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A Propositional Theory of Truth

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Abstract The liar and kindred paradoxes show that we can derive contradictions if our language possesses sentences lending themselves to paradox and we reason classically from schema (T) about truth:

$$\mathbf{S} \text{ is true iff } p,$$

where the letter p is to be replaced with a sentence and the letter **S** with a name of that sentence. This article presents a theory of truth that keeps (T) at the expense of classical logic. The theory is couched in a language that possesses paradoxical sentences. It incorporates all the instances of the analogue of (T) for that language and also includes other platitudes about truth. The theory avoids contradiction because its logical framework is an appropriately constructed nonclassical propositional logic. The logic and the theory are different from others that have been proposed for keeping (T), and the methods used in the main proofs are novel.

1 Introduction

The semantic paradoxes make it difficult to construct a coherent and plausible formal theory of truth, for they show that apparent platitudes about truth can lead to contradiction. The main platitude here is schema

$$\mathbf{S} \text{ is true iff } p. \tag{T}$$

To get an instance of the schema, we must replace the letter p with a declarative sentence and the letter **S** with a name of that sentence. Schema (T) seems to be a principle that characterizes the concept of truth and that should be incorporated in any theory of truth.

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As is well known, the simplest semantic paradox is the liar. In a characteristic version, it concerns the sentence (L):

(L) is not true.

One instance of (T) is the biconditional

(L) is true iff (L) is not true.

Assume that (L) is true. Then, because of the biconditional, it is not true. Hence, by reductio ad absurdum, we can deny the assumption: (L) is not true. Consequently, because of the biconditional again, it is true—a contradiction.

Another paradox is Curry's. It concerns sentences such as (C):

If (C) is true, then $\perp$.

Here, $\perp$ abbreviates a contradiction. One instance of (T) is the equivalence

(C) is true iff (if (C) is true, then $\perp$).

Assume that (C) is true. Then, because of the equivalence, if (C) is true, then $\perp$. So, by *modus ponens*, $\perp$. Hence, by conditional proof, if (C) is true, then $\perp$. Thus, because of the equivalence again, (C) is true. Therefore, by *modus ponens*, $\perp$.

One treatment of the paradoxes keeps (T) in its unrestricted form at the expense of classical logic. We endorse the biconditionals "(L) is true iff (L) is not true" and "(C) is true iff (if (C) is true, then $\perp$)" and deviate from classical logic. That treatment comprises two distinct approaches. One of them accepts contradictions "p and not-p." This approach has been connected mainly with the work of Priest (see, e.g., [10]), but others, such as Beall [3], have also done work along the same lines. The other approach does not accept contradictions and, in some way or other, disables the inference from the problematic instances of (T) (like "(L) is true iff (L) is not true") to a contradiction. Perhaps the most well-known advocate of that approach is Field (see, e.g., [4]), but others, such as Zardini [12], have also made important contributions to it.

This article presents a theory of truth that falls within the approach exemplified by Field. The theory is couched in a language that possesses paradoxical sentences, like (L) and (C). It incorporates all the instances of the analogue of (T) for that language and also incorporates other platitudes about truth. It does not include any contradiction, though. The theory avoids contradiction because its logical framework is an appropriately constructed nonclassical propositional logic. Both the logic and the theory are presented in a model-theoretic, rather than proof-theoretic, way.

The theory of truth (and its extensions in a first-order framework) should be compared philosophically with other theories that have been proposed for keeping (T) in its unrestricted form at the expense of classical logic, but such a comparison goes beyond the scope of the present article. Let me note, however, that the main difference between Field's logic and that presented here is that his logic validates the substitution of equivalents but does not include the law of noncontradiction, while the logic to be presented includes the law of noncontradiction but does not validate all inferences of the form "... **A** ... ; **A** $\leftrightarrow$ **B**. Hence, ... **B** ...."

Unless we restrict (T) or reject, in at least some cases, the equivalence between a sentence **A** and the conjunction **A** $\wedge$ **A**, accepting both the law of noncontradiction and the substitution of equivalents leads to contradiction (see Section 2 below and [4, p. 9, fn. 8]).[1] It seems that in the literature on truth, whenever one was faced with a choice between the law and the substitution, one opted for the substitution. I think that, in this way, we underestimate the law of noncontradiction and overestimate the need for a sentence to be intersubstitutable with an attribution of truth to it. It would take a separate article to argue for that, but let me offer an outline of the argument.

We have a practice of never assenting to both a sentence and its negation. As is well known, endorsing the law of noncontradiction, $\neg[p \wedge \neg p]$, is neither necessary nor sufficient for following the practice. Field follows it without accepting the law, whereas Priest does just the opposite. But it has not been realized that those of us who follow the practice ought to endorse the law. We need to explain, for an arbitrary sentence **A**, why we are unwilling to assert both **A** and not-**A**. Unless we explain that, we do not rule out the possibility that our attitude may be due to epistemic reasons: perhaps we believe that **A** and not-**A** can jointly be true, but that one can never have adequate evidence for assenting to both. If we endorse the law of noncontradiction, then we can offer an explanation. We may say: it is not the case that **A** and not-**A**. Of course, one can offer an alternative explanation: anything follows from a sentence and its negation. But that principle, in contradistinction to the law, possesses no immediate obviousness, so one should not just assert it without support.

One way to support it is to appeal to this simple argument: assume both **A** and not-**A** and take any other sentence **B**; from **A** we infer **A**-or-**B**, and then from this disjunction and not-**A** we derive **B**. The problem here is that, within the scope of assuming both **A** and not-**A**, the inference from **A**-or-**B** and not-**A** to **B** is not rational. Supporters of classical logic will balk at the suggestion that a classical pattern of reasoning may not always be rational or that its rationality may depend on the assumptions in whose context it is used. In fact, when we are thinking within the scope of some assumptions, they constrain what we may accept and even how we may reason. When we infer from a disjunction **S**-or-**S**$'$ and the negation of **S** to **S**$'$, the idea is that, with **S** being false, the disjunction cannot hold unless **S**$'$ is true. When, however, we have assumed both **A** and not-**A** and inferred from **A** to **A**-or-**B**, we can take it that the disjunction is sustained by **A**, so we no longer have reason to conclude **B**; the disjunction is supported by **A** and needs no support from **B**.[2]

One may vary the simple argument and avoid reasoning within the scope of contradictory assumptions. One may argue as follows: for any sentence **B**, **A**-or-**B** follows from **A**; **B** follows from **A**-or-**B** and not-**A**; hence **B** follows from **A** and not-**A**. But as long as we leave room for the possibility that **A** and not-**A** may both be true, we should not claim that **B** follows from **A**-or-**B** and not-**A**. For the idea behind the claim is, once more, that, with **A** being false, the disjunction cannot hold unless **B** is true, and that idea would prove wrong if **A** should turn out to be both true and false. When we do not deny that **A** and not-**A** are both true, and we are still arguing that anything follows from a sentence and its negation, we have not yet excluded the possibility that **A** and not-**A** may both be true.

Another way to argue for the principle that anything follows from a sentence and its negation is to appeal to a traditional concept of *following from* and point out that it is not possible for **A** and not-**A** to be true while **B** is false. But this is so because it is not possible for both **A** and not-**A** to be true. And whoever accepts that should also accept that **A** and not-**A** are not both true, which does not differ significantly from accepting $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ and thus endorsing the law of noncontradiction.

On the other hand, the need for a theory to allow replacing a sentence with an attribution of truth to it and conversely has been overestimated. We know that we cannot validly make such replacements in modal contexts. It may necessarily be the case that every rose is a flower, but it is not necessarily the case that the sentence "Every rose is a flower" is true, since the sentence could have meant something other than what it actually means and been false. This may seem a superficial remark, correct at the level of sentences but not at that of propositions. Surely, a sentence expressing a proposition and an attribution of truth to that proposition can replace each other in modal contexts. But for those who are *existentialists* about propositions in the sense of Plantinga [7], things are not so simple. We existentialists believe that at least some propositions are contingent entities; in particular, a proposition expressed in a sentence that contains a nonempty name would not exist if the bearer of the name did not exist. So the proposition that Virgil is a poet, as well as its negation, would not exist if Virgil had not existed. In that case, the proposition would not be true (nor would it have any other property). As for Virgil himself, he would not be a poet, just because he would not even have existed. Thus, (Virgil is not a poet) iff (the proposition that Virgil is not a poet is true), but it could have been that Virgil was not a poet and the proposition was not true, whereas of course it could not have been that the proposition was true and was not true.[3] Existentialism is a controversial doctrine, but at least it shows that the problems with intersubstituting an attribution of truth with what is described as true run deep. At any rate, as we cannot validly replace a sentence with an attribution of truth to it and conversely in all contexts, it will not be a very negative aspect of a theory if it disallows such replacements in some more contexts than those frequently recognized. The theory to be presented disallows them in some contexts involving conjunction or implication, though in none involving only negation and disjunction.

The next section develops the logic presupposed by the theory of truth, while Section 3 turns to the theory itself. Section 4 is devoted to proving the Central Theorem, to the effect that the theory has models (of the kind specified in Section 2) and so is consistent. The methods used in the proof are different from what one can find in the literature, and one may reasonably expect that they can also be used for different theories of truth in different logical frameworks. Section 5 indicates how I have extended the work presented in this article.

2 A Logic

The symbols of our language are the sentential letters $p_1, p_2, p_3, \dots$, the predicate letter T, the individual constants $a_1, a_2, a_3, \dots$, and the connectives $\neg$, $\vee$, $\wedge$, and $\rightarrow$. The atomic well-formed formulas (wffs) are the sentential letters and, for every individual constant **a**, the combination $T\mathbf{a}$. If **A** and **B** are wffs, so are $\neg\mathbf{A}$, $[\mathbf{A} \vee \mathbf{B}]$, $[\mathbf{A} \wedge \mathbf{B}]$, and $[\mathbf{A} \rightarrow \mathbf{B}]$. Every wff either is atomic or is built from atomic wffs through a finite number of applications of connectives as just shown. So our language is a standard propositional language except that it possesses a predicate letter and individual constants.

I use bold letters as metalinguistic variables. **A**, **B**, **C**, **D**, and **E** (with or without primes and subscripts) will range over the wffs of our language, whereas **a**, **b**, **c**, **d**, and **e** range over the individual constants. The atomic wffs Ta_1, Ta_2, and so on will be called T-*attributions*. As usual, $[\mathbf{A} \leftrightarrow \mathbf{B}]$ is defined as $[[\mathbf{A} \rightarrow \mathbf{B}] \wedge [\mathbf{B} \rightarrow \mathbf{A}]]$. Brackets will be omitted according to standard conventions.[4]

Our language is geared to the needs of the theory of truth which will be presented later. That is why it possesses individual constants and a predicate letter. As far as the logic is concerned, though, there will be no semantic difference between T-attributions and sentential letters.

The semantics involves three values, the numbers 0, 1/2, and 1, which are distributed among wffs in accordance with tables like the truth tables of classical semantics. The treatment of the values 1 and 0 in the tables will be similar to the treatment of truth and falsehood, respectively, in classical semantics. Indeed, our tables will incorporate the classical ones, if the latter are set out in terms of the numerical values 1 and 0. Having value 1, however, is not the same as being true, and having value 0 is not the same as being false.

Our semantics, including the distribution of numerical values to wffs, is a way of defining a class of valid inferences and wffs. In other words, it aims to set apart the inferences and wffs which comprise the logic that is being constructed. It is also a way of defining a theory of truth. The values are not implicitly identified with any interesting properties of statements. So the word "semantics" is here used conventionally for model theory; it has nothing to do with meaning or other semantic notions. The notion of truth will of course emerge explicitly in our object-language when we turn from the logic to the theory of truth. It would be problematic to identify 1 and 0 with truth and falsehood. For one thing, it would be incoherent to propose a nonclassical logic for statements about truth, but use classical logic (as I do) when reasoning about the assignment of numerical values. And, as we shall see, there would be other problems as well.

One difference from classical semantics is that some connectives are not value-functional. The value of a wff $\mathbf{A} \wedge \mathbf{B}$ or a wff $\mathbf{A} \rightarrow \mathbf{B}$ is not always determined by the values of **A** and **B**. So the semantics is nondeterministic in Avron's sense (see [2]).[5]

There are two kinds of models, valuations$_*$ and valuations. The former include the latter. A valuation$_*$ is an assignment, to each wff, of one of the three values. The connectives are governed by Tables 1–4.

There are additional rules for cases (i) and (ii) in Tables 3 and 4, respectively. To set them out, we need some terminology. We say that **A** is a *deep conjunct* of **B** if and only if **A** is not a conjunction or double negation (does not have the form $\mathbf{C} \wedge \mathbf{D}$ or the form $\neg\neg\mathbf{C}$) and there is an occurrence $\mathcal{O}$ of **A** in **B** such that every symbol, in **B** but outside $\mathcal{O}$, in whose scope $\mathcal{O}$ lies is either a $\wedge$ or a $\neg$ in a string of $\neg$'s, where that string is not immediately preceded by another $\neg$, consists of an even

Table 1 Negation.

$\mathbf{A}$	$\neg\mathbf{A}$
1	0
1/2	1/2
0	1

Table 2 Disjunction.

A **B**	**A** $\vee$ **B**
at least one has 1	1
both have $1/2$, or the one has $1/2$ and the other 0	$1/2$
both have 0	0

Table 3 Conjunction.

A **B**	**A** $\wedge$ **B**
both have 1	1
both have $1/2$, or the one has $1/2$ and the other 1	$1/2$ or 0 (i)
at least one has 0	0

Table 4 The conditional.

A	**B**	**A** $\to$ **B**
1	1	1
1	$1/2$	$1/2$
1	0	0
$1/2$	1	1
$1/2$	$1/2$	1 or $1/2$ (ii)
$1/2$	0	$1/2$
0	1	1
0	$1/2$	1
0	0	1

number of $\neg$'s and is immediately followed either by $\mathcal{O}$ or by a symbol that is not a $\neg$ and lies before $\mathcal{O}$. Note that any wff that is not a conjunction or double negation counts as a deep conjunct of itself. A conjunction is built out of its deep conjuncts through repeated application of $\wedge$ and possibly also $\neg\neg$. The wffs $[\mathbf{C} \wedge \mathbf{D}] \wedge \mathbf{E}$ and $\mathbf{C} \wedge [\mathbf{E} \wedge \mathbf{D}]$ have the same deep conjuncts: those of **C**, those of **D**, and those of **E**. Again, $p_1 \wedge [p_1 \wedge p_1]$ has only one deep conjunct. And the deep conjuncts of $p_1 \wedge \neg\neg[p_2 \wedge \neg\neg\neg p_3]$ are p_1, p_2, and $\neg p_3$. A wff **B** gets 1 in a valuation$_*$ V if and only if all its deep conjuncts have 1 there. And if every deep conjunct of **B**, or even just one, has 0 in V, then **B** gets 0 there.

We say that $\mathbf{B}' \to \mathbf{A}'$ is a *contrapositive* of $\mathbf{A} \to \mathbf{B}$ if and only if one wff in the pair $\{\mathbf{A}, \mathbf{A}'\}$ is the negation of the other and, also, one wff in the pair $\{\mathbf{B}, \mathbf{B}'\}$ is the negation of the other. So $\neg p_1 \to \neg p_2$ has four contrapositives: $p_2 \to p_1$, $\neg\neg p_2 \to p_1$, $p_2 \to \neg\neg p_1$, and $\neg\neg p_2 \to \neg\neg p_1$.

We have two additional rules for conjunctions that fall under case (i): such a conjunction must get the value $1/2$ if it has only one deep conjunct; and there are no conjunctions **C** and **D** such that every deep conjunct of **C** is a deep conjunct of **D**, **C** gets 0, but **D** gets $1/2$. We have two additional rules for conditionals in case (ii): there are no wffs **A**, **B**, and **C** such that **A**, **B**, and **C** get $1/2$, $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{C}$ get 1, but $\mathbf{A} \to \mathbf{C}$ gets $1/2$; and there are no wffs **A** and **B** such that **A** and **B** get $1/2$,

$\mathbf{A} \to \mathbf{B}$ gets 1, but a contrapositive of $\mathbf{A} \to \mathbf{B}$ gets $1/2$. Any assignment that does not conform with those rules is not a valuation$_*$.

As usual, the value of a wff $\mathbf{A}$ in a valuation$_*$ V will be designated as $|\mathbf{A}|_V$. By saying that $\mathbf{A}$ *implies* $\mathbf{B}$, we mean that, for every valuation$_*$ V, $|\mathbf{A}|_V \leq |\mathbf{B}|_V$. For instance, $\mathbf{A}$ and $\neg\neg\mathbf{A}$ imply each other.

A valuation, now, is also an assignment, to each wff, of one of the values 1, $1/2$, and 0. The tables remain the same, and the additional rules still hold, but we have two more rules, one for $\wedge$ and one for $\to$. Rule (α) says that if $\mathbf{A} \wedge \mathbf{B}$ falls under case (i), and there is no valuation$_*$ in which all its deep conjuncts have 1, and there is also no valuation$_*$ in which they all have 0, then $\mathbf{A} \wedge \mathbf{B}$ must get 0. And rule (β) says that if $\mathbf{A} \to \mathbf{B}$ falls under (ii), and $\mathbf{A}$ implies $\mathbf{B}$, then $\mathbf{A} \to \mathbf{B}$ must get 1. Any valuation$_*$ that does not conform with those rules is not a valuation.

In the case of $\wedge$, it should be noted that if the deep conjuncts of $\mathbf{A} \wedge \mathbf{B}$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$, then $\mathbf{A} \wedge \mathbf{B}$ possesses more than one deep conjunct. For if under such circumstances it possessed just one, that deep conjunct would get $1/2$ in all valuations$_*$. But no wff gets $1/2$ in all valuations$_*$, since all wffs have integral values in the valuations$_*$ in which the atomic wffs have such values. Also, if every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{D}$, and $\mathbf{D}$ has only one deep conjunct, then $\mathbf{C}$, too, has only one deep conjunct. And if every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{D}$, and the deep conjuncts of $\mathbf{C}$ do not all have 1, and do not all have 0, in any valuation$_*$, then the deep conjuncts of $\mathbf{D}$ do not all have 1, and do not all have 0, in any valuation$_*$. Thus, when we are constructing a valuation and want to ensure that the second additional rule governing $\wedge$ in valuations$_*$ applies to all conjunctions in case (i), it suffices to ensure that it applies to the conjunctions that do not have to get $1/2$ because of possessing only one deep conjunct and do not have to get 0 because of (α).

As a result of the tables, if $\mathbf{A} \leftrightarrow \mathbf{B}$ gets 1 in a valuation$_*$, then the value of $\mathbf{A}$ there is the same as the value of $\mathbf{B}$; but if, in a valuation$_*$ or even a valuation, $\mathbf{A}$ and $\mathbf{B}$ both have $1/2$, then $\mathbf{A} \leftrightarrow \mathbf{B}$ may or may not get 1 there.

Our focus will be on valuations, and validity is defined in terms of them. Valuations$_*$ served as a prerequisite for formulating two of the rules that govern valuations. 1 is the only designated value. An inference from premises $\mathbf{A}_1, \ldots, \mathbf{A}_j$ to conclusion $\mathbf{B}$ is *valid* ($\mathbf{A}_1, \ldots, \mathbf{A}_j \vDash \mathbf{B}$) if and only if $\mathbf{B}$ has 1 in every valuation in which $\mathbf{A}_1, \ldots, \mathbf{A}_j$ have 1. A wff $\mathbf{A}$ is *valid* ($\vDash \mathbf{A}$) if and only if it has 1 in every valuation.

We will sometimes talk about the *opposite* of a wff $\mathbf{A}$. If $\mathbf{A}$ does not begin with a $\neg$, then its opposite is $\neg\mathbf{A}$. But if $\mathbf{A}$ is $\neg\mathbf{B}$, then its opposite is $\mathbf{B}$. We will also say that 1 is the *opposite* of the value 0, and 0 is the opposite of 1; $1/2$ is the opposite of itself. So a wff and its opposite get opposite values.

