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Locally Finite Reducts of Heyting Algebras and Canonical Formulas

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Abstract The variety of Heyting algebras has two well-behaved locally finite reducts, the variety of bounded distributive lattices and the variety of implicative semilattices. The variety of bounded distributive lattices is generated by the $\to$-free reducts of Heyting algebras, while the variety of implicative semilattices is generated by the $\vee$-free reducts. Each of these reducts gives rise to canonical formulas that generalize Jankov formulas and provide an axiomatization of all superintuitionistic logics (si-logics for short).

The $\vee$-free reducts of Heyting algebras give rise to the $(\wedge, \to)$-canonical formulas that we studied in an earlier work. Here we introduce the $(\wedge, \vee)$-canonical formulas, which are obtained from the study of the $\to$-free reducts of Heyting algebras. We prove that every si-logic is axiomatizable by $(\wedge, \vee)$-canonical formulas. We also discuss the similarities and differences between these two kinds of canonical formulas.

One of the main ingredients of these formulas is a designated subset D of pairs of elements of a finite subdirectly irreducible Heyting algebra A. When $D = A^2$, we show that the $(\wedge, \vee)$-canonical formula of A is equivalent to the Jankov formula of A. On the other hand, when $D = \emptyset$, the $(\wedge, \vee)$-canonical formulas produce a new class of si-logics we term stable si-logics. We prove that there are continuum many stable si-logics and that all stable si-logics have the finite model property. We also compare stable si-logics to splitting and subframe si-logics.

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1 Introduction

We recall that superintuitionistic logics (si-logics for short) are extensions of the intuitionistic propositional calculus **IPC**. Consistent si-logics are also known as *intermediate logics* because they are situated between **IPC** and the classical propositional calculus **CPC**. The problem of axiomatizing a si-logic is a central topic in the area. One of the first general methods of axiomatizing large classes of si-logics was developed by Jankov [19]. For each finite subdirectly irreducible Heyting algebra A, Jankov designed a formula that encodes the structure of A. The main property of the Jankov formula $\chi(A)$ is that a Heyting algebra B refutes $\chi(A)$ if and only if A is isomorphic to a subalgebra of a homomorphic image of B. In [20] Jankov utilized this method to show that there are continuum many si-logics; in fact, continuum many si-logics axiomatized by Jankov formulas. However, not every si-logic is axiomatizable by Jankov formulas. The si-logics axiomatizable by Jankov formulas are exactly the splitting logics and their joins in the lattice of si-logics.

The reason that there exist si-logics that are not axiomatizable by Jankov formulas is that the variety of Heyting algebras is not locally finite. Indeed, as follows from Citkin [12], if L is a si-logic whose corresponding variety V_L of Heyting algebras is locally finite (such logics are called *locally tabular*), then each extension of L is axiomatized over L by Jankov formulas. In fact, L itself is axiomatized over **IPC** by Jankov formulas (see Tomaszewski [25], Bezhanishvili [8], [9]).

Although the variety of Heyting algebras is not locally finite, it has well-behaved locally finite reducts. For example, as Diego has shown in [14] the variety of implicative semilattices is locally finite, and it is generated by the $\vee$-free reducts of Heyting algebras. This allows us to generalize Jankov's method as follows.

For a finite subdirectly irreducible Heyting algebra A, we can design a Jankov-like formula that encodes fully the structure of the $\vee$-free reduct of A, and only partially the behavior of $\vee$. In other words, if B is a Heyting algebra and $h : A \to B$ is a map that preserves all Heyting operations except $\vee$, then h may still preserve $\vee$ for some elements of A. This can be encoded in the formula by postulating that $\vee$ is preserved for only those pairs of elements of A that belong to some designated subset D of A^2. This results in the formula that has properties similar to the Jankov formula of A, but captures the behavior of A not with respect to Heyting homomorphisms, but rather morphisms that preserve the $\vee$-free reduct of A. We call this new formula the *canonical formula* of A because of its connection to Zakharyaschev's canonical formulas. As is shown in [3], a Heyting algebra B refutes the canonical formula of A if and only if there is an embedding of A into a homomorphic image of B that preserves the $\vee$-free reduct of A and only preserves $\vee$ for the pairs of elements from D. In particular, if $D = A^2$, then the embedding is a Heyting embedding and the canonical formula of A is the Jankov formula of A. The key result of [3] is that canonical formulas axiomatize all si-logics.

As we pointed out, this algebraic approach to canonical formulas has a model-theoretic counterpart. Model-theoretic analogues of Jankov formulas were developed by de Jongh [13] for si-logics and by Fine [17] for modal logics. Zakharyaschev generalized Fine's approach, developed the model-theoretic theory of canonical formulas (in [28] for si-logics and in [27], [29] for modal logics), and showed that each superintuitionistic and transitive modal logic is axiomatizable by canonical formulas.

The link between the algebraic and model-theoretic approaches to canonical formulas is via duality. In the intuitionistic case, there is a well-known duality between Heyting algebras and descriptive Kripke frames, which can be described as special ordered topological spaces, known as *Esakia spaces*. Esakia duality (see [15]) establishes that the category of Heyting algebras and Heyting homomorphisms is dually equivalent to the category of Esakia spaces and Esakia morphisms. In order to account for morphisms between Heyting algebras that preserve the $\vee$-free reduct, a generalized Esakia duality was developed in [3]. On the dual side, this results in the notion of partial Esakia morphisms, which are closely related to Zakharyaschev's subreductions. Moreover, Zakharyaschev's key notion of the closed domain condition corresponds on the algebra side to preserving $\vee$ for the elements from the designated subset D of A^2. Consequently, the canonical formulas described above are algebraic counterparts of Zakharyaschev's canonical formulas. Each si-logic may be axiomatized either by Zakharyaschev's canonical formulas or by their algebraic counterparts.

Now, Heyting algebras have another very natural locally finite reduct, the variety of bounded distributive lattices, which is generated by the $\to$-free reducts of Heyting algebras. In this paper we develop an algebraic theory of canonical formulas based on this locally finite reduct of Heyting algebras. In order to distinguish between these two variants of canonical formulas, we call the canonical formulas based on the $(\wedge, \to)$-reducts the $(\wedge, \to)$-canonical formulas, and the canonical formulas based on the $(\wedge, \vee)$-reducts the $(\wedge, \vee)$-canonical formulas.

For a finite subdirectly irreducible Heyting algebra A and $D \subseteq A^2$, we design the $(\wedge, \vee)$-canonical formula of A that encodes fully the structure of the $\to$-free reduct of A, and only partially the behavior of $\to$. We prove that a Heyting algebra B refutes the $(\wedge, \vee)$-canonical formula of A if and only if there is a bounded lattice embedding of A into a subdirectly irreducible homomorphic image of B that preserves $\to$ for the pairs of elements from D. One of our main results shows that each si-logic is axiomatizable by $(\wedge, \vee)$-canonical formulas, which is parallel to what happens in the theory of $(\wedge, \to)$-canonical formulas. The main idea of the proof for the $(\wedge, \vee)$-case is similar to that of the $(\wedge, \to)$-case, but uses the $\to$-free locally finite reduct of Heyting algebras instead of the $\vee$-free locally finite reduct. However, there are a number of subtle differences as well, which will be discussed in the body of the paper.

We provide a dual characterization of $(\wedge, \vee)$-canonical formulas. While the $(\wedge, \to)$-case is based on the generalized Esakia duality, the $(\wedge, \vee)$-case is based on Priestley duality for bounded distributive lattices (see [21], [22]). As in the $(\wedge, \to)$-case, we show that Jankov formulas are a particular case of $(\wedge, \vee)$-canonical formulas, and are obtained by letting $D = A^2$. Another extreme case is when $D = \emptyset$. In the case of $(\wedge, \to)$-canonical formulas, setting $D = \emptyset$ produces the algebraic counterpart of subframe formulas, which axiomatize subframe logics (see Fine [18]). In the $(\wedge, \vee)$-case, $D = \emptyset$ produces a new class of si-logics, which we term *stable si-logics*. As in the case of subframe logics, we prove that all stable si-logics have the finite model property. We show that there are continuum many stable si-logics, and we give examples showing that the classes of stable, subframe, and join-splitting si-logics are incomparable. Among more prominent si-logics, the logic of weak excluded middle **KC** and the Gödel–Dummett logic **LC** are examples of stable si-logics. More examples are given in the body of the paper.

As we pointed out, the theory of canonical formulas for si-logics has its counterpart for modal logics. Zakharyaschev [27], [29] developed the theory of canonical formulas for transitive modal logics using model-theoretic means. The algebraic counterpart of Zakharyaschev's canonical formulas for transitive modal logics was developed in [4], and in [5] it was generalized to weakly transitive modal logics. The canonical formulas of [4] and [5] are the modal logic version of $(\wedge, \to)$-canonical formulas. The modal logic version of $(\wedge, \vee)$-canonical formulas is developed in Bezhanishvili, Bezhanishvili, and Iemhoff [7].

The paper is organized as follows. In Section 2 we recall Priestley duality for bounded distributive lattices and Esakia duality for Heyting algebras. We also recall the generalized Esakia duality and $(\wedge, \to)$-canonical formulas. In Section 3 we introduce the $(\wedge, \vee)$-canonical formulas and prove that every si-logic is axiomatized by $(\wedge, \vee)$-canonical formulas. In Section 4 we develop a dual approach to $(\wedge, \vee)$-canonical formulas. In Section 5 we show how to obtain Jankov formulas as a particular case of $(\wedge, \vee)$-canonical formulas. In Section 6 we introduce stable si-logics, show that all stable si-logics have the finite model property, give their axiomatization, and prove that there are continuum many stable si-logics. Finally, in Section 7 we give several examples of si-logics axiomatized by $(\wedge, \vee)$-canonical formulas, and we show that the classes of stable, subframe, and join-splitting si-logics are incomparable.

2 Preliminaries

In this preliminary section we present the main technical tools that will be used throughout the paper, including Priestley duality for bounded distributive lattices and Esakia duality for Heyting algebras. We also briefly discuss the generalized Esakia duality for $(\wedge, \to)$-morphisms between Heyting algebras and $(\wedge, \to)$-canonical formulas.

2.1 Priestley duality Let X be a poset (partially ordered set). We denote the partial order on X by $\leq$. For $Y \subseteq X$, we recall that the *down-set* of Y is the set $\downarrow Y = \{x \in X : \exists y \in Y \text{ with } x \leq y\}$. The *up-set* of Y is defined dually and is denoted by $\uparrow Y$. If Y is a singleton set $\{y\}$, then we use $\downarrow y$ and $\uparrow y$ instead of $\downarrow\{y\}$ and $\uparrow\{y\}$, respectively. We call $U \subseteq X$ an *up-set* if $x \in U$ and $x \leq y$ imply $y \in U$. A *down-set* of X is defined dually. For posets X and Y, a map $f : X \to Y$ is *order-preserving* if $x \leq y$ implies that $f(x) \leq f(y)$ for all $x, y \in X$.

