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Regular Ultrapowers at Regular Cardinals

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Abstract In earlier work by the first and second authors, the equivalence of a finite square principle $\square^{\text{fin}}_{\lambda,D}$ with various model-theoretic properties of structures of size λ and regular ultrafilters was established. In this paper we investigate the principle $\square^{\text{fin}}_{\lambda,D}$—and thereby the above model-theoretic properties—at a regular cardinal. By Chang's two-cardinal theorem, $\square^{\text{fin}}_{\lambda,D}$ holds at regular cardinals for all regular filters D if we assume the generalized continuum hypothesis (GCH). In this paper we prove in ZFC that, for certain regular filters that we call *doubly*$^{+}$ *regular*, $\square^{\text{fin}}_{\lambda,D}$ holds at regular cardinals, with no assumption about GCH. Thus we get new positive answers in ZFC to Open Problems 18 and 19 in Chang and Keisler's book *Model Theory*.

1 Introduction

In Kennedy and Shelah [7], [8] the equivalence of the following finite square principle $\square^{\text{fin}}_{\lambda,D}$ with various model-theoretic properties of regular reduced powers of models was established:

$\square^{\text{fin}}_{\lambda,D}$: D is a filter on a cardinal λ, and there exist finite sets C^{ξ}_{α} and integers n_{ξ} for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each ξ, α,

- (i) $C^{\xi}_{\alpha} \subseteq \alpha + 1$;
- (ii) if $B \subset \lambda^+$ is a finite set of ordinals and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C^{\xi}_{\alpha}\} \in D$;
- (iii) $\beta \in C^{\xi}_{\alpha}$ implies $C^{\xi}_{\beta} = C^{\xi}_{\alpha} \cap (\beta + 1)$;
- (iv) $|C^{\xi}_{\alpha}| < n_{\xi}$.

The model-theoretic properties were the following. First, if D is an ultrafilter, then $\square^{\text{fin}}_{\lambda,D}$ is equivalent to $\mathcal{M}^{\lambda}/D$ being λ^{++}-universal for each model $\mathcal{M}$ in a vocabulary of size at most λ. To formulate the second model-theoretic property, let us

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say that two models are EF_α*-equivalent* if the second player (i.e., the *isomorphism* player) has a winning strategy in the Ehrenfeucht–Fraïssé game of length α on the two models.[1] Now $\Box^{\mathrm{fin}}_{\lambda,D}$ is equivalent to $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ being EF_{λ^+}-equivalent for any elementarily equivalent models $\mathcal{M}$ and $\mathcal{N}$ (without loss of generality of cardinality at most λ^+) in a vocabulary of size at most λ. The existence of such ultrafilters and models is related to Open Problems 18 and 19 in Chang and Keisler's book on model theory [1].

The consistency of the failure of $\Box^{\mathrm{fin}}_{\lambda,D}$ for a regular filter at a singular strong limit cardinal λ was proved in [8, Corollary 6] relative to the consistency of a supercompact cardinal. In Kennedy, Shelah, and Väänänen [9, Corollary 8] this was improved to the failure of $\Box^{\mathrm{fin}}_{\lambda,D}$ for a regular (ultra)filter D at a singular strong limit cardinal λ relative to the consistency of a strongly compact cardinal. The failure of $\Box^{\mathrm{fin}}_{\lambda,D}$ for an ultrafilter implies the failure of λ^{++}-universality of $\mathcal{M}^\lambda/D$ for some $\mathcal{M}$, as well as the failure of isomorphism of some regular ultrapowers $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$. Thus [9] answered negatively the following problems listed in [1] modulo large cardinal assumptions.

Problem 18 ([1]) Let $|M|, |N|, |L| \leq \alpha$, and let D be a regular ultrafilter over α. If $\mathcal{M} \equiv \mathcal{N}$, then $\prod_D \mathcal{M} \cong \prod_D \mathcal{N}$.

Problem 19 ([1]) If D is a regular ultrafilter of α, then for all infinite $\mathcal{M}$, $\prod_D \mathcal{M}$ is α^{++}-universal.

The use of large cardinals is justified by [7], [8], and [12] as the failure of $\Box^{\mathrm{fin}}_{\lambda,D}$, for a singular strong limit λ implies the failure of $\Box_\lambda$, which implies the consistency of large cardinals.

In this paper we investigate the principle $\Box^{\mathrm{fin}}_{\lambda,D}$, and thereby the above model-theoretic problems, at a *regular* cardinal. The following result is proved in Hyttinen [4, Theorem 3.3]. Assume that κ is regular and that $\lambda^{<\kappa} = \lambda$. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are structures for a finite vocabulary such that $\mathcal{M}$ and $\mathcal{N}$ are EF_α-equivalent for each $\alpha < \kappa$. Suppose that D is a filter on $\xi \times \lambda$, $\xi \leq \lambda$, extending $F' \times F$, where F' is a κ-descendingly incomplete filter on ξ and F is a κ-semigood filter on λ (the concept is defined in [4, Definition 3.2]). Then $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ are EF_{λ^+}-equivalent. For $\kappa = \omega$ this, combined with the existence proof of semigood filters in [4, Lemma 3.5], yields filters D with $\Box^{\mathrm{fin}}_{\lambda,D}$. The structure of the paper is the following. In Section 2 we prove weaker versions of $\Box^{\mathrm{fin}}_{\lambda,D}$ in the case where the filter D extends the club filter on λ. Naturally this case is in spirit quite far from the case of regular D, which is our prime interest. However, this result is useful in the sequel. Note that there are many regular (ultra)filters extending the club filter. In Section 3 we define the concept of *doubly*$^+$ *regular* filter and show that such filters D on regular $\lambda > \aleph_0$ satisfy $\Box^{\mathrm{fin}}_{\lambda,D}$. Thus we get new positive answers in ZFC to the above Problem 18 (with isomorphism replaced, in the absence of $2^\lambda = \lambda^+$, by EF_{λ^+}-equivalence) and the above Problem 19. In Section 4 we prove results to the effect that not all regular filters are doubly regular. In Section 5 we compare our concept of double regularity to Keisler's concept of goodness of a filter. In Section 6 we present some open questions.