Eschewing value-functionality for $\to$ in case (ii) of Table 4 will give us a flexibility we need in order to accommodate all instances of schema (T). As for $\wedge$, permitting the value 0 in case (i) allows us to give that value to various contradictions and other conjunctions with incompatible conjuncts. For example, we want to be able to assign 0 to a contradiction $\mathbf{A} \wedge \neg\mathbf{A}$ even if $\mathbf{A}$ and $\neg\mathbf{A}$ have $1/2$. We also want to be able to give 0 to a conjunction $[\mathbf{A} \wedge \mathbf{B}] \wedge [\neg\mathbf{A} \vee \neg\mathbf{B}]$ even if none of $\mathbf{A}$, $\mathbf{B}$, and $\neg\mathbf{A} \vee \neg\mathbf{B}$ has 0. On the other hand, if $\mathbf{A}$ has $1/2$, then we should give the same value to the conjunction $\mathbf{A} \wedge \mathbf{A}$. Uttering an unembedded conjunction is equivalent to uttering the two conjuncts: asserting the conjunction hardly differs from asserting the conjuncts, assuming the conjunction by way of a hypothesis is like assuming the conjuncts, and so on. Hence, saying $\mathbf{A} \wedge \mathbf{A}$ is equivalent to saying $\mathbf{A}$ twice. So $\mathbf{A} \wedge \mathbf{A}$ is repetitive in a strong sense, and we are precluded from giving it a value other than that of $\mathbf{A}$. Thus, value-functionality is eschewed for $\wedge$ too.[6]

Once we have abandoned value-functionality, the rules that accompany the tables restore the validity of various classical principles which we would otherwise lose just because of the flexibility we allowed ourselves in cases (i) and (ii). There is no reason to deviate from classical logic in respects that do not serve the purpose of accommodating schema (T) or other platitudes about truth. The point of the particular rules will become clear once we see some inferences and wffs that are valid and some that are not.

Modus ponens is validated. (For any $\mathbf{A}$ and $\mathbf{B}$) $\mathbf{A} \to \mathbf{B}, \mathbf{A} \vDash \mathbf{B}$. Thus, if the conditional $\mathbf{A} \to \mathbf{B}$ is valid, then so is the inference from $\mathbf{A}$ to $\mathbf{B}$. On the other hand, it may be that the inference is valid but the conditional is not. For the validity of the inference turns only on what the value of $\mathbf{B}$ is when $\mathbf{A}$ gets 1, whereas the validity of the conditional depends also on what is the case when $\mathbf{A}$ gets $1/2$.

Generally, if $\vDash \mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i \to \mathbf{B}$, then $\mathbf{A}_1, \ldots, \mathbf{A}_i \vDash \mathbf{B}$. But it may be that $\mathbf{A}_1, \ldots, \mathbf{A}_i \vDash \mathbf{B}$ yet $\nvDash \mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i \to \mathbf{B}$. For example, $\nvDash [\mathbf{A} \to \mathbf{B}] \wedge \mathbf{A} \to \mathbf{B}$; that is, it is not the case that, for every $\mathbf{A}$ and every $\mathbf{B}$, the conditional $[\mathbf{A} \to \mathbf{B}] \wedge \mathbf{A} \to \mathbf{B}$ is valid. For it is easy to construct a valuation V in which p_1 gets $1/2$, p_2 gets 0, and $[p_1 \to p_2] \wedge p_1$ gets $1/2$. (Say that V gives $1/2$ to every conjunction $\mathbf{E}_1 \wedge \mathbf{E}_2$ in case (i) of Table 3 unless the deep conjuncts of $\mathbf{E}_1 \wedge \mathbf{E}_2$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$.) Then, $[p_1 \to p_2] \wedge p_1 \to p_2$ will get $1/2$ in V.

$\vDash \mathbf{A} \to \mathbf{A}$. A wff of the form $\mathbf{A} \to \mathbf{A}$ is as tautological as any one can be. So it should be validated. This is ensured by rule (β). Owing to that rule, $\mathbf{A} \to \mathbf{A}$ gets 1 in a valuation even if $\mathbf{A}$ has $1/2$. Generally, the effect of (β) is to validate various conditionals in which it is valid to infer the consequent from the antecedent. Other such conditionals are those of the form $\mathbf{A} \to \neg\neg\mathbf{A}$ and those of the form $\neg\neg\mathbf{A} \to \mathbf{A}$. It validates them because it makes them get 1 in a valuation even when their antecedent and consequent have $1/2$. On the other hand, it may be valid to infer $\mathbf{B}$ from $\mathbf{A}$ although $\mathbf{A}$ does not imply $\mathbf{B}$; then (β) does not validate $\mathbf{A} \to \mathbf{B}$.

$\vDash \mathbf{A} \to \mathbf{A} \vee \mathbf{B}$, $\vDash \mathbf{B} \to \mathbf{A} \vee \mathbf{B}$, and $\vDash \neg\mathbf{A} \wedge \neg\mathbf{B} \to \neg[\mathbf{A} \vee \mathbf{B}]$. Thanks to the table of $\to$ and to rule (β), in order to show the validity of a conditional $\mathbf{C} \to \mathbf{D}$, it suffices to show that, in every valuation$_*$ V, $|\mathbf{C}|_V \leq |\mathbf{D}|_V$.

$\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \to \mathbf{C}$, $\mathbf{B} \to \mathbf{C} \vDash \mathbf{C}$. Yet $\nvDash [\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \to \mathbf{C}] \wedge [\mathbf{B} \to \mathbf{C}] \to \mathbf{C}$. For there is a valuation where p_1 and p_2 get $1/2$, p_3 gets 0, but $[p_1 \vee p_2] \wedge [p_1 \to p_3] \wedge [p_2 \to p_3]$ gets $1/2$. Also, $\vDash \neg[\mathbf{A} \vee \mathbf{B}] \to \neg\mathbf{A}$, and $\vDash \neg[\mathbf{A} \vee \mathbf{B}] \to \neg\mathbf{B}$.

Of course, $\vDash \mathbf{A} \vee \mathbf{A} \to \mathbf{A}$, $\vDash \mathbf{A} \vee \mathbf{B} \to \mathbf{B} \vee \mathbf{A}$, and $\vDash [\mathbf{A} \vee \mathbf{B}] \vee \mathbf{C} \to \mathbf{A} \vee [\mathbf{B} \vee \mathbf{C}]$. $\nvDash \mathbf{A} \vee \neg\mathbf{A}$; it is not the case that, for every $\mathbf{A}$, the disjunction $\mathbf{A} \vee \neg\mathbf{A}$ is valid. The law of excluded middle is not validated. For if $\mathbf{A}$ has $1/2$ in a valuation, then $\mathbf{A} \vee \neg\mathbf{A}$ also has $1/2$ there.

$\mathbf{A} \vee \mathbf{B}, \neg\mathbf{A} \vDash \mathbf{B}$. But $\nvDash [\mathbf{A} \vee \mathbf{B}] \wedge \neg\mathbf{A} \to \mathbf{B}$. For instance, there is a valuation in which p_1 has $1/2$, p_2 has 0, $[p_1 \vee p_2] \wedge \neg p_1$ gets $1/2$, and so $[p_1 \vee p_2] \wedge \neg p_1 \to p_2$ gets $1/2$ too. However, $\vDash \mathbf{A} \vee \mathbf{B} \to [\neg\mathbf{A} \to \mathbf{B}]$. For, in every valuation$_*$ V, $|\mathbf{A} \vee \mathbf{B}|_V \leq |\neg\mathbf{A} \to \mathbf{B}|_V$. On the other hand, $\nvDash \neg\mathbf{A} \to [\mathbf{A} \vee \mathbf{B} \to \mathbf{B}]$. There is a valuation V in which p_1 has 0, p_2 has $1/2$, and $p_1 \vee p_2 \to p_2$ gets $1/2$. (Say that, in case (ii) of Table 4, V gives $1/2$ to every conditional $\mathbf{C} \rightarrow \mathbf{D}$ unless $\mathbf{C}$ implies $\mathbf{D}$.) So $\neg p_1 \rightarrow [p_1 \vee p_2 \rightarrow p_2]$ gets $1/2$ in V.

$\mathbf{A} \vee \mathbf{B}, \mathbf{A} \rightarrow \mathbf{C}, \mathbf{B} \rightarrow \mathbf{D} \vDash \mathbf{C} \vee \mathbf{D}$. But $\nvDash [\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \rightarrow \mathbf{C}] \wedge [\mathbf{B} \rightarrow \mathbf{D}] \rightarrow \mathbf{C} \vee \mathbf{D}$. For example, there is a valuation where p_1 and p_2 have $1/2$, p_3 and p_4 have 0, and $[p_1 \vee p_2] \wedge [p_1 \rightarrow p_3] \wedge [p_2 \rightarrow p_4]$ gets $1/2$.

$\mathbf{A}, \mathbf{B} \vDash \mathbf{A} \wedge \mathbf{B}$. Yet $\nvDash \mathbf{A} \rightarrow [\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B}]$. For it is easy to construct a valuation V in which p_1 gets 1, p_2 gets $1/2$, and $p_1 \wedge p_2$ gets 0. (Say that V gives 0 to every conjunction $\mathbf{E}_1 \wedge \mathbf{E}_2$ in case (i) of Table 3 unless $\mathbf{E}_1 \wedge \mathbf{E}_2$ has only one deep conjunct.) Then, $p_1 \rightarrow [p_2 \rightarrow p_1 \wedge p_2]$ will get $1/2$ in V. Also, $\vDash \neg\mathbf{A} \rightarrow \neg[\mathbf{A} \wedge \mathbf{B}]$, $\vDash \neg\mathbf{B} \rightarrow \neg[\mathbf{A} \wedge \mathbf{B}]$, $\vDash \mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{A}$, and $\vDash \mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B}$.

As is presumably desirable, $\vDash \mathbf{A} \rightarrow \mathbf{A} \wedge \mathbf{A}$. One of the factors guaranteeing this validity is the rule that, when in a valuation$_*$ we have case (i) of Table 3, $\mathbf{A} \wedge \mathbf{B}$ must get $1/2$ if it has only one deep conjunct. Without some such rule it could be that a sentential letter $\mathbf{p}$ had $1/2$ in a valuation$_*$ while $\mathbf{p} \wedge \mathbf{p}$, or $\mathbf{p} \wedge [\mathbf{p} \wedge \mathbf{p}]$, had 0. Now, assume that $\mathbf{A}$ gets $1/2$ in a valuation$_*$ V. $\mathbf{A}$ can be seen as having the form $\underbrace{\neg \cdots \neg}_{n \text{ times}} \mathbf{B}$, where n is an even number, possibly zero, and $\mathbf{B}$ is not a double negation. $\mathbf{B}$ has $1/2$ in V. If it is not a conjunction, then $\mathbf{A} \wedge \mathbf{A}$ has only one deep conjunct and so gets $1/2$ in the valuation$_*$. But $\mathbf{A} \wedge \mathbf{A}$ also gets $1/2$ there, and not 0, if $\mathbf{B}$ is a conjunction. For then $\mathbf{B}$ comes under case (i) (i.e., either both conjuncts of $\mathbf{B}$ have $1/2$ in V or the one has $1/2$ and the other 1) and every deep conjunct of $\mathbf{A} \wedge \mathbf{A}$ is a deep conjunct of $\mathbf{B}$. Generally, in every valuation$_*$ V, $|\mathbf{A}|_V = |\mathbf{A} \wedge \mathbf{A}|_V$.

$\vDash \mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B} \wedge \mathbf{A}$ and $\vDash [\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C} \rightarrow \mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$. It is here that we can clearly see the effect of the rule that in no valuation$_*$ are there conjunctions $\mathbf{C}$ and $\mathbf{D}$, falling under case (i), such that every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{D}$, $\mathbf{C}$ gets 0, but $\mathbf{D}$ gets $1/2$. For this rule does not allow the consequent of any conditional $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B} \wedge \mathbf{A}$ or $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C} \rightarrow \mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$ to get 0 in a valuation$_*$ in which the antecedent has $1/2$. Let us begin with $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B} \wedge \mathbf{A}$. Note that $\mathbf{B} \wedge \mathbf{A}$ has the same deep conjuncts as $\mathbf{A} \wedge \mathbf{B}$. Suppose that $\mathbf{A} \wedge \mathbf{B}$ gets $1/2$ in a valuation$_*$ V. Then, $\mathbf{A} \wedge \mathbf{B}$ comes under case (i), as does $\mathbf{B} \wedge \mathbf{A}$. So $\mathbf{B} \wedge \mathbf{A}$ cannot get 0 in V; it must get $1/2$. Generally, in every valuation$_*$ V, $|\mathbf{A} \wedge \mathbf{B}|_V = |\mathbf{B} \wedge \mathbf{A}|_V$.

The case of $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C} \rightarrow \mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$ is more complicated. Here we must note that every deep conjunct of $\mathbf{B} \wedge \mathbf{C}$ is a deep conjunct of $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}$, and $\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$ has the same deep conjuncts as $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}$. Suppose that $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}$ gets $1/2$ in a valuation$_*$ V. Then, $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}$ comes under case (i) of Table 3, none of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ has 0 in V, and at least one of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ has $1/2$ there (they do not all have 1). So either (I) $\mathbf{B} \wedge \mathbf{C}$ comes under case (i) or (II) both $\mathbf{B}$ and $\mathbf{C}$ have 1 in V. In (I) $\mathbf{B} \wedge \mathbf{C}$ must get $1/2$, and not 0, in V; in (II) $\mathbf{B} \wedge \mathbf{C}$ gets 1 there, but then $\mathbf{A}$ has $1/2$. In either (I) or (II), $\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$ comes under case (i). Then it cannot get 0 in V; it must get $1/2$. So, in every valuation$_*$ V, $|[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}|_V \leq |\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]|_V$. We can similarly see that, also in every valuation$_*$ V, $|\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]|_V \leq |[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}|_V$.

$\vDash \neg[\mathbf{A} \wedge \neg\mathbf{A}]$. A conjunction of the form $\mathbf{A} \wedge \neg\mathbf{A}$ is a straightforward contradiction. Denying it seems as natural as endorsing a straightforward tautology of the form $\mathbf{A} \rightarrow \mathbf{A}$. Of course, when we diverge from classical logic, we are bound to deviate from principles that seem natural. Yet not all principles are equally central to logical orthodoxy. If we do not validate the wffs of the form $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$, we abandon the law of noncontradiction. That is one of the most central laws in standard logic; abandoning it marks as radical a deviation as abandoning $\mathbf{A} \rightarrow \mathbf{A}$. Such radicalism is not required by the project of providing a logical framework for a theory of truth containing all biconditionals $T\mathbf{a} \leftrightarrow \mathbf{A}$ where T means "true" and the constant **a** is a name of **A**. The history of logic attests to how central the law of noncontradiction has been.[7]

The validity of $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ is ensured by rule (α). It is clear that if **A** has 1 or 0 in a valuation, $\mathbf{A} \wedge \neg\mathbf{A}$ gets 0 there. What is due to (α) is that $\mathbf{A} \wedge \neg\mathbf{A}$ not only can, but must, get the value 0 if **A** and $\neg\mathbf{A}$ have $1/2$. The deep conjuncts of $\mathbf{A} \wedge \neg\mathbf{A}$ are the deep conjuncts of **A** and those of $\neg\mathbf{A}$. If in a valuation$_*$ V they all had 1, then both **A** and $\neg\mathbf{A}$ would get 1 in V. And if all the deep conjuncts had 0 in V, then both **A** and $\neg\mathbf{A}$ would get 0 there. Thus $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ gets 1 in all valuations. Rule (α) also ensures that $\models \neg[\mathbf{A} \wedge [\mathbf{B} \wedge [\neg\mathbf{A} \vee \neg\mathbf{B}]]]$. Generally, the effect of the rule is to make various conjunctions whose conjuncts clearly seem incompatible take on the value 0 in all valuations. In this way, it validates the negations of those conjunctions.

It may be asked why the rule in question did not assume the simpler form "If there is no valuation$_*$ in which all the deep conjuncts of $\mathbf{A} \wedge \mathbf{B}$ have 1, then $\mathbf{A} \wedge \mathbf{B}$ must get 0." What is the purpose of the clause "there is no valuation$_*$ in which all the deep conjuncts have 0"?

Some wffs, such as $\neg[p_1 \vee \neg p_1]$ and $\neg[[p_1 \rightarrow \neg p_1] \rightarrow \neg p_1]$, get $1/2$ in some valuations but do not get 1 in any valuation$_*$. Conjunctions in which such a wff is the single deep conjunct give rise to a problem. If the rule assumed the simpler form, then it would prescribe the value 0 for $\neg[p_1 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_1]$ in the valuations that give $1/2$ to $\neg[p_1 \vee \neg p_1]$, and so it would clash with our rule that conjunctions in case (i) of Table 3 get $1/2$ if they have only one deep conjunct. But even if we ignore our rule about only one deep conjunct, it would be unmotivated to validate the negation of $\neg[p_1 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_1]$ when we do not validate the negation of $\neg[p_1 \vee \neg p_1]$. The problem extends to conjunctions that have more than one deep conjunct, including some whose deep conjuncts each get 1 in a valuation$_*$; $\neg[p_2 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_2]$ is such a conjunction. Its deep conjuncts, which are also its conjuncts, are $\neg[p_2 \vee \neg p_1]$ and $\neg[p_1 \vee \neg p_2]$. Each one of them has 1 in a valuation$_*$, but there is no valuation$_*$ where both have 1. So if the rule under discussion assumed the simpler form, we would give 0 to the conjunction in the valuations that give $1/2$ to $\neg[p_2 \vee \neg p_1]$ and $\neg[p_1 \vee \neg p_2]$. Since there is also no valuation$_*$ where one of the conjuncts has 1 and the other $1/2$, the conjunction would get 0 in all valuations, and so its negation would be validated. But, as we will see, we want substitution to be validity-preserving: if in a valid wff we replace a sentential letter with some wff, then the result of the replacement should also be valid. If we replace p_2 with p_1 in the negation of $\neg[p_2 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_2]$, then we get the negation of $\neg[p_1 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_1]$. So we would end up validating the latter negation.

We therefore need to strengthen the condition "there is no valuation$_*$ in which all the deep conjuncts of $\mathbf{A} \wedge \mathbf{B}$ have 1" in the rule. The idea here is to replace a condition that is analogous to the ordinary concept of incompatibility with one ("there is no valuation$_*$ in which all the deep conjuncts of $\mathbf{A} \wedge \mathbf{B}$ have 1 and there is no valuation$_*$ in which they all have 0") which is analogous to a standard notion of contradictoriness. According to that notion, two statements are *contradictory* if and only if they cannot both be true and they cannot both be false (see, e.g., Sainsbury [11, pp. 19–23]). In this way, the conjunction $\neg[p_1 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_1]$, as well as $\neg[p_1 \vee \neg p_1] \wedge \neg[p_2 \vee \neg p_2]$ and $\neg[p_2 \vee \neg p_1] \wedge \neg[p_1 \vee \neg p_2]$, does not have to get

0 in the valuations that give its conjuncts $1/2$. The condition chosen here is merely *analogous* to that standard notion of contradictoriness, since it may concern more than two deep conjuncts and replaces truth and falsehood with the numerical values 1 and 0. It is an open question whether there is any weaker condition that avoids the problem explained in the preceding paragraph.

It may also be asked why the possibility of two values for $\mathbf{A} \wedge \mathbf{B}$, $1/2$ and 0, arises not only when both $\mathbf{A}$ and $\mathbf{B}$ have $1/2$, but also when one of them has $1/2$ and the other has 1. Why not stipulate that $\mathbf{A} \wedge \mathbf{B}$ should get $1/2$ in the latter case? The problem is that some conjunctions whose negations we want to validate would sometimes get $1/2$. Two examples are $p_1 \wedge [p_2 \wedge [\neg p_1 \vee \neg p_2]]$ and $p_1 \wedge [p_2 \wedge \neg[p_1 \wedge p_2]]$. It is good to give either conjunction 0 in all valuations, since intuitively its deep conjuncts are incompatible and even contradictory. If we adopted the suggested stipulation, but p_1 and p_2 had 1 and $1/2$ respectively in a valuation V, then $\neg p_1 \vee \neg p_2$ and $\neg[p_1 \wedge p_2]$ would get $1/2$. Then, $p_2 \wedge [\neg p_1 \vee \neg p_2]$ and $p_2 \wedge \neg[p_1 \wedge p_2]$ might well get $1/2$. We could not force either of those conjunctions to get 0, since there is no problem with giving 1, in a valuation$_*$, to both p_2 and $\neg p_1 \vee \neg p_2$ or with giving 1 to both p_2 and $\neg[p_1 \wedge p_2]$. And then $p_1 \wedge [p_2 \wedge [\neg p_1 \vee \neg p_2]]$ and $p_1 \wedge [p_2 \wedge \neg[p_1 \wedge p_2]]$ would have $1/2$ in V rather than 0.