A poset X is a *Priestley space* if X is a compact space and for each $x, y \in X$, from $x \not\leq y$ it follows that there is a clopen (closed and open) up-set U of X such that $x \in U$ and $y \notin U$. For Priestley spaces X and Y, a map $f : X \to Y$ is a *Priestley morphism* if f is continuous and order-preserving. Let Pries be the category of Priestley spaces and Priestley morphisms. Let also Dist be the category of bounded distributive lattices and bounded lattice homomorphisms. By Priestley duality (see [21], [22]), Dist is dually equivalent to Pries.

It is a consequence of Priestley duality that onto bounded lattice homomorphisms dually correspond to 1 to 1 Priestley morphisms, and 1 to 1 bounded lattice homomorphisms dually correspond to onto Priestley morphisms. This yields that homomorphic images of $D \in$ Dist dually correspond to closed subsets of the Priestley dual X of D, while bounded sublattices of D dually correspond to Priestley quasiorders on X, where we recall that a quasiorder Q on X is a *Priestley quasiorder* if it contains $\leq$ and for each $x, y \in X$, from $x \not{Q} y$ it follows that there is a clopen subset U of X which is a Q-up-set (i.e., $x \in U$ and $x Q y$ imply $y \in U$), $x \in U$, and $y \notin U$ (see, e.g., Priestley [23] and [6]).

2.2 Esakia duality We recall that a *Heyting algebra* is a bounded distributive lattice with an additional binary operation $\to$ that is a residual of $\wedge$. For Heyting algebras A and B, a *Heyting homomorphism* is a bounded lattice homomorphism $h : A \to B$ such that $h(a \to b) = h(a) \to h(b)$ for each $a, b \in A$. Let Heyt be the category of Heyting algebras and Heyting homomorphisms. It is well known (see, e.g., Rasiowa and Sikorski [24, Chapter IX] or Chagrov and Zakharyaschev [11, Chapter 7]) that Heyting algebras provide an adequate algebraic semantics for si-logics. In fact, there is a dual isomorphism between the (complete) lattice of si-logics and the (complete) lattice of varieties of Heyting algebras.

An *Esakia space* is a Priestley space X such that $\downarrow U$ is clopen for each clopen U of X. For Esakia spaces X and Y, a map f is an *Esakia morphism* if it is a Priestley morphism that satisfies the following property: for each $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. Order-preserving maps between posets are called *bounded morphisms* (or *p-morphisms*) if they satisfy the property above. Thus, an Esakia morphism is a Priestley morphism that is a bounded morphism. Let Esa be the category of Esakia spaces and Esakia morphisms. By Esakia duality (see [15]), Heyt is dually equivalent to Esa. This duality is a restricted version of Priestley duality.

It follows from Esakia duality that onto Heyting homomorphisms dually correspond to 1 to 1 Esakia morphisms, and 1 to 1 Heyting homomorphisms dually correspond to onto Esakia morphisms. In particular, homomorphic images of $A \in$ Heyt correspond to closed up-sets of the Esakia dual X of A, while Heyting subalgebras correspond to special equivalence relations on X called *Esakia equivalence relations* (see, e.g., Bezhanishvili, Bezhanishvili, Gabelaia, et al. [6]). This difference between Priestley duality and Esakia duality in describing homomorphic images and subalgebras is explained by the fact that Heyt is not a full subcategory of Dist, and so additional conditions are required when describing dually homomorphic images and subalgebras of Heyting algebras.

We recall that a Heyting algebra A is *subdirectly irreducible* if it has a least non-trivial congruence. It is well known (see, e.g., Balbes and Dwinger [1, p. 179, Theorem 5]) that A is subdirectly irreducible if and only if $A - \{1\}$ has the largest element s, called the *second largest element* of A. A Heyting algebra A is *well connected* if $a \vee b = 1$ implies that $a = 1$ or $b = 1$ for each $a, b \in A$. Obviously each subdirectly irreducible Heyting algebra is well connected, but there exist infinite well-connected Heyting algebras that are not subdirectly irreducible. On the other hand, a finite Heyting algebra is subdirectly irreducible if and only if it is well connected.

We call an Esakia space X *rooted* if there exists $x \in X$, called the *root* of X, such that $X = \uparrow x$, and we call X *strongly rooted* if X is rooted and the root x of X is an isolated point. By Esakia [16] (see also Bezhanishvili and Bezhanishvili [2]), a Heyting algebra A is well connected if and only if its dual Esakia space X is rooted, and A is subdirectly irreducible if and only if X is strongly rooted.

2.3 Generalized Esakia duality In the theory of $(\wedge, \to)$-canonical formulas, we work with maps between Heyting algebras that only preserve part of the Heyting algebra structure. For Heyting algebras A and B, a map $h : A \to B$ is called a $(\wedge, \to)$-*homomorphism* if $h(a \wedge b) = h(a) \wedge h(b)$ and $h(a \to b) = h(a) \to h(b)$ for each $a, b \in A$. Note that $h(1) = 1$ for each $(\wedge, \to)$-homomorphism h, however h may not preserve 0. If $h(0) = 0$, then we call h a $(\wedge, \to, 0)$-*homomorphism*. Such maps that only preserve part of the signature of Heyting algebras are dually described by means of special partial maps between Esakia spaces.

Let X and Y be Esakia spaces, and let $f : X \to Y$ be a partial map. We let $\mathrm{dom}(f)$ denote the domain of f, and we call f a *partial Esakia morphism* if the following conditions are satisfied.

1. If $x, z \in \mathrm{dom}(f)$ and $x \leq z$, then $f(x) \leq f(z)$.
2. If $x \in \mathrm{dom}(f)$, $y \in Y$, and $f(x) \leq y$, then there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$ and $f(z) = y$.
3. For $x \in X$, we have $x \in \mathrm{dom}(f)$ if and only if there exists $y \in Y$ such that $f[{\uparrow}x] = {\uparrow}y$.
4. $f[{\uparrow}x]$ is closed for each $x \in X$.
5. If U is a clopen up-set of Y, then $X - {\downarrow}f^{-1}(Y - U)$ is a clopen up-set of X.

Let $\mathsf{Esa}^{\mathsf{P}}$ be the category of Esakia spaces and partial Esakia morphisms. We note that the composition in this category is not the usual function composition of partial maps (see [3, Section 3.3] for details). Let also $\mathsf{Heyt}^{(\wedge,\to)}$ be the category of Heyting algebras and $(\wedge, \to)$-homomorphisms. By [3, Theorem 3.27], $\mathsf{Heyt}^{(\wedge,\to)}$ is dually equivalent to $\mathsf{Esa}^{\mathsf{P}}$, and this dual equivalence restricts to Esakia duality between Heyt and Esa.

In order to characterize dually $(\wedge, \to, 0)$-homomorphisms, we call a partial Esakia morphism f between Esakia spaces X and Y a *well partial Esakia morphism* if for each $x \in X$ there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$. Let $\mathsf{Esa}^{\mathsf{W}}$ be the (nonfull) subcategory of $\mathsf{Esa}^{\mathsf{P}}$ consisting of Esakia spaces and well partial Esakia morphisms. Let also $\mathsf{Heyt}^{(\wedge,\to,0)}$ be the (nonfull) subcategory of $\mathsf{Heyt}^{(\wedge,\to)}$ consisting of Heyting algebras and $(\wedge, \to, 0)$-homomorphisms. By [3, Theorem 3.33], the dual equivalence of $\mathsf{Heyt}^{(\wedge,\to)}$ and $\mathsf{Esa}^{\mathsf{P}}$ restricts to a dual equivalence of $\mathsf{Heyt}^{(\wedge,\to,0)}$ and $\mathsf{Esa}^{\mathsf{W}}$, which further restricts to Esakia duality.

2.4 The closed domain condition A $(\wedge, \to)$-homomorphism $h : A \to B$ between Heyting algebras may not preserve $\vee$. Whether or not $h(a \vee b) = h(a) \vee h(b)$ for some $a, b \in A$ turns out to be closely related to the so-called *closed domain condition* (CDC), which is one of the main tools in Zakharyaschev's model-theoretic development of canonical formulas.

Let X and Y be Esakia spaces, let $f : X \to Y$ be a partial Esakia morphism, and let $x \in X$. Since $f[{\uparrow}x]$ is a closed subset of Y, for each $y \in f[{\uparrow}x]$, there exists a minimal element z of $f[{\uparrow}x]$ such that $z \leq y$. We let $\min f[{\uparrow}x]$ denote the set of minimal elements of X, and $\mathfrak{D}$ be a (possibly empty) set of antichains in Y. Then we say that f *satisfies* (CDC) for $\mathfrak{D}$ if $x \notin \mathrm{dom}(f)$ implies that $\min f[{\uparrow}x] \notin \mathfrak{D}$.

Let A, B be Heyting algebras, let $h : A \to B$ be a $(\wedge, \to)$-homomorphism, and let $a, b \in A$. Also, let X, Y be Esakia duals of A, B, let $f : Y \to X$ be the dual partial Esakia morphism of h, and let $\varphi(a), \varphi(b)$ be the clopen up-sets of X corresponding to a, b. We let

$$\mathfrak{D}_{\varphi(a),\varphi(b)} = \{\text{antichains } \mathfrak{d} \text{ in } \varphi(a) \cup \varphi(b) : \mathfrak{d} \cap (\varphi(a) - \varphi(b)) \neq \emptyset \\ \text{and } \mathfrak{d} \cap (\varphi(b) - \varphi(a)) \neq \emptyset\}.$$

Then, by [3, Lemma 3.40], $h(a \vee b) = h(a) \vee h(b)$ if and only if $f : Y \to X$ satisfies (CDC) for $\mathfrak{D}_{\varphi(a),\varphi(b)}$.

2.5 $(\wedge, \to)$-canonical formulas Let A be a finite subdirectly irreducible Heyting algebra, let s be the second largest element of A, and let D be a subset of A^2. For each $a \in A$, we introduce a new variable p_a and define the *canonical formula* $\alpha(A, D, \bot)$ associated with A and D as

$$\begin{aligned} \alpha(A, D, \bot) = \Big[& \bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \\ & \wedge \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\} \\ & \wedge \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \\ & \wedge \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}\Big] \to p_s. \end{aligned}$$

Thus, $\alpha(A, D, \bot)$ encodes the $(\wedge, \to, 0)$-structure of A fully and the behavior of $\vee$ only partially, on the designated subset D of A^2. The key result, first obtained by Zakharyaschev [28] by model-theoretic means, and in [3] by algebraic means, is that each si-logic L is axiomatizable by canonical formulas. Moreover, if L is finitely axiomatizable, then L is axiomatizable by finitely many canonical formulas.