2 Filters Extending the Club Filter

We can get provable cases of a weaker form of $\square^{\text{fin}}_{\lambda,D}$, when D extends the club filter. This will prove useful in the next section, where we will use Theorem 1 in the proof of Theorem 5. The original $\square^{\text{fin}}_{\lambda,D}$ is equivalent to reduced powers of elementarily equivalent models of cardinality λ being EF_{λ^+}-equivalent. The weaker form (which we prove below) will give the EF_{λ^+}-equivalence of reduced powers of models of power λ that are not just elementarily equivalent but even EF_{λ}-equivalent.

Theorem 1 *Suppose that*

(a) *λ is regular $> \aleph_0$,*
(b) *D is a filter on λ,*
(c) *D extends the club filter.*

If $\mathcal{M}$ and $\mathcal{N}$ are EF_λ-equivalent, then $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ are EF_{λ^+}-equivalent.

Proof If $\alpha < \lambda^+$, λ regular, let $\{u^i_\alpha : i < \lambda\}$ be a continuously increasing sequence of subsets of α such that $|u^i_\alpha| < \lambda$ for all $i < \lambda$ and $\alpha = \bigcup_{i<\lambda} u^i_\alpha$. Let

$$D_\alpha = \left\{i < \lambda : \forall \beta \in u^i_\alpha (u^i_\beta = u^i_\alpha \cap \beta)\right\}. \tag{1}$$

It is easy to see that D_α is a club of λ (recall that λ is regular).

Now we can proceed, as in [7], to prove that if M and N are EF_λ-equivalent, then M^λ/D and N^λ/D are EF_{λ^+}-equivalent.

Let L be a finite vocabulary, and for each $i < \lambda$, let $\mathcal{M}_i$ and $\mathcal{N}_i$ be EF_λ-equivalent L-structures. We show that II has a winning strategy in the game EF_{λ^+} on the models $\mathcal{M} = \prod_D \mathcal{M}_i$ and $\mathcal{N} = \prod_D \mathcal{N}_i$.

The crucial idea of the proof is the following: When the Ehrenfeucht–Fraïssé game $EF_{\lambda^+}(\mathcal{M}, \mathcal{N})$ is played, the players are actually playing λ Ehrenfeucht–Fraïssé games simultaneously, namely, the games $EF_\lambda(\mathcal{M}_i, \mathcal{N}_i)$, $i < \lambda$.

For each $i < \lambda$, let σ_i be a winning strategy for II in the game EF_λ on the models $\mathcal{M}_i$ and $\mathcal{N}_i$. A *good* position is a sequence $\langle (f_\beta, g_\beta) : \beta < \alpha\rangle$ for some $\alpha < \lambda^+$, together with a club $C \subseteq D_\alpha$, such that for all $\beta < \alpha$ we have $f_\beta \in \prod_i M_i$, $g_\beta \in \prod_i N_i$, and if $i \in C$, then

$$\left\langle \left(f_\eta(i), g_\eta(i)\right) : \eta \in u^i_\alpha \right\rangle$$

is a play according to σ_i on the models $\mathcal{M}_i$ and $\mathcal{N}_i$. In a good position the equivalence classes of the functions f_β and g_β determine a partial isomorphism of the reduced products. Suppose that α rounds have been played and that we are in a good position. Let $\varphi_\gamma([f_{\beta_1}], \ldots, [f_{\beta_k}])$ be an atomic formula holding in $\prod_i \mathcal{M}_i/D$, where $\beta_1 < \cdots < \beta_k < \alpha$, and let $A = \{i \in D_\alpha : \{\beta_1, \ldots, \beta_k\} \subseteq u^i_\alpha\}$. By assumption, $A \in D$. Since also $B = \{i < \lambda : \mathcal{M}_i \models \varphi_\gamma(f_{\beta_1}(i), \ldots, f_{\beta_k}(i))\} \in D$, we have $A \cap B \in D$. For $i \in A \cap B$, we have $\beta_1, \ldots, \beta_k \in u^i_\alpha$; hence

$$u^i_{\beta_j} = u^i_\alpha \cap \beta_j.$$

Since we are in a good position, $\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\alpha\rangle$ is a play according to winning strategy σ_i. Hence $\langle (f_\epsilon(\xi), g_\epsilon(\xi)) : \epsilon \in u^i_\alpha\rangle$ determines a partial isomorphism of the structures $\mathcal{M}_i$ and $\mathcal{N}_i$. Since this was the case for all $i \in A \cap B \in D$, we get $\prod_\epsilon \mathcal{N}_\epsilon/D \models \varphi_\gamma([g_{\beta_1}], \ldots, [g_{\beta_k}])$.