Further, it may be asked why the definition of "deep conjunct" involves double negation. Why did we not stipulate, more simply, that $\mathbf{A}$ is a *deep conjunct* of $\mathbf{B}$ if and only if $\mathbf{A}$ is not a conjunction and there is an occurrence $\mathcal{O}$ of $\mathbf{A}$ in $\mathbf{B}$ such that every symbol, in $\mathbf{B}$ but outside $\mathcal{O}$, in whose scope $\mathcal{O}$ lies is a $\wedge$? The problem is that if we adopted the simpler stipulation, we would end up giving 0 in every valuation to $p_1 \wedge [\neg p_1 \wedge p_2]$ and to $[p_1 \wedge p_2] \wedge [\neg p_1 \wedge \neg p_2]$, but not to $p_1 \wedge \neg\neg[\neg p_1 \wedge p_2]$ or to $\neg\neg[p_1 \wedge p_2] \wedge \neg\neg[\neg p_1 \wedge \neg p_2]$. For (in whichever of the two ways we may define "deep conjunct") p_1, $\neg p_1$, and p_2 do not all have 1, and do not all have 0, in any valuation$_*$, whereas p_1 and $\neg\neg[\neg p_1 \wedge p_2]$ do not both have 1 in any valuation$_*$, but there is a valuation$_*$ where both have 0. Similarly, there is a valuation$_*$ where both $\neg\neg[p_1 \wedge p_2]$ and $\neg\neg[\neg p_1 \wedge \neg p_2]$ get 0. But, as a referee pointed out, any reason for validating the negation of $p_1 \wedge [\neg p_1 \wedge p_2]$ is also a reason for validating the negation of $p_1 \wedge \neg\neg[\neg p_1 \wedge p_2]$.

Since we validate $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ and do not permit a conjunction $\mathbf{A} \wedge \mathbf{A}$ to have 0 in a valuation where $\mathbf{A}$ has $1/2$, we abandon the substitution of equivalents: $\ldots\mathbf{B}\ldots, \mathbf{A} \leftrightarrow \mathbf{B} \nvDash \ldots\mathbf{A}\ldots$; that is, it is not the case that, for every wff $\ldots\mathbf{B}\ldots$ and every wff $\mathbf{A}$, the inference from $\ldots\mathbf{B}\ldots$ and $\mathbf{A} \leftrightarrow \mathbf{B}$ to $\ldots\mathbf{A}\ldots$ is valid. The reason is that some valuations give 1 to certain biconditionals of the form $\mathbf{A} \leftrightarrow \neg\mathbf{A}$. Indeed, it is crucial that there should be such valuations, for we want to allow valuations that give 1 to all the statements about truth contained in the theory to be presented in the next section. The theory contains all biconditionals $T\mathbf{c} \leftrightarrow \mathbf{C}$ where T means "true" and the constant $\mathbf{c}$ is a name of $\mathbf{C}$. If $\mathbf{C}$ is a liar sentence, then the relevant biconditional has the form $\mathbf{A} \leftrightarrow \neg\mathbf{A}$. If, now, a valuation V gives 1 to $\mathbf{A} \leftrightarrow \neg\mathbf{A}$, then it must assign the value $1/2$ to $\mathbf{A}$. Thus V will give 1 to $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$, like any valuation, but not to $\neg[\mathbf{A} \wedge \mathbf{A}]$, for it cannot give 0 to $\mathbf{A} \wedge \mathbf{A}$. The value of $\neg[\mathbf{A} \wedge \mathbf{A}]$ in V will be $1/2$.

Although $\vDash \mathbf{A} \rightarrow \mathbf{A}$, it is not the case that, for every $\mathbf{A}$ and every valid wff $\top$, $\vDash \mathbf{A} \rightarrow \mathbf{A} \wedge \top$. Whatever $\top$ may be, there is a valuation that assigns $1/2$ to p_1 and 0 to $p_1 \wedge \top$ and so gives $1/2$ to $p_1 \rightarrow p_1 \wedge \top$. That may seem odd, but there are independent reasons for not validating such conditionals. For example, we should not validate $\neg[p_3 \vee \neg p_3] \rightarrow \neg[p_3 \vee \neg p_3] \wedge \neg[\neg p_3 \wedge \neg\neg p_3]$. The antecedent will have $1/2$ in any valuation that gives $1/2$ to p_3. But we ought always to give 0 to the consequent; it has the form $\neg[\mathbf{A} \vee \mathbf{B}] \wedge \neg[\neg\mathbf{A} \wedge \neg\mathbf{B}]$, so its conjuncts are intuitively contradictory. Then, since we should not validate that conditional, and we want substitution to be validity-preserving, we should not validate $p_1 \rightarrow p_1 \wedge \neg[p_2 \wedge \neg p_2]$ either. And it would be unmotivated to validate $p_1 \rightarrow p_1 \wedge \top$ for some choices of $\top$ and not for others.

$\mathbf{A} \wedge [\mathbf{B} \vee \mathbf{C}] \vDash [\mathbf{A} \wedge \mathbf{B}] \vee [\mathbf{A} \wedge \mathbf{C}]$. But $\nvDash \mathbf{A} \wedge [\mathbf{B} \vee \mathbf{C}] \rightarrow [\mathbf{A} \wedge \mathbf{B}] \vee [\mathbf{A} \wedge \mathbf{C}]$. For instance, there is a valuation V in which p_1, p_2, and p_3 have $1/2$, $p_1 \wedge [p_2 \vee p_3]$ gets $1/2$ too, but $p_1 \wedge p_2$ and $p_1 \wedge p_3$ get 0. (Say that, in case (i) of Table 3, V gives 0 to every conjunction $\mathbf{E}_1 \wedge \mathbf{E}_2$ if and only if either the deep conjuncts of $\mathbf{E}_1 \wedge \mathbf{E}_2$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$ or $\mathbf{E}_1 \wedge \mathbf{E}_2$ has more than one deep conjunct and one of its deep conjuncts is p_2 or p_3.) Then $p_1 \wedge [p_2 \vee p_3] \rightarrow [p_1 \wedge p_2] \vee [p_1 \wedge p_3]$ gets $1/2$ in V. Likewise, $[\mathbf{A} \wedge \mathbf{B}] \vee [\mathbf{A} \wedge \mathbf{C}] \vDash \mathbf{A} \wedge [\mathbf{B} \vee \mathbf{C}]$, but $\nvDash [\mathbf{A} \wedge \mathbf{B}] \vee [\mathbf{A} \wedge \mathbf{C}] \rightarrow \mathbf{A} \wedge [\mathbf{B} \vee \mathbf{C}]$.

Also, $\mathbf{A} \vee [\mathbf{B} \wedge \mathbf{C}] \vDash [\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \vee \mathbf{C}]$ and $[\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \vee \mathbf{C}] \vDash \mathbf{A} \vee [\mathbf{B} \wedge \mathbf{C}]$, but $\nvDash \mathbf{A} \vee [\mathbf{B} \wedge \mathbf{C}] \rightarrow [\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \vee \mathbf{C}]$ and $\nvDash [\mathbf{A} \vee \mathbf{B}] \wedge [\mathbf{A} \vee \mathbf{C}] \rightarrow \mathbf{A} \vee [\mathbf{B} \wedge \mathbf{C}]$. There is a valuation V in which p_1, p_2, and p_3 have $1/2$, $p_2 \wedge p_3$ gets 0, and $[p_1 \vee p_2] \wedge [p_1 \vee p_3]$ also gets 0. The value of $p_1 \vee [p_2 \wedge p_3] \rightarrow [p_1 \vee p_2] \wedge [p_1 \vee p_3]$ in V is $1/2$. And there is a valuation V' in which p_1 has 0, p_2 and p_3 have $1/2$, $p_2 \wedge p_3$ gets 0, but $[p_1 \vee p_2] \wedge [p_1 \vee p_3]$ gets $1/2$. The value of $[p_1 \vee p_2] \wedge [p_1 \vee p_3] \rightarrow p_1 \vee [p_2 \wedge p_3]$ in V' is $1/2$.

As regards De Morgan's laws, we have already seen that $\vDash \neg\mathbf{A} \wedge \neg\mathbf{B} \rightarrow \neg[\mathbf{A} \vee \mathbf{B}]$. Similarly, $\vDash \neg\mathbf{A} \vee \neg\mathbf{B} \rightarrow \neg[\mathbf{A} \wedge \mathbf{B}]$. For, in every valuation$_*$ V, $|\neg\mathbf{A} \vee \neg\mathbf{B}|_V \leq |\neg[\mathbf{A} \wedge \mathbf{B}]|_V$. Also, $\neg[\mathbf{A} \vee \mathbf{B}] \vDash \neg\mathbf{A} \wedge \neg\mathbf{B}$. Yet $\nvDash \neg[\mathbf{A} \vee \mathbf{B}] \rightarrow \neg\mathbf{A} \wedge \neg\mathbf{B}$. For example, there is a valuation in which p_1 has $1/2$, so $\neg[p_1 \vee \neg p_1]$ gets $1/2$ too, but of course $\neg p_1 \wedge \neg\neg p_1$ has 0, and so $\neg[p_1 \vee \neg p_1] \rightarrow \neg p_1 \wedge \neg\neg p_1$ gets $1/2$. Moreover, $\neg[\mathbf{A} \wedge \mathbf{B}] \nvDash \neg\mathbf{A} \vee \neg\mathbf{B}$; it is not the case that, for all $\mathbf{A}$ and $\mathbf{B}$, inferring $\neg\mathbf{A} \vee \neg\mathbf{B}$ from $\neg[\mathbf{A} \wedge \mathbf{B}]$ is valid. If that were the case, then all disjunctions $\mathbf{A} \vee \neg\mathbf{A}$ would be valid. For $\vDash \neg[\mathbf{A} \wedge \neg\mathbf{A}]$, and $\neg\mathbf{A} \vee \neg\neg\mathbf{A} \vDash \mathbf{A} \vee \neg\mathbf{A}$.

Similarly, if, for all $\mathbf{A}$ and $\mathbf{B}$, inferring $\mathbf{A} \vee \mathbf{B}$ from $\neg[\neg\mathbf{A} \wedge \neg\mathbf{B}]$ were valid, then all disjunctions $\mathbf{A} \vee \neg\mathbf{A}$ would be valid. For $\vDash \neg[\neg\mathbf{A} \wedge \neg\neg\mathbf{A}]$. Thus $\neg[\neg\mathbf{A} \wedge \neg\mathbf{B}] \nvDash \mathbf{A} \vee \mathbf{B}$. On the other hand, $\neg[\neg\mathbf{A} \vee \neg\mathbf{B}] \vDash \mathbf{A} \wedge \mathbf{B}$. Yet $\nvDash \neg[\neg\mathbf{A} \vee \neg\mathbf{B}] \rightarrow \mathbf{A} \wedge \mathbf{B}$. There is a valuation in which p_1 and p_2 have $1/2$, but $p_1 \wedge p_2$ gets 0, and so the value of $\neg[\neg p_1 \vee \neg p_2] \rightarrow p_1 \wedge p_2$ is $1/2$. Also, since inferring from a conditional to a contrapositive of it is valid (as we will see), $\vDash \mathbf{A} \vee \mathbf{B} \rightarrow \neg[\neg\mathbf{A} \wedge \neg\mathbf{B}]$ and $\vDash \mathbf{A} \wedge \mathbf{B} \rightarrow \neg[\neg\mathbf{A} \vee \neg\mathbf{B}]$.

$\vDash \neg\mathbf{A} \rightarrow [\mathbf{A} \rightarrow \mathbf{B}]$, $\vDash \mathbf{B} \rightarrow [\mathbf{A} \rightarrow \mathbf{B}]$, $\vDash \neg[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow \mathbf{A}$, and $\vDash \neg[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow \neg\mathbf{B}$. Given these validities, our conditional can fairly be described as a material conditional. Also, $\mathbf{A}, \neg\mathbf{B} \vDash \neg[\mathbf{A} \rightarrow \mathbf{B}]$. But $\nvDash \mathbf{A} \wedge \neg\mathbf{B} \rightarrow \neg[\mathbf{A} \rightarrow \mathbf{B}]$. It is easy to construct a valuation V in which p_1, p_2, and $p_1 \wedge \neg p_2$ get $1/2$, but $p_1 \rightarrow p_2$ gets 1. (Say that V gives $1/2$ to every conjunction $\mathbf{E}_1 \wedge \mathbf{E}_2$ in case (i) of Table 3 unless the deep conjuncts of $\mathbf{E}_1 \wedge \mathbf{E}_2$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$, and that V gives 1 to every conditional in case (ii) of Table 4.) The value of $p_1 \wedge \neg p_2 \rightarrow \neg[p_1 \rightarrow p_2]$ in V is $1/2$.

$\mathbf{A} \rightarrow \mathbf{B}, \mathbf{B} \rightarrow \mathbf{C} \vDash \mathbf{A} \rightarrow \mathbf{C}$. For if $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{C}$ get 1 in a valuation V, then $|\mathbf{A}|_V \leq |\mathbf{B}|_V$ and $|\mathbf{B}|_V \leq |\mathbf{C}|_V$. So $|\mathbf{A}|_V \leq |\mathbf{C}|_V$. This guarantees that $\mathbf{A} \rightarrow \mathbf{C}$ gets 1 in V unless $\mathbf{A}$ and $\mathbf{C}$ have $1/2$ in V. But, if they have $1/2$, then

$\mathbf{B}$ also has $1/2$ there, and one of the additional rules for conditionals in case (ii) ensures that, once more, $\mathbf{A} \to \mathbf{C}$ gets 1 in V. The purpose of the rule is to help validate the inference from $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{C}$ to $\mathbf{A} \to \mathbf{C}$. On the other hand, $\nvDash [\mathbf{A} \to \mathbf{B}] \wedge [\mathbf{B} \to \mathbf{C}] \to [\mathbf{A} \to \mathbf{C}]$. For there is a valuation in which, say, p_1, p_2, and p_3 have 1, $1/2$, and 0, respectively, while $[p_1 \to p_2] \wedge [p_2 \to p_3]$ gets $1/2$. Then $[p_1 \to p_2] \wedge [p_2 \to p_3] \to [p_1 \to p_3]$ gets $1/2$.

$\vDash [\mathbf{A} \to \mathbf{B}] \to [\mathbf{B}' \to \mathbf{A}']$, where $\mathbf{B}' \to \mathbf{A}'$ is a contrapositive of $\mathbf{A} \to \mathbf{B}$. For, in every valuation$_*$ V, $|\mathbf{A} \to \mathbf{B}|_V \leq |\mathbf{B}' \to \mathbf{A}'|_V$. If $\mathbf{A} \to \mathbf{B}$ gets 1 in V, then $|\mathbf{A}|_V \leq |\mathbf{B}|_V$. So $|\mathbf{B}'|_V \leq |\mathbf{A}'|_V$. This guarantees that $\mathbf{B}' \to \mathbf{A}'$ gets 1 in V unless $\mathbf{B}'$ and $\mathbf{A}'$ have $1/2$ there. If they have $1/2$, then the value of $\mathbf{A}$ and $\mathbf{B}$ is also $1/2$, and one of the additional rules for conditionals in case (ii) ensures that, again, $\mathbf{B}' \to \mathbf{A}'$ gets 1 in V. Clearly, the purpose of the rule is just to help validate the inference from $\mathbf{A} \to \mathbf{B}$ to $\mathbf{B}' \to \mathbf{A}'$. And it cannot be that, in V, $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B}' \to \mathbf{A}'$ get $1/2$ and 0, respectively. For if $\mathbf{B}'$ and $\mathbf{A}'$ have 1 and 0, then $\mathbf{A}$ and $\mathbf{B}$ have 1 and 0, so $\mathbf{A} \to \mathbf{B}$ gets 0.

$\mathbf{A} \to \neg\mathbf{A} \nvDash \neg\mathbf{A}$ and $\neg\mathbf{A} \to \mathbf{A} \nvDash \mathbf{A}$. There is a valuation in which p_1 has $1/2$, but both $p_1 \to \neg p_1$ and $\neg p_1 \to p_1$ get 1. On the other hand, $\mathbf{A} \to \mathbf{B} \wedge \neg\mathbf{B} \vDash \neg\mathbf{A}$ and $\neg\mathbf{A} \to \mathbf{B} \wedge \neg\mathbf{B} \vDash \mathbf{A}$. However, $\nvDash [\mathbf{A} \to \mathbf{B} \wedge \neg\mathbf{B}] \to \neg\mathbf{A}$. For instance, $p_1 \to p_2 \wedge \neg p_2$ does not imply $\neg p_1$: there are valuations$_*$ in which $p_1 \to p_2 \wedge \neg p_2$ has 1, but $\neg p_1$ has $1/2$, and there are also valuations$_*$ in which $p_1 \to p_2 \wedge \neg p_2$ gets $1/2$, but $\neg p_1$ gets 0. Thus it is easy to construct a valuation V in which p_1 has $1/2$, so $p_1 \to p_2 \wedge \neg p_2$ also has $1/2$, and $[p_1 \to p_2 \wedge \neg p_2] \to \neg p_1$ gets $1/2$. (Say that, in case (ii) of Table 4, V gives $1/2$ to every conditional $\mathbf{C} \to \mathbf{D}$ unless $\mathbf{C}$ implies $\mathbf{D}$.) Likewise, $\nvDash [\neg\mathbf{A} \to \mathbf{B} \wedge \neg\mathbf{B}] \to \mathbf{A}$.

Of course, $\vDash \neg\mathbf{A} \vee \mathbf{B} \to [\mathbf{A} \to \mathbf{B}]$. But $\mathbf{A} \to \mathbf{B} \nvDash \neg\mathbf{A} \vee \mathbf{B}$. For example, there is a valuation in which p_1 has $1/2$, so $\neg p_1 \vee p_1$ gets $1/2$ too, but $p_1 \to p_1$ gets 1. Again, $\neg[\mathbf{A} \to \mathbf{B}] \vDash \mathbf{A} \wedge \neg\mathbf{B}$. But $\nvDash \neg[\mathbf{A} \to \mathbf{B}] \to \mathbf{A} \wedge \neg\mathbf{B}$. There is a valuation where p_1 has 1, p_2 has $1/2$, $p_1 \wedge \neg p_2$ gets 0, and so $\neg[p_1 \to p_2] \to p_1 \wedge \neg p_2$ gets $1/2$.

Up to now, we have considered only inferences whose premises are wffs. We can now touch upon inferences whose premises, or at least some of them, are other inferences. To be precise, we can consider inferences of the form "$\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k, \mathbf{A}_1, \ldots, \mathbf{A}_n$; hence, $\mathbf{B}$" ($k \geq 1$, $n \geq 0$). Here, $\mathbf{A}_1, \ldots, \mathbf{A}_n$ and $\mathbf{B}$ are wffs, but each one of $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k$ is an inference in which the premises and conclusion are wffs. For example, we make an inference whose premise is another inference whenever we conclude $\mathbf{C} \to \mathbf{D}$ once we have inferred $\mathbf{D}$ from $\mathbf{C}$.[8]

In order to evaluate inferences whose premises include other inferences, we need to extend our concept of validity to them. Here is how: the inference from $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k, \mathbf{A}_1, \ldots, \mathbf{A}_n$ to $\mathbf{B}$ is *valid* ($\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k, \mathbf{A}_1, \ldots, \mathbf{A}_n \vDash \mathbf{B}$) if and only if $\mathbf{B}$ has value 1 in every valuation in which $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k$ are 1-preserving and $\mathbf{A}_1, \ldots, \mathbf{A}_n$ have 1. An inference from the wffs $\mathbf{C}_1, \ldots, \mathbf{C}_j$ to the wff $\mathbf{D}$ is 1-*preserving in* a valuation V just in case either $\mathbf{C}_1, \ldots, \mathbf{C}_j$ do not all have 1 in V or $\mathbf{D}$ has 1 there.