If $D = A^2$, then $\alpha(A, D, \bot)$ encodes the entire Heyting structure of A, and so $\alpha(A, A^2, \bot)$ is the Jankov formula $\chi(A)$ (see [3, Section 5.3]). On the other hand, for $D = \emptyset$, the formulas $\alpha(A, \emptyset, \bot)$ axiomatize cofinal subframe si-logics, while the formulas $\alpha(A, \emptyset)$ axiomatize subframe si-logics, where $\alpha(A, \emptyset)$ is obtained from $\alpha(A, \emptyset, \bot)$ by deleting the conjunct $\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ (see [3, Section 5.4]).

3 $(\wedge, \vee)$-Canonical Formulas

In this section we introduce $(\wedge, \vee)$-canonical formulas and show that every si-logic is axiomatizable by $(\wedge, \vee)$-canonical formulas. We start by the following simple observation which will be useful throughout. Let A and B be Heyting algebras. If B is subdirectly irreducible and A is a subalgebra of B, then A does not have to be subdirectly irreducible. However, it is elementary to see that if B is well connected and A is a bounded sublattice of B, then A is also well connected. (In fact, it is sufficient for A to just be a join-subsemilattice of B containing 1_B.) In particular, if B is well connected and A is a finite bounded sublattice of B, then A is subdirectly irreducible. (This is because a finite Heyting algebra is subdirectly irreducible if and only if it is well connected.)

Definition 3.1 Let A be a finite subdirectly irreducible Heyting algebra, let s be the second largest element of A, and let D be a subset of A^2. For each $a \in A$, introduce a new variable p_a, and set

$$\begin{aligned} \Gamma = \ & (p_0 \leftrightarrow \bot) \wedge (p_1 \leftrightarrow \top) \\ & \wedge \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \end{aligned}$$

$$\wedge \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}$$
$$\wedge \bigwedge \{p_{a\to b} \leftrightarrow p_a \to p_b : (a, b) \in D\}$$

and

$$\Delta = \bigvee \{p_a \to p_b : a, b \in A \text{ with } a \not\leq b\}.$$

Then define the $(\wedge, \vee)$-*canonical formula* $\gamma(A, D)$ associated with A and D as

$$\gamma(A, D) = \Gamma \to \Delta.$$

Thus, $\gamma(A, D)$ encodes the bounded lattice structure of A fully and the behavior of $\to$ only partially, on the designated subset D of A^2.

Remark 3.2 If we compare the $(\wedge, \vee)$-canonical formula $\gamma(A, D)$ to the $(\wedge, \to)$-canonical formula $\alpha(A, D, \bot)$ of [3], we see that the antecedent of $\gamma(A, D)$ encodes the bounded lattice structure of A and the implications in D, while the antecedent of $\alpha(A, D, \bot)$ encodes the meet-semilattice structure of A (including 0) and the joins in D.

The consequent of $\gamma(A, D)$ is more complicated than that of $\alpha(A, D, \bot)$. The intention in both cases is that the formula is "pre-true" on the algebra. For $\alpha(A, D, \bot)$, since the formula encodes implications of entire A, this can simply be expressed by introducing a variable for the second largest element s of A. On the other hand, for $\gamma(A, D)$ we need a more complicated consequent because the formula encodes implications only from the designated subset D of A^2.

Lemma 3.3 *Let A be a finite subdirectly irreducible Heyting algebra, let s be the second largest element of A, and let D be a subset of A^2. Then $A \not\models \gamma(A, D)$.*

Proof Define a valuation ν on A by $\nu(p_a) = a$ for each $a \in A$. Then

$$\nu\big(\gamma(A, D)\big) = \nu(\Gamma \to \Delta) = \nu(\Gamma) \to \nu(\Delta).$$

It is straightforward to verify that $\nu(\Gamma) = 1$ and $\nu(\Delta) = s$. Therefore,

$$\nu\big(\gamma(A, D)\big) = 1 \to s = s.$$

Thus, $A \not\models \gamma(A, D)$. □

Theorem 3.4 *Let A be a finite subdirectly irreducible Heyting algebra, let $D \subseteq A^2$, and let B be a Heyting algebra. Then $B \not\models \gamma(A, D)$ if and only if there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \rightarrowtail C$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$.*

Proof First suppose that there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \rightarrowtail C$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$. By Lemma 3.3, the valuation $\nu(p_a) = a$ for each $a \in A$ refutes $\gamma(A, D)$ on A. Define a valuation μ on C by $\mu(p_a) = h(\nu(p_a)) = h(a)$ for each $a \in A$. We have $\mu(p_0) = h(0_A) = 0_C$ and $\mu(p_1) = h(1_A) = 1_C$. Also,

$$\mu(p_{a\wedge b}) = h(a \wedge b) = h(a) \wedge h(b) = \mu(p_a) \wedge \mu(p_b)$$

and

$$\mu(p_{a\vee b}) = h(a \vee b) = h(a) \vee h(b) = \mu(p_a) \vee \mu(p_b).$$

If $(a, b) \in D$, then

$$\mu(p_{a\to b}) = h(a \to b) = h(a) \to h(b) = \mu(p_a) \to \mu(p_b).$$

Thus, $\mu(\Gamma) = 1_C$. Let $a, b \in A$ with $a \not\leq b$. Since h is 1–1, we have $h(a) \not\leq h(b)$. Therefore, $\mu(p_a \to p_b) = \mu(p_a) \to \mu(p_b) = h(a) \to h(b) \neq 1_C$. As C is subdirectly irreducible, it has the second largest element s_C. So $h(a) \to h(b) \leq s_C$, and hence $\mu(\Delta) \leq s_C$. Thus, $\mu(\gamma(A, D)) = \mu(\Gamma) \to \mu(\Delta) \leq s_C$, and so μ refutes $\gamma(A, D)$ on C. Since C is a homomorphic image of B, we conclude that $B \not\models \gamma(A, D)$.

Conversely, suppose that $B \not\models \gamma(A, D)$. It is well known (see, e.g., Wronski [26, Lemma 1]) that if $b \neq 1_B$, then there exists a subdirectly irreducible Heyting algebra C and an onto Heyting homomorphism $f : B \twoheadrightarrow C$ such that $f(b) = s_C$, where s_C is the second largest element of C. Therefore, since $B \not\models \gamma(A, D)$, there exists a valuation μ on B such that $\mu(\gamma(A, D)) \neq 1_B$, and so there exists a subdirectly irreducible Heyting algebra C and an onto Heyting homomorphism $f : B \twoheadrightarrow C$ such that $f(\mu(\gamma(A, D))) = s_C$. Thus, $\nu = f \circ \mu$ is a valuation on C such that $\nu(\gamma(A, D)) = s_C$. It follows that $\nu(\gamma(A, D)) = \nu(\Gamma \to \Delta) = \nu(\Gamma) \to \nu(\Delta) = s_C$. Since s_C is the second largest element, this implies that $\nu(\Gamma) = 1_C$ and $\nu(\Delta) = s_C$. We define $h : A \to C$ by $h(a) = \nu(p_a)$ for each $a \in A$, and we show that h is a bounded lattice embedding such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$.

Let $a, b \in A$. Since $\nu(\Gamma) = 1_C$ and $\nu(\Gamma) \leq \nu(p_{a \wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b))$, we obtain that $\nu(p_{a \wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b)) = 1_C$. Therefore, $\nu(p_{a \wedge b}) = \nu(p_a) \wedge \nu(p_b)$. By a similar argument, $\nu(p_{a \vee b}) = \nu(p_a) \vee \nu(p_b)$, $\nu(p_0) = \nu(\bot)$, $\nu(p_1) = \nu(\top)$, and if $(a, b) \in D$, then $\nu(p_{a \to b}) = \nu(p_a) \to \nu(p_b)$. But $\nu(p_a) = h(a)$ for each $a \in A$. Therefore, for each $a, b \in A$, we have $h(a \wedge b) = h(a) \wedge h(b)$, $h(a \vee b) = h(a) \vee h(b)$, $h(0) = 0_C$, $h(1) = 1_C$, and if $(a, b) \in D$, then $h(a \to b) = h(a) \to h(b)$. Thus, h is a bounded lattice homomorphism such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$.

To see that h is 1–1 it suffices to show that $a \not\leq b$ in A implies that $h(a) \not\leq h(b)$ in C. If $a \not\leq b$, then $h(a) \to h(b) = \nu(p_a) \to \nu(p_b) = \nu(p_a \to p_b) \leq \nu(\Delta) = s_C$. Therefore, $h(a) \not\leq h(b)$. Thus, h is 1–1. We conclude that C is a subdirectly irreducible homomorphic image of B and that $h : A \rightarrowtail C$ is a bounded lattice embedding such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$. □

Remark 3.5 Theorem 3.4 plays the same role in the theory of $(\wedge, \vee)$-canonical formulas as [3, Theorem 5.3] in the theory of $(\wedge, \to)$-canonical formulas, but it is weaker in that the C in the theorem is required to be subdirectly irreducible, while in [3, Theorem 5.3] it is not. The proof of Theorem 3.4 heavily depends on the fact that C is subdirectly irreducible. In fact, the next example shows that if we do not assume that C is subdirectly irreducible, then the proof is no longer valid. Let $A = \{0, s, 1\}$ be the three-element chain, let $D = \emptyset$, and let $B = \{0, a, \neg a, 1\}$ be the four-element Boolean algebra (see Figure 1). Clearly B is not subdirectly irreducible. Define $h : A \to B$ by $h(0) = 0$, $h(s) = a$, and $h(1) = 1$. Then h is a bounded lattice embedding of A into B. Therefore, there exists a non-subdirectly irreducible homomorphic image C of B ($C = B$) and a bounded lattice embedding $h : A \rightarrowtail C$. The condition $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$ is vacuous because $D = \emptyset$. On the other hand, as $\mu((p_1 \to p_s) \vee (p_s \to p_0)) = a \vee \neg a = 1$, we see that $\mu(\Delta) = 1$, and so μ does not refute $\gamma(A, D)$ on B.

The next lemma is well known and is based on the fact that the variety of bounded distributive lattices is locally finite.