The strategy of II is to keep the position of the game *good* and thereby win the game. So suppose β rounds have been played and II has been able to keep the position *good*. Then for all $\gamma < \beta$ there is a club $C_\gamma \subseteq D_\gamma$ such that for $i \in C_\gamma$, $\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\gamma \rangle$ is a play according to σ_i.

Case 1: $\beta = \bigcup \beta$. Let $C = \bigcap_{\gamma<\beta} C_\gamma$. Since λ is regular, this is still a club. We show that $\langle (f_\gamma, g_\gamma) : \gamma < \beta \rangle$ is good. Let $i \in C$. Let us look at $\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\beta \rangle$. Since $i \in D_\beta$, every initial segment of this play is a play according to σ_i. Hence so is the entire play $\langle (f_\gamma, g_\gamma) : \gamma < \beta \rangle$. We have shown that II can maintain a good position.

Case 2: $\beta = \delta + 1$. Let $C \subseteq \bigcap_{\gamma\le\beta} C_\gamma$ such that $\delta \in u^i_\beta$ for $i \in C$. Now suppose I plays f_δ. We show that II can play g_δ so that $\langle (f_\gamma, g_\gamma) : \gamma < \beta \rangle$ remains good. Let $i \in C$. Let us look at $\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\delta \rangle$. This is a play according to the strategy σ_i. Since $i \in D_\beta$ and $\delta \in u^i_\beta$, $u^i_\delta = u^i_\beta \cap \delta$, so after the moves $\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\delta \rangle$ II can play one more move in EF_λ on $\mathcal{M}_i$ and $\mathcal{N}_i$ with I playing the element $f_\delta(i)$. Let $g_\delta(i)$ be the answer of II in this game according to σ_i. The values $g_\delta(i)$, $i \in C$, constitute the function g_δ mod D. We have shown that II can maintain a good position. $\square$

We do not know whether the conditions (a)–(c) of Theorem 1 are necessary for the conclusion.

Remark 2 We point out some variants of Theorem 1.

1. We can define a version $\square^\gamma_{\lambda,D}$ of $\square^{\text{fin}}_{\lambda,D}$ which is equivalent to: "If $\mathcal{M}$ and $\mathcal{N}$ are EF_γ-equivalent, then $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ are EF_{λ^+}-equivalent":

   $\square^\gamma_{\lambda,D}$: D is a filter on a cardinal λ, and there exist finite sets C^ξ_α and ordinals $\gamma_\xi < \gamma$ for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each ξ, α

   (i) $C^\xi_\alpha \subseteq \alpha + 1$;

   (ii) if $B \subset \lambda^+$ is a set of ordinals with $\text{otp}(B) < \gamma$ and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C^\xi_\alpha\} \in D$;

   (iii) $\beta \in C^\xi_\alpha$ implies $C^\xi_\beta = C^\xi_\alpha \cap (\beta + 1)$;

   (iv) $\text{otp}(C^\xi_\alpha) < \gamma_\xi$.

   If clauses (a), (b), and (c) of Theorem 1 are assumed, then $\square^\lambda_{\lambda,D}$.
2. We can also define a version $\square^{<\delta}_{\lambda,D}$ of $\square^{\text{fin}}_{\lambda,D}$ which is equivalent to: "If $\mathcal{M}$ and $\mathcal{N}$ are EF_γ-equivalent for all $\gamma < \delta$, then $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ are EF_{λ^+}-equivalent." If clauses (a), (b), and $(c)^+$ of Theorem 1 are assumed, then $\square^{<\lambda}_{\lambda,D}$ holds, where $(c)^+$ says that (c) holds and there are functions f_α, $\alpha \le \lambda^+$, such that $\alpha < \beta \le \lambda^+$ implies $\{i < \lambda : f_\alpha(i) < f_\beta(i)\} \in D$. (For $D =$ the club filter, this is the so-called assumption of the existence of the λ^+th canonical function; see, e.g., Jech [5, p. 445].)
3. Note that
   $$\square^{\text{fin}}_{\lambda,D} \Rightarrow \square^\gamma_{\lambda,D} \Rightarrow \square^{<\lambda}_{\lambda,D} \Rightarrow \square^\lambda_{\lambda,D}$$
   for $\gamma < \lambda$.
4. We get a variant of Theorem 1 also by showing, assuming (a), (b), and (c), that $\prod_D \mathcal{M}_i$ and $\prod_D \mathcal{N}_i$ are EF_{λ^+}-equivalent, if for all $\beta < \lambda$:
   $$\{i < \lambda : \mathcal{M}_i \text{ and } \mathcal{N}_i \text{ are } EF_\beta\text{-equivalent}\} \in D.$$

5. We can weaken clause (c) of the theorem to the assumption that D is unreasonable (see Shelah [14]) in the following sense: There is a partition $\{w_i : i < \lambda\}$ of λ such that $\bigcup_{i \in E} w_i \in D$ for every club E of λ.