$\mathbf{A} \vee \mathbf{B}$, $(\mathbf{A}; \mathbf{C})$, $(\mathbf{B}; \mathbf{D}) \vDash \mathbf{C} \vee \mathbf{D}$, where $(\mathbf{A}; \mathbf{C})$ is the inference from $\mathbf{A}$ to $\mathbf{C}$ and $(\mathbf{B}; \mathbf{D})$ is the inference from $\mathbf{B}$ to $\mathbf{D}$. On the other hand, $(\mathbf{A}; \mathbf{B}) \nvDash \mathbf{A} \to \mathbf{B}$; that is, it is not the case that, for all $\mathbf{A}$ and $\mathbf{B}$, inferring $\mathbf{A} \to \mathbf{B}$ once one has inferred $\mathbf{B}$ from $\mathbf{A}$ is valid. For instance, inferring $p_1 \to p_2$ from $(p_1; p_2)$ is not valid. There is a valuation V in which p_1 has $1/2$, so $(p_1; p_2)$ is 1-preserving, but p_2 has $1/2$ or 0, and $p_1 \to p_2$ gets $1/2$. However, $\mathbf{A} \vee \neg\mathbf{A}$, $(\mathbf{A}; \mathbf{B}) \vDash \mathbf{A} \to \mathbf{B}$. Also, $(\mathbf{A}; \neg\mathbf{A}) \nvDash \neg\mathbf{A}$

and $(\mathbf{A}; \mathbf{B} \wedge \neg\mathbf{B}) \not\models \neg\mathbf{A}$. This is shown by V again. For, in V, $(p_1; \neg p_1)$ and $(p_1; p_2 \wedge \neg p_2)$ are 1-preserving, but $\neg p_1$ gets $1/2$. Still, $\mathbf{A} \vee \neg\mathbf{A}$, $(\mathbf{A}; \neg\mathbf{A}) \models \neg\mathbf{A}$ and $\mathbf{A} \vee \neg\mathbf{A}$, $(\mathbf{A}; \mathbf{B} \wedge \neg\mathbf{B}) \models \neg\mathbf{A}$.

If we wish to treat disjunction analogously to conjunction, we can allow that $\mathbf{A} \vee \mathbf{B}$ may get either $1/2$ or 1 in a valuation$_*$ when $\mathbf{A}$ and $\mathbf{B}$ have $1/2$ or the one has $1/2$ and the other 0. We can then define a concept of deep disjunct by analogy with that of deep conjunct and stipulate that, in any valuation where $\mathbf{A}$ and $\mathbf{B}$ have such values, $\mathbf{A} \vee \mathbf{B}$ will get 1 if there is no valuation$_*$ in which all its deep disjuncts have 0. We may here add "and there is also no valuation$_*$ in which they all have 1." If we make those changes, we will validate $\mathbf{A} \vee \neg\mathbf{A}$ for every $\mathbf{A}$. On the other hand, we will cease validating the inference from $p_1 \vee \neg p_1$, $p_1 \rightarrow p_2$, and $\neg p_1 \rightarrow p_2$ to p_2. For some valuations will give $1/2$ to p_1 and p_2 but 1 to all of $p_1 \vee \neg p_1$, $p_1 \rightarrow p_2$, and $\neg p_1 \rightarrow p_2$. So it will not be generally valid to infer from $\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \rightarrow \mathbf{C}$, and $\mathbf{B} \rightarrow \mathbf{C}$ to $\mathbf{C}$. We will also cease validating the inference from $p_1 \wedge [p_2 \vee \neg p_2]$ to $[p_1 \wedge p_2] \vee [p_1 \wedge \neg p_2]$. For some valuations will give 1 to both p_1 and $p_2 \vee \neg p_2$ but $1/2$ to p_2 and also $1/2$ to $p_1 \wedge p_2$, $p_1 \wedge \neg p_2$, and their disjunction. So it will not be always valid to infer from $\mathbf{A} \wedge [\mathbf{B} \vee \mathbf{C}]$ to $[\mathbf{A} \wedge \mathbf{B}] \vee [\mathbf{A} \wedge \mathbf{C}]$. It seems that overall no significant gain would stem from making those changes.

It is worth showing that substitution is validity-preserving. In other words, replacing a sentential letter with a wff, or an individual constant with another individual constant, throughout a valid wff or a valid inference yields a wff or inference that is also valid. Sentential letters and individual constants are schematic letters. So wffs are schemas, and inferences are patterns of reasoning. If we begin with a wff or inference, and in it we replace a sentential letter with a wff, or an individual constant with another one, then the result is either a more restricted schema or pattern than what we began with or an alphabetic variant of what we began with. For example, when in a wff $\mathbf{A}$ we replace a sentential letter with a wff that is not a sentential letter, the result can be seen as a schema that is more restricted than $\mathbf{A}$. When in $\mathbf{A}$ we replace an individual constant with another one $\mathbf{c}$, the result can be seen as a schema that either is more restricted than $\mathbf{A}$ or is just an alphabetic variant of $\mathbf{A}$ (depending on whether $\mathbf{c}$ already occurred in $\mathbf{A}$). If we validate a schema or a pattern of reasoning, then we should also validate its alphabetic variants, as well as the more restricted schemas or patterns. So substitution should be validity-preserving. That it is will be a corollary of the next theorem.

For any $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, $\mathbf{A}[\mathbf{B}/\mathbf{C}]$ will be the result of substituting $\mathbf{B}$ for $\mathbf{C}$ throughout $\mathbf{A}$. Similarly, if $\mathbf{\Gamma}$ is an inference, then $\mathbf{\Gamma}[\mathbf{B}/\mathbf{C}]$ will be the result of substituting $\mathbf{B}$ for $\mathbf{C}$ throughout $\mathbf{\Gamma}$.

Theorem 1 *If V is a valuation, $\mathbf{E}$ is an atomic wff, and $\mathbf{E}'$ is any wff, then there is a valuation which, for every $\mathbf{A}$, gives $\mathbf{A}$ the value that V gives to $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$.*

Proof For any valuation$_*$ V, let $V^{[\mathbf{E}'/\mathbf{E}]}$ be the assignment of values to wffs which, for every wff $\mathbf{A}$, gives $\mathbf{A}$ the value that V gives to $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$. We can prove that $V^{[\mathbf{E}'/\mathbf{E}]}$ is a valuation$_*$. Of course, $V^{[\mathbf{E}'/\mathbf{E}]}$ does not assign 1 to $\mathbf{A}$ and $1/2$ to $\neg\mathbf{A}$. It does not go against the table of any connective. So how can it fail to be a valuation$_*$? Three possibilities arise.

(i) Perhaps, there are $\mathbf{A}$ and $\mathbf{B}$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives $1/2$ to both, or $1/2$ to the one and 1 to the other, and then gives 0 to $\mathbf{A} \wedge \mathbf{B}$, yet $\mathbf{A} \wedge \mathbf{B}$ has only one deep conjunct, $\mathbf{D}$. But then, V will be assigning $1/2$ to both $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ and

$\mathbf{B}[\mathbf{E}'/\mathbf{E}]$, or $1/2$ to the one and 1 to the other, and will also be assigning 0 to $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$. Note that the deep conjuncts of $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ are just the deep conjuncts of $\mathbf{D}[\mathbf{E}'/\mathbf{E}]$. Moreover, V will be assigning $1/2$ to $\mathbf{D}[\mathbf{E}'/\mathbf{E}]$; for if it assigned it 1, it would assign 1 to both $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ and $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$, and if it assigned it 0, it would assign 0 to both $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ and $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$. Now, $\mathbf{D}[\mathbf{E}'/\mathbf{E}]$, like any wff, can be seen as having the form $\underbrace{\neg \cdots \neg}_{n \text{ times}} \mathbf{C}$ where n is an even number, possibly zero, and $\mathbf{C}$ is not a double negation. $\mathbf{C}$ has $1/2$ in V. The deep conjuncts of $\mathbf{D}[\mathbf{E}'/\mathbf{E}]$ are those of $\mathbf{C}$. If $\mathbf{C}$ is not a conjunction, then $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ has only one deep conjunct. This goes against the fact that V gives 0 to $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$. If, on the other hand, $\mathbf{C}$ is a conjunction, then V will be assigning $1/2$ to both conjuncts of $\mathbf{C}$ or $1/2$ to the one and 1 to the other. And then the values 0 and $1/2$ of $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ and $\mathbf{C}$, respectively, go against the fact that every deep conjunct of $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ is a deep conjunct of $\mathbf{C}$.

(ii) Perhaps, there are conjunctions $\mathbf{C}$ and $\mathbf{C}'$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives them 0 and $1/2$, respectively, yet they both come under case (i) of Table 3, and every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{C}'$. But then, V will be assigning 0 to $\mathbf{C}[\mathbf{E}'/\mathbf{E}]$ and either $1/2$ to both its conjuncts or $1/2$ to the one and 1 to the other, and V will also be assigning $1/2$ to $\mathbf{C}'[\mathbf{E}'/\mathbf{E}]$ and either $1/2$ to both its conjuncts or $1/2$ to the one and 1 to the other. This goes against the fact that every deep conjunct of $\mathbf{C}[\mathbf{E}'/\mathbf{E}]$ will be a deep conjunct of $\mathbf{C}'[\mathbf{E}'/\mathbf{E}]$. And that is a fact because any wff $\mathbf{A}$ is a deep conjunct of $\mathbf{C}[\mathbf{E}'/\mathbf{E}]$ if and only if it is a deep conjunct of $\mathbf{D}[\mathbf{E}'/\mathbf{E}]$ for some deep conjunct $\mathbf{D}$ of $\mathbf{C}$, and any wff $\mathbf{A}'$ is a deep conjunct of $\mathbf{C}'[\mathbf{E}'/\mathbf{E}]$ if and only if it is a deep conjunct of $\mathbf{D}'[\mathbf{E}'/\mathbf{E}]$ for some deep conjunct $\mathbf{D}'$ of $\mathbf{C}'$.

(iii) Perhaps, there are wffs $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives $1/2$ to $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, 1 to $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{C}$, but $1/2$ to $\mathbf{A} \to \mathbf{C}$. Or there are wffs $\mathbf{A}$ and $\mathbf{B}$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives $1/2$ to $\mathbf{A}$ and $\mathbf{B}$, 1 to $\mathbf{A} \to \mathbf{B}$, but $1/2$ to a contrapositive of $\mathbf{A} \to \mathbf{B}$. It is clear that either case goes against the fact that V conforms with the additional rules governing conditionals in valuations$_*$.

Thus, for any valuation$_*$ V, $V^{[\mathbf{E}'/\mathbf{E}]}$ is also a valuation$_*$. As a consequence, we see that if $\mathbf{D}_1, \ldots, \mathbf{D}_i$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$, then $\mathbf{D}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{D}_i[\mathbf{E}'/\mathbf{E}]$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$. For if there is a valuation$_*$ V in which all of $\mathbf{D}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{D}_i[\mathbf{E}'/\mathbf{E}]$ get 1 (or one in which all get 0), then all of $\mathbf{D}_1, \ldots, \mathbf{D}_i$ get 1 (or 0) in $V^{[\mathbf{E}'/\mathbf{E}]}$. We similarly see that if $\mathbf{A}$ implies $\mathbf{B}$, then $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ implies $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$.

We can further prove that, for any valuation V, $V^{[\mathbf{E}'/\mathbf{E}]}$ is also a valuation. We already know it is a valuation$_*$. How can it fail to be a valuation? Two possibilities arise.

(iv) Perhaps, there are wffs $\mathbf{A}$ and $\mathbf{B}$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives $1/2$ to both, or $1/2$ to the one and 1 to the other, and gives $1/2$ to $\mathbf{A} \wedge \mathbf{B}$, yet the deep conjuncts $\mathbf{D}_1, \ldots, \mathbf{D}_i$ of $\mathbf{A} \wedge \mathbf{B}$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$. But then, V will be assigning $1/2$ to both $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ and $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$, or $1/2$ to the one and 1 to the other, and will also be assigning $1/2$ to $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$. Moreover, a wff is a deep conjunct of $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ if and only if it is a deep conjunct of at least one of $\mathbf{D}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{D}_i[\mathbf{E}'/\mathbf{E}]$.

Also, $\mathbf{D}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{D}_i[\mathbf{E}'/\mathbf{E}]$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$. Thus it is not the case that, in some valuation$_*$, the deep conjuncts of $\mathbf{D}_1[\mathbf{E}'/\mathbf{E}], \ldots,$ and the deep conjuncts of $\mathbf{D}_i[\mathbf{E}'/\mathbf{E}]$ all have 1 or all have 0. So the value 1/2 of $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \wedge \mathbf{B}[\mathbf{E}'/\mathbf{E}]$ goes against the fact that the deep conjuncts of that conjunction do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$.

(v) Perhaps, there are wffs $\mathbf{A}$ and $\mathbf{B}$ such that $V^{[\mathbf{E}'/\mathbf{E}]}$ gives 1/2 to both and also gives 1/2 to $\mathbf{A} \to \mathbf{B}$, yet $\mathbf{A}$ implies $\mathbf{B}$. Then, V will be assigning 1/2 to $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$, $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$, and $\mathbf{A}[\mathbf{E}'/\mathbf{E}] \to \mathbf{B}[\mathbf{E}'/\mathbf{E}]$, although $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ will imply $\mathbf{B}[\mathbf{E}'/\mathbf{E}]$. □

Corollary *Let* $\mathbf{E}$ *be an atomic wff, and let* $\mathbf{E}'$ *be any wff. If* $\vDash \mathbf{A}$*, then* $\vDash \mathbf{A}[\mathbf{E}'/\mathbf{E}]$*. If* $\mathbf{A}_1, \ldots, \mathbf{A}_j \vDash \mathbf{B}$*, then* $\mathbf{A}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{A}_j[\mathbf{E}'/\mathbf{E}] \vDash \mathbf{B}[\mathbf{E}'/\mathbf{E}]$*. If* $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k, \mathbf{A}_1, \ldots, \mathbf{A}_n \vDash \mathbf{B}$*, then* $\mathbf{\Gamma}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{\Gamma}_k[\mathbf{E}'/\mathbf{E}], \mathbf{A}_1[\mathbf{E}'/\mathbf{E}], \ldots, \mathbf{A}_n[\mathbf{E}'/\mathbf{E}] \vDash \mathbf{B}[\mathbf{E}'/\mathbf{E}]$.

Proof If there is a valuation V in which $\mathbf{A}[\mathbf{E}'/\mathbf{E}]$ gets a value other than 1, then there is a valuation, namely, $V^{[\mathbf{E}'/\mathbf{E}]}$, in which $\mathbf{A}$ gets a value other than 1. Thus if $\nvDash \mathbf{A}[\mathbf{E}'/\mathbf{E}]$, then $\nvDash \mathbf{A}$. A similar point applies to inferences. □

The corollary does not only cover the case of replacing a sentential letter with a wff, but also the case of replacing an individual constant with another individual constant. For, in our language, substituting $\mathbf{b}$ for $\mathbf{a}$ amounts to substituting $T\mathbf{b}$ for $T\mathbf{a}$.

3 Truth

At the level of the logic just expounded, the predicate letter T and the individual constants had no particular meaning. In the theory of truth that will be presented now, they take on their intended meanings: T means "true," and the individual constants are names of the wffs of our language. I assume that we have a function $\mathcal{R}$ from the set of the individual constants onto the set of the wffs. Each constant $\mathbf{a}$ is a name of $\mathcal{R}(\mathbf{a})$.[9] $\mathcal{R}$ can be any function from the constants onto the wffs. So $\mathcal{R}(\mathbf{a})$ may contain $\mathbf{A}$. For example, it may be the liar sentence $\neg T\mathbf{a}$. Or again, $\mathcal{R}(\mathbf{a})$ may be a Curry sentence $T\mathbf{a} \to \mathbf{B} \wedge \neg\mathbf{B}$. Or $\mathcal{R}(\mathbf{a})$ and $\mathcal{R}(\mathbf{b})$ may be $\neg T\mathbf{b}$ and $T\mathbf{a}$, respectively.

The theory will be specified model-theoretically. I will select a set $\mathcal{S}$ of valuations and define the theory as $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ is the class of the wffs that have the designated value, 1, in all valuations in $\mathcal{S}$, and $\mathcal{C}_2$ is the class of the simple inferences where the conclusion has 1 in every valuation $V \in \mathcal{S}$ in which the premises have 1. An inference is *simple* if and only if the premises and conclusion are wffs (and not other inferences). The members of $\mathcal{C}_1$ in which T occurs are the statements and principles about truth that are sanctioned by the theory. The members of $\mathcal{C}_2$ in which T occurs are the simple inferences involving truth that the theory approves. (Of course, $\mathcal{C}_1$ contains the valid wffs of our logic irrespective of whether T occurs in them. Likewise, $\mathcal{C}_2$ contains all valid simple inferences.)

Let $\mathcal{R}(a_1), \mathcal{R}(a_2), \ldots$ be $\mathbf{A}_1, \mathbf{A}_2, \ldots$. The class $\mathcal{C}_1$ should contain all biconditionals $Ta_i \leftrightarrow \mathbf{A}_i$. (To be precise, it should contain all conjunctions of the form $[Ta_i \to \mathbf{A}_i] \wedge [\mathbf{A}_i \to Ta_i]$.) It is those biconditionals, more than anything else, that characterize the concept of truth. An attribution of truth to a sentence is equivalent to that sentence. So even $Ta_i \leftrightarrow \neg Ta_i$ should belong to $\mathcal{C}_1$ if a_i names a liar sentence $\neg Ta_i$. Thus $\mathcal{S}$ should contain only valuations that give 1 to all of $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$. This is our basic stipulation about $\mathcal{S}$.

There are other statements and inferences about truth which we would like to include in our theory because they seem obvious. Some are automatically included once we make the basic stipulation about $\mathcal{S}$. One example is the wffs of the form $Ta_i \rightarrow \neg Ta_n$ where $\mathbf{A}_n$ is $\neg\mathbf{A}_i$. If $\mathbf{A}_i$ is true, then its negation is not true. Any valuation that assigns 1 to $Ta_i \rightarrow \mathbf{A}_i$ and to $Ta_n \rightarrow \neg\mathbf{A}_i$ will also, by contraposition and transitivity, give 1 to $Ta_i \rightarrow \neg Ta_n$. Other statements about truth may not be automatically included, though. For example, there seems to be no guarantee that each valuation assigning 1 to all of $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$ will give 1 to $\neg[Ta_i \wedge Ta_n]$ for all $\mathbf{A}_i$ and $\mathbf{A}_n$ that are related as explained.

All valuations give 0 to conjunctions of the form $\mathbf{A} \wedge \neg\mathbf{A}$ and thus give 1 to $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$. But we also want the valuations in $\mathcal{S}$ to assign 1 to $\neg[Ta_i \wedge Ta_n]$ if the wff $\mathbf{A}_n$ is the negation of the wff $\mathbf{A}_i$; for valuations intended to articulate the concept of truth should assign the designated value to the claim that those two wffs are not both true. Just as the law of noncontradiction is a basic principle of logic, so it is a basic principle about truth (very akin to noncontradiction) that a statement and its negation are not both true. The inference from $\neg[\mathbf{A}_i \wedge \mathbf{A}_n]$, $Ta_i \leftrightarrow \mathbf{A}_i$, and $Ta_n \leftrightarrow \mathbf{A}_n$ to $\neg[Ta_i \wedge Ta_n]$ is not valid in our logic unless $\mathbf{A}_i$ meets special conditions. I stipulate that $\mathcal{S}$ should contain only valuations V that have the following property.