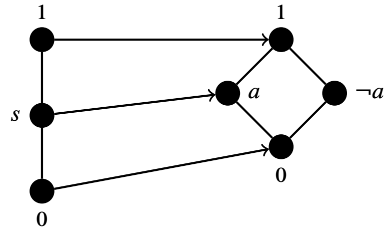

Figure 1

Lemma 3.6 *Let B be a Heyting algebra, and let $B \not\models \varphi(p_1, \ldots, p_n)$. Then there exists a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi(p_1, \ldots, p_n)$. In addition, if B is well connected, then A is subdirectly irreducible.*

Proof Since $B \not\models \varphi(p_1, \ldots, p_n)$, there exist $b_1, \ldots, b_n \in B$ such that $\varphi(b_1, \ldots, b_n) \neq 1_B$. Let $\mathrm{Sub}_B(\varphi)$ be the set of subpolynomials of $\varphi(b_1, \ldots, b_n)$, and let A be the bounded sublattice of B generated by $\mathrm{Sub}_B(\varphi)$. Since the variety of bounded distributive lattices is locally finite, A is finite. Therefore, A is a finite Heyting algebra, where $a \to_A b = \bigvee\{c \in A : a \wedge c \leq b\}$ for each $a, b \in A$. As $a \to b = \bigvee\{d \in B : a \wedge d \leq b\}$, it is easy to see that $a \to_A b \leq a \to b$ and that $a \to_A b = a \to b$ whenever $a \to b \in A$. Since for $\psi, \chi \in \mathrm{Sub}_B(\varphi)$, if $\psi \to \chi \in \mathrm{Sub}_B(\varphi)$, then $\psi \to_A \chi = \psi \to \chi$, we see that the value of $\varphi(b_1, \ldots, b_n)$ in A is the same as the value of $\varphi(b_1, \ldots, b_n)$ in B. As $\varphi(b_1, \ldots, b_n) \neq 1_B$, we conclude that $\varphi(b_1, \ldots, b_n) \neq 1_A$. Thus, A is a finite Heyting algebra that is a bounded sublattice of B and refutes $\varphi(p_1, \ldots, p_n)$. Finally, if B is well connected, then so is A, and as A is finite, A is subdirectly irreducible. □

Theorem 3.7 *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist $(A_1, D_1), \ldots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each subdirectly irreducible Heyting algebra B, the following conditions are equivalent.*

1. *We have $B \not\models \varphi(p_1, \ldots, p_n)$.*
2. *There is $i \leq m$ and a bounded lattice embedding $h : A_i \rightarrowtail B$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$.*
3. *There is $i \leq m$, a subdirectly irreducible homomorphic image C of B, and a bounded lattice embedding $h : A_i \rightarrowtail C$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$.*

Proof Let $F(n)$ be the free n-generated Heyting algebra, and let $g_1, \ldots, g_n$ be the generators of $F(n)$. If **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$, then $F(n) \not\models \varphi(p_1, \ldots, p_n)$. Therefore, $\varphi(g_1, \ldots, g_n) \neq 1$ in $F(n)$. Let $\mathrm{Sub}_{F(n)}(\varphi)$ be the set of subpolynomials of $\varphi(g_1, \ldots, g_n)$ in $F(n)$, and let S be the bounded sublattice of $F(n)$ generated by $\mathrm{Sub}_{F(n)}(\varphi)$. By Lemma 3.6, S is a finite Heyting algebra and $\varphi(g_1, \ldots, g_n) \neq 1$ in S. It is well known (see, e.g., [8], [11]) that $F(n)$ is well connected. Therefore, by Lemma 3.6, S is subdirectly irreducible.

Consider the subdirectly irreducible Heyting algebras that are bounded lattice images of S and refute φ. For each such A_i, let φ be refuted in A_i on $a_1, \ldots, a_n$, and let $\mathrm{Sub}_{A_i}(\varphi)$ be the set of subpolynomials of $\varphi(a_1, \ldots, a_n)$ in A_i. We set $D_i = \{(a, b) \in [\mathrm{Sub}_{A_i}(\varphi)]^2 : a \to b \in \mathrm{Sub}_{A_i}(\varphi)\}$. Let $(A_1, D_1), \ldots, (A_m, D_m)$ be the list of all such pairs.

Let B be a subdirectly irreducible Heyting algebra. We show that the three conditions of the theorem are equivalent.

(2) $\Rightarrow$ (3): This is obvious because we can take $C = B$.

(3) $\Rightarrow$ (1): Suppose that there is $i \leq m$, a subdirectly irreducible homomorphic image C of B, and a bounded lattice embedding $h : A_i \rightarrowtail C$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$. Since $\varphi(a_1, \ldots, a_n) \neq 1$ in A_i and $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$, we also have $\varphi(h(a_1), \ldots, h(a_n)) \neq 1$ in C. Because C is a homomorphic image of B, there exist $b_1, \ldots, b_n \in B$ such that $\varphi(b_1, \ldots, b_n) \neq 1$ in B. Thus, $B \not\models \varphi(p_1, \ldots, p_n)$.

(1) $\Rightarrow$ (2): Suppose that $B \not\models \varphi(p_1, \ldots, p_n)$. Then there exist $b_1, \ldots, b_n \in B$ such that $\varphi(b_1, \ldots, b_n) \neq 1$ in B. Let $B(n)$ be the Heyting subalgebra of B generated by $b_1, \ldots, b_n$, let $\mathrm{Sub}_{B(n)}(\varphi)$ be the set of subpolynomials of $\varphi(b_1, \ldots, b_n)$ in $B(n)$, and let $S_{B(n)}$ be the bounded sublattice of $B(n)$ generated by $\mathrm{Sub}_{B(n)}(\varphi)$. Then $S_{B(n)}$ is a finite Heyting algebra. As $S_{B(n)}$ is a finite bounded sublattice of B and B is subdirectly irreducible, $S_{B(n)}$ is also subdirectly irreducible.

Since $B(n)$ is an n-generated Heyting algebra, $B(n)$ is a homomorphic image of $F(n)$. Let $f : F(n) \twoheadrightarrow B(n)$ be the corresponding Heyting homomorphism. Then $f(g_1) = b_1, \ldots, f(g_n) = b_n$ and $\mathrm{Sub}_{B(n)}(\varphi) = f[\mathrm{Sub}_{F(n)}(\varphi)]$. Therefore, the restriction of f to S is an onto bounded lattice homomorphism. Thus, $S_{B(n)}$ is a bounded lattice image of S, and hence there is $i \leq m$ such that $S_{B(n)} = A_i$. Let $h : A_i \to B$ be the identity map. Then $h : A_i \rightarrowtail B$ is a bounded lattice embedding. Moreover, if $(a, b) \in D_i$, then $a \to b \in \mathrm{Sub}_{B(n)}(\varphi)$. Therefore, $a \to_{A_i} b = a \to_{B(n)} b = a \to_B b$. Thus, $h(a \to b) = h(a) \to h(b)$. Consequently, there is $i \leq m$ and a bounded lattice embedding $h : A_i \rightarrowtail B$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$.

$$
\begin{tikzcd}
S \arrow[r, tail] \arrow[d, "f \restriction S"] & F(n) \arrow[d, "f"] & \\
S_{B(n)} \arrow[r, tail] \arrow[rr, tail, bend right, "h"'] & B(n) \arrow[r, tail] & B
\end{tikzcd}
$$

$\square$

Remark 3.8 Theorem 3.7 plays the same role in the theory of $(\wedge, \vee)$-canonical formulas as [3, Theorem 5.7] in the theory of $(\wedge, \to)$-canonical formulas, but it is weaker in that the B in the theorem is required to be subdirectly irreducible, while in [3, Theorem 5.7] it is arbitrary. On the other hand, this makes the proof of Theorem 3.7 somewhat simpler than that of [3, Theorem 5.7].

Combining Theorems 3.4 and 3.7 yields the following.

Corollary 3.9 *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible Heyting algebra,* $D_i \subseteq A_i^2$*, and* *for each subdirectly irreducible Heyting algebra B, we have*

$$B \models \varphi(p_1, \ldots, p_n) \quad \text{iff} \quad B \models \bigwedge_{i=1}^{m} \gamma(A_i, D_i).$$

Proof Suppose that **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$. By Theorem 3.7, there exist $(A_1, D_1), \ldots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each subdirectly irreducible Heyting algebra B, we have $B \not\models \varphi(p_1, \ldots, p_n)$ if and only if there is $i \leq m$, a subdirectly irreducible homomorphic image C of B, and a bounded lattice embedding $h : A_i \rightarrowtail C$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D_i$. By Theorem 3.4, this is equivalent to the existence of $i \leq m$ such that $B \not\models \gamma(A_i, D_i)$. Thus, $B \models \varphi(p_1, \ldots, p_n)$ if and only if $B \models \bigwedge_{i=1}^{m} \gamma(A_i, D_i)$. □

We are ready to provide a new axiomatization of si-logics, which is an alternative to Zakharyaschev's axiomatization.

Theorem 3.10 *Each si-logic L is axiomatizable by $(\wedge, \vee)$-canonical formulas. Moreover, if L is finitely axiomatizable, then L is axiomatizable by finitely many $(\wedge, \vee)$-canonical formulas.*

Proof Let L be a si-logic. Then L is obtained by adding $\{\varphi_i : i \in I\}$ to **IPC** as new axioms. Therefore, **IPC** $\nvdash \varphi_i$ for each $i \in I$. By Corollary 3.9, for each $i \in I$, there exist $(A_{i1}, D_{i1}), \ldots, (A_{im_i}, D_{im_i})$ such that the A_{ij} are finite subdirectly irreducible Heyting algebras, $D_{ij} \subseteq A_{ij}^2$, and for each subdirectly irreducible Heyting algebra B, we have $B \models \varphi_i$ if and only if $B \models \bigwedge_{j=1}^{m_i} \gamma(A_{ij}, D_{ij})$. Thus, $B \models L$ if and only if $B \models \{\varphi_i : i \in I\}$, which happens if and only if $B \models \{\bigwedge_{j=1}^{m_i} \gamma(A_{ij}, D_{ij}) : i \in I\}$. Consequently, since each si-logic is determined by the class of its subdirectly irreducible Heyting algebras, $L = \mathbf{IPC} + \{\bigwedge_{j=1}^{m_i} \gamma(A_{ij}, D_{ij}) : i \in I\}$, and so L is axiomatizable by $(\wedge, \vee)$-canonical formulas. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many $(\wedge, \vee)$-canonical formulas. □

4 Closed Domain Condition for →

Let A, B be Heyting algebras, and let X, Y be their dual Esakia spaces. As we saw in Section 2.4, a $(\wedge, \to)$-homomorphism $h : A \to B$ satisfies $h(a \vee b) = h(a) \vee h(b)$ for $a, b \in A$ if and only if the dual partial Esakia morphism $f : Y \to X$ satisfies (CDC) for $\mathfrak{D}_{\varphi(a),\varphi(b)}$. In this section, for a Priestley morphism between Esakia spaces, we introduce the closed domain condition for →, (CDC$_\to$), and show that a bounded lattice homomorphism $h : A \to B$ satisfies $h(a \to b) = h(a) \to h(b)$ for $a, b \in A$ if and only if its Priestley dual $f : Y \to X$ satisfies (CDC$_\to$) for $\mathfrak{D}_{\varphi(a),\varphi(b)}$.

Definition 4.1 Let X, Y be Esakia spaces, let $f : X \to Y$ be a Priestley morphism, and let D be a clopen subset of Y. We say that f satisfies the *closed domain condition* (CDC$_\to$) *for D* if

$$\uparrow f(x) \cap D \neq \emptyset \Rightarrow f[\uparrow x] \cap D \neq \emptyset.$$

Let $\mathfrak{D}$ be a collection of clopen subsets of Y. We say that $f : X \to Y$ satisfies the *closed domain condition* (CDC$_\to$) *for* $\mathfrak{D}$ if f satisfies (CDC$_\to$) for each $D \in \mathfrak{D}$.

Lemma 4.2 *Let X, Y be Esakia spaces, let $f : X \to Y$ be a Priestley morphism, and let U, V be clopen up-sets of Y. We let $D_{U,V} = U - V$. Then $D_{U,V}$ is a clopen subset of Y and the following two conditions are equivalent:*

1. *f satisfies* $(\mathrm{CDC}_{\to})$ *for $D_{U,V}$;*
2. *$f^{-1}(U) \to f^{-1}(V) \subseteq f^{-1}(U \to V)$.*

Proof That $D_{U,V}$ is a clopen subset of Y is obvious.

(1) $\Rightarrow$ (2): Suppose that $x \notin f^{-1}(U \to V)$. Then $f(x) \notin U \to V$. Therefore, $\uparrow f(x) \cap U \not\subseteq V$, which means that $\uparrow f(x) \cap D_{U,V} \neq \emptyset$. Applying (1) yields $f[\uparrow x] \cap D_{U,V} \neq \emptyset$. This means that $\uparrow x \cap (f^{-1}(U) - f^{-1}(V)) \neq \emptyset$. Thus, $\uparrow x \cap f^{-1}(U) \not\subseteq f^{-1}(V)$, implying that $x \notin f^{-1}(U) \to f^{-1}(V)$.

(2) $\Rightarrow$ (1): Suppose that $\uparrow f(x) \cap D_{U,V} \neq \emptyset$. Then $\uparrow f(x) \cap U \not\subseteq V$. Therefore, $f(x) \notin U \to V$, so $x \notin f^{-1}(U \to V)$. Applying (2) yields $x \notin f^{-1}(U) \to f^{-1}(V)$, so $\uparrow x \cap f^{-1}(U) \not\subseteq f^{-1}(V)$. This implies that $f[\uparrow x] \cap U \not\subseteq V$, so $f[\uparrow x] \cap D_{U,V} \neq \emptyset$. Thus, f satisfies $(\mathrm{CDC}_{\to})$ for $D_{U,V}$. $\square$

For Heyting algebras A, B, a bounded lattice homomorphism $h : A \to B$, and $a, b \in A$, we always have that $h(a \to b) \leq h(a) \to h(b)$. Therefore, as an immediate consequence of Lemma 4.2, we obtain the following.

Lemma 4.3 *Let A, B be Heyting algebras, let $h : A \to B$ be a bounded lattice homomorphism, and let $a, b \in A$. Let also X, Y be the Esakia duals of A, B, and let $f : Y \to X$ be the Priestley dual of h. Then the following two conditions are equivalent:*

1. *$h(a \to b) = h(a) \to h(b)$;*
2. *f satisfies* $(\mathrm{CDC}_{\to})$ *for $D_{\varphi(a),\varphi(b)}$.*

Since the Esakia dual of a subdirectly irreducible Heyting algebra is a strongly rooted Esakia space, the Esakia dual of a finite subdirectly irreducible Heyting algebra is a rooted poset, the Esakia dual of a subdirectly irreducible homomorphic image of a Heyting algebra A is a strongly rooted closed up-set of the Esakia dual of A, and the Priestley dual of a bounded sublattice of A is an onto Priestley morphism from the Esakia dual of A, the above observation yields the following dual reading of Theorems 3.4 and 3.7.

Theorem 4.4

1. *Let A be a finite subdirectly irreducible Heyting algebra, and let X be the Esakia dual of A. For $D \subseteq A^2$, let $\mathfrak{D} = \{D_{\varphi(a),\varphi(b)} : (a, b) \in D\}$. Then X is a finite rooted poset and for each Esakia space Y, we have $Y \not\models \gamma(A, D)$ if and only if there is a strongly rooted closed up-set Z of Y and an onto Priestley morphism $f : Z \twoheadrightarrow X$ satisfying* $(\mathrm{CDC}_{\to})$ *for $\mathfrak{D}$.*
2. *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist $(A_1, D_1), \ldots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each strongly rooted Esakia space Y, we have $Y \not\models \varphi(p_1, \ldots, p_n)$ if and only if there is $i \leq m$, a strongly rooted closed up-set Z of Y, and an onto Priestley morphism $f : Z \twoheadrightarrow X_i$ satisfying* $(\mathrm{CDC}_{\to})$ *for $\mathfrak{D}_i$, where X_i is the dual rooted poset of A_i and $\mathfrak{D}_i = \{D_{\varphi(a),\varphi(b)} : (a, b) \in D_i\}$.*

Remark 4.5 When comparing the dual approaches to the $(\wedge, \to)$- and $(\wedge, \vee)$-cases, we see that in the $(\wedge, \to)$-case we work with partial p-morphisms whose duals are $(\wedge, \to)$-homomorphisms, and the closed domain condition (CDC) provides the means for the dual to also preserve $\vee$. On the other hand, in the $(\wedge, \vee)$-case we work with order-preserving maps whose duals are bounded lattice homomorphisms, and the closed domain condition ($\mathrm{CDC}_{\to}$) provides the means for the dual to also preserve $\to$. In the end, both approaches provide the same result, that all si-logics are axiomatizable either by $(\wedge, \to)$-canonical formulas or by $(\wedge, \vee)$-canonical formulas. However, both the algebra and geometry of the two approaches are different.

5 Jankov Formulas

Given the $(\wedge, \vee)$-canonical formula $\gamma(A, D)$, there are two obvious extreme cases to consider: when $D = A^2$ and when $D = \emptyset$. In this section we show that if $D = A^2$, then $\gamma(A, D)$ is equivalent to the Jankov formula $\chi(A)$. In the next section we discuss in detail the case when $D = \emptyset$.

Let A be a finite subdirectly irreducible Heyting algebra with the second largest element s. We recall that the *Jankov formula* of A is

$$\begin{aligned} \chi(A) = \Big[& \bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \\ & \wedge \bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \\ & \wedge \bigwedge \{p_{a \to b} \leftrightarrow p_a \to p_b : a, b \in A\} \\ & \wedge \bigwedge \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \Big] \to p_s. \end{aligned}$$

Thus, the Jankov formula encodes the entire Heyting structure of A. As proved by Jankov in [19] (cf. [26]), for a finite subdirectly irreducible Heyting algebra A and a Heyting algebra B, we have $B \not\models \chi(A)$ if and only if A is isomorphic to a Heyting subalgebra of a homomorphic image of B. We show that $\chi(A)$ is equivalent to $\gamma(A, A^2)$.

Theorem 5.1 *Let A be a finite subdirectly irreducible Heyting algebra, and let B be a Heyting algebra.*

1. *$B \not\models \gamma(A, A^2)$ if and only if A is isomorphic to a Heyting subalgebra of a homomorphic image of B.*
2. *$B \models \chi(A)$ if and only if $B \models \gamma(A, A^2)$.*

Proof (1) First assume that there is a Heyting embedding $h : A \rightarrowtail C$ into a homomorphic image C of B. The same argument as in the proof of Lemma 3.3 gives that the valuation $\nu(p_a) = a$ for each $a \in A$ refutes $\gamma(A, A^2)$ on A. Define a valuation μ on C by $\mu(p_a) = h(\nu(p_a)) = h(a)$ for each $a \in A$. Since $D = A^2$, the same argument as in the proof of Theorem 3.4 gives that $\mu(p_0) = 0_C$, $\mu(p_1) = 1_C$, $\mu(p_{a \wedge b}) = \mu(p_a) \wedge \mu(p_b)$, $\mu(p_{a \vee b}) = \mu(p_a) \vee \mu(p_b)$, and $\mu(p_{a \to b}) = \mu(p_a) \to \mu(p_b)$. Thus, $\mu(\Gamma) = 1_C$. Let $a, b \in A$ with $a \not\leq b$. Then $a \to b \neq 1_A$, and so $a \to b \leq s$, where s is the second largest element of A. So

$$\mu(p_a \to p_b) = \mu(p_a) \to \mu(p_b) = h(a) \to h(b) = h(a \to b) \leq h(s),$$

and hence $\mu(\Delta) \leq h(s)$. Therefore, $\mu(\gamma(A, D)) = \mu(\Gamma) \to \mu(\Delta) \leq h(s) < 1_C$, where $h(s)$ is strictly less than 1_C because h is 1–1. Consequently, μ refutes $\gamma(A, A^2)$ on C. Since C is a homomorphic image of B, we conclude that

$B \not\models \gamma(A, A^2)$. Conversely, if $B \not\models \gamma(A, A^2)$, then since $D = A^2$, the same argument as in the proof of Theorem 3.4 produces a Heyting embedding $h : A \to C$ into a subdirectly irreducible homomorphic image C of B.

(2) Apply Jankov's theorem and (1). □

It is well known that Jankov formulas axiomatize the so-called join-splitting si-logics (see, e.g., [11]; a short account can be found in [3, Section 5.3]). Therefore, Theorem 5.1 yields that the $(\wedge, \vee)$-canonical formulas $\gamma(A, A^2)$ axiomatize join-splitting si-logics.

6 Stable Superintuitionistic Logics

In this section we study the class of si-logics axiomatized by the $(\wedge, \vee)$-canonical formulas $\gamma(A, D)$, where $D = \emptyset$. We term such si-logics *stable*. These are the si-logics L whose corresponding variety V_L of Heyting algebras is stable in the sense that if A, B are subdirectly irreducible Heyting algebras, $B \in \mathsf{V}_L$, and A is isomorphic to a bounded sublattice of B, then $A \in \mathsf{V}_L$. In terms of Esakia spaces, L is stable if for strongly rooted Esakia spaces X and Y, whenever $X \models L$ and Y is an image of X under a Priestley morphism, then $Y \models L$. Stable si-logics play a role in the theory of $(\wedge, \vee)$-canonical formulas that is similar to the role of subframe logics in the theory of $(\wedge, \to)$-canonical formulas.

To simplify notation, we denote $\gamma(A, \emptyset)$ by $\gamma(A)$. It is an immediate consequence of Theorem 3.4 that given a finite subdirectly irreducible Heyting algebra A, a Heyting algebra B refutes $\gamma(A)$ if and only if there is a subdirectly irreducible homomorphic image C of B and a bounded lattice embedding $h : A \rightarrowtail C$. For a subdirectly irreducible B, utilizing that $D = \emptyset$, we can actually prove that $B \not\models \gamma(A)$ if and only if there is a bounded lattice embedding $h : A \rightarrowtail B$. For this we require some auxiliary lemmas.

Lemma 6.1 *Let X, Y, Z be finite posets, let X be rooted, and let Z be an up-set of Y. If there is an onto order-preserving map $f : Z \to X$, then f extends to an onto order-preserving map $g : Y \to X$.*

Proof Let x be the root of X. Define $g : Y \to X$ by $g(y) = f(y)$ if $y \in Z$ and $g(y) = x$ otherwise.

$$\begin{tikzcd} Z \arrow[r, hook, "\mathrm{id}"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

Clearly g is a well-defined map extending f, and it is onto since f is onto. To see that g is order-preserving, let $y, z \in Y$ with $y \leq z$. First suppose that $y \in Z$. Then $z \in Z$ as Z is an up-set of Y. Since f is order-preserving, $f(y) \leq f(z)$. Therefore, by the definition of g, we have $g(y) \leq g(z)$. On the other hand, if $y \in Y - Z$, then $g(y) = x$. As x is the root of X, we have $x \leq u$ for each $u \in X$. Thus, $x \leq g(z)$ for each $z \in Y$, which implies that $g(y) \leq g(z)$. Consequently, g is an onto order-preserving map extending f. □

Since in the finite case posets dually correspond to Heyting algebras, rooted posets to subdirectly irreducible Heyting algebras, up-sets to homomorphic images, and onto order-preserving maps to bounded lattice embeddings, we have the following dual reading of Lemma 6.1.

Lemma 6.2 *Let A, B, C be finite Heyting algebras, let A be subdirectly irreducible, let $f : B \twoheadrightarrow C$ be an onto Heyting homomorphism, and let $h : A \rightarrowtail C$ be a bounded lattice embedding. Then there exists a bounded lattice embedding $g : A \rightarrowtail B$ such that $f \circ g = h$.*

$$\begin{tikzcd}
C & B \arrow[l, two heads, "f"'] \\
& A \arrow[u, tail, "g"'] \arrow[ul, tail, "h"]
\end{tikzcd}$$

Although Lemma 6.2 follows from Lemma 6.1 by duality, we also sketch a direct algebraic proof. Since f is a Heyting homomorphism from a finite Heyting algebra B onto a finite Heyting algebra C, the kernel $f^{-1}(1_C)$ of f is a principal filter. Suppose that the filter is generated by $b \in B$. Then C is isomorphic to the interval $[0, b]$ in B. Without loss of generality, we identify C with $[0, b]$. Then $f(x) = x \wedge b$ for each $x \in C$. In addition, we may assume that h is a lattice embedding of A into B. Define $g : A \to B$ by $g(1_A) = 1_B$ and $g(a) = h(a)$ if $a \neq 1_A$. Clearly g is a well-defined map and as h is 1–1, so is g. It follows from the definition of g that $g(0_A) = 0_B$ and $g(1_A) = 1_B$. Since h is a lattice homomorphism, $g(a \wedge b) = g(a) \wedge g(b)$ and $g(a \vee b) = g(a) \vee g(b)$ as long as $a \vee b \neq 1_A$. But since A is finite and subdirectly irreducible, if $a \vee b = 1_A$, then $a = 1_A$ or $b = 1_A$. Therefore, $g(a \vee b) = g(a) \vee g(b)$ for each $a, b \in A$. Thus, g is a bounded lattice embedding, and it follows from the definition of g that $f \circ g = h$.

Theorem 6.3 *Let A be a finite subdirectly irreducible Heyting algebra, and let B be a subdirectly irreducible Heyting algebra. Then $B \not\models \gamma(A)$ if and only if there is a bounded lattice embedding of A into B.*

Proof First suppose that there is a bounded lattice embedding $h : A \rightarrowtail B$. By Lemma 3.3, the valuation $\nu(p_a) = a$ for each $a \in A$ refutes $\gamma(A)$ on A. Define a valuation μ on B by $\mu(p_a) = h(\nu(p_a)) = h(a)$ for each $a \in A$. Then, as in the proof of Theorem 3.4, we have $\mu(p_0) = 0_B$, $\mu(p_1) = 1_B$, $\mu(p_{a \wedge b}) = \mu(p_a) \wedge \mu(p_b)$, and $\mu(p_{a \vee b}) = \mu(p_a) \vee \mu(p_b)$. Thus, $\mu(\Gamma) = 1_B$. Let $a, b \in A$ with $a \not\leq b$. Since h is 1–1, we have $h(a) \not\leq h(b)$. Therefore, $\mu(p_a \to p_b) = \mu(p_a) \to \mu(p_b) = h(a) \to h(b) \neq 1_B$. As B is subdirectly irreducible, it has the second largest element s_B. So $h(a) \to h(b) \leq s_B$, and hence $\mu(\Delta) \leq s_B$. Thus, $\mu(\gamma(A)) = \mu(\Gamma) \to \mu(\Delta) \leq s_B$, and so $B \not\models \gamma(A)$.

Conversely, if $B \not\models \gamma(A)$, then by Lemma 3.6, there is a finite Heyting algebra S which is a bounded sublattice of B and refutes $\gamma(A)$. Since B is subdirectly irreducible and S is finite, S is also subdirectly irreducible. Next, the same proof as in Theorem 3.4 yields a subdirectly irreducible homomorphic image C of S and a bounded lattice embedding $h : A \rightarrowtail C$. By Lemma 6.2, there is a bounded lattice embedding $g : A \rightarrowtail S$. Since S is a bounded sublattice of B, we conclude that g is a bounded lattice embedding of A into B. □

Remark 6.4 The reason Theorem 6.3 holds only for $D = \emptyset$ is that if $D \neq \emptyset$, then the bounded lattice embedding $g : A \rightarrowtail B$ constructed in the proof of Lemma 6.2 may not preserve implications from D even if $h : A \rightarrowtail C$ preserves them.

The dual reading of Theorem 6.3 is the following.

Theorem 6.5 *Let A be a finite subdirectly irreducible Heyting algebra, and let X be its dual finite rooted poset. For a strongly rooted Esakia space Y, we have $Y \not\models \gamma(A)$ if and only if there is an onto Priestley morphism $f : Y \twoheadrightarrow X$.*

We are ready to introduce stable si-logics.

Definition 6.6

1. Let V be a variety of Heyting algebras. We call V *stable* if for any subdirectly irreducible Heyting algebras A, B and a bounded lattice embedding $h : A \rightarrowtail B$, from $B \in \mathsf{V}$ it follows that $A \in \mathsf{V}$.
2. Let L be a si-logic. We call L *stable* if for any strongly rooted Esakia spaces X, Y and an onto Priestley morphism $f : Y \twoheadrightarrow X$, from $Y \models L$ it follows that $X \models L$.

The next theorem is an easy consequence of the dual correspondence between subdirectly irreducible Heyting algebras and strongly rooted Esakia spaces on the one hand and bounded lattice embeddings and onto Priestley morphisms on the other.

Theorem 6.7 *Let L be a si-logic, and let V_L be its corresponding variety of Heyting algebras. Then L is stable if and only if V_L is stable.*

As a consequence of Theorem 6.7 and Lemma 3.6, we obtain that all stable si-logics have the finite model property.

Theorem 6.8 *Each stable si-logic has the finite model property.*

Proof Let L be a stable si-logic, and let $L \nvdash \varphi$. Then there exists a subdirectly irreducible $B \in \mathsf{V}_L$ such that $B \not\models \varphi$. By Lemma 3.6, there exists a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi$. Moreover, as B is subdirectly irreducible, so is A. Since L is stable, by Theorem 6.7, V_L is stable. Thus, $A \in \mathsf{V}_L$, and as A is finite and $A \not\models \varphi$, we conclude that L has the finite model property. □

In order to axiomatize stable si-logics, we recall the theory of frame-based formulas of [8] and [9]. Although the theory was developed for frames, as was pointed out in [5], dualizing frame-based formulas yields algebra-based formulas that we use here. Let $\leq$ be a reflexive and transitive relation on the class $\mathsf{HA}_{\mathsf{si}}$ of subdirectly irreducible Heyting algebras. In [8] and [9], the class is restricted to (the dual spaces of) finitely generated subdirectly irreducible Heyting algebras, but for our purposes this restriction is not essential. For $A, B \in \mathsf{HA}_{\mathsf{si}}$, write $A < B$ if $A \leq B$ and $B \not\leq A$. For a Heyting algebra A, let $|A|$ denote the cardinality of A.

Definition 6.9 We call $\leq$ an *algebra order* if the following two conditions are satisfied.

1. If $A, B \in \mathsf{HA}_{\mathsf{si}}$, B is finite, and $A < B$, then $|A| < |B|$.
2. If $A \in \mathsf{HA}_{\mathsf{si}}$ is finite, then there exists a formula $\zeta(A)$ such that for each $B \in \mathsf{HA}_{\mathsf{si}}$, we have $A \leq B$ if and only if $B \not\models \zeta(A)$.

The formula $\zeta(A)$ is called the *algebra-based formula* of A for $\leq$.

The following criterion of axiomatizability of si-logics by algebra-based formulas follows from [9, Theorem 3.9] (see also [8, Theorem 3.4.12] and [5, Theorem 7.2]).

Theorem 6.10 *Let L be a si-logic, and let $\leq$ be an algebra order on $\mathsf{HA}_{\mathsf{si}}$. Then L is axiomatized by algebra-based formulas for $\leq$ if and only if*

(a) $(\mathsf{V}_L)_{\mathsf{si}}$ *is a down-set of* $\mathsf{HA}_{\mathsf{si}}$*;*
(b) for each $B \in \mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}$, there exists a finite $A \in \mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}$ such that $A \leq B$.

If (a) and (b) are satisfied, then L is axiomatized by the algebra-based formulas of the $\leq$-minimal elements of $\mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}$.

We are ready to prove that stable si-logics are axiomatized by formulas of the form $\gamma(A)$.

Theorem 6.11 *A si-logic L is stable if and only if there is a family $\{A_i : i \in I\}$ of finite subdirectly irreducible Heyting algebras such that L is axiomatized by $\{\gamma(A_i) : i \in I\}$.*

Proof First suppose that there is a family $\{A_i : i \in I\}$ of finite subdirectly irreducible Heyting algebras such that $L = \mathbf{IPC} + \{\gamma(A_i) : i \in I\}$. Let A and B be subdirectly irreducible Heyting algebras, let $h : A \rightarrowtail B$ be a bounded lattice embedding, and let $B \in \mathsf{V}_L$. If $A \notin \mathsf{V}_L$, then there exists $i \in I$ such that $A \not\models \gamma(A_i)$. By Theorem 6.3, there exists a bounded lattice embedding $h_i : A_i \rightarrowtail A$. Therefore, $h \circ h_i$ is a bounded lattice embedding of A_i into B. Applying Theorem 6.3 again yields $B \not\models \gamma(A_i)$, so $B \notin \mathsf{V}_L$. The obtained contradiction proves that V_L is stable. Now apply Theorem 6.7 to conclude that L is stable.

Conversely, suppose that L is stable. Define $\leq$ on $\mathsf{HA}_{\mathsf{si}}$ by $A \leq B$ if there is a bounded lattice embedding from A into B. It is straightforward to see that $\leq$ is reflexive and transitive. To see that $\leq$ is an algebra order, observe that condition (1) of Definition 6.9 is satisfied trivially. For condition (2), if $A, B \in \mathsf{HA}_{\mathsf{si}}$ with A finite, Theorem 6.3 yields that $A \leq B$ if and only if $B \not\models \gamma(A)$. Therefore, $\leq$ is an algebra order on $\mathsf{HA}_{\mathsf{si}}$ and $\gamma(A)$ is the algebra-based formula of A for $\leq$. It is left to verify that $\leq$ satisfies conditions (a) and (b) of Theorem 6.10. Since L is stable, by Theorem 6.7, V_L is stable. Therefore, $(\mathsf{V}_L)_{\mathsf{si}}$ is a down-set of $\mathsf{HA}_{\mathsf{si}}$, and so $\leq$ satisfies condition (a). For condition (b), let $B \in \mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}$. Then $B \not\models L$, and so $B \not\models \varphi$ for some theorem φ of L. By Lemma 3.6, there is a finite subdirectly irreducible Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi$. This implies that $A \in \mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}$ and $A \leq B$. Thus, $\leq$ satisfies condition (b), and hence, by Theorem 6.10, the family $\{\gamma(A) : A$ is a $\leq$-minimal element of $\mathsf{HA}_{\mathsf{si}} - (\mathsf{V}_L)_{\mathsf{si}}\}$ axiomatizes L. □

We conclude this section by showing that the cardinality of the stable si-logics is that of continuum. For a finite rooted poset $\mathfrak{F}$, we let $\mathfrak{F}^+$ denote the dual finite subdirectly irreducible Heyting algebra of $\mathfrak{F}$. For finite rooted posets $\mathfrak{F}$ and $\mathfrak{G}$, we write $\mathfrak{F} \leq \mathfrak{G}$ if $\mathfrak{F}$ is an order-preserving image of $\mathfrak{G}$. Clearly $\mathfrak{F} \leq \mathfrak{G}$ if and only if there is a bounded lattice embedding from $\mathfrak{F}^+$ into $\mathfrak{G}^+$. Therefore, if $\mathsf{HA}_{\mathsf{fsi}}$ denotes the class of finite subdirectly irreducible Heyting algebras, then $\leq$ corresponds to the restriction to $\mathsf{HA}_{\mathsf{fsi}}$ of the order $\leq$ on $\mathsf{HA}_{\mathsf{si}}$ defined in the proof of Theorem 6.11. Consider the sequence $\Delta = \{\mathfrak{F}_n : n \in \omega\}$ of finite rooted posets, where $\mathfrak{F}_n$ is shown in Figure 2. This sequence together with the proof of Lemma 6.12 was suggested to the authors by I. Hodkinson.

Lemma 6.12 *The sequence Δ forms an $\leq$-antichain.*

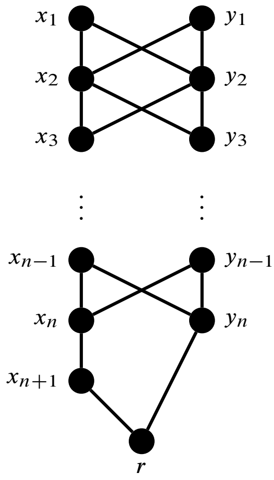

Figure 2

Proof Obviously if $n > m$, then $|\mathfrak{F}_n| > |\mathfrak{F}_m|$, and so $\mathfrak{F}_n \not\leq \mathfrak{F}_m$. Suppose that $n \leq m$. If $\mathfrak{F}_n \leq \mathfrak{F}_m$, then there is an order-preserving onto map $f : \mathfrak{F}_m \twoheadrightarrow \mathfrak{F}_n$. Therefore, the maximum of $\mathfrak{F}_m$ is mapped onto the maximum of $\mathfrak{F}_n$. As each point of depth two in $\mathfrak{F}_m$ is related to each maximal point in $\mathfrak{F}_m$, the points of depth two in $\mathfrak{F}_m$ can only be mapped to the points of depth two in $\mathfrak{F}_n$. Repeating this argument n times yields that the up-set of $\mathfrak{F}_n$ consisting of points of depth $\leq n$ is the image of the up-set of $\mathfrak{F}_m$ consisting of points of depth $\leq n$. For $x_{n+1} \in \mathfrak{F}_n$, there is $y \in \mathfrak{F}_m$ such that $f(y) = x_{n+1}$. Then y is underneath all the points of $\mathfrak{F}_m$ of depth $\leq n$. Therefore, there is $z \in \mathfrak{F}_m$ such that $y \leq z$ and $f(z) = y_n$. But as $x_{n+1} \not\leq y_n$, we have $f(y) \not\leq f(z)$. The obtained contradiction proves that $\mathfrak{F}_n \not\leq \mathfrak{F}_m$. Thus, Δ is an $\leq$-antichain. □

That there are continuum many stable si-logics can now be derived from Lemma 6.12 by a standard argument. It was first used by Jankov [20] for proving that there are continuum many si-logics axiomatized by Jankov formulas (join-splitting si-logics). A similar argument is used in [11, Theorem 11.19] for proving that there are continuum many subframe and cofinal subframe si-logics, as well as in [9, Theorem 3.14] (see also [8, Theorem 3.4.18]) for proving that there are continuum many si-logics axiomatized by frame-based formulas.

Theorem 6.13 *The cardinality of the stable si-logics is that of continuum.*

Proof Let Δ be the sequence of posets shown in Figure 2. For each $I \subseteq \Delta$, let $L_I = \mathbf{IPC} + \{\gamma(\mathfrak{F}^+) : \mathfrak{F} \in I\}$. By Theorem 6.11, each L_I is a stable si-logic. Let $I, J \subseteq \Delta$ be nonempty, and let $I \neq J$. Without loss of generality, we may assume that there is $\mathfrak{F} \in I$ such that $\mathfrak{F} \notin J$. As $\mathfrak{F} \not\models \gamma(\mathfrak{F}^+)$, we have $\mathfrak{F} \not\models L_I$. Let $\mathfrak{G} \in J$. By Lemma 6.12, $\mathfrak{G} \not\leq \mathfrak{F}$. This, by Theorem 6.5, means that $\mathfrak{F} \models \gamma(\mathfrak{G}^+)$. Therefore, $\mathfrak{F} \models L_J$. Thus, $L_I \neq L_J$. Consequently, as there are continuum many nonempty subsets of Δ, the cardinality of the stable si-logics is that of continuum. □

Remark 6.14 In Bezhanishvili and de Jongh [10] a new class of formulas, called ONNILLI, is described syntactically. It is shown that each formula in the class is preserved under order-preserving images of posets, and that for a finite subdirectly irreducible Heyting algebra A, the formula $\gamma(A)$ is equivalent to a formula in ONNILLI.

7 Examples

In this final section we show that several well-known si-logics are stable. We also give examples showing that there is no containment between the classes of stable, splitting, and subframe si-logics. We start by recalling some well-known si-logics (see, e.g., [11, p. 112, Table 4.1]).

1. The logic of weak excluded middle $\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$.
2. The Gödel–Dummett logic $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$.
3. $\mathbf{BD}_n = \mathbf{IPC} + \mathbf{bd}_n$, where $\mathbf{bd}_1 = p_1 \vee \neg p_1$ and $\mathbf{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n)$.
4. $\mathbf{LC}_n = \mathbf{LC} + \mathbf{bd}_n$.
5. $\mathbf{BW}_n = \mathbf{IPC} + \mathbf{bw}_n$, where $\mathbf{bw}_n = \bigvee_{i=0}^{n}(p_i \to \bigvee_{j\neq i} p_j)$.
6. $\mathbf{BTW}_n = \mathbf{IPC} + \mathbf{btw}_n$, where

$$\mathbf{btw}_n = \bigvee_{0\leq i<j\leq n}^{n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}\Big(\neg p_i \to \bigvee_{j\neq i} p_j\Big).$$

In particular, $\mathbf{LC}_1 = \mathbf{CPC}$, $\mathbf{BW}_1 = \mathbf{LC}$, and $\mathbf{BTW}_1 = \mathbf{KC}$. To describe Esakia spaces of each of these si-logics, we need the following well-known definition (see, e.g., [11, Chapter 2]).

Definition 7.1 Let X be a strongly rooted Esakia space, and let $n \in \omega$.

1. The *depth* of X is n, denoted $d(X) = n$, if X contains a chain consisting of n elements and does not contain a chain consisting of $n + 1$ elements.
2. The *width* of X is n, denoted $w(X) = n$, if X contains an antichain consisting of n elements and does not contain an antichain consisting of $n + 1$ elements.
3. The *cofinal width* (or *top width*) of X is n, denoted $w_c(X) = n$, if the maximum of X contains exactly n elements.

We call an Esakia space *linear* if it is a chain. The next theorem is well known (see, e.g., [11]).

Theorem 7.2

1. **KC** *is the logic of all strongly rooted Esakia spaces that have a largest element.*
2. **LC** *is the logic of all linear strongly rooted Esakia spaces.*
3. $\mathbf{LC}_n$ *is the logic of all linear strongly rooted Esakia spaces of depth* $\leq n$.
4. $\mathbf{BD}_n$ *is the logic of all strongly rooted Esakia spaces of depth* $\leq n$.
5. $\mathbf{BW}_n$ *is the logic of all strongly rooted Esakia spaces of width* $\leq n$.
6. $\mathbf{BTW}_n$ *is the logic of all strongly rooted Esakia spaces of cofinal width* $\leq n$

In fact, each of these logics is the logic of the corresponding class of finite rooted posets.

We are ready to show that some of these si-logics are stable.

Theorem 7.3

1. *For each $n \in \omega$, the logic* $\mathbf{LC}_n$ *is stable. Consequently,* $\mathbf{CPC}$ *is stable.*
2. *For each $n \in \omega$, the logic* $\mathbf{BW}_n$ *is stable. Consequently,* $\mathbf{LC}$ *is stable.*
3. *For each $n \in \omega$, the logic* $\mathbf{BTW}_n$ *is stable. Consequently,* $\mathbf{KC}$ *is stable.*

Proof (1) Let X and Y be strongly rooted Esakia spaces, and let $f : X \twoheadrightarrow Y$ be an onto Priestley morphism. It is straightforward to see that if X is linear, then so is Y. Let $d(X) \le n$. If $d(Y) > n$, then there exists a chain Z in Y consisting of $n+1$ elements. But then $f^{-1}(Z)$ is a chain in X consisting of at least $n + 1$ elements. This means that $d(X) > n$, which is a contradiction. Therefore, the class of strongly rooted linear Esakia spaces of depth $\le n$ is closed under onto Priestley morphisms. Thus, by Definition 6.6.2, $\mathbf{LC}_n$ is stable. In particular, since $\mathbf{CPC} = \mathbf{LC}_1$, we have that $\mathbf{CPC}$ is stable.

(2) Let X and Y be strongly rooted Esakia spaces, let $w(X) \le n$, and let $f : X \twoheadrightarrow Y$ be an onto Priestley morphism. If $w(Y) > n$, then there exists an antichain Z in Y consisting of $n + 1$ elements. As f is order-preserving, selecting a representative from each $f^{-1}(z)$, $z \in Z$, produces an antichain in X consisting of at least $n + 1$ elements. This contradicts to $w(X) \le n$. Therefore, the class of strongly rooted Esakia spaces of width $\le n$ is closed under onto Priestley morphisms. Thus, by Definition 6.6.2, $\mathbf{BW}_n$ is stable. In particular, since $\mathbf{LC} = \mathbf{BW}_1$, we have that $\mathbf{LC}$ is stable.

(3) Let X and Y be strongly rooted Esakia spaces, let $w_c(X) \le n$, and let $f : X \twoheadrightarrow Y$ be an onto Priestley morphism. If $w_c(Y) > n$, then there exists $Z \subseteq \max(Y)$ consisting of $n + 1$ elements. As f is order-preserving, $\max(f^{-1}(Z)) \subseteq \max(X)$ consists of at least $n + 1$ elements. This contradicts to $w_c(X) \le n$. Therefore, the class of strongly rooted Esakia spaces of cofinal width $\le n$ is closed under onto Priestley morphisms. Thus, by Definition 6.6.2, $\mathbf{BTW}_n$ is stable. In particular, since $\mathbf{KC} = \mathbf{BTW}_1$, we have that $\mathbf{KC}$ is stable. □

Theorem 7.4

1. *There exist stable si-logics that are not join-splitting logics.*
2. *There exist stable si-logics that are not subframe logics.*
3. *If $n \ge 2$, then* $\mathbf{BD}_n$ *is not stable.*
4. *There exist splitting and subframe si-logics that are not stable.*

Proof (1) By Theorem 7.3.3, $\mathbf{BTW}_n$ is stable for each n. On the other hand, by [11, Proposition 9.50] (see also [8, Theorem 3.4.31]), $\mathbf{BTW}_n$ is not axiomatizable by Jankov formulas for $n \ge 4$, hence is not a join-splitting logic.

(2) By Theorem 7.3.3, $\mathbf{KC}$ is a stable logic. On the other hand, it is well known that $\mathbf{KC}$ is not a subframe logic.

(3) It is easy to see that for each n, the chain $\mathfrak{C}_{n+1}$ of depth $n + 1$ is an order-preserving image of the n-fork $\mathfrak{F}_n$ (see Figure 3). Since $\mathfrak{F}_n \models \mathbf{bd}_2$ but $\mathfrak{C}_{n+1} \not\models \mathbf{bd}_n$, we conclude that $\mathbf{BD}_n$ is not stable for $n \ge 2$.

(4) Observe that each $\mathbf{BD}_n$ is both a splitting logic and a subframe logic and apply (3). □

We conclude the paper by giving a convenient axiomatization of some of the stable si-logics described above. For a finite rooted poset $\mathfrak{F}$, we recall that we use $\mathfrak{F}^+$ for denoting the dual subdirectly irreducible Heyting algebra of $\mathfrak{F}$. For each n, the n-fork $\mathfrak{F}_n$, the n-fork with an adjoint top $\mathfrak{H}_n$, and the n-chain $\mathfrak{C}_n$ are shown in Figure 3.

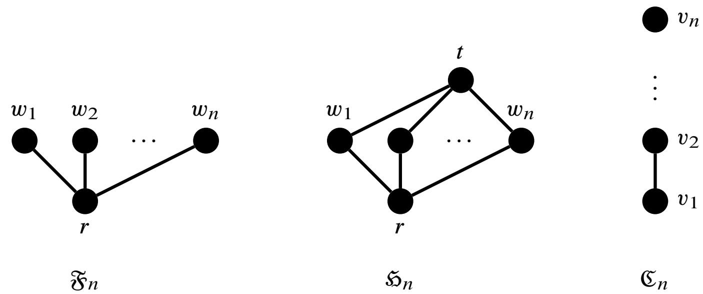

Figure 3

Theorem 7.5

1. $\mathbf{BTW}_n = \mathbf{IPC} + \gamma(\mathfrak{F}_{n+1}^+)$.
2. $\mathbf{KC} = \mathbf{IPC} + \gamma(\mathfrak{F}_2^+)$.
3. $\mathbf{BW}_n = \mathbf{IPC} + \gamma(\mathfrak{F}_{n+1}^+) + \gamma(\mathfrak{H}_{n+1}^+)$.
4. $\mathbf{LC} = \mathbf{IPC} + \gamma(\mathfrak{F}_2^+) + \gamma(\mathfrak{H}_2^+)$.
5. $\mathbf{LC}_n = \mathbf{LC} + \gamma(\mathfrak{C}_{n+1}^+)$.
6. $\mathbf{CPC} = \mathbf{LC} + \gamma(\mathfrak{C}_2^+)$.

Proof In all of these cases, the si-logics on both sides of the equality are stable, hence have the finite model property by Theorem 6.8. That the logics on the left-hand side are stable follows from Theorem 7.3, and that the logics on the right-hand side are stable follows from Theorem 6.11. So in order to prove these equalities, it is sufficient to show that the finite rooted posets of these logics coincide.

(1) Let X be a finite rooted poset. If $X \not\models \mathbf{BTW}_n$, then $w_c(X) > n$. Therefore, there is an antichain $x_1, \ldots, x_{n+1}$ in $\max(X)$. Define $f : X \to \mathfrak{F}_{n+1}$ by

$$f(x) = \begin{cases} w_k, & \text{if } x = x_k \text{ for some } k \leq n+1, \\ r, & \text{otherwise.} \end{cases}$$

It is easy to see that f is an onto order-preserving map, so $\mathfrak{F}_{n+1}$ is an order-preserving image of X. Thus, by Theorem 6.5, $X \not\models \gamma(\mathfrak{F}_{n+1}^+)$. Conversely, if $X \not\models \gamma(\mathfrak{F}_{n+1}^+)$, then by Theorem 6.5, there is an order-preserving map f from X onto $\mathfrak{F}_{n+1}$. Therefore, $\max(f^{-1}(\{w_1, \ldots, w_{n+1}\})) \subseteq \max(X)$ and $\max(f^{-1}(\{w_1, \ldots, w_{n+1}\}))$ consists of at least $n+1$ elements. Thus, $X \not\models \mathbf{BTW}_n$. Consequently, the finite rooted posets of $\mathbf{BTW}_n$ and $\mathbf{IPC} + \gamma(\mathfrak{F}_{n+1}^+)$ coincide, hence $\mathbf{BTW}_n = \mathbf{IPC} + \gamma(\mathfrak{F}_{n+1}^+)$.

(2) follows from (1) because $\mathbf{KC} = \mathbf{BTW}_1$.

(3) Let X be a finite rooted poset. If $X \not\models \mathbf{BW}_n$, then there is an antichain $x_1, \ldots, x_{n+1}$ in X. Let Z be the up-set $\uparrow\{x_1, \ldots, x_{n+1}\}$. If $Z = \{x_1, \ldots, x_{n+1}\}$, then the same argument as in (1) produces an onto order-preserving map $f : X \to \mathfrak{F}_{n+1}$. On the other hand, if Z properly contains $\{x_1, \ldots, x_{n+1}\}$, then define

$f : X \to \mathfrak{H}_{n+1}$ by

$$f(x) = \begin{cases} w_k, & \text{if } x = x_k \text{ for some } k \le n+1, \\ t & \text{if } x \in Z - \{x_1, \dots, x_{n+1}\}, \\ r, & \text{otherwise.} \end{cases}$$

It is easy to see that f is an onto order-preserving map. Therefore, either $\mathfrak{F}_{n+1}$ or $\mathfrak{H}_{n+1}$ is an order-preserving image of X. Thus, by Theorem 6.5, $X \not\models \gamma(\mathfrak{F}^+_{n+1})$ or $X \not\models \gamma(\mathfrak{H}^+_{n+1})$. Conversely, if $X \not\models \gamma(\mathfrak{F}^+_{n+1})$ or $X \not\models \gamma(\mathfrak{H}^+_{n+1})$, then by Theorem 6.5, either $\mathfrak{F}_{n+1}$ or $\mathfrak{H}_{n+1}$ is an order-preserving image of X. But then there exists an antichain in X consisting of at least $n+1$ elements. Therefore, $X \not\models \mathbf{BW}_n$. Thus, the finite rooted posets of $\mathbf{BW}_n$ and $\mathbf{IPC} + \gamma(\mathfrak{F}^+_{n+1}) + \gamma(\mathfrak{H}^+_{n+1})$ coincide, hence $\mathbf{BW}_n = \mathbf{IPC} + \gamma(\mathfrak{F}^+_{n+1}) + \gamma(\mathfrak{H}^+_{n+1})$.

(4) follows from (3) because $\mathbf{LC} = \mathbf{BW}_1$.

(5) Let X be a finite rooted poset. Suppose that $X \not\models \mathbf{LC}_n$. If X is not linear, then $X \not\models \mathbf{LC}$. So suppose that X is linear. Then $d(X) > n$. Therefore, there exists a chain $x_1 < x_2 < \cdots < x_{n+1}$ of $n+1$ elements in X. Define $f : X \to \mathfrak{C}_{n+1}$ by

$$f(x) = \begin{cases} v_{n+1}, & \text{if } x \ge x_{n+1}, \\ v_k, & \text{if } x_k \le x < x_{k+1} \text{ for } k \le n, \\ v_1, & \text{if } x \le x_1. \end{cases}$$

It is easy to see that f is an onto order-preserving map. Thus, by Theorem 6.5, $X \not\models \gamma(\mathfrak{C}^+_{n+1})$. Conversely, if $X \not\models \mathbf{LC} + \gamma(\mathfrak{C}^+_{n+1})$, then ether X is not linear, in which case $X \not\models \mathbf{LC}_n$, or else $X \not\models \gamma(\mathfrak{C}^+_{n+1})$. In the latter case, by Theorem 6.5, $\mathfrak{C}_{n+1}$ is an order-preserving image of X. But as X is linear, this implies that there is a chain of $n+1$ elements in X, which means that $X \not\models \mathbf{LC}_n$. Thus, the finite rooted posets of $\mathbf{LC}_n$ and $\mathbf{LC} + \gamma(\mathfrak{C}^+_{n+1})$ coincide, hence $\mathbf{LC}_n = \mathbf{LC} + \gamma(\mathfrak{C}^+_{n+1})$.

(4) follows from (3) because $\mathbf{CPC} = L_1$. □

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