3 Doubly Regular Filters

We define the concept of a *doubly regular* filter, give examples of such on regular cardinals, and prove that $\square^{\text{fin}}_{\lambda,D}$ holds for such filters. Recall that a family of sets is a *regular family* if finite intersections of members of the family are nonempty, but all infinite intersections are empty, a filter is called *μ-regular* if it contains a regular family of size μ, and a filter on λ is called *regular* if it is λ-regular.

Definition 3 Suppose that D is a filter on a regular cardinal λ.

1. D is called *doubly regular* if there are pairwise disjoint sets $u_i \subseteq \lambda$, $i < \lambda$, each of cardinality λ, and regular filters D_i on u_i such that for all $A \subseteq \lambda$:
$$\left[\forall^\infty i < \lambda(A \cap u_i \in D_i)\right] \Rightarrow A \in D$$
(" $\forall^\infty i < \lambda$" means "for all but boundedly many i").
2. The filter D is called *doubly$^+$ regular* if the above holds with "$\forall^\infty i < \lambda$" replaced by "for a club of i."

Let us make some easy observations about doubly regular filters.

Observation 4

1. *A doubly regular filter is necessarily regular. Let $\{A^\alpha_i : \alpha < \lambda\}$ be a regular family in D_i. Let*
$$B^\alpha = \bigcup_{i<\lambda} A^\alpha_i.$$
Then $\{B^\alpha : \alpha < \lambda\}$ is a regular family in D. We will show in Theorem 7 below that the converse need not be true.
2. *A doubly$^+$ regular filter is always doubly regular.*
3. *It is easy to construct doubly $(^+)$ regular filters. Indeed, if the sets $u_i \subseteq \lambda$, $i < \lambda$, are disjoint, each of cardinality λ, $\lambda = \bigcup_i u_i$, and we have regular filters D_i on u_i, then the set $\{A \subseteq \lambda : \forall^\infty i < \lambda(A \cap u_i \in D_i)\}$ is a doubly regular filter on λ, and the larger set $\{A \subseteq \lambda :$ for a club of $i < \lambda(A \cap u_i \in D_i)\}$ is a doubly$^+$ regular filter on λ. Both double regularity and double$^+$ regularity are closed under extensions of the filter, so we get also ultrafilter examples of both.*

Here is the main point of doubly$^+$ regular filters, at least from the point of view of this paper.

Theorem 5 *If D is a doubly$^+$ regular filter on a regular cardinal $\lambda > \aleph_0$, then $\square^{\text{fin}}_{\lambda,D}$ holds.*

Proof Let the sets u_i and the filters D_i be as in Definition 3. Let D^* be the club filter of λ, and let
$$D' = \left\{A \subseteq \lambda : \{i < \lambda : A \cap u_i \in D_i\} \in D^*\right\}.$$
We prove $\square^{\text{fin}}_{\lambda,D'}$. From this $\square^{\text{fin}}_{\lambda,D}$ follows, as $D' \subseteq D$. It suffices to prove that if $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$, $\alpha < \lambda$, are elementarily equivalent, with a vocabulary of size at most λ, then $\mathcal{M} = \prod_{D'} \mathcal{M}_\alpha$ and $\mathcal{N} = \prod_{D'} \mathcal{N}_\alpha$ are EF_{λ^+}-equivalent. Note that

(a) $\mathcal{M} \cong \prod_{i<\lambda} \mathcal{M}^i/D^*$, where $\mathcal{M}^i = \prod_{\alpha\in u_i} \mathcal{M}_\alpha/D_i$;
(b) $\mathcal{N} \cong \prod_{i<\lambda} \mathcal{N}^i/D^*$, where $\mathcal{N}^i = \prod_{\alpha\in u_i} \mathcal{N}_\alpha/D_i$.

Since each D_i is λ-regular, the models $\mathcal{M}^i$ and $\mathcal{N}^i$ are EF_λ-equivalent by Shelah [13, Theorem VI.1.8]. By Theorem 1 the models $\mathcal{M}$ and $\mathcal{N}$ are now EF_{λ^+}-equivalent. □

4 On Regular But Nondoubly Regular Filters

Nonregular uniform filters do not necessarily exist. If there is a nonregular uniform ultrafilter on ω_1, then $V \neq L$ by Prikry [11], $0^\#$ exists by Ketonen [10], and in fact ω_2 is a limit of measurable cardinals in the Dodd–Jensen core model by Deiser and Donder [2]. We show that we can always construct a regular but nondoubly regular filter. In this sense, double regularity is easier to avoid than regularity.

If E is an equivalence relation on λ, we denote the set of all E-classes by λ/E, and the E-class of i by i/E.

First we give an equivalent condition for double regularity, one that better fits our present purpose.

Lemma 6 *A filter D is doubly regular if and only if there is an equivalence relation E of λ and $\bar{u} = \langle u_\alpha : \alpha \in \lambda\rangle$ such that*

(DR-a) *$\{u_\epsilon : \epsilon \sim_E i\}$ is a regular family of subsets of i/E for each $i < \lambda$;*
(DR-b) *if $S \subseteq \lambda$ and $|S| < \lambda$, then $\bigcup\{i/E : i \in S\} = \emptyset \mod D$;*
(DR-c) *$|i/E| = \lambda$ for all $i < \lambda$;*
(DR-d) *if f is a function such that $\mathrm{dom}(f) = \lambda/E$ and $f(i/E) \sim_E i$ for all $i \in \lambda/E$, then $\bigcup_{i\in\lambda/E} u_{f(i)} \notin D$.*

The proof is easy.

Theorem 7 *If $2^\lambda = \lambda^+$, then there is a regular ultrafilter on λ which is not doubly regular.*

Proof Let $\{B_\alpha : \alpha \in \lambda^+\}$ list $\mathcal{P}(\lambda)$. Let $\{(E^\alpha, \bar{u}_\alpha) : \alpha < \lambda^+\}$ list potential candidates for double regularity; that is, E and $\bar{u} = \langle u_\zeta : \zeta < \lambda\rangle$ such that $\{u_\zeta : \zeta < i/E\}$ is a regular family on i/E for each $i < \lambda$. This is only place where we use $2^\lambda = \lambda^+$.

We construct by induction sets $\mathcal{D}_\alpha$, $\alpha < \lambda^+$, such that the following conditions will hold:

(C-a) $\mathcal{D}_\alpha \subseteq \mathcal{P}(\lambda)$ is $\subseteq$-continuously increasing;
(C-b) $|\mathcal{D}_\alpha| = \lambda$;
(C-c) $\mathcal{D}_\alpha$ is closed under finite intersections (we use $\mathrm{Fil}(\mathcal{D}_\alpha)$ to denote the filter $\mathcal{D}_\alpha$ generates);
(C-d) $\mathcal{D}_0$ contains a regular family (so necessarily, $u \in [\lambda]^{<\lambda}$ implies $u = \emptyset \mod D$);
(C-e) if $\alpha = 2\beta + 1$, then $B_\beta \in \mathcal{D}_\alpha$ or $(\lambda \setminus B_\beta) \in \mathcal{D}_\alpha$;
(C-f) if $\alpha = 2\beta + 2$, then either there is $S \in [\lambda]^{<\lambda}$ such that $\bigcup_{\epsilon\in S} \epsilon/E_\beta \neq \emptyset \mod \mathrm{Fil}(\mathcal{D}_\alpha)$, or, letting $\bar{u}_\beta = \langle u_{\beta,\epsilon} : \epsilon < \lambda\rangle$, there is f such that $\mathrm{dom}(f) = \lambda/E_\beta$, $f(i/E_\beta) \sim_{E_\beta} i$ for all $i \in \lambda/E_\beta$, and $\bigcup_{i\in\lambda/E_\beta} u_{\beta,f(i)} \in \mathcal{D}_a$.

Here is the construction.

Case 1: $\alpha = 0$. Let E be a regular family on λ. (We can construct a regular family on λ in the standard way. Let J be the set of finite subsets of λ. The family $\{\{X \in J : \beta \in X\} : \beta < \lambda\}$ is a regular family on J and hence gives rise to one on λ.) We extend E to $\mathcal{D}_0$ by closing under finite intersections.

Case 2: $\alpha = 2\beta + 1$. We make a choice between $B_\beta \in \mathcal{D}_\alpha$ and $(\lambda \setminus B_\beta) \in \mathcal{D}_\alpha$ so that $\emptyset \notin \mathrm{Fil}(\mathcal{D}_\alpha)$.

Case 3: $\alpha = 2\beta + 2$. Let $\{C_l^\alpha : l < \lambda\}$ list $\mathcal{D}_{2\beta+1}$. If there is $S \in [\lambda]^{<\lambda}$ such that $\bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \mod \mathrm{Fil}(\mathcal{D}_{2\beta+1})$, we let $\mathcal{D}_{2\beta+2} = \mathcal{D}_{2\beta+1}$. So let us assume the following.

($\star$) For all $S \in [\lambda]^{<\lambda}$ we have $\bigcup_{\epsilon \in S} \epsilon/E_\beta = \emptyset \mod \mathrm{Fil}(\mathcal{D}_{2\beta+1})$.

We prove the following auxiliary.

Subclaim *There are $(\epsilon_i, \gamma_i), i < \lambda$ such that*

(a) $\epsilon_i \in \lambda \setminus \{\epsilon_j : j < i\}$,
(b) $\gamma_i \sim_{E_\beta} \epsilon_i$,
(c) $u_{\beta,\gamma_i} \not\supseteq C_i^\alpha \cap \epsilon_i/E_\beta$.

Let us first suppose that the subclaim is true and that we have such a sequence $(\epsilon_i, \gamma_i), i < \lambda$. Choose f by letting $f(\epsilon_i) = \gamma_i$. So $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f(i)}$ is a subset of λ, which includes no element of $\mathcal{D}_{2\beta+1}$. So we let

$$\mathcal{D}_\alpha = \mathcal{D}_{2\beta+1} \cup \left\{A \setminus \bigcup_{i \in \lambda/E_\beta} u_{\beta,f(i)} : A \in \mathcal{D}_{2\beta+1}\right\}.$$

This is clearly closed under finite intersections and does not contain $\emptyset$, and every set in $\mathcal{D}_\alpha$ has cardinality λ.

Let us then prove the subclaim. Let $i < \lambda$, and let

$$W_1 = \bigcup_{j<i} \epsilon_j/E_\beta.$$

By our assumption ($\star$), $W_1 = \emptyset \mod \mathrm{Fil}(\mathcal{D}_{2\beta+1})$. Choose ξ_i from the nonempty set $(\lambda \setminus W_1) \cap C_{\alpha,i}$. Then pick ϵ_i so that $\xi_i \sim_{E_\beta} \epsilon_i$. Finally, let

$$W_2 = \{\gamma < \lambda : \gamma \sim_{E_\beta} \epsilon_i \text{ and } \xi_i \in u_{\beta,\gamma}\}.$$

Since $\mathcal{A}^\beta$ is a regular family, the set W_2 is finite. So there is $\gamma_i \in u_{\beta,\epsilon_i} \setminus W_2$. This ends the construction of the sequence $(\epsilon_i, \gamma_i), i < \lambda$, and thereby finishes the proof of the subclaim.

Finishing the proof Now that we have constructed the sequence $\mathcal{D}_\alpha, \alpha < \lambda^+$, we can let

$$D = \bigcup_{\alpha<\lambda^+} \mathcal{D}_\alpha.$$

This is an ultrafilter on λ. It is regular by (C-d). Now we can easily see that D is not doubly regular. Suppose E_β and $\bar{u}_\beta$ witnesses that D is doubly regular. Let us look at the construction of $\mathcal{D}_{2\beta+2}$. In the first case we assumed that there is $S \in [\lambda]^{<\lambda}$ with $\bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \mod \mathrm{Fil}(\mathcal{D}_{2\beta+1})$. So $\bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \mod D$, and (DR-b) is violated. In the second case we found f such that $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f(i)} = \emptyset \mod \mathrm{Fil}(\mathcal{D}_\alpha)$. Hence $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f(i)} = \emptyset \mod D$, and (DR-d) is violated. $\square$

Note that double$^+$ regularity of D implies $\square^{\text{fin}}_{\lambda,D}$ on a regular cardinal $\lambda > \aleph_0$ (see Theorem 5), but in light of Theorem 7, not conversely, as GCH implies $\square^{\text{fin}}_{\lambda,D}$ for regular D and regular λ (see [7, Lemma 4]).

Theorem 7 has the assumption that $2^\lambda = \lambda^+$, which may fail for all λ. We present next a slightly different construction under a different assumption, one that is always satisfied by a multitude of cardinals λ.

Theorem 8 *Assume the following two conditions.*

(A1) $\operatorname{cof}(\lambda) > \aleph_0$ *or* $\lambda > 2^{\aleph_0}$.

(A2) There is $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ *of cardinality* 2^λ *such that* $|\{A \cap i : A \in \mathcal{A}\}| \leq \lambda$ *for all* $i < \lambda$.

Then there is a regular but not doubly regular filter on λ.

Note that a family $\mathcal{A}$, as in (A2), always exists if $\lambda = 2^{<\lambda}$. Hence condition (A2) can be replaced by $\lambda = \beth_\alpha$, α limit.

Proof Let $\langle (E_\beta, \bar{u}_\beta) : \beta < 2^\lambda \rangle$ list all pairs where E_β is an equivalence relation on λ and $\bar{u}^i_\beta = \langle u_{\beta,\epsilon} : \epsilon \sim_{E_\beta} i \rangle$ is a regular family of subsets of i/E_β for each $i < \lambda$. Let $\{B_\alpha : \alpha < 2^\lambda\}$ list $\mathcal{P}(\lambda)$.

We construct a sequence $(I_\alpha, \mathcal{D}_\alpha), \alpha < 2^\lambda$ such that

1. $|I_\alpha| \leq |\alpha|$, $I_\alpha \subseteq \mathcal{P}(\lambda)$, (I_α) is continuously increasing;
2. $\mathcal{D}_\alpha$ is the filter $\mathcal{D}[I_\alpha] = \{A \subseteq \lambda : \exists J \in [I_\alpha]^{<\aleph_0} \exists S \in [\lambda]^{<\lambda} (\bigcap J \subseteq A \cup S)\}$;
3. $\mathcal{D}_{2\beta+1} = \mathcal{D}_{2\beta} \cup \{B_\beta\}$ or $\mathcal{D}_{2\beta+1} = \mathcal{D}_{2\beta} \cup \{\lambda \setminus B_\beta\}$;
4. $\mathcal{D}_{2\beta+2}$ satisfies:
   (a) there is some $W \in [\lambda]^{<\lambda}$ such that $\bigcup_{i \in W} i/E_\beta \neq \emptyset \bmod \mathcal{D}_{2\beta+1}$, or
   (b) there is an f such that $f(i/E_\beta) \in i/E_\beta$ for all i and $\lambda \setminus \bigcup\{u_{\beta,f(x)} : x \in \lambda/E_\beta\} \in I_\beta$, or
   (c) $|\{X \in \lambda/E_\beta : |X \cap B| = \lambda\}| < \lambda$ for some $B \in \mathcal{D}_{2\beta+1}$.

The construction now follows. Let us look at the case $\alpha = 2\beta + 2$. If we cannot form $\mathcal{D}_\alpha$ as required, then

(N1) if $W \in [\lambda]^{<\lambda}$, then $\bigcup_{i \in W} i/E_\beta = \emptyset \bmod \mathcal{D}_{2\beta+1}$;

(N2) if f is a function such that $\operatorname{dom}(f) = \lambda/E_\beta$ and $f(i/E_\beta) \sim_{E_\beta} i$ for all $i < \lambda$, and
$$A_{\beta,f} = \bigcup\{u_{\beta,f(x)} : x \in \lambda/E_\beta\},$$
then $\emptyset \in \mathcal{D}(I_{2\beta+1} \cup \{\lambda \setminus A_{\beta,f}\})$;

(N3) for $B \in \mathcal{D}_{2\beta+1}$, $|\{X \in \lambda/E_\beta : |X \cap B| = \lambda\}| = \lambda$.

We derive a contradiction. This will ensure that $\mathcal{D}_\alpha$ can be found. Let $\langle x_{\beta,i} : i < \lambda \rangle$ list λ/E_β. By our choice of $\mathcal{A}$, there are one-one functions $b_i : \{A \cap i : A \in \mathcal{A}\} \to x_{\beta,i}$ for each $i < \lambda$. If $s \subseteq \lambda$, let g_s be a function such that $\operatorname{dom}(g_s) = \lambda/E_\beta$ and
$$g_s(x_{\beta,i}) = b_i(s \cap i)$$
so that $g_s(x_{\beta,i}) \in x_{\beta,i}$. By (N2) there are
$$J_{\beta,s} \in [I_{2\beta+1}]^{<\aleph_0}, \qquad W_{\beta,s} \in [\lambda]^{<\lambda}$$
such that
$$\bigcap_{B \in J_{\beta,s}} B \subseteq A_{\beta,g_s} \cup W_{\beta,s}.$$

Since $|\mathcal{A}| = 2^\lambda$, there are $J_* \in [I_{2\beta+1}]^{<\aleph_0}$ and $\mu < \lambda$ such that if

$$\mathcal{A}_1 = \{s \in \mathcal{A} : J_{\beta,s} = J_*, |W_{\beta,s}| = \mu\},$$

then $|\mathcal{A}_1| = 2^\lambda$. Let $B_* = \bigcap J_* \in \mathcal{D}_{2\beta+1}$. By (N3),

$$\left|\{j < \lambda : |x_{\beta,j} \cap B_*| = \lambda\}\right| = \lambda. \tag{2}$$

Claim *There are $s_n \in \mathcal{A}_1$, $n < \omega$, and $i < \omega$ such that $s_n \cap i \neq s_m \cap i$ for all $n < m < \omega$.*

Case 1: $\mathrm{cof}(\lambda) > \aleph_0$. Pick distinct $s_n \in \mathcal{A}_1$, $n < \omega$. Since $\mathrm{cof}(\lambda) > \aleph_0$, there is $i < \lambda$ such that $s_n \cap i \neq s_m \cap i$ for all $n < m < \omega$.

Case 2: $\mathrm{cof}(\lambda) = \aleph_0$, $\lambda > 2^{\aleph_0}$. Pick distinct $s_\xi \in \mathcal{A}_1$, $\xi < (2^{\aleph_0})^+$. Let $C \subseteq \lambda$ be cofinal, $|C| = \aleph_0$. Let $\chi : [(2^{\aleph_0})^+]^2 \to C$ be defined by $\chi(\{\xi, \zeta\}) = \min\{c \in C : s_\xi \cap c \neq s_\zeta \cap c\}$. By the Erdős–Rado theorem $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$, there is $i \in C$ and an uncountable $H \subseteq (2^{\aleph_0})^+$ such that $\chi \restriction [H]^2$ has constant value i.

The claim is proved. By (2), there is $j > i$ such that $|B_* \cap x_{\beta,j}| = \lambda$. With the notation of (N2),

$$A_{\beta,g_{s_n}} \cap x_{\beta,j} = u_{\beta,b_j(s_n \cap j)}$$

and the sets $u_{\beta,b_j(s_n \cap j)}$ are distinct because b_j is one-one. By regularity,

$$\bigcap_n u_{\beta,b_j(s_n \cap j)} = \emptyset. \tag{3}$$

Let $W = \bigcup\{W_{\beta,s_n} : n < \omega\}$. Clearly, $|W| = \mu$. Now

$$B_* \cap x_{\beta,j} \subseteq u_{\beta,b_j(s_n \cap j)} \cup W.$$

This contradicts $|B_* \cap x_{\beta,j}| = \lambda$, since $|W| = \mu$ and (3) gives

$$B_* \cap x_{\beta,j} \subseteq \bigcap_n (u_{\beta,b_j(s_n \cap j)} \cup W) = W.$$

$\square$

If we start with a model of GCH, we can use Easton forcing (see [3]) to obtain a model in which 2^λ is—for all regular λ—anything not ruled out by the conditions $\kappa \leq \lambda \Rightarrow 2^\kappa \leq 2^\lambda$ and $\mathrm{cof}(2^\lambda) > \lambda$. In the arising forcing extension $V[G]$ the tree $({}^{<\lambda}2)^V$, λ regular, has cardinality λ and 2^λ branches. Hence we have in $V[G]$ a set $\mathcal{A}_\lambda$ of cardinality 2^λ—for all regular λ—such that $\forall i < \lambda(|\{A \cap i : A \in \mathcal{A}_\lambda\}| \leq \lambda)$, which is exactly the assumption (A2) of Theorem 8.

5 Good Ultrafilters

Keisler [6] introduced the concept of κ-goodness of ultrafilters and proved that if $2^\lambda = \lambda^+$ and if D is a λ^+-good (i.e., good) countably incomplete ultrafilter on λ, then $\prod_D \mathcal{M}_i \cong \prod_D \mathcal{N}_i$ for any models $\mathcal{M}_i \equiv \mathcal{N}_i$ of cardinality at most λ^+ in a vocabulary of cardinality at most λ. This raises the question whether there is a connection between goodness and double regularity. It turns out that these concepts are independent of each other.

Proposition 9 *Suppose that $\lambda > \aleph_0$. There is a doubly regular ultrafilter on λ which is not good. If $2^\lambda = \lambda^+$, then there is a good countably incomplete ultrafilter on λ which is not doubly regular.*

Proof For the first claim, let D_1 be a doubly regular ultrafilter on λ (exists by Observation 4), and let D_2 be a countably incomplete ultrafilter of ω which is not $\aleph_2$-good (exists by [6, Theorem 5.1]). Let $D = D_1 \times D_2$. This is an ultrafilter on the set $\lambda \times \omega$ of size λ. Since D_2 is not λ^+-good, neither is D (see [13, Chapter VI, Lemma 3.7]). Double regularity is inherited from D_1 as follows. Suppose that we have pairwise disjoint sets u_i, $i < \lambda$, on λ, each of cardinality λ, and regular filters F_i on u_i such that for all $A \subseteq \lambda$:

$$\left[\forall^\infty i < \lambda (A \cap u_i \in F_i)\right] \to A \in D_1.$$

Let $G_i \subseteq F_i$ be a regular family on u_i. Let $u_i^* = u_i \times \omega$ and $G_i^* = \{A \times \omega : A \in G_i\}$. Let F_i^* be the filter on u_i^* generated by $\{A \times \omega : A \in F_i\}$. Now G_i^* is a regular family $\subseteq F_i^*$, and if $A \subseteq \lambda \times \omega$, then

$$\left[\forall^\infty i < \lambda (A \cap u_i^* \in F_i^*)\right] \to A \in D_1 \times D_2.$$

This ends the proof that D is doubly regular.

For the second claim we use a combination of the construction of the proof of Theorem 7 and Keisler's construction of a good ultrafilter in [6, Theorem 4.4]. The construction of Keisler as presented in [1, Chapter 6, p. 387] proceeds in stages, generating a continuously increasing sequence F_α, $\alpha < 2^\lambda$, of filters such that the following condition holds (for unexplained terminology we refer to [1, Chapter 6, p. 387]). For the first (in a fixed well-ordering) monotone $f : [\lambda]^{<\aleph_0} \to F_\alpha$ for which there is no additive extension $[\lambda]^{<\aleph_0} \to F_\alpha$, there is an additive extension $g : [\lambda]^{<\aleph_0} \to F_{\alpha+1}$. To make sure that such g and $F_{\alpha+1}$ always exist, an auxiliary sequence is simultaneously defined, namely, a descending sequence Π_α, $\alpha < 2^\lambda$, of partitions of λ, starting from a carefully chosen initial set Π_0 with $|\Pi_0| = 2^\lambda$. There is no problem in interleaving the inductive construction of the filters F_α into the construction in the proof of Theorem 7. The resulting ultrafilter is good but not doubly regular. $\square$

6 Concluding Remarks

We proved that $\square^{\text{fin}}_{\lambda,D}$ holds if λ is a regular cardinal and D is a doubly regular filter. This naturally raises the question whether $\square^{\text{fin}}_{\lambda,D}$ can fail at a regular cardinal for some regular, but not doubly regular, filter. We know that it can fail at a singular cardinal (see [8]).

Conjecture 1 Consistently, $\square^{\text{fin}}_{\lambda,D}$ fails for some regular $\lambda > \omega$ and some regular filter λ generated by λ sets.

Conjecture 2 If D is a regular ultrafilter on $\aleph_1$ such that $\neg\square^{\text{fin}}_{\aleph_1,D}$, then for any increasing continuous $\langle \alpha_i : i < \omega_1 \rangle$ with $\alpha_i < \omega_1$, there is $A \in D$ such that $A \cap [\alpha_i, \alpha_{i+1})$ is finite for all $i < \omega_1$.

Note that if

$$D = \left\{A \subseteq \omega_1 : \forall^\infty i < \lambda\big(A \cap [\alpha_i, \alpha_{i+1}) \in D_i\big)\right\},$$

D_i ultrafilter on $[\alpha_i, \alpha_{i+1})$, then the answer to Conjecture 2 is positive. This may indicate that looking for counterexamples for $\square^{\text{fin}}_{\aleph_1,D}$ can be hard.

Note

1. The usual elementary equivalence in a finite relational vocabulary is thus EF_n-equivalence for all $n < \omega$, and $L_{\infty\omega}$-equivalence is the same as EF_ω-equivalence. For models of cardinality at most κ, EF_κ-equivalence is equivalent to isomorphism.

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