(Prop) *For any* $\mathbf{C}$ *and* $\mathbf{D}$, $\mathbf{C} \wedge \mathbf{D}$ *gets* 0 *in* V *if there are sequences* $\mathbf{E}_1, \ldots, \mathbf{E}_m$ *and* $\mathbf{E}'_1, \ldots, \mathbf{E}'_m$ *of wffs* ($m \geq 2$) *such that for every* i ($1 \leq i < m$): $\mathbf{E}_i$ *is a* T*-attribution* Ta_{j_i} *prefixed with zero, one, or more negation signs;* $\mathbf{E}'_i$ *is also a* T*-attribution,* $Ta_{j'_i}$, *prefixed with zero, one, or more negation signs;* $\mathbf{E}_1$ *and* $\mathbf{E}'_1$ *are deep conjuncts of* $\mathbf{C} \wedge \mathbf{D}$*;* $\mathbf{E}_{i+1}$ *results from* $\mathbf{E}_i$ *by replacing* Ta_{j_i} *with* $\mathbf{A}_{j_i}$*;* $\mathbf{E}'_{i+1}$ *results from* $\mathbf{E}'_i$ *by replacing* $Ta_{j'_i}$ *with* $\mathbf{A}_{j'_i}$*; and, in every valuation,* $\mathbf{E}_m$ *and* $\mathbf{E}'_m$ *have opposite values.*

(Prop) guarantees that if $\mathbf{A}_n$ is $\neg\mathbf{A}_i$, then $Ta_i \wedge Ta_n$ gets 0 and so $\neg[Ta_i \wedge Ta_n]$ gets 1: take $m = 2$, and let Ta_i, $\mathbf{A}_i$, Ta_n, $\neg\mathbf{A}_i$ be $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}'_1$, $\mathbf{E}'_2$, respectively.

(Prop) also guarantees that $Ta_{i+} \wedge Ta_{n+}$ gets 0, and so $\neg[Ta_{i+} \wedge Ta_{n+}]$ gets 1, if $\mathbf{A}_{i+}$ is Ta_i while $\mathbf{A}_{n+}$ is Ta_n. The attribution of truth to $\mathbf{A}_i$ and the attribution of truth to $\neg\mathbf{A}_i$ are not both true. Further, (Prop) ensures that $\neg[Ta_i \wedge \neg Ta_d]$ gets 1 if $\mathbf{A}_d$ is $\neg\neg\mathbf{A}_i$. It cannot be that $\mathbf{A}_i$ is true but its double negation is not. (Take $m = 2$, and let Ta_i, $\mathbf{A}_i$, $\neg Ta_d$, $\neg\neg\neg\mathbf{A}_i$ be $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}'_1$, $\mathbf{E}'_2$, respectively.) (Prop) also ensures that $\neg[Ta_{i+} \wedge \neg Ta_{d+}]$ gets 1 if $\mathbf{A}_{d+}$ is Ta_d. It cannot be that the attribution of truth to $\mathbf{A}_i$ is true but the attribution of truth to the double negation of $\mathbf{A}_i$ is not true.

However, a valuation that has (Prop) may not give 0 to the wffs $Ta_i \wedge [Ta_n \vee [\mathbf{B} \wedge \neg\mathbf{B}]]$, $Ta_i \wedge [Ta_n \vee \neg[\mathbf{B} \rightarrow \mathbf{B}]]$, and the like. We can modify the property so as to secure that value for all such wffs. We can do that by replacing the clause "$\mathbf{E}_1$ and $\mathbf{E}'_1$ are deep conjuncts of $\mathbf{C} \wedge \mathbf{D}$" in (Prop) with "each one of $\mathbf{E}_1$ and $\mathbf{E}'_1$ is equivalent to a deep conjunct of $\mathbf{C} \wedge \mathbf{D}$," where two wffs are *equivalent* if and only if, in each valuation, the one has the same value as the other. For any $\mathbf{B}$, Ta_n is equivalent to $Ta_n \vee [\mathbf{B} \wedge \neg\mathbf{B}]$. In this way, the negation of $Ta_i \wedge [Ta_n \vee [\mathbf{B} \wedge \neg\mathbf{B}]]$ is certain to be in our theory. Still, that way will not secure the value 0 for $Ta_i \wedge [Ta_n \vee [Ta_j \wedge Ta_m]]$, where $\mathbf{A}_m$ is $\neg\mathbf{A}_j$, $j \neq n$, and $m \neq n$, and it is an open question how to secure 0 for that conjunction. It is desirable to have $\neg[Ta_i \wedge [Ta_n \vee [\mathbf{B} \wedge \neg\mathbf{B}]]]$ and

$\neg[Ta_i \wedge [Ta_n \vee [Ta_j \wedge Ta_m]]]$ in our theory, but at any rate if we have the one, then we should also have the other. Thus I do not modify (Prop).

So, a valuation belongs to $\mathcal{S}$ if and only if it gives 1 to all biconditionals $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$ and also has the property (Prop). I do not claim that if we select $\mathcal{S}$ in that way, then our theory $(\mathcal{C}_1 \cup \mathcal{C}_2)$ contains all the statements and inferences about truth that seem obvious. For example, there seems to be no guarantee that the valuations in $\mathcal{S}$ will give 1 to every conditional $Ta_k \to Ta_i \vee Ta_j$ where $\mathbf{A}_k$ is $\mathbf{A}_i \vee \mathbf{A}_j$. (On the other hand, if Ta_k gets 1 in a valuation V in $\mathcal{S}$, then $Ta_i \vee Ta_j$ also gets 1 there. For $T\mathbf{a}$ has 1 in V if and only if $\mathcal{R}(\mathbf{a})$ has 1 in V; and a disjunction has 1 in a valuation if and only if one of its disjuncts has 1 there. Thus the inference from Ta_k to $Ta_i \vee Ta_j$ makes it into $\mathcal{C}_2$.) Nevertheless, thanks especially to the biconditionals $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$, the theory is quite close to our pretheoretic conception of truth.

Or rather, that is so if $\mathcal{S}$, as defined, is not empty. If it is empty, then $\mathcal{C}_1$ will contain all wffs, and $\mathcal{C}_2$ will contain all simple inferences, in our language. In this case, our theory of truth will not be particularly interesting. Happily, things are different. We can prove that there are valuations with the features required for membership in $\mathcal{S}$. Indeed, any assignment of values (1, 1/2, 0) to sentential letters can be extended to a valuation in $\mathcal{S}$.

Central Theorem *For each assignment K of values to one or more sentential letters, there is a valuation that gives* 1 *to all of* $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$, *incorporates K, and has (Prop).*

Proving the Central Theorem will be our task in Section 4. The proof is complicated. One difficulty in constructing the desirable valuation is of course that, on the one hand, the values of compound wffs depend on the values of shorter ones, but, on the other, the value of any atomic wff Ta_i should be the same as the value of the possibly compound $\mathbf{A}_i$. We cannot use a fixed-point construction of the kind developed by Kripke [6] here. There is a property of monotonicity that is crucial for that construction: if all atomic wffs that have integral values in a model M keep those values in a model M', then all wffs that have integral values in M keep those values in M'. Tables 1–4 of Section 2 do not guarantee that property, and the reason is that value 0 is possible in case (i) of Table 3 for conjunction and value 1 is possible in case (ii) in Table 4 of the conditional.

We can see here how some paradoxical wffs lead us to eschew value-functionality in case (ii). Say that a_j names the wff $Ta_j \to \neg Ta_j$. We want to give 1 to the biconditional $Ta_j \leftrightarrow [Ta_j \to \neg Ta_j]$. Suppose that we have selected the tables that will govern the connectives, except for rows (i) and (ii), and we have determined that $\mathbf{A} \wedge \mathbf{B}$ does not get 1 unless both $\mathbf{A}$ and $\mathbf{B}$ get 1. Then the only way to ensure value 1 for the biconditional is to assign $1/2$ to Ta_j and to $\neg Ta_j$, consider that $Ta_j \to \neg Ta_j$ gets $1/2$, and consider that each one of the conditionals that make up the biconditional gets 1 while its antecedent and consequent have $1/2$.

If $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_n} \vDash \mathbf{A}_j$, then the inference from $Ta_{i_1}, \ldots, Ta_{i_n}$ to Ta_j belongs to $\mathcal{C}_2$; for, in any valuation in $\mathcal{S}$, $T\mathbf{a}$ gets 1 just in case $\mathcal{R}(\mathbf{a})$ gets 1. Also, if $\vDash \mathbf{A}_i \to \mathbf{A}_j$, then the conditional $Ta_i \to Ta_j$ belongs to $\mathcal{C}_1$. For every valuation in $\mathcal{S}$ gives 1 to $Ta_i \to \mathbf{A}_i$ and $\mathbf{A}_j \to Ta_j$, so by transitivity it also gives 1 to $Ta_i \to Ta_j$.

We have seen that the inference from Ta_k to $Ta_i \vee Ta_j$, where $\mathbf{A}_k$ is $\mathbf{A}_i \vee \mathbf{A}_j$, belongs to $\mathscr{C}_2$. Likewise, the converse inference, from $Ta_i \vee Ta_j$ to Ta_k, belongs to $\mathscr{C}_2$. Our theory also contains the inference from $Ta_i \wedge Ta_j$ to Ta_l, where $\mathbf{A}_l$ is $\mathbf{A}_i \wedge \mathbf{A}_j$, and that from Ta_l to $Ta_i \wedge Ta_j$. Moreover, the inference from $Ta_i \to Ta_j$ to Ta_m, where $\mathbf{A}_m$ is $\mathbf{A}_i \to \mathbf{A}_j$, and the inference from Ta_m to $Ta_i \to Ta_j$, make it into the theory because, by transitivity, $Ta_i \to Ta_j$ gets 1 in any valuation V in $\mathscr{S}$ if and only if $\mathbf{A}_i \to \mathbf{A}_j$ gets 1 in V.

Perhaps unexpectedly, our theory of truth leaves no room for sentences that are neither true nor false. (Do not see the numerical values; see what the theory says.) Of course, we have no predicate F for falsity, but we can equate the falsity of a sentence with the truth of its negation. This equation is usually accepted by those who believe that some sentences are neither true nor false. Let $\mathbf{A}_i$ be any wff, and let $\mathbf{A}_n$ be $\neg\mathbf{A}_i$. We have that $\mathscr{C}_1$ contains the conditional $\neg Ta_i \to Ta_n$, which effectively tells us that if $\mathbf{A}_i$ is not true, then it is false. For every valuation in $\mathscr{S}$ gives 1 to $\mathbf{A}_i \to Ta_i$ and $\mathbf{A}_n \to Ta_n$, so by contraposition and transitivity it also gives 1 to $\neg Ta_i \to Ta_n$. The theory also contains $\neg Ta_n \to Ta_i$: if $\mathbf{A}_i$ is not false, then it is true. This is partly due to the fact that $\models \neg\neg\mathbf{A}_i \to \mathbf{A}_i$. And, just as $\neg[Ta_i \wedge Ta_n]$ belongs to the theory, so does $\neg[\neg Ta_i \wedge \neg Ta_n]$: it is not the case that $\mathbf{A}_i$ is not true and not false. To see that, apply (Prop), taking $m = 2$ and letting $\neg Ta_i$, $\neg\mathbf{A}_i$, $\neg Ta_n$, $\neg\neg\mathbf{A}_i$ be $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}'_1$, $\mathbf{E}'_2$, respectively.

On the other hand, our theory does not characterize every sentence as being either true or false. It may not contain the disjunction $Ta_i \vee Ta_n$ ($\mathbf{A}_i$ is true or false). The logic does not validate the inference from $\neg[\neg Ta_i \wedge \neg Ta_n]$ to $Ta_i \vee Ta_n$. For example, the theory does not contain the disjunction if $\mathbf{A}_i$ is a liar sentence $\neg Ta_i$. For every valuation in $\mathscr{S}$, in order to give 1 to $Ta_i \leftrightarrow \neg Ta_i$, must assign $1/2$ to Ta_i. So it also assigns $1/2$ to $\neg\neg Ta_i$, which is $\mathbf{A}_n$, and to Ta_n. Thus $Ta_i \vee Ta_n$ gets $1/2$ and not 1.

Here we see another problem that would result if we identified values 1 and 0 with truth and falsehood. We base our theory of truth and falsehood on the valuations in $\mathscr{S}$. Thus, since a liar sentence (like other paradoxical sentences) does not get either 1 or 0 in any one of those valuations, the identification would lead us to say that the sentence is not true and not false. But that is something that the theory denies.

One may ask why we want our theory to contain the biconditionals $Ta_i \leftrightarrow \mathbf{A}_i$ and do not rest content with the inferences from Ta_i to $\mathbf{A}_i$ and from $\mathbf{A}_i$ to Ta_i. After all, why say that it is the biconditionals, rather than the inferences, that characterize the concept of truth? First, if we rested content with the inferences, then our theory would become poorer. Second, we would risk losing conditionals such as $Ta_i \to \neg Ta_n$ (if $\mathbf{A}_i$ is true, then it is not false) and $\neg Ta_i \to Ta_n$. We would also risk losing the conditionals $Ta_i \to Ta_j$ where $\models \mathbf{A}_i \to \mathbf{A}_j$, for example, $Ta_i \to Ta_d$ where $\mathbf{A}_d$ is $\neg\neg\mathbf{A}_i$. It would require special maneuvers to keep such conditionals. The point of having them becomes clear when we move from the propositional to a first-order theory of truth. We would like the latter to contain principles like "If a sentence is true, then it is not false" and "If a sentence is true, then its double negation is true too." It would be very odd if the theory contained the general principles but lacked the corresponding conditionals for particular sentences. A propositional theory of truth should resemble such a first-order theory as far as this is allowed by the lack of quantifiers. The first-order theories alluded to in Section 5 endorse those principles, as well as "If a sentence is not true, then it is false" and others. And third, it seems that if we abandoned the biconditionals, then we would lose the inference from $Ta_i \rightarrow Ta_j$ to Ta_m, where $\mathbf{A}_m$ is $\mathbf{A}_i \rightarrow \mathbf{A}_j$, and the inference from Ta_m to $Ta_i \rightarrow Ta_j$.

Iteration of T can make a difference. If $\mathbf{A}_l$ is a liar sentence $\neg Ta_l$, then every valuation V in $\mathcal{S}$ gives $1/2$ to $Ta_l \wedge Ta_l$. But if $\mathbf{A}_k$ is Ta_l, then by (Prop) V gives 0 to $Ta_k \wedge Ta_l$. So $\neg[Ta_k \wedge Ta_l]$ belongs to the theory, but $\neg[Ta_l \wedge Ta_l]$ does not, and in fact, for any wff $\mathbf{B}$, the theory contains the inference from $\neg[Ta_l \wedge Ta_l]$ to $\mathbf{B}$. The wff Ta_k can be seen as implicitly involving an iteration of T; it is like "the sentence a_l *is true* is true."

If we want, we can add another component, $\mathcal{C}_3$, to our theory. $\mathcal{C}_3$ is a class of inferences of the form "$\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k, \mathbf{B}_1, \ldots, \mathbf{B}_n$; hence, $\mathbf{C}$" ($k \geq 1$, $n \geq 0$), where each one of $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k$ is a simple inference. An inference of that form belongs to $\mathcal{C}_3$ if and only if $\mathbf{C}$ has 1 in every valuation $V \in \mathcal{S}$ in which $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_k$ are 1-preserving and $\mathbf{B}_1, \ldots, \mathbf{B}_n$ have 1. For example, the third component will contain, for all $\mathbf{C}$, $\mathbf{D}$, $i \geq 1$, and $j \geq 1$, the inference from $\mathbf{C} \vee \mathbf{D}$, $(\mathbf{C}; \mathbf{A}_i)$, and $(\mathbf{D}; \mathbf{A}_j)$ to $Ta_i \vee Ta_j$.

Now, it is not the case that our theory contains all wffs of the form $\neg[Ta_k \wedge Ta_h]$ where, for some $\mathbf{B}$ and $\mathbf{C}$, $\mathbf{A}_k$ is $\mathbf{B} \vee \mathbf{C}$ while $\mathbf{A}_h$ is $\neg\mathbf{B} \wedge \neg\mathbf{C}$. Proving that some such wffs have a value other than 1 in a valuation in $\mathcal{S}$ will have to await the proof of the Central Theorem. But, as we can already see, (Prop) does not guarantee that all wffs $Ta_k \wedge Ta_h$, where $\mathbf{A}_k$ and $\mathbf{A}_h$ are as described, get the value 0. For there are many $\mathbf{B}$ and $\mathbf{C}$ where it is not the case that $\mathbf{B} \vee \mathbf{C}$ and $\neg\mathbf{B} \wedge \neg\mathbf{C}$ have opposite values in all valuations. For instance, there are valuations in which p_1 and p_2 have $1/2$, so $p_1 \vee p_2$ has $1/2$, but $\neg p_1 \wedge \neg p_2$ gets 0. If the clause "in every valuation, $\mathbf{E}_m$ and $\mathbf{E}'_m$ have opposite values" in (Prop) were replaced with "there is no valuation in which $\mathbf{E}_m$ and $\mathbf{E}'_m$ both have 1 or both have 0," then the value 0 would be guaranteed for all wffs $Ta_k \wedge Ta_h$ where $\mathbf{A}_k$ and $\mathbf{A}_h$ are as described. But it is not clear to me how the proof of the Central Theorem should then be modified. Admittedly, since $\models \neg[[\mathbf{B} \vee \mathbf{C}] \wedge [\neg\mathbf{B} \wedge \neg\mathbf{C}]]$, it would be preferable if our theory contained all wffs $\neg[Ta_k \wedge Ta_h]$.

Also, it is not the case that our theory contains all wffs of the form $\neg[\mathbf{A}_i \wedge \neg Ta_i]$. For example, as we will see after the proof of the Central Theorem, the theory does not contain $\neg[p_1 \wedge \neg Ta_m]$, where $\mathbf{A}_m$ is p_1. (Prop) does not guarantee the value 0 for $p_1 \wedge \neg Ta_m$. It may appear that our theory should preferably contain the wffs $\neg[\mathbf{A}_i \wedge \neg Ta_i]$, but appearances are misleading. Say that $\mathbf{A}_l$ is the liar sentence $\neg Ta_l$. Then, $\neg Ta_l \wedge \neg Ta_l$ has the form $\mathbf{A}_i \wedge \neg Ta_i$, but also has the form $\mathbf{B} \wedge \mathbf{B}$. Our theory is to contain just the wffs that get 1 in certain preselected valuations. These valuations must assign $1/2$ to the liar sentence. How should we treat the conjunction $\neg Ta_l \wedge \neg Ta_l$? I think that its repetitive character leaves no room for giving it any value other than that of $\neg Ta_l$. Thus, the relevant valuations rightly give it $1/2$, so that $\neg[\neg Ta_l \wedge \neg Ta_l]$ is not in our theory. But then $\neg[p_1 \wedge \neg Ta_m]$, too, should be left out. For the sentential letters, like p_1, are schematic letters, and $\neg[\neg Ta_l \wedge \neg Ta_l]$ is an instance of $\neg[p_1 \wedge \neg Ta_m]$. It would be unmotivated if we incorporated the schematic principle in our theory, but left out some of its instances.

Likewise, it is not the case that our theory contains all wffs of the form $\neg[Ta_i \wedge \neg\mathbf{A}_i]$. For example, it does not contain $\neg[Ta_m \wedge \neg p_1]$, where $\mathbf{A}_m$ is p_1. (Prop) and our logic do not guarantee the value 0 for conjunctions $Ta_i \wedge \neg\mathbf{A}_i$ except in special circumstances (if, for instance, $\mathbf{A}_i$ is Ta_i itself). That is, I think, acceptable. Take the liar sentence $\neg Ta_l$ again. $Ta_l \wedge \neg\neg Ta_l$ has the form $Ta_i \wedge \neg \mathbf{A}_i$ but also the form $\mathbf{B} \wedge \neg\neg\mathbf{B}$. Since $\vDash \mathbf{B} \leftrightarrow \neg\neg\mathbf{B}$, it might initially seem that we ought to give the same value to a conjunction $\mathbf{B} \wedge \neg\neg\mathbf{B}$ as we give to $\mathbf{B}$. In particular, once we assign $1/2$ to Ta_l, as do the valuations that determine which wffs and which inferences make it to our theory, should we give the conjunction $Ta_l \wedge \neg\neg Ta_l$ the same value or should we give it 0 because it has the form $Ta_i \wedge \neg\mathbf{A}_i$? The answer is not clear. Our rules are such that we give $1/2$ to the conjunction because it has only one deep conjunct. If, however, we had adopted the simpler definition of "deep conjunct" discussed in Section 2, we would allow the conjunction to get either $1/2$ or 0. At any rate, $\neg[Ta_l \wedge \neg\neg Ta_l]$ does not make it to our theory. Finally, as before, it would be unmotivated if we incorporated the wff $\neg[Ta_m \wedge \neg p_1]$ in the theory but left out $\neg[Ta_l \wedge \neg\neg Ta_l]$, which is an instance of $\neg[Ta_m \wedge \neg p_1]$.

4 Proving the Central Theorem

4.1 In this and the next three subsections, I present some definitions and prove some lemmas that will be used in Subsections 4.5 and 4.6, which are the main parts of Section 4. The Central Theorem is proved in Subsection 4.6.

Let C be a class of sets of wffs which meets the following four conditions: each $S \in C$ is an unordered pair; for each $S \in C$, every member of S is a T-attribution or the negation of a T-attribution; $\{T\mathbf{a}, T\mathbf{b}\} \in C$ if and only if $\{\neg T\mathbf{a}, \neg T\mathbf{b}\} \in C$; and $\{T\mathbf{a}, \neg T\mathbf{b}\} \in C$ if and only if $\{\neg T\mathbf{a}, T\mathbf{b}\} \in C$. We will say that C *involves* a T-attribution just in case a pair in C contains that T-attribution. And we will say that C *connects* the wff $\mathbf{B}$ *with* (or *and*) the wff $\mathbf{C}$ just in case there are wffs $\mathbf{A}_1, \ldots, \mathbf{A}_n$ ($n \geq 2$) such that $\mathbf{A}_1$ is $\mathbf{B}$, $\mathbf{A}_n$ is $\mathbf{C}$, and for every i ($1 \leq i < n$) $\{\mathbf{A}_i, \mathbf{A}_{i+1}\} \in C$.[10] Clearly, C connects $\mathbf{B}$ with $\mathbf{B}$ if $\mathbf{B}$ belongs to a member of C; C connects $\mathbf{B}$ with $\mathbf{C}$ if and only if it connects $\mathbf{C}$ with $\mathbf{B}$; and if C connects $\mathbf{B}$ with $\mathbf{C}$ and also connects $\mathbf{C}$ with $\mathbf{D}$, then it connects $\mathbf{B}$ with $\mathbf{D}$. Moreover, C connects $T\mathbf{a}$ and $T\mathbf{b}$ if and only if it connects $\neg T\mathbf{a}$ and $\neg T\mathbf{b}$. This we can see if we assume that $\mathbf{A}_1$ and $\mathbf{A}_n$ are $T\mathbf{a}$ and $T\mathbf{b}$, respectively (or $\neg T\mathbf{a}$ and $\neg T\mathbf{b}$, respectively), and take the opposites of $\mathbf{A}_1, \ldots, \mathbf{A}_n$. Likewise, C connects $T\mathbf{a}$ and $\neg T\mathbf{b}$ if and only if it connects $\neg T\mathbf{a}$ and $T\mathbf{b}$.

Furthermore, we will say that C *combines* a T-attribution $T\mathbf{a}$ *with* (or *and*) a T-attribution $T\mathbf{b}$ if and only if C either connects $T\mathbf{a}$ with $T\mathbf{b}$ or connects $T\mathbf{a}$ with $\neg T\mathbf{b}$. Of course, C combines $T\mathbf{a}$ with $T\mathbf{a}$ if $T\mathbf{a}$ belongs to a member of C; C combines $T\mathbf{a}$ with $T\mathbf{b}$ just in case it combines $T\mathbf{b}$ with $T\mathbf{a}$; and if C combines $T\mathbf{a}$ with $T\mathbf{b}$ and also combines $T\mathbf{b}$ with $T\mathbf{c}$, then it combines $T\mathbf{a}$ with $T\mathbf{c}$. So combination by C divides the T-attributions that C involves into equivalence classes. A T-attribution $T\mathbf{a}$ will be called a *C-associate* of a T-attribution $T\mathbf{b}$ if and only if C combines $T\mathbf{a}$ with $T\mathbf{b}$. And a class C will be called *appropriate* just in case it meets the four conditions set out at the beginning of the preceding paragraph and also does not connect any T-attribution with its negation. We will use appropriate classes in order to make sure that the valuations we are constructing assign the same value to wffs to which we want them to assign the same value.

Given any appropriate class C, a *C-valuation* will be a valuation that, for each $S \in C$, gives the same value to both wffs in S. Thus, if V is a C-valuation, and C

connects $T\mathbf{a}$ with $T\mathbf{b}$, then V assigns the same value to $T\mathbf{a}$ and $T\mathbf{b}$. And if C connects $T\mathbf{a}$ with $\neg T\mathbf{b}$, then V assigns opposite values to $T\mathbf{a}$ and $T\mathbf{b}$. Of course, every valuation is a $\varnothing$-valuation. By saying that $\mathbf{A}$ *C-implies* $\mathbf{B}$, we will mean that, for every C-valuation V, $|\mathbf{A}|_V \leq |\mathbf{B}|_V$. And by saying that $\mathbf{A}$ and $\mathbf{B}$ are *C-equivalent*, we will mean that, in each C-valuation, $\mathbf{A}$ has the same value as $\mathbf{B}$. Thus, $\mathbf{A}$ and $\mathbf{B}$ are C-equivalent just in case they C-imply each other. Of course, C-implying is transitive, and if $\mathbf{A}$ C-implies $\mathbf{B}$, then $\neg\mathbf{B}$ C-implies $\neg\mathbf{A}$. Also, $\mathbf{A}$ and $\mathbf{B}$ are C-equivalent if and only if $\neg\mathbf{A}$ and $\neg\mathbf{B}$ are C-equivalent; and $\mathbf{A}$ is C-equivalent to $\neg\mathbf{B}$ if and only if $\neg\mathbf{A}$ is C-equivalent to $\mathbf{B}$.

Lemma 1 *Take any appropriate class C, any C-valuation V, and any wffs $\mathbf{A}_1, \ldots, \mathbf{A}_n$ ($n \geq 2$). If, for each i, $1 \leq i \leq n$, $\mathbf{A}_i$ gets $1/2$ in V or ($\mathbf{A}_i$ gets 1 in V and, for some j where $1 \leq j \leq n$ and $j \neq i$, C-implies either $\mathbf{A}_j$ or $\neg\mathbf{A}_j$) or ($\neg\mathbf{A}_i$ gets 1 in V and, for some j where $1 \leq j \leq n$ and $j \neq i$, C-implies either $\mathbf{A}_j$ or $\neg\mathbf{A}_j$), then either all of $\mathbf{A}_1, \ldots, \mathbf{A}_n$ get $1/2$ in V or there are i and j among $1, \ldots, n$ such that $i \neq j$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_j$ or to $\neg\mathbf{A}_j$.*

Proof If not all of $\mathbf{A}_1, \ldots, \mathbf{A}_n$ get $1/2$ in V, then for some i ($1 \leq i \leq n$) $\mathbf{A}_i$ or $\neg\mathbf{A}_i$ gets 1. So for some j ($1 \leq j \leq n$, $j \neq i$) $\mathbf{A}_j$ or $\neg\mathbf{A}_j$ gets 1 and is C-implied by whichever of $\mathbf{A}_i$ and $\neg\mathbf{A}_i$ has 1. Thus for some k ($1 \leq k \leq n$, $k \neq j$) $\mathbf{A}_k$ or $\neg\mathbf{A}_k$ gets 1 and is C-implied by whichever of $\mathbf{A}_j$ and $\neg\mathbf{A}_j$ has 1, and so forth. Since $1, \ldots, n$ are finitely many, the sequence $i, j, k, \ldots$ must somewhere form the circle $\ldots, h, h', \ldots, h, \ldots$. In other words, there are h and h' ($1 \leq h \leq n$, $1 \leq h' \leq n$, $h \neq h'$) such that $\mathbf{A}_{h'}$ or $\neg\mathbf{A}_{h'}$ gets 1 and is C-implied by whichever of $\mathbf{A}_h$ and $\neg\mathbf{A}_h$ has 1, and again $\mathbf{A}_h$ or $\neg\mathbf{A}_h$ gets 1 and is C-implied by whichever of $\mathbf{A}_{h'}$ and $\neg\mathbf{A}_{h'}$ has 1. Thus, the one out of $\mathbf{A}_h$ and $\neg\mathbf{A}_h$ which has 1 (in V) is C-equivalent to the one out of $\mathbf{A}_{h'}$ and $\neg\mathbf{A}_{h'}$ which has 1. So $\mathbf{A}_h$ is C-equivalent to $\mathbf{A}_{h'}$, or $\mathbf{A}_h$ is C-equivalent to $\neg\mathbf{A}_{h'}$. □

There follow more definitions. A set $\mathcal{I}$ of wffs of the form $\mathbf{A} \to \mathbf{B}$ is *C-insertable between* a wff $\mathbf{C}$ *and* a wff $\mathbf{D}$ if and only if there are wffs $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, and for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to $\mathcal{I}$. Clearly, when $\mathcal{J}$ is a subset of $\mathcal{I}$ and is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, $\mathcal{I}$ too is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. If $\mathcal{J} \subseteq \mathcal{I}$, then we will say that $\mathcal{I}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ *in the presence of at least one wff in $\mathcal{J}$* just in case there are wffs $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to $\mathcal{I}$, and there is at least one wff $\mathbf{A} \to \mathbf{B}$ belonging to $\mathcal{J}$ where, for some i, $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A} \to \mathbf{B}$.

4.2 Now, suppose that we have T-attributions $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ and wffs $\mathbf{A}_1, \ldots, \mathbf{A}_l$ ($l \geq 1$). For each j ($1 \leq j \leq l$) $\mathcal{A}_j$ will be $\{\mathbf{A}_j \to T\mathbf{a}_j, T\mathbf{a}_j \to \mathbf{A}_j, \neg T\mathbf{a}_j \to \neg\mathbf{A}_j, \neg\mathbf{A}_j \to \neg T\mathbf{a}_j\}$. Let C be any appropriate class, and consider the set $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$. It is easy to see that if that set is C-insertable both between $\mathbf{B}$ and $\mathbf{C}$ and between $\mathbf{C}$ and $\mathbf{D}$, then it is C-insertable between $\mathbf{B}$ and $\mathbf{D}$. But we can also see the following.

Lemma 2 *If $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then it is also C-insertable between $\mathbf{D}'$ and $\mathbf{C}'$, where $\mathbf{D}' \to \mathbf{C}'$ is any contrapositive of $\mathbf{C} \to \mathbf{D}$.*

Proof If the set $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then there are wffs $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, and for every i ($1 \leq i < n$)

either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to that set. Consider the wffs $\mathbf{D}', \neg\mathbf{E}_n, \ldots, \neg\mathbf{E}_1, \mathbf{C}'$. The wff $\mathbf{D}'$ C-implies $\neg\mathbf{E}_n$, and $\neg\mathbf{E}_1$ C-implies $\mathbf{C}'$. Take any i, $1 \leq i < n$. If $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$, then $\neg\mathbf{E}_{i+1}$ C-implies $\neg\mathbf{E}_i$. Otherwise, for some j, $\neg\mathbf{E}_{i+1} \to \neg\mathbf{E}_i$ is $\neg T\mathbf{a}_j \to \neg\mathbf{A}_j$, $\neg\mathbf{A}_j \to \neg T\mathbf{a}_j$, $\neg\neg\mathbf{A}_j \to \neg\neg T\mathbf{a}_j$, or $\neg\neg T\mathbf{a}_j \to \neg\neg\mathbf{A}_j$. If it is $\neg\neg\mathbf{A}_j \to \neg\neg T\mathbf{a}_j$, then insert $\mathbf{A}_j$ and $T\mathbf{a}_j$, in that order, between $\neg\mathbf{E}_{i+1}$ and $\neg\mathbf{E}_i$ in $\mathbf{D}', \neg\mathbf{E}_n, \ldots, \neg\mathbf{E}_1, \mathbf{C}'$. If it is $\neg\neg T\mathbf{a}_j \to \neg\neg\mathbf{A}_j$, then insert $T\mathbf{a}_j$ and $\mathbf{A}_j$, in that order, between $\neg\mathbf{E}_{i+1}$ and $\neg\mathbf{E}_i$. If $\neg\mathbf{E}_{i+1} \to \neg\mathbf{E}_i$ is $\neg T\mathbf{a}_j \to \neg\mathbf{A}_j$ or $\neg\mathbf{A}_j \to \neg T\mathbf{a}_j$, then insert nothing. By making all such insertions, we end up with a sequence of wffs which shows that $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is C-insertable between $\mathbf{D}'$ and $\mathbf{C}'$. □

Sets like $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ will play a major role in the rules about $\to$ in the valuations we construct. When defining an assignment of values to wffs, we can stipulate that if $\mathbf{C}$ and $\mathbf{D}$ have $1/2$, then $\mathbf{C} \to \mathbf{D}$ will get 1 or $1/2$ depending on whether $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is or is not, respectively, C-insertable between $\mathbf{C}$ and $\mathbf{D}$. This stipulation guarantees that the assignment will conform with (β) and the other two additional rules for conditionals in case (ii) of Table 4.

Lemma 3 *If $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ ($\varnothing$ if $l = 1$) is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$, or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$.*

Proof Suppose that $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. Then there are $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, and for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$. If, for some i, $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, and, for some j, $\mathbf{E}_j \to \mathbf{E}_{j+1}$ is $T\mathbf{a}_l \to \mathbf{A}_l$, then take the smallest number k such that $\mathbf{E}_k \to \mathbf{E}_{k+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$ or $T\mathbf{a}_l \to \mathbf{A}_l$. If $\mathbf{E}_k \to \mathbf{E}_{k+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$ and not $T\mathbf{a}_l \to \mathbf{A}_l$, then consider the largest number h such that $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $T\mathbf{a}_l \to \mathbf{A}_l$. Then the wffs $\mathbf{E}_1, \ldots, \mathbf{E}_k, \mathbf{E}_{h+2}, \ldots, \mathbf{E}_n$ (if $h + 1 < n$) or the wffs $\mathbf{E}_1, \ldots, \mathbf{E}_k$ (if $h+1 = n$) show that $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. If, on the other hand, $\mathbf{E}_k \to \mathbf{E}_{k+1}$ is $T\mathbf{a}_l \to \mathbf{A}_l$, then consider the largest number h such that $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$. Then the wffs $\mathbf{E}_1, \ldots, \mathbf{E}_k, \mathbf{E}_{h+2}, \ldots, \mathbf{E}_n$ (if $h+1 < n$) or the wffs $\mathbf{E}_1, \ldots, \mathbf{E}_k$ (if $h+1 = n$) show that $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. Finally, if there is no i where $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, or there is no j where $\mathbf{E}_j \to \mathbf{E}_{j+1}$ is $T\mathbf{a}_l \to \mathbf{A}_l$, then of course $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$.

We can similarly see that if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$; and that if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$.

Now, suppose that there are $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, and for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$. Also suppose that $\mathbf{A}_l$ does not C-imply $T\mathbf{a}_l$, $\neg\mathbf{A}_l$ does not C-imply $\neg T\mathbf{a}_l$, and neither $\mathbf{A}_l \to T\mathbf{a}_l$ nor $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$. Finally, assume that there are k and h where $\mathbf{E}_k \to \mathbf{E}_{k+1}$ and $\mathbf{E}_h \to \mathbf{E}_{h+1}$ are $\mathbf{A}_l \to T\mathbf{a}_l$ and $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$, respectively.

Then there are, in particular, k^* and h^* such that $k^* < h^*$ and either (α) $\mathbf{E}_{k^*} \to \mathbf{E}_{k^*+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, $\mathbf{E}_{h^*} \to \mathbf{E}_{h^*+1}$ is $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$, and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ is C-insertable between $\mathbf{E}_{k^*+1}$ and $\mathbf{E}_{h^*}$, or (β) $\mathbf{E}_{k^*} \to \mathbf{E}_{k^*+1}$ is $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$, $\mathbf{E}_{h^*} \to \mathbf{E}_{h^*+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ is C-insertable between $\mathbf{E}_{k^*+1}$ and $\mathbf{E}_{h^*}$. So, by Lemma 2, $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ is C-insertable between $\mathbf{E}_{k^*}$ and $\mathbf{E}_{h^*+1}$. Thus there are wffs $\mathbf{E}'_1, \ldots, \mathbf{E}'_m$ such that the first k^* wffs in $\mathbf{E}'_1, \ldots, \mathbf{E}'_m$ are $\mathbf{E}_1, \ldots, \mathbf{E}_{k^*}$ and so $\mathbf{E}'_1$ is $\mathbf{C}$, the last $n - h^*$ wffs in $\mathbf{E}'_1, \ldots, \mathbf{E}'_m$ are $\mathbf{E}_{h^*+1}, \ldots, \mathbf{E}_n$ and so $\mathbf{E}'_m$ is $\mathbf{D}$, for every i ($1 \leq i < m$) either $\mathbf{E}'_i$ C-implies $\mathbf{E}'_{i+1}$ or $\mathbf{E}'_i \to \mathbf{E}'_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$, the numbers k such that $\mathbf{E}'_k \to \mathbf{E}'_{k+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$ are fewer than the numbers k such that $\mathbf{E}_k \to \mathbf{E}_{k+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, and the numbers h such that $\mathbf{E}'_h \to \mathbf{E}'_{h+1}$ is $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$ are fewer than the numbers h such that $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$. Since decreasing the membership in a finite set will eventually empty it, there are $\mathbf{E}^+_1, \ldots, \mathbf{E}^+_r$ ($r \geq 2$) such that $\mathbf{E}^+_1$ is $\mathbf{C}$, $\mathbf{E}^+_r$ is $\mathbf{D}$, for every i ($1 \leq i < r$) either $\mathbf{E}^+_i$ C-implies $\mathbf{E}^+_{i+1}$ or $\mathbf{E}^+_i \to \mathbf{E}^+_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$, and either there is no k where $\mathbf{E}^+_k \to \mathbf{E}^+_{k+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$ or there is no h where $\mathbf{E}^+_h \to \mathbf{E}^+_{h+1}$ is $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$. But then, $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. That is also the case if $\mathbf{A}_l$ C-implies $T\mathbf{a}_l$, if $\neg\mathbf{A}_l$ C-implies $\neg T\mathbf{a}_l$, and if either $\mathbf{A}_l \to T\mathbf{a}_l$ or $\neg\mathbf{A}_l \to \neg T\mathbf{a}_l$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$.

We can similarly see that if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$, then so is $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$. The overall conclusion is that $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. □

4.3 Given any appropriate class C, we can easily construct C-valuations in which no wff gets $1/2$. To see that, consider all T-attributions $T\mathbf{a}$ such that C involves $T\mathbf{a}$ and, if $T\mathbf{b}$ is any other T-attribution that C combines with $T\mathbf{a}$, then $\mathbf{a}$ has a smaller subscript than $\mathbf{b}$; that is, $T\mathbf{a}$ has the smallest subscript in its equivalence class of T-attributions that C involves. For any such $T\mathbf{a}$, assign it 1 or 0, and then give the same value to every T-attribution with which C connects $T\mathbf{a}$, and give the opposite value to every T-attribution with whose negation C connects $T\mathbf{a}$. Of course C does not connect $T\mathbf{a}$ with both $T\mathbf{b}$ and $\neg T\mathbf{b}$, since otherwise it would connect $T\mathbf{b}$ with $\neg T\mathbf{b}$. Also assign 1 or 0 to each sentential letter and to each T-attribution that C does not involve. As the reader can demonstrate, the resulting valuation, in which no wff gets $1/2$, is a C-valuation.

When we define an assignment of values to wffs, we can adopt the following rule about conjunction, which consists of two clauses.

(Con) (a) *When* $\mathbf{A}$ *and* $\mathbf{B}$ *have* $1/2$*, or the one has* $1/2$ *and the other* 1*, then* $\mathbf{A} \wedge \mathbf{B}$ *gets* 0 *if there is no* C*-valuation in which the deep conjuncts of* $\mathbf{A} \wedge \mathbf{B}$ *all have*

1 *or all have* 0.

(b) *When* $\mathbf{A}$ *and* $\mathbf{B}$ *have* $1/2$*, or the one has* $1/2$ *and the other* 1*, then* $\mathbf{A} \wedge \mathbf{B}$ *gets* $1/2$ *if there is such a* C*-valuation.*

We will say that (Con), or (Con)(a) or (Con)(b), is *followed with respect to* C. Note that, by following (Con) with respect to C, we conform with rule (α) and the other two additional rules for conjunctions in case (i) of Table 3. In the first place, if $\mathbf{A} \wedge \mathbf{B}$ comes under case (i) and possesses only one deep conjunct, then following (Con) with respect to C will give $1/2$ to $\mathbf{A} \wedge \mathbf{B}$. For if the single deep conjunct did not have 1 or 0 in any C-valuation, it would have $1/2$ in all C-valuations. In the second place, if the conjunctions $\mathbf{C}$ and $\mathbf{D}$ come under case (i), and every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{D}$, then following (Con) with respect to C will not give 0 to $\mathbf{C}$ and $1/2$ to $\mathbf{D}$. For if the deep conjuncts of $\mathbf{C}$ do not all have 1 in any C-valuation and do not all have 0 in any C-valuation, then the same must be true of the deep conjuncts of $\mathbf{D}$.

4.4 If C and C' are classes of sets of wffs, then we will say that C' is an *extension of* C just in case there are appropriate classes $C_1, \ldots, C_n$ ($n \geq 2$) of sets of wffs such that C_1 is C, C_n is C', and for each i ($1 \leq i < n$) either C_{i+1} is C_i or, for some distinct T-attributions $T\mathbf{a}$ and $T\mathbf{b}$ where C_i does not combine $T\mathbf{a}$ and $T\mathbf{b}$, C_{i+1} is $C_i \cup \{\{T\mathbf{a}, T\mathbf{b}\}, \{\neg T\mathbf{a}, \neg T\mathbf{b}\}\}$ or $C_i \cup \{\{T\mathbf{a}, \neg T\mathbf{b}\}, \{\neg T\mathbf{a}, T\mathbf{b}\}\}$. Of course, every appropriate class is an extension of itself. If C' is an extension of C, then both C and C' are appropriate.

It is clear that if C' is an extension of C, then every C'-valuation is a C-valuation. Hence, if a wff $\mathbf{A}$ C-implies $\mathbf{B}$, then $\mathbf{A}$ also C'-implies $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ are C-equivalent, then they are also C'-equivalent. And if a valuation V follows (Con)(a) with respect to C', then it follows (Con)(a) with respect to C too. On the other hand, we are not entitled to claim that if V follows (Con) with respect to C', then it follows (Con) with respect to C too.

Lemma 4 *If* C *is an appropriate class while* $T\mathbf{a}$ *and* $T\mathbf{b}$ *are distinct wffs that are not combined by* C*, then the classes* $C \cup \{\{T\mathbf{a}, T\mathbf{b}\}, \{\neg T\mathbf{a}, \neg T\mathbf{b}\}\}$ *and* $C \cup \{\{T\mathbf{a}, \neg T\mathbf{b}\}, \{\neg T\mathbf{a}, T\mathbf{b}\}\}$ *are also appropriate.*

Proof Let C' be $C \cup \{\{T\mathbf{a}, T\mathbf{b}\}, \{\neg T\mathbf{a}, \neg T\mathbf{b}\}\}$. Of the conditions that C' must meet in order to be appropriate, the only one that is not obvious is that it should not connect a T-attribution with its negation. So suppose that it does: there are $\mathbf{A}_1, \ldots, \mathbf{A}_n$ ($n > 1$) such that $\mathbf{A}_1$ is $T\mathbf{c}$, $\mathbf{A}_n$ is $\neg T\mathbf{c}$, and for every k ($1 \leq k < n$) $\{\mathbf{A}_k, \mathbf{A}_{k+1}\} \in C'$. As C does not connect $T\mathbf{c}$ and $\neg T\mathbf{c}$, there is at least one pair $\{\mathbf{A}_l, \mathbf{A}_{l+1}\}$ that belongs to $C' - C$.

Say there are two such pairs. Then, there will be two pairs $\{\mathbf{A}_l, \mathbf{A}_{l+1}\}$ and $\{\mathbf{A}_{l'}, \mathbf{A}_{l'+1}\}$ such that $l < l'$, both pairs belong to $C' - C$, and for no k, $l < k < l'$, does $\{\mathbf{A}_k, \mathbf{A}_{k+1}\}$ belong to $C' - C$. Since C does not combine $T\mathbf{a}$ with $T\mathbf{b}$ and connects neither $T\mathbf{a}$ with $\neg T\mathbf{a}$ nor $T\mathbf{b}$ with $\neg T\mathbf{b}$, $\mathbf{A}_{l+1}$ must be $\mathbf{A}_{l'}$ (whether or not $l + 1 = l'$) and so $\mathbf{A}_l$ will be $\mathbf{A}_{l'+1}$. Then we can remove the section $\mathbf{A}_l, \ldots, \mathbf{A}_{l'}$ from $\mathbf{A}_1, \ldots, \mathbf{A}_n$ and get a shorter sequence of wffs showing that C' connects $T\mathbf{c}$ and $\neg T\mathbf{c}$. As a finite sequence of wffs cannot be shortened ad infinitum, there must be wffs $\mathbf{B}_1, \ldots, \mathbf{B}_m$ such that $\mathbf{B}_1$ is $T\mathbf{c}$, $\mathbf{B}_m$ is $\neg T\mathbf{c}$, for every k ($1 \leq k < m$) $\{\mathbf{B}_k, \mathbf{B}_{k+1}\} \in C'$, and there is just one pair $\{\mathbf{B}_l, \mathbf{B}_{l+1}\}$ that belongs to $C' - C$. It will be that $m > 2$, for $\{T\mathbf{c}, \neg T\mathbf{c}\} \notin C' - C$.

Since C connects neither $T\mathbf{a}$ with $\neg T\mathbf{b}$ nor $T\mathbf{b}$ with $\neg T\mathbf{a}$, we have that $1 < l$ and $l + 1 < m$. But then C connects $T\mathbf{c}$ with $\mathbf{B}_l$ and also connects $\mathbf{B}_{l+1}$ with $\neg T\mathbf{c}$. So it connects $\mathbf{B}_l$ with the opposite of $\mathbf{B}_{l+1}$. In other words, it combines $T\mathbf{a}$ with $T\mathbf{b}$, contrary to the hypothesis of the lemma.

We can similarly show that $C \cup \{\{T\mathbf{a}, \neg T\mathbf{b}\}, \{\neg T\mathbf{a}, T\mathbf{b}\}\}$ does not connect a T-attribution with its negation. □

4.5 We will now prove two theorems, numbered 2 and 3, that pave the way for the Central Theorem. Theorem 2 suffices to show that if $T\mathbf{a}$ is any T-attribution and $\mathbf{A}$ is the wff named by $\mathbf{a}$ (or it is any wff, for that matter), then there is a valuation in which $T\mathbf{a} \leftrightarrow \mathbf{A}$ gets 1. Indeed, a simpler version of Theorem 2 that involved a single T-attribution $T\mathbf{a}$ and mentioned no appropriate class would suffice to show that. The idea is to define a valuation V which gives $1/2$ to $T\mathbf{a}$ and is so constructed that if $\mathbf{A}$ has $1/2$ in V, then $T\mathbf{a} \leftrightarrow \mathbf{A}$ gets 1 there, but if $\mathbf{A}$ has an integral value in V, then it also has that value in some valuations that give the same value to $T\mathbf{a}$. In its current version, Theorem 2 involves many T-attributions so that it can be used in proving Theorem 3. It also mentions an appropriate class C and an assignment Q of values. These are needed in the proofs of both Theorem 3 and the Central Theorem. (In addition, Q is needed because the Central Theorem mentions an assignment K of values to sentential letters.) Theorem 3 shows that if $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ are finitely many T-attributions and $\mathbf{A}_1, \ldots, \mathbf{A}_l$ are the wffs named by $\mathbf{a}_1, \ldots, \mathbf{a}_l$, respectively (or they are any wffs, for that matter), then there is a valuation in which all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ get 1. It is proved through an induction that involves repeated application of Theorem 2. Finally, in the Central Theorem we show that there is a valuation in which all the infinitely many biconditionals $Ta_1 \leftrightarrow \mathcal{R}(a_1), Ta_2 \leftrightarrow \mathcal{R}(a_2), \ldots$ get 1. It is proved through an induction that involves repeated application of Theorem 3.[11]

Theorems 2 and 3 both rest on various assumptions about $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$, $\mathbf{A}_1, \ldots, \mathbf{A}_l$, Q, and C. We assume that $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ ($l \geq 1$) are distinct T-attributions, while $\mathbf{A}_1, \ldots, \mathbf{A}_l$ are any wffs. Q is an assignment of values to zero, one, or more atomic wffs other than $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$. C is an appropriate class that does not combine any one of $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ with any T-attribution receiving a value in Q. Q conforms with C; in other words, if $T\mathbf{b}$ and $T\mathbf{c}$ receive values in Q, then they receive the same value if C connects $T\mathbf{b}$ with $T\mathbf{c}$, and they receive opposite values if C connects $T\mathbf{b}$ with $\neg T\mathbf{c}$. Moreover, if C connects $T\mathbf{a}_i$ with $T\mathbf{a}_j$ ($1 \leq i \leq l$, $1 \leq j \leq l$), then $\mathbf{A}_i$ and $\mathbf{A}_j$ are C-equivalent, and if C connects $T\mathbf{a}_i$ with $\neg T\mathbf{a}_j$, then $\mathbf{A}_i$ and $\neg\mathbf{A}_j$ are C-equivalent. (So, also, if C connects $\neg T\mathbf{a}_i$ with $\neg T\mathbf{a}_j$, then $\neg\mathbf{A}_i$ and $\neg\mathbf{A}_j$ are C-equivalent, and if C connects $\neg T\mathbf{a}_i$ with $T\mathbf{a}_j$, then $\neg\mathbf{A}_i$ and $\mathbf{A}_j$ are C-equivalent.)

Theorem 2 also rests on some assumptions about V. We assume that V is a C-valuation in which we incorporate Q; we give $1/2$ to all atomic wffs that receive no value in Q and are not C-associates of any T-attribution receiving a value in Q; we follow (Con) with respect to C; and if $\mathbf{C}$ and $\mathbf{D}$ get $1/2$, then we give 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$ is or is not (respectively) C-insertable between $\mathbf{C}$ and $\mathbf{D}$.

Theorem 2 *All of* $\mathbf{A}_1, \ldots, \mathbf{A}_l$ *get* $1/2$ *in* V *or (an* $\mathbf{A}_j$ *from among* $\mathbf{A}_1, \ldots, \mathbf{A}_l$ *gets* 1 *in* V*, but also has* 1 *in every* C*-valuation that assigns* 1 *to* $T\mathbf{a}_j$*, incorporates* Q*, follows (Con)(a) with respect to* C*, gives* 1 *to all of* $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$

other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$*, and gives* 1 *to* $\mathbf{E} \to \mathbf{E}'$ *if it gives* 1/2 *to* $\mathbf{E}$ *and to* $\mathbf{E}'$ *and* $\mathbf{E}$ C*-implies* $\mathbf{E}'$*) or (an* $\mathbf{A}_j$ *from among* $\mathbf{A}_1, \ldots, \mathbf{A}_l$ *gets* 0 *in* V*, but also has* 0 *in every* C*-valuation that assigns* 0 *to* $T\mathbf{a}_j$*, incorporates* Q*, follows (Con)(a) with respect to* C*, gives* 1 *to all of* $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ *other than* $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$*, and gives* 1 *to* $\mathbf{E} \to \mathbf{E}'$ *if it gives* 1/2 *to* $\mathbf{E}$ *and to* $\mathbf{E}'$ *and* $\mathbf{E}$ C*-implies* $\mathbf{E}'$*) or there are* i *and* j *among* $1, \ldots, l$ *such that* $i \neq j$ *and* $\mathbf{A}_i$ *is* C*-equivalent to* $\mathbf{A}_j$ *or to* $\neg\mathbf{A}_j$*.*

Proof We will first focus on $\mathbf{A}_l$ and demonstrate that $\mathbf{A}_l$ gets 1/2 in V or (it gets 1 in V, but also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns 1/2 to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 0 in V, with $l > 1$, and is C-implied by $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$) or (it gets 0 in V, but also has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns 1/2 to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 1 in V, with $l > 1$, and C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$). Note that $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ have 1/2 in V, since they receive no value in Q and are not C-associates of any T-attribution receiving a value in Q.

We know from Lemma 3 that if V assigns 1/2 to $\mathbf{C}$ and $\mathbf{D}$, then it gives 1 or 1/2 to $\mathbf{C} \to \mathbf{D}$ depending on whether or not the following disjunctive condition is met: $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ ($\varnothing$ if $l = 1$) is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$. We can distinguish four cases.

(i) $\mathbf{A}_l$ contains no part $\mathbf{C} \to \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get 1/2 in V and either $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ or $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$. Then we can prove inductively that, for every well-formed part $\mathbf{B}$ of $\mathbf{A}_l$, $\mathbf{B}$ gets 1/2 in V or (it gets 1 in V, but also has 1 in every C-valuation that incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns 1/2 to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 0 in V, but also has 0 in every C-valuation that incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns 1/2 to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$). Once we have proved that, what we are trying to demonstrate about $\mathbf{A}_l$ follows as a corollary. The interesting cases in the induction are two. (α) $\mathbf{B}$ is a conjunction $\mathbf{C} \wedge \mathbf{D}$ and gets 0 in V while $\mathbf{C}$ and $\mathbf{D}$ have 1/2 or the one has 1/2 and the other 1. Then, as $\mathbf{B}$ gets 0 in V by application of (Con), it also has 0 in every C-valuation that follows (Con)(a) with respect to C. For it cannot get 1 in such a valuation V', since its deep conjuncts do not all have 1 in any C-valuation. So it will have 0 in V' either because one of its conjuncts has 0 or because it falls under case (i) of Table 3 so that (Con)(a) applies. (β) $\mathbf{B}$ is $\mathbf{C} \to \mathbf{D}$ and gets 1 in V while $\mathbf{C}$ and $\mathbf{D}$ have 1/2. Then $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ is C-insertable between $\mathbf{C}$

and $\mathbf{D}$. In this case, $\mathbf{B}$ also has 1 in every C-valuation that gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$ and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$. For, in such a valuation, all the members of $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$ get 1, and moreover $\mathbf{E}_i \to \mathbf{E}_{i+1}$ gets 1 if $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$. Thus by transitivity $\mathbf{C} \to \mathbf{D}$, too, will get 1.

(ii) $\mathbf{A}_l$ contains at least one part $\mathbf{C} \to \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$, but $\mathbf{A}_l$ contains no part $\mathbf{C} \to \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \to \mathbf{A}_l, \neg\mathbf{A}_l \to \neg T\mathbf{a}_l\}$. Then we can prove inductively that, for every well-formed part $\mathbf{B}$ of $\mathbf{A}_l$, $\mathbf{B}$ gets $1/2$ in V or (it gets 1 in V, but also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 0 in V, but also has 0 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$). The only case in the induction that differs from (i) above is that in which $\mathbf{B}$ is $\mathbf{C} \to \mathbf{D}$ and gets 1 in V while $\mathbf{C}$ and $\mathbf{D}$ have $1/2$. Then $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$. In this case, $\mathbf{B}$ also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \to \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$. For, in such a valuation V', $\mathbf{A}_l \to T\mathbf{a}_l$ and $\neg T\mathbf{a}_l \to \neg\mathbf{A}_l$ get 1, so all the members of $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$ get 1. Moreover, as in (i) above, $\mathbf{E}_i \to \mathbf{E}_{i+1}$ gets 1 in V' if $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$.

We have assumed that $\mathbf{A}_l$ contains a part $\mathbf{C} \to \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and there are $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) where $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \to \mathbf{E}_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \to T\mathbf{a}_l, \neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$, and for some i $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$ or $\neg T\mathbf{a}_l \to \neg\mathbf{A}_l$. Suppose, in addition, that $\mathbf{A}_l$ gets 0 in V. If there is a number i such that $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, then let k be the smallest such number. Let h be the largest number among $1, \ldots, k$ such that either $h = 1$ or $\mathbf{E}_{h-1} \to \mathbf{E}_h$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\neg T\mathbf{a}_l \to \neg\mathbf{A}_l\}$. Then $\mathbf{E}_k$, that is, $\mathbf{A}_l$, is C-implied by $\mathbf{E}_h$. But $\mathbf{A}_l$ is not C-implied by any one of $\mathbf{C}$, $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l-1}$, and $\neg T\mathbf{a}_1, \ldots, \neg T\mathbf{a}_{l-1}$. For all these wffs have $1/2$ in V, $\mathbf{A}_l$ has 0 there, and V is a C-valuation. Nor is $\mathbf{A}_l$ C-implied by $\neg\mathbf{A}_l$, as they have 0 and 1, respectively, in V. Hence $\mathbf{E}_h$ is $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$, and so $l > 1$. If, on the other hand, there is no i where $\mathbf{E}_i \to \mathbf{E}_{i+1}$ is $\mathbf{A}_l \to T\mathbf{a}_l$, then let k be the largest number such that $\mathbf{E}_{k-1} \to \mathbf{E}_k$ is $\neg T\mathbf{a}_l \to \neg\mathbf{A}_l$. And let h be the smallest number among $k, \ldots, n$ such that either $h = n$ or $\mathbf{E}_h \to \mathbf{E}_{h+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1}$. Then $\mathbf{E}_k$, that is, $\neg\mathbf{A}_l$, C-implies $\mathbf{E}_h$. But $\neg\mathbf{A}_l$, which gets 1 in V, does not C-imply any one of $\mathbf{D}$, $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l-1}$, and $\neg T\mathbf{a}_1, \ldots, \neg T\mathbf{a}_{l-1}$. Hence $\mathbf{E}_h$ is $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$, and so $l > 1$. And since $\neg\mathbf{A}_l$ C-implies $\mathbf{E}_h$, $\neg\mathbf{E}_h$

C-implies $\mathbf{A}_l$. So, once again, $\mathbf{A}_l$ is C-implied by $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$.

Therefore, $\mathbf{A}_l$ gets $1/2$ in V or (it gets 1 in V, but also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \rightarrow \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 0 in V, with $l > 1$, and is C-implied by $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$).

(iii) $\mathbf{A}_l$ contains at least one part $\mathbf{C} \rightarrow \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \rightarrow \mathbf{A}_l, \neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \rightarrow \mathbf{A}_l, \neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l\}$, but $\mathbf{A}_l$ contains no part $\mathbf{C} \rightarrow \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \rightarrow T\mathbf{a}_l, \neg T\mathbf{a}_l \rightarrow \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \rightarrow T\mathbf{a}_l, \neg T\mathbf{a}_l \rightarrow \neg\mathbf{A}_l\}$. Then we can prove inductively that, for every well-formed part $\mathbf{B}$ of $\mathbf{A}_l$, $\mathbf{B}$ gets $1/2$ in V or (it gets 1 in V, but also has 1 in every C-valuation that assigns 0 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \rightarrow \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 0 in V, but also has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \rightarrow \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$). The only difference from the induction in (ii) above consists in the fact that $T\mathbf{a}_l \rightarrow \mathbf{A}_l$ and $\neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l$ get 1 when $T\mathbf{a}_l$ is assigned 0.

We have assumed that $\mathbf{A}_l$ contains a part $\mathbf{C} \rightarrow \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and there are $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) where $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, for every i ($1 \leq i < n$) either $\mathbf{E}_i$ C-implies $\mathbf{E}_{i+1}$ or $\mathbf{E}_i \rightarrow \mathbf{E}_{i+1}$ belongs to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \rightarrow \mathbf{A}_l, \neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l\}$, and for some i $\mathbf{E}_i \rightarrow \mathbf{E}_{i+1}$ is $T\mathbf{a}_l \rightarrow \mathbf{A}_l$ or $\neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l$. Proceeding as in (ii), we can show that if $\mathbf{A}_l$ gets 1 in V, then $l > 1$ and $\mathbf{A}_l$ C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$. Thus, in the end, $\mathbf{A}_l$ gets $1/2$ in V or (it gets 0 in V, but also has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l-1} \leftrightarrow \mathbf{A}_{l-1}$, and assigns 1 to $\mathbf{E} \rightarrow \mathbf{E}'$ if it assigns $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (it gets 1 in V, with $l > 1$, and C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$).

(iv) $\mathbf{A}_l$ contains at least one part $\mathbf{C} \rightarrow \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{\mathbf{A}_l \rightarrow T\mathbf{a}_l, \neg T\mathbf{a}_l \rightarrow \neg\mathbf{A}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{\mathbf{A}_l \rightarrow T\mathbf{a}_l, \neg T\mathbf{a}_l \rightarrow \neg\mathbf{A}_l\}$, and $\mathbf{A}_l$ also contains at least one part $\mathbf{C} \rightarrow \mathbf{D}$ such that $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V and $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l-1} \cup \{T\mathbf{a}_l \rightarrow \mathbf{A}_l, \neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l\}$ is C-insertable between $\mathbf{C}$ and $\mathbf{D}$ in the presence of at least one of the wffs in $\{T\mathbf{a}_l \rightarrow \mathbf{A}_l, \neg\mathbf{A}_l \rightarrow \neg T\mathbf{a}_l\}$. Then because of the former kind of part, it is the case, as in (ii) above, that if $\mathbf{A}_l$ gets 0 in V, then $l > 1$ and $\mathbf{A}_l$ is C-implied by $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$. And because of the latter kind of part, it is the case, as in (iii), that if $\mathbf{A}_l$ gets 1 in V, then $l > 1$ and $\mathbf{A}_l$ C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$. This ends (iv).

Now assume that no $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_l$ gets 1 in V, but also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_j$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$. Also assume that no $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_l$ gets 0 in V, but also has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_j$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$. Then $\mathbf{A}_l$ gets $1/2$ in V or ($\mathbf{A}_l$ gets 1 in V, with $l > 1$, and C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$) or ($\neg\mathbf{A}_l$ gets 1 in V, with $l > 1$, and C-implies $\mathbf{A}_1$ or $\neg\mathbf{A}_1$ or $\cdots$ or $\mathbf{A}_{l-1}$ or $\neg\mathbf{A}_{l-1}$).

The reasoning from the beginning of the proof up to this point can be repeated with any k from among $1, \ldots, l-1$ in place of l and with $\{1, \ldots, l\} - \{k\}$ in place of $\{1, \ldots, l-1\}$. Thus we can conclude, within the scope of the assumptions made in the preceding paragraph, that, for each i, $1 \le i \le l$, $\mathbf{A}_i$ gets $1/2$ in V or ($\mathbf{A}_i$ gets 1 in V and, for some j where $1 \le j \le l$ and $j \ne i$, C-implies either $\mathbf{A}_j$ or $\neg\mathbf{A}_j$) or ($\neg\mathbf{A}_i$ gets 1 in V and, for some j where $1 \le j \le l$ and $j \ne i$, C-implies either $\mathbf{A}_j$ or $\neg\mathbf{A}_j$). Hence, by Lemma 1, either all of $\mathbf{A}_1, \ldots, \mathbf{A}_l$ get $1/2$ in V or there are i and j among $1, \ldots, l$ such that $i \ne j$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_j$ or to $\neg\mathbf{A}_j$. □

Theorem 3 *There are an extension C' of C and a C'-valuation V such that if all T-attributions among $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$ that get $1/2$ in V are $T\mathbf{a}_{k_1}, \ldots, T\mathbf{a}_{k_r}$, then: $C' - C$ involves only T-attributions from among $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$; for any k_h and $k_{h'}$ where $h, h' \in \{1, \ldots, r\}$ and $\mathbf{a}_{k_h}$ is other than $\mathbf{a}_{k_{h'}}$, C' connects $T\mathbf{a}_{k_h}$ and $T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_{k_{h'}}$ are C'-equivalent, and C' connects $T\mathbf{a}_{k_h}$ and $\neg T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_{k_{h'}}$ are C'-equivalent; V incorporates Q; V assigns $1/2$ to all atomic wffs that are not in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_l\}$, receive no value in Q, and are not C'-associates of any T-attribution in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_l\}$ and any T-attribution receiving a value in Q; $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ get 1 in V; V follows (Con) with respect to C'; and if V assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ ($\varnothing$ if $r = 0$) is or is not, respectively, C'-insertable between $\mathbf{C}$ and $\mathbf{D}$.*

Proof This is by induction on l.

(i) $l = 1$. Take the set $S = \{T\mathbf{b} : T\mathbf{b}$ receives a value in Q, and if $T\mathbf{c}$ is other than $T\mathbf{b}$, is a C-associate of $T\mathbf{b}$, and receives a value in Q, then $\mathbf{b}$ has a smaller subscript than $\mathbf{c}\} \cup \{\mathbf{p} : \mathbf{p}$ is a sentential letter receiving a value in $Q\}$. Of course, no two T-attributions in S are C-associates of each other, and so no two T-attributions in S have a C-associate in common. Also, C does not connect $T\mathbf{b}$ with both $T\mathbf{c}$ and $\neg T\mathbf{c}$, for otherwise it would connect $T\mathbf{c}$ with $\neg T\mathbf{c}$.

Consider the valuation V in which: we incorporate Q as regards the members of S; for every $T\mathbf{b}$ in S and for every C-associate $T\mathbf{c}$ of $T\mathbf{b}$, we give $T\mathbf{c}$ the value of $T\mathbf{b}$ if C connects $T\mathbf{b}$ with $T\mathbf{c}$, and we give it the opposite value if C connects $T\mathbf{b}$ with $\neg T\mathbf{c}$; we assign $1/2$ to all remaining atomic wffs; we follow (Con) with respect to C; and if $\mathbf{C}$ and $\mathbf{D}$ get $1/2$, then we give 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_l$ is or is not (respectively) C-insertable between $\mathbf{C}$ and $\mathbf{D}$. Note that $T\mathbf{a}_l$ gets $1/2$.

V is a C-valuation. To see that, suppose it is not. Then either (α) $\{T\mathbf{d}, T\mathbf{e}\} \in C$ and we have assigned distinct values to $T\mathbf{d}$ and $T\mathbf{e}$, or (β)

$\{T\mathbf{d}, \neg T\mathbf{e}\} \in C$ and we have assigned distinct values to $T\mathbf{d}$ and $\neg T\mathbf{e}$. In case (α), as $T\mathbf{d}$ and $T\mathbf{e}$ do not both have $1/2$ in V, neither took on a value as a remaining atomic wff. (It cannot be that only one of them took on a value as such a wff.) Thus, C combines $T\mathbf{d}$ with a T-attribution, perhaps $T\mathbf{d}$ itself, belonging to S, and C also combines $T\mathbf{e}$ with a T-attribution, perhaps $T\mathbf{e}$ itself, belonging to S. (C combines $T\mathbf{d}$ with $T\mathbf{d}$, as it does with every T-attribution it involves.) Since C does not combine any two T-attributions belonging to S, it combines $T\mathbf{d}$, as well as $T\mathbf{e}$, with the same T-attribution, $T\mathbf{b}$, in S. Then, C must be connecting $T\mathbf{b}$ with a member of $\{T\mathbf{d}, T\mathbf{e}\}$ and also connecting $T\mathbf{b}$ with the negation of the other member of $\{T\mathbf{d}, T\mathbf{e}\}$. So C connects a T-attribution with its negation, which is impossible given that C is appropriate. Case (β) is similar.

It is easy to see that, since Q conforms with C, V incorporates the whole of Q. Thus all the conditions on which Theorem 2 rested are satisfied. So, by that theorem, $\mathbf{A}_l$ gets $1/2$ in V or ($\mathbf{A}_l$ gets 1 in V, but also has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or ($\mathbf{A}_l$ gets 0 in V, but also has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_l$, incorporates Q, follows (Con)(a) with respect to C, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$).

In the first case (i.e., $\mathbf{A}_l$ gets $1/2$ in V), $T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$ has 1 in V, and, more generally, C and V have all the desirable properties. (C counts as an extension of itself.) In the second case, consider the valuation V' in which: we incorporate Q as regards the members of S; we assign 1 to $T\mathbf{a}_l$; for every $T\mathbf{b}$ that either belongs to S or is $T\mathbf{a}_l$ and for every C-associate $T\mathbf{c}$ of $T\mathbf{b}$, we give $T\mathbf{c}$ the value of $T\mathbf{b}$ if C connects $T\mathbf{b}$ with $T\mathbf{c}$, and we give it the opposite value if C connects $T\mathbf{b}$ with $\neg T\mathbf{c}$; we assign $1/2$ to all atomic wffs that are other than $T\mathbf{a}_l$, do not belong to S, and are not C-associates of either $T\mathbf{a}_l$ or any member of S; we follow (Con) with respect to C; and if $\mathbf{C}$ and $\mathbf{D}$ get $1/2$, then we give 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether or not $\mathbf{C}$ C-implies $\mathbf{D}$. As with V, we can see that V' is a C-valuation. ($S \cup \{T\mathbf{a}_l\}$ now plays the role that S on its own played in the corresponding proof for V.) And, again like V, V' incorporates Q. Thus $\mathbf{A}_l$ gets 1 in V', as does $T\mathbf{a}_l \leftrightarrow \mathbf{A}_l$. More generally, C and V' have the desirable properties. Finally, the third case (i.e., $\mathbf{A}_l$ gets 0 in V) is like the second.

(ii) Let $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$, $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$, Q, and C be just as in the assumptions on which Theorems 2 and 3 rest, but with $l+1$ in place of l. Supposing that what we are trying to prove holds for l, we must show that there are an extension C' of C and a C'-valuation V which have the following properties if all T-attributions among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ that get $1/2$ in V are $T\mathbf{a}_{k'_1}, \ldots, T\mathbf{a}_{k'_s}$: $C' - C$ involves only T-attributions from among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$; for any k'_h and $k'_{h'}$ where $h, h' \in \{1, \ldots, s\}$ and $\mathbf{a}_{k'_h}$ is other than $\mathbf{a}_{k'_{h'}}$, C' connects $T\mathbf{a}_{k'_h}$ and $T\mathbf{a}_{k'_{h'}}$ if and only if $\mathbf{A}_{k'_h}$ and $\mathbf{A}_{k'_{h'}}$ are C'-equivalent, and C' connects $T\mathbf{a}_{k'_h}$ and $\neg T\mathbf{a}_{k'_{h'}}$ if and only if $\mathbf{A}_{k'_h}$ and $\neg\mathbf{A}_{k'_{h'}}$ are C'-equivalent; V incorporates Q; V assigns $1/2$ to all atomic wffs that are not in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\}$, receive no value in Q, and are not C'-associates of any T-attribution in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\}$ and any

T-attribution receiving a value in Q; $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ get 1 in V; V follows (Con) with respect to C'; and if V assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k'_1} \cup \cdots \cup \mathcal{A}_{k'_s}$ ($\varnothing$ if $s = 0$) is or is not C'-insertable between $\mathbf{C}$ and $\mathbf{D}$.

The set S is defined as in (i) above. Consider the valuation V in which: we incorporate Q as regards the members of S; for every $T\mathbf{b}$ in S and for every C-associate $T\mathbf{c}$ of $T\mathbf{b}$, we give $T\mathbf{c}$ the value of $T\mathbf{b}$ if C connects $T\mathbf{b}$ with $T\mathbf{c}$, and we give it the opposite value if C connects $T\mathbf{b}$ with $\neg T\mathbf{c}$; we assign $1/2$ to all remaining atomic wffs; we follow (Con) with respect to C; and if $\mathbf{C}$ and $\mathbf{D}$ get $1/2$, then we give 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{l+1}$ is or is not C-insertable between $\mathbf{C}$ and $\mathbf{D}$. So all of $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ get $1/2$ in V. As in (i) above, V is a C-valuation and incorporates the whole of Q.

Thus, by Theorem 2, all of $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ get $1/2$ in V or (an $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ has 1 in every C-valuation that assigns 1 to $T\mathbf{a}_j$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or (an $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ has 0 in every C-valuation that assigns 0 to $T\mathbf{a}_j$, incorporates Q, follows (Con)(a) with respect to C, gives 1 to all of $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1$, $\ldots$, $T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$, and gives 1 to $\mathbf{E} \to \mathbf{E}'$ if it gives $1/2$ to $\mathbf{E}$ and to $\mathbf{E}'$ and $\mathbf{E}$ C-implies $\mathbf{E}'$) or there are i and j among $1, \ldots, l+1$ such that $i \neq j$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_j$ or to $\neg\mathbf{A}_j$.

In the first case (i.e., if all of $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ get $1/2$ in V) the biconditionals $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ have 1 in V. Suppose that there are no i and j among $1, \ldots, l+1$ such that $i \neq j$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_j$ or to $\neg\mathbf{A}_j$. Then, more generally, C and V have the properties we must show (with C counting as an extension of itself).

In the second case (an $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ has 1 $\ldots$) suppose again that there are no i and k among $1, \ldots, l+1$ such that $i \neq k$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_k$ or to $\neg\mathbf{A}_k$. Then C does not combine $T\mathbf{a}_j$ with any other T-attribution $T\mathbf{a}_k$ from among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$; for if it did, $\mathbf{A}_j$ would be C-equivalent to $\mathbf{A}_k$ or to $\neg\mathbf{A}_k$. Let us take the T-attributions $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$, the corresponding wffs from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$, the assignment $Q \cup \{\langle T\mathbf{a}_j, 1\rangle\}$, and the class C in the roles played in the formulation of Theorem 3 by $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$, $\mathbf{A}_1, \ldots, \mathbf{A}_l$, Q, and C, respectively. Then we know by the inductive hypothesis that there are an extension C' of C and a C'-valuation V' where if all T-attributions among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$ that get $1/2$ in V' are $T\mathbf{a}_{k_1}, \ldots, T\mathbf{a}_{k_r}$, then: $C' - C$ involves only T-attributions from among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$; for any k_h and $k_{h'}$ where $h, h' \in \{1, \ldots, r\}$ and $\mathbf{a}_{k_h}$ is other than $\mathbf{a}_{k_{h'}}$, C' connects $T\mathbf{a}_{k_h}$ and $T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_{k_{h'}}$ are C'-equivalent, and C' connects $T\mathbf{a}_{k_h}$ and $\neg T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_{k_{h'}}$ are C'-equivalent; V' incorporates $Q \cup \{\langle T\mathbf{a}_j, 1\rangle\}$; V' assigns $1/2$ to all atomic wffs that are not in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$, receive no value in $Q \cup \{\langle T\mathbf{a}_j, 1\rangle\}$, and are not C'-associates of any T-attribution in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$ and any T-attribution receiving a value in $Q \cup \{\langle T\mathbf{a}_j, 1\rangle\}$; $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$ get

1 in V'; V' follows (Con) with respect to C'; and if V' assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ ($\varnothing$ if $r = 0$) is or is not (respectively) C'-insertable between $\mathbf{C}$ and $\mathbf{D}$. $\mathbf{A}_j$ must have 1 in V', so $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$ gets 1 there. More generally, C' and V' have the properties we must show. The third case (an $\mathbf{A}_j$ from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$ has 0 ...) is like the second.

Next we have the case in which there are i and j among $1, \ldots, l + 1$ such that $i \neq j$ and $\mathbf{A}_i$ is C-equivalent to $\mathbf{A}_j$. Then, it may or may not be that C combines $T\mathbf{a}_i$ with $T\mathbf{a}_j$. If it does, it connects $T\mathbf{a}_i$ with $T\mathbf{a}_j$. For if it connected $T\mathbf{a}_i$ with $\neg T\mathbf{a}_j$, then $\mathbf{A}_i$ and $\neg\mathbf{A}_j$ would be C-equivalent, and so $\mathbf{A}_j$ and $\neg\mathbf{A}_j$ would be C-equivalent. In that case, $\mathbf{A}_j$ would have $1/2$ in every C-valuation, which we know to be impossible because there are C-valuations in which no wff gets $1/2$. If C combines $T\mathbf{a}_i$ with $T\mathbf{a}_j$, then let C^+ be C. Otherwise, let C^+ be $C \cup \{\{T\mathbf{a}_i, T\mathbf{a}_j\}, \{\neg T\mathbf{a}_i, \neg T\mathbf{a}_j\}\}$. We know from Lemma 4 that C^+ is appropriate.

A number of points should be noted here. First, since C does not combine any one of $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ with a T-attribution receiving a value in Q, C^+ does not combine any T-attribution in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$ with one receiving a value in Q. To see that, suppose that there are $\mathbf{B}_1, \ldots, \mathbf{B}_n$ such that $\mathbf{B}_1 \in \{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$, $\mathbf{B}_n$ is either a T-attribution receiving a value in Q or the negation of such a T-attribution, and for every k ($1 \leq k < n$) $\{\mathbf{B}_k, \mathbf{B}_{k+1}\} \in C^+$. Then, for some number k, $\{\mathbf{B}_k, \mathbf{B}_{k+1}\}$ will be $\{T\mathbf{a}_i, T\mathbf{a}_j\}$ or $\{\neg T\mathbf{a}_i, \neg T\mathbf{a}_j\}$. Let h be the largest such number. As $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ do not receive values in Q, $h + 1 < n$. So C will connect $T\mathbf{a}_i$ or $T\mathbf{a}_j$ with $\mathbf{B}_n$ or the opposite of $\mathbf{B}_n$.

Second, since Q conforms with C, it also conforms with C^+. Suppose that $T\mathbf{c}$ and $T\mathbf{d}$ receive values in Q, and there are $\mathbf{B}_1, \ldots, \mathbf{B}_n$ such that $\mathbf{B}_1$ is $T\mathbf{c}$, $\mathbf{B}_n$ is $T\mathbf{d}$, and for every k ($1 \leq k < n$) $\{\mathbf{B}_k, \mathbf{B}_{k+1}\} \in C^+$. In that case, if for some k $\{\mathbf{B}_k, \mathbf{B}_{k+1}\}$ is $\{T\mathbf{a}_i, T\mathbf{a}_j\}$ or $\{\neg T\mathbf{a}_i, \neg T\mathbf{a}_j\}$, then (as in the preceding paragraph) C will connect $T\mathbf{a}_i$ or $T\mathbf{a}_j$ with $T\mathbf{d}$ or $\neg T\mathbf{d}$. As in fact C does not combine any one of $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ with a T-attribution receiving a value in Q, for every k $\{\mathbf{B}_k, \mathbf{B}_{k+1}\} \in C$. In other words, C connects $T\mathbf{c}$ with $T\mathbf{d}$, and so Q gives them the same value. We can likewise see that if $T\mathbf{c}$ and $T\mathbf{d}$ receive values in Q, and C^+ connects $T\mathbf{c}$ with $\neg T\mathbf{d}$, then C connects $T\mathbf{c}$ with $\neg T\mathbf{d}$, and so Q gives opposite values to $T\mathbf{c}$ and $T\mathbf{d}$.

Third, for any $h, h' \in \{1, \ldots, l+1\} - \{j\}$, if C^+ connects $T\mathbf{a}_h$ with $T\mathbf{a}_{h'}$, then $\mathbf{A}_h$ and $\mathbf{A}_{h'}$ are C^+-equivalent, and if C^+ connects $T\mathbf{a}_h$ with $\neg T\mathbf{a}_{h'}$, then $\mathbf{A}_h$ and $\neg\mathbf{A}_{h'}$ are C^+-equivalent. For example, say that C^+ is not C and there are $\mathbf{B}_1, \ldots, \mathbf{B}_n$ such that $\mathbf{B}_1$ is $T\mathbf{a}_h$, $\mathbf{B}_n$ is $T\mathbf{a}_{h'}$, for some k ($1 < k$ and $k+1 < n$) $\{\mathbf{B}_k, \mathbf{B}_{k+1}\}$ is $\{T\mathbf{a}_i, T\mathbf{a}_j\}$ or $\{\neg T\mathbf{a}_i, \neg T\mathbf{a}_j\}$, and for every k' ($1 \leq k' < k$, