J]$,*

$$\Big\|\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p_k'}}}|A_j|^{\frac{p'}{p_k'}}\Big\|_{LH_{p_k'1,w_k}} \le 1. \tag{5.11}$$

Proof. The inequality (5.11) follows from Lemma 2.3, $\||A_j|^{p'/p_k'}\|_{L^{p_k'}} = \|A_j\|_{L^{p'}}^{p'/p_k'}$, and $\|A_j\|_{L^{p'}} \le w(2^j)$. □

Lemma 5.3. *For every $z \in \overline{S}$, we have $F_J(z,\cdot) \in LH_{p_0'1,w_0} + LH_{p_1'1,w_1}$. Moreover,* $\sup_{z\in\overline{S}} \|F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} < \infty$.

Proof. We decompose $F_J(z,\cdot) = F_{J,0}(z,\cdot) + F_{J,1}(z,\cdot)$, where

$$F_{J,0}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,0}(z,\cdot), \qquad F_{J,1}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,1}(z,\cdot). \tag{5.12}$$

Combining Lemma 5.2 and

$$\left|G_{j,0}(z,\cdot)\right| = \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}} \left|A_j(\cdot)\right|^{\frac{p'}{p_0'}} u_j^{\mathrm{Re}(z)} \chi_{\{u_j \le 1\}} \le \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}} \left|A_j(\cdot)\right|^{\frac{p'}{p_0'}}$$

we have

$$\left\|F_{J,0}(z,\cdot)\right\|_{LH_{p_0'1,w_0}} \le \sum_{j=-J}^{J} |\lambda_j| \|G_{j,0}\|_{LH_{p_0'1,w_0}} \le \sum_{j=-J}^{J} |\lambda_j| \lesssim \|f\|_{LH_{p'1,w}} < \infty.$$

Therefore, $F_{J,0}(z,\cdot) \in LH_{p_0'1,w_0}$. Similarly, we also have $F_{J,1}(z) \in LH_{p_1'1,w_1}$ with

$$\left\|F_{J,1}(z,\cdot)\right\|_{LH_{p_1'1,w_1}} \lesssim \|f\|_{LH_{p'1,w}}.$$

Since $F_J(z,\cdot) = F_{J,1}(z,\cdot) + F_{J,2}(z,\cdot)$, we have $F_J(z) \in LH_{p_0'1,w_0} + LH_{p_1'1,w_1}$ and

$$\begin{aligned} \left\|F_J(z,\cdot)\right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} &\le \left\|F_{J,0}(z,\cdot)\right\|_{LH_{p_0'1,w_0}} + \left\|F_{J,1}(z,\cdot)\right\|_{LH_{p_1'1,w_1}} \\ &\lesssim \|f\|_{LH_{p'1,w}}. \end{aligned} \tag{5.13}$$

Thus, $\sup_{z\in\overline{S}} \|F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} \lesssim \|f\|_{LH_{p'1,w}} < \infty$. □

Lemma 5.4. *The function $F_J : \overline{S} \to LH_{p_0'1,w_0} + LH_{p_1'1,w_1}$ is continuous.*

Proof. Let $z \in \overline{S}$. We will show that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}} \left\|F_J(z+h,\cdot) - F_J(z,\cdot)\right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} = 0. \tag{5.14}$$

Let $F_{J,0}$ and $F_{J,1}$ be defined by (5.12). For every $h \in \mathbb{C}$ satisfying $z + h \in \mathbb{C}$, we have

$$\begin{aligned} &\left\|F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)\right\|_{LH_{p_0'1,w_0}} \\ &\le \sum_{j=-J}^{J} |\lambda_j| \left\|G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)\right\|_{LH_{p_0'1,w_0}} \\ &\le \sum_{j=-J}^{J} \frac{|\lambda_j|}{w_0(2^j)} \left\|G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)\right\|_{L^{p_0'}}. \end{aligned}$$

Since $\lim_{h\to 0} G_{j,0}(z+h,x) - G_{j,0}(z,x) = G_{j,0}(z,x)\lim_{h\to 0}(u_j(x)^h - 1) = 0$,

$$\begin{aligned} &\big|G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)\big| \\ &\quad \le \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}}(u_j^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}} + u_j^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}}) \\ &\quad \le 2\frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}}, \end{aligned}$$

and $|A_j|^{\frac{p'}{p'_0}} \in L^{p'_0}$, because of the dominated convergence theorem we have

$$\lim_{h\to 0}\big\|G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)\big\|_{L^{p'_0}} = 0.$$

Consequently,

$$\lim_{h\to 0}\big\|F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)\big\|_{LH_{p'_0 1,w_0}} = 0. \tag{5.15}$$

By a similar argument, we also have

$$\lim_{h\to 0}\big\|F_{J,1}(z+h,\cdot) - F_{J,1}(z,\cdot)\big\|_{LH_{p'_1 1,w_0}} = 0. \tag{5.16}$$

Combining (5.15), (5.16), and

$$\begin{aligned} &\big\|F_J(z+h,\cdot) - F_J(z,\cdot)\big\|_{LH_{p'_0 1,w_0}+LH_{p'_1 1,w_1}} \\ &\quad \le \big\|F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)\big\|_{LH_{p'_0 1,w_0}} \\ &\qquad + \big\|F_{J,1}(z+h,\cdot) - F_{J,1}(z,\cdot)\big\|_{LH_{p'_1 1,w_1}}, \end{aligned}$$

we get (5.14). □

Lemma 5.5. *The function $F_J : S \to LH_{p'_0 1,w_0} + LH_{p'_1 1,w_1}$ is holomorphic.*

Proof. Let $\varepsilon \in (0,1/2)$. It suffices to show that

$$F'_J(z) := \sum_{j=-J}^{J}\lambda_j G_j(z,\cdot)\log\big(u_j(\cdot)\big) \in LH_{p'_0 1,w_0} + LH_{p'_1 1,w_1} \tag{5.17}$$

for every $z \in S$ and

$$\lim_{\substack{h\to 0\\ z+h\in S}}\Big\|\frac{F_J(z+h) - F_J(z)}{h} - F_J(z)\Big\|_{LH_{p'_0 1,w_0}+LH_{p'_1 1,w_1}} = 0 \tag{5.18}$$

for every $z \in S_\varepsilon := \{\tilde z \in S : \varepsilon < \mathrm{Re}(\tilde z) < 1-\varepsilon\}$. We define

$$\begin{aligned} F'_{J,0}(z,\cdot) &:= \sum_{j=-J}^{J}\lambda_j G_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big), \\ F'_{J,1}(z,\cdot) &:= \sum_{j=-J}^{J}\lambda_j G_{j,1}(z,\cdot)\log\big(u_j(\cdot)\big). \end{aligned} \tag{5.19}$$

Since

$$\begin{aligned}\big|G_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big)\big| &= \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}}u_j(\cdot)^{\mathrm{Re}(z)}\big|\log\big(u_j(\cdot)\big)\big|\chi_{\{u_j\le 1\}}\\ &\le \frac{w_0(2^j)}{e\,\mathrm{Re}(z)w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}},\end{aligned}$$

by Lemma 5.2, we have

$$\begin{aligned}\big\|F'_{J,0}(z,\cdot)\big\|_{LH_{p'_0 1,w_0}} &\le \sum_{j=-J}^{J}|\lambda_j|\bigg\|\frac{w_0(2^j)}{e\,\mathrm{Re}(z)w(2^j)^{\frac{p'}{p'_0}}}|A_j|^{p'/p'_0}\bigg\|_{LH_{p'_0 1,w_0}}\\ &\le \frac{1}{e\,\mathrm{Re}(z)}\sum_{j=-J}^{J}|\lambda_j|\\ &\lesssim \|f\|_{LH_{p'1,w}}<\infty,\end{aligned}$$

so $F'_{J,0}(z,\cdot)\in LH_{p'_0 1,w_0}$. By a similar argument, we have $F'_{J,1}(z,\cdot)\in LH_{p'_1 1,w_1}$. Since $F'_J(z,\cdot)=F'_{J,0}(z,\cdot)+F'_{J,1}(z,\cdot)$, we conclude that $F'_J(z,\cdot)\in LH_{p'_0 1,w_0}+LH_{p'_1 1,w_1}$.

Now, we prove (5.18). Let $z\in S_\varepsilon$ and $h\in C$ be such that $z+h\in S$ and $|h|<\frac{\varepsilon}{2}$. Since

$$\begin{aligned}\big|G_{j,0}(z)\big| &= \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}u_j(\cdot)^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}}\\ &\le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}\chi_{\{u_j\le 1\}},\end{aligned}$$

we have

$$\begin{aligned}&\bigg|\frac{G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)}{h}-G_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big)\bigg|\\ &\quad=\big|G_{j,0}(z,\cdot)\big|\bigg|\frac{u_j(\cdot)^h-1-h\log(u_j(\cdot))}{h}\bigg|\\ &\quad\le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}u_j(\cdot)^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}}|h|\big(\log\big(u_j(\cdot)\big)\big)^2\\ &\qquad\times\sum_{k=2}^{\infty}\frac{|h\log(u_j(\cdot))|^{k-2}}{k!}\\ &\quad\le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}u_j(\cdot)^{\varepsilon}\chi_{\{u_j\le 1\}}|h|\big(\log\big(u_j(\cdot)\big)\big)^2e^{-|h|\log(u_j(\cdot))}. \qquad (5.20)\end{aligned}$$

Combining (5.20) and $\sup_{0<t\le 1} t^{\varepsilon/2}(\log t)^2 = \frac{16}{e^2\varepsilon^2}$, we get

$$\begin{aligned}
&\left|\frac{G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)}{h} - G_{j,0}(z,\cdot)\log\bigl(u_j(\cdot)\bigr)\right| \\
&\quad \le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} u_j(\cdot)^{\varepsilon/2}\chi_{\{u_j\le 1\}}|h|\bigl(\log\bigl(u_j(\cdot)\bigr)\bigr)^2 \\
&\quad \le C_\varepsilon |h| \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}. \qquad (5.21)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|\frac{F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)}{h} - F'_{J,0}(z,\cdot)\right\|_{LH_{p'_0 1,w_0}} \\
&\le C_\varepsilon |h| \sum_{j=-J}^{J} |\lambda_j| \left\|\frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}\right\|_{LH_{p'_0 1,w_0}} \le C_\varepsilon |h| \sum_{j=-J}^{J} |\lambda_j|.
\end{aligned}$$

Consequently,

$$\lim_{h\to 0}\left\|\frac{F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)}{h} - F'_{J,0}(z,\cdot)\right\|_{LH_{p'_0 1,w_0}} = 0. \qquad (5.22)$$

Likewise,

$$\lim_{h\to 0}\left\|\frac{F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)}{h} - F'_{J,1}(z,\cdot)\right\|_{LH_{p'_1 1,w_1}} = 0. \qquad (5.23)$$

Thus, (5.18) follows from (5.22) and (5.23). □

Lemma 5.6. *For each $k \in \{0,1\}$, the function $t \in \mathbb{R} \mapsto F_J(k+it) \in LH_{p'_k 1,w_k}$ is bounded and continuous.*

Proof. First, we will show that

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\bigl\|F_J(k+it)\bigr\|_{LH_{p'_k 1,w_k}} < \infty. \qquad (5.24)$$

For each $k \in \{0,1\}$, $t \in \mathbb{R}$, and $j \in [-J,J]\cap\mathbb{Z}$, we have

$$\bigl|G_j(k+it)\bigr| = \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|u_j^{it}||A_j|^{\frac{p'}{p'_k}} = \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|A_j|^{\frac{p'}{p'_k}}, \qquad (5.25)$$

so, by virtue of Lemma 5.2, we have

$$\bigl\|F_J(k+it)\bigr\|_{LH_{p'_k 1,w_k}} \le \sum_{j=-J}^{J}|\lambda_j| \lesssim \|f\|_{LH_{p'_k 1,w_k}}.$$

Therefore,

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\bigl\|F_J(k+it)\bigr\|_{LH_{p' 1,w}} \lesssim \|f\|_{LH_{p' 1,w}} < \infty.$$

Now, we prove the continuity of $t \in \mathbb{R} \mapsto F_J(k+it)$. Let $t_0 \in \mathbb{R}$. As a result of Lemma 2.3, we have

$$\begin{aligned}
&\left\|F_J(k+it,\cdot)-F_J(k+it_0,\cdot)\right\|_{LH_{p'_k 1,w_k}} \\
&\quad\le \sum_{j=-J}^{J}\left\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\right\|_{LH_{p'_k 1,w_k}} \\
&\quad\le \sum_{j=-J}^{J}\frac{\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{L^{p'_k}}}{w_k(2^j)}. \qquad (5.26)
\end{aligned}$$

Since $|A_j|^{\frac{p'}{p'_k}} \in L^{p'_k}$, $|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)| \le 2\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}}|A_j|^{\frac{p'}{p'_k}}$, and

$$\lim_{t\to t_0} G_j(k+it,x)-G_j(k+it,x) = G_j(k+it_0,x)\lim_{t\to t_0} u_j(x)^{t-t_0}-1=0,$$

because of the dominated convergence theorem we have

$$\lim_{t\to t_0}\left\|G_j(k+it)-G_j(k+it_0)\right\|_{L^{p'_k}}=0. \qquad (5.27)$$

By combining (5.26) and (5.27), we get

$$\lim_{t\to t_0}\left\|F_J(k+it,\cdot)-F_J(k+it,\cdot)\right\|_{LH_{p'_k 1,w_k}}=0,$$

as desired. □

From Lemmas 5.3–5.6, it follows that $F_J \in \mathcal{F}(LH_{p'_0 1,w_0}, LH_{p'_1 1,w_1})$. Consequently, $F_J(\theta) \in [LH_{p'_0 1,w_0}, LH_{p'_1 1,w_1}]_\theta$. Thus, it remains to prove (4.17).

Proof of (4.17). Let $J, K \in \mathbb{N}$ with $J > K$. From (5.25) and Lemma 5.2, it follows that

$$\begin{aligned}
&\left\|F_J(\theta,\cdot)-F_K(\theta,\cdot)\right\|_{[LH_{p'_0 1,w_0},LH_{p'_1 1,w_1}]_\theta} \\
&\quad\le \max_{k=0,1}\sup_{t\in\mathbb{R}}\left\|F_J(k+it,\cdot)-F_K(k+it,\cdot)\right\|_{LH_{p'_k 1,w_k}} \\
&\quad\le \max_{k=0,1}\sup_{t\in\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|\left\|G_j(k+it,\cdot)\right\|_{LH_{p'_k 1,w_k}} \\
&\quad= \max_{k=0,1}\sup_{t\in\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|\left\|\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}}|A_j(\cdot)|^{\frac{p'}{p'_k}}\right\|_{LH_{p'_k 1,w_k}} \le \sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|. \qquad (5.28)
\end{aligned}$$

Since $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^1$, we see that

$$\lim_{\substack{J,K\to\infty\\ J>K}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|=0. \qquad (5.29)$$

Thus, (4.17) follows from (5.28) and (5.29). □

5.2. Proofs of (4.33) and (4.34). Let $F_J(z)$ be defined by (4.32). For each $j \in [-J, J] \cap \mathbb{Z}$, we define

$$u_j := |\lambda_j|^{\frac{q'}{q'_1} - \frac{q'}{q'_0}} \frac{w_0(2^j)^{-1} w_1(2^j)}{w(2^j)^{\frac{p'}{p'_1} - \frac{p'}{p'_0}}} |A_j|^{\frac{p'}{p'_1} - \frac{p'}{p'_0}}. \tag{5.30}$$

For each $j \in [-J, J] \cap \mathbb{Z}$ and $z \in \overline{S}$, define

$$G_j(z, \cdot) := \operatorname{sgn}(\lambda_j) |\lambda_j|^{q'(\frac{1-z}{q'_0} + \frac{z}{q'_1})} \frac{w_0(2^j)^{1-z} w_1(2^j)^z}{w(2^j)^{p'(\frac{1-z}{p'_0} + \frac{z}{p'_1})}} \operatorname{sgn}\big(A_j(\cdot)\big) \big|A_j(\cdot)\big|^{p'(\frac{1-z}{p'_0} + \frac{z}{p'_1})}. \tag{5.31}$$

Lemma 5.7. *For every $z \in \overline{S}$ and $j \in [-J, J] \cap \mathbb{Z}$, define*

$$G_{j,0}(z, \cdot) := G_j(z, \cdot) \chi_{\{u_j \le 1\}}, \qquad G_{j,1}(z, \cdot) := G_j(z, \cdot) - G_{j,0}(z, \cdot). \tag{5.32}$$

Then, for each $k \in \{0, 1\}$, we have

$$\big|G_{j,k}(z, \cdot)\big| \le |\lambda_j|^{\frac{q'}{q'_k}} \frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} \big|A_j(\cdot)\big|^{\frac{p'}{p'_0}}. \tag{5.33}$$

Proof. We prove (5.33) only for $k = 0$. We leave the case $k = 1$ to the reader because there are no differences. Since $\operatorname{Re}(z) \ge 0$, we have

$$\big|G_{j,1}(z, \cdot)\big| = |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} \big|A_j(\cdot)\big|^{\frac{p'}{p'_0}} u_j^{\operatorname{Re}(z)} \chi_{\{u_j \le 1\}} \le |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} \big|A_j(\cdot)\big|^{\frac{p'}{p'_0}},$$

as desired. □

Lemma 5.8. *Let $k \in \{0, 1\}$ and $j \in [-J, J] \cap \mathbb{Z}$. Then $\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} |A_j|^{p'/p'_k}$ is a $(p'_k, w_k, 2^j)$-block.*

Proof. Since $\operatorname{supp}(A_j) \subseteq B(2^j)$, we have $\operatorname{supp}(\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} |A_j|^{p'/p'_k}) \subseteq B(2^j)$. Moreover, from $\|A_j\|_{L^{p'}} \le w(2^j)$ it follows that

$$\left\| \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} |A_j|^{p'/p'_k} \right\|_{L^{p'_k}} = \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} \|A_j\|_{L^{p'}}^{p'/p'_k} \le w_k(2^j).$$

Thus, $\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} |A_j|^{p'/p'_k}$ is a $(p'_k, w_k, 2^j)$-block. □

Lemma 5.9. *For every $z \in \overline{S}$, we have $F_J(z, \cdot) \in LH_{p'_0 q'_0, w_0} + LH_{p'_1 q'_1, w_1}$. Moreover, $\sup_{z \in \overline{S}} \|F_J(z, \cdot)\|_{LH_{p'_0 q'_0, w_0} + LH_{p'_1 q'_1, w_1}} < \infty$.*

Proof. We define

$$F_{J,0}(z, \cdot) := \sum_{j=-J}^{J} G_{j,0}(z, \cdot), \qquad F_{J,1}(z, \cdot) := \sum_{j=-J}^{J} G_{j,1}(z, \cdot). \tag{5.34}$$

We use (2.2), (4.31), and Lemma 5.8 to obtain

$$\begin{aligned}\left\|F_{J,0}(z,\cdot)\right\|_{LH_{p_0'q_0',w_0}} &\le \left\|\sum_{j=-J}^{J}\left|G_{j,0}(z,\cdot)\right|\right\|_{LH_{p_0'q_0',w_0}}\\ &\le \left\|\sum_{j=-J}^{J}|\lambda_j|^{\frac{q'}{q_0'}}\frac{w_0(2^j)}{w(2^j)^{p'/p_0'}}|A_j|^{p'/p_0'}\right\|_{LH_{p_0'1,w_0}}\\ &\le \left(\sum_{j=-J}^{J}\left(|\lambda_j|^{q'/q_0'}\right)^{q_0'}\right)^{1/q_0'} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'}<\infty.\end{aligned}$$

Therefore, $F_{J,0}(z,\cdot)\in LH_{p_0'q_0',w_0}$. Likewise, $F_{J,1}(z)\in LH_{p_1'q_1',w_1}$ with

$$\left\|F_{J,1}(z,\cdot)\right\|_{LH_{p_1'q_1',w_1}} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_1'}.$$

Consequently, $F_J(z,\cdot)\in LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}$ and

$$\begin{aligned}\left\|F_J(z,\cdot)\right\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} &\le \left\|F_{J,0}(z,\cdot)\right\|_{LH_{p_0'q_0',w_0}}+\left\|F_{J,1}(z,\cdot)\right\|_{LH_{p_1'q_1',w_1}}\\ &\lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'}+\|f\|_{LH_{p'q'.w}}^{q'/q_1'}. \end{aligned}\tag{5.35}$$

Thus, $\sup_{z\in\overline{S}}\|F_J(z,\cdot)\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'}+\|f\|_{LH_{p'q'.w}}^{q'/q_1'}<\infty.$ □

Lemma 5.10. *The function $F_J:\overline{S}\to LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}$ is continuous.*

Proof. Let $z\in\overline{S}$. For $h\in\mathbb{C}$ with $z+h\in\overline{S}$, we have

$$\begin{aligned}&\left\|F_J(z+h,\cdot)-F_J(z,\cdot)\right\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}}\\ &\quad\le \left\|F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)\right\|_{LH_{p_0'q_0',w_0}}\\ &\qquad+\left\|F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)\right\|_{LH_{p_1'q_1',w_1}}.\end{aligned}\tag{5.36}$$

Hence, it suffices to show that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}}\left\|F_{J,k}(z+h,\cdot)-F_{J,k}(z,\cdot)\right\|_{LH_{p_k'q_k',w_k}}=0\tag{5.37}$$

for each $k\in\{0,1\}$. By Lemma 2.3, we have

$$\begin{aligned}\left\|F_{J,k}(z+h,\cdot)-F_{J,k}(z,\cdot)\right\|_{LH_{p_k'q_k',w_k}} &\le \sum_{j=-J}^{J}\left\|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\right\|_{LH_{p_k'q_k',w_k}}\\ &\le \sum_{j=-J}^{J}\frac{\|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\|_{L^{p_k'}}\cdot}{w_k(2^j)}.\end{aligned}\tag{5.38}$$

According to Lemma 5.7, we have

$$\begin{aligned}\big|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\big| &\le \big|G_{j,k}(z+h,\cdot)\big|+\big|G_{j,k}(z,\cdot)\big|\\ &\le 2|\lambda_j|^{\frac{q'}{q'_k}}\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_k}}.\end{aligned}$$

Since $\lim_{h\to 0} G_{j,k}(z+h,x)-G_{j,k}(z,x) = G_{j,k}(z,x)\lim_{h\to 0}(u_j(x)^h-1)=0$ and $|A_j|^{\frac{p'}{p'_k}}\in L^{p'_k}$, by the dominated convergence theorem we have

$$\lim_{h\to 0}\big\|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\big\|_{L^{p'_k}}=0. \tag{5.39}$$

Thus, we obtain (5.37) by combining (5.38) and (5.39). $\square$

Lemma 5.11. *The function $F_J : S\to LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}$ is holomorphic.*

Proof. Let $\varepsilon\in(0,1/2)$ and $S_\varepsilon:=\{z\in S:\varepsilon<\mathrm{Re}(z)<1-\varepsilon\}$. We will show that

$$F'_J(z):=\sum_{j=-J}^{J}G_j(z,\cdot)\log\big(u_j(\cdot)\big)\in LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1} \tag{5.40}$$

for every $z\in S$ and

$$\lim_{\substack{h\to 0\\ z+h\in S}}\Big\|\frac{F_J(z+h)-F_J(z)}{h}-F'_J(z)\Big\|_{LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}}=0 \tag{5.41}$$

for every $z\in S_\varepsilon$. We define

$$\begin{aligned} &F'_{J,0}(z,\cdot):=\sum_{j=-J}^{J}\lambda_jG_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big) \qquad \text{and}\\ &F'_{J,1}(z,\cdot):=\sum_{j=-J}^{J}\lambda_jG_{j,1}(z,\cdot)\log\big(u_j(\cdot)\big).\end{aligned} \tag{5.42}$$

Since

$$\begin{aligned}\big|G_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big)\big| &= |\lambda_j|^{\frac{q'}{q'_0}}\frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}}u_j(\cdot)^{\mathrm{Re}(z)}\big|\log\big(u_j(\cdot)\big)\big|\chi_{\{u_j\le 1\}}\\ &\le |\lambda_j|^{\frac{q'}{q'_0}}\frac{w_0(2^j)}{e\,\mathrm{Re}(z)w(2^j)^{\frac{p'}{p'_0}}}\big|A_j(\cdot)\big|^{\frac{p'}{p'_0}},\end{aligned}$$

by Lemma 5.8, we have

$$\big\|F'_{J,0}(z,\cdot)\big\|_{LH_{p'_0q'_0,w_0}}\le\frac{1}{e\,\mathrm{Re}(z)}\Big(\sum_{j=-J}^{J}|\lambda_j|^{q'}\Big)^{1/q'_0}\lesssim\frac{\|f\|_{LH_{p'q',w}}^{q'/q'_0}}{e\,\mathrm{Re}(z)}<\infty,$$

so $F'_{J,0}(z,\cdot)\in LH_{p'_0q'_0,w_0}$. Similarly, $F'_{J,1}(z,\cdot)\in LH_{p'_1q'_1,w_1}$. Since

$$F'_J(z,\cdot)=F'_{J,0}(z,\cdot)+F'_{J,1}(z,\cdot),$$

we see that $F'_J(z,\cdot)\in LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}$.

The proof of (5.41) is obtained in a way similar to that of (5.18). Let $z \in S_\varepsilon$ and $h \in C$ be such that $z+h \in S$ and $|h| < \frac{\varepsilon}{2}$. By a calculation similar to that in (5.21), we obtain

$$\begin{aligned}\Big|\frac{G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)}{h} &- G_{j,0}(z,\cdot)\log\big(u_j(\cdot)\big)\Big| \\ &\le C_\varepsilon |h| |\lambda_j|^{\frac{q'}{q_0'}} \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p_0'}}}{w(2^j)^{\frac{p'}{p_0'}}}.\end{aligned}$$

The last inequality, (4.31), and Lemma 5.8 imply that

$$\begin{aligned}\Big\|\frac{F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)}{h} - F_{J,0}'(z,\cdot)\Big\|_{LH_{p_0'q_0',w_0}} &\le C_\varepsilon |h| \Big(\sum_{j=-J}^{J} |\lambda_j|^{q'}\Big)^{\frac{1}{q_0'}} \\ &\lesssim C_\varepsilon |h| \|f\|_{LH_{p'q',w}}^{\frac{q'}{q_0'}}.\end{aligned}$$

Consequently,

$$\lim_{h\to 0}\Big\|\frac{F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)}{h} - F_{J,0}'(z,\cdot)\Big\|_{LH_{p_0'q_0',w_0}} = 0. \tag{5.43}$$

Similarly,

$$\lim_{h\to 0}\Big\|\frac{F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)}{h} - F_{J,1}'(z,\cdot)\Big\|_{LH_{p_1'q_1',w_1}} = 0. \tag{5.44}$$

Thus, (5.41) follows from (5.43) and (5.44). □

Lemma 5.12. *For each $k \in \{0,1\}$, the function $t \in \mathbb{R} \mapsto F_J(k+it) \in LH_{p_k'q_k',w_k}$ is bounded and continuous.*

Proof. First, we will prove that

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\big\|F_J(k+it)\big\|_{LH_{p_k'q_k',w_k}} < \infty. \tag{5.45}$$

Let $k \in \{0,1\}$, $t \in \mathbb{R}$, and $j \in [-J,J]\cap\mathbb{Z}$. By virtue of Lemma 5.8 and

$$\begin{aligned}\big|G_j(k+it)\big| &= |\lambda_j|^{\frac{q'}{q_k'}} \frac{w_k(2^j)}{w(2^j)^{p'/p_k'}} |u_j^{it}||A_j|^{\frac{p'}{p_k'}} \\ &= |\lambda_j|^{\frac{q'}{q_k'}} \frac{w_k(2^j)}{w(2^j)^{p'/p_k'}} |A_j|^{\frac{p'}{p_k'}}\end{aligned} \tag{5.46}$$

we have

$$\begin{aligned}\big\|F_J(k+it)\big\|_{LH_{p_k'q_k',w_k}} &\le \Big(\sum_{j=-J}^{J}|\lambda_j|^{q'}\Big)^{1/q_k'} \\ &\lesssim \|f\|_{LH_{p'q',w}}^{q'/q_k'}.\end{aligned}$$

Therefore,

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\big\|F_J(k+it)\big\|_{LH_{p_k'q_k',w_k}} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_k'} < \infty.$$

The proof of the continuity of $t\in\mathbb{R}\mapsto F_J(k+it)$ goes as follows. Let $t_0\in\mathbb{R}$. According to Lemma 2.3, we have

$$\begin{aligned}
&\big\|F_J(k+it,\cdot)-F_J(k+it_0,\cdot)\big\|_{LH_{p_k'q_k',w_k}}\\
&\quad\le \sum_{j=-J}^{J}|\lambda_j|\big\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\big\|_{LH_{p_k'q_k',w_k}}\\
&\quad\le \sum_{j=-J}^{J}|\lambda_j|\frac{\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{L^{p_k'}}}{w_k(2^j)}. \qquad (5.47)
\end{aligned}$$

Since $|A_j|^{\frac{p'}{p_k'}}\in L^{p_k'}$, $|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)|\le 2|\lambda_j|^{\frac{q'}{q_k'}}\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p_k'}}}|A_j|^{\frac{p'}{p_k'}}$, and

$$\lim_{t\to t_0}G_j(k+it,x)-G_j(k+it,x)=G_j(k+it_0,x)\lim_{t\to t_0}\big(u_j(x)^{t-t_0}-1\big)=0,$$

by the dominated convergence theorem we have

$$\lim_{t\to t_0}\big\|G_j(k+it)-G_j(k+it_0)\big\|_{L^{p_k'}}=0. \qquad (5.48)$$

By combining (5.47) and (5.48), we get

$$\lim_{t\to t_0}\big\|F_J(k+it,\cdot)-F_J(k+it,\cdot)\big\|_{LH_{p_k'q_k',w_k}}=0,$$

as desired. □

According to Lemmas 5.9–5.12, we have $F_J\in\mathcal{F}(LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1})$. Therefore, $F_J(\theta,\cdot)\in[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta$. Thus, it remains to prove (4.34).

Proof of (4.34). Let $J,K\in\mathbb{N}$ with $J>K$. As a result of (5.46) and Lemma 5.46, we have

$$\begin{aligned}
&\big\|F_J(\theta,\cdot)-F_K(\theta,\cdot)\big\|_{[LH_{p_0'q_0',w_0},LH_{p_1'q_1',w_1}]_\theta}\\
&\quad\le \max_{k=0,1}\sup_{t\in\mathbb{R}}\big\|F_J(k+it,\cdot)-F_K(k+it,\cdot)\big\|_{LH_{p_k'q_k',w_k}}\\
&\quad= \max_{k=0,1}\sup_{t\in\mathbb{R}}\Big\|\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}G_j(k+it,\cdot)\Big\|_{LH_{p_k'q_k',w_k}}\\
&\quad\le \Big(\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|^{q'}\Big)^{\frac{1}{q_k'}}. \qquad (5.49)
\end{aligned}$$

Since $\{\lambda_j\}_{j=-\infty}^{\infty}\in\ell^{q'}$, the right-hand side of (5.49) tends to zero as $J,K\to\infty$, which implies (4.17). □

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Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo, 192-0397, Japan.
E-mail address: dennyivanalhakim@gmail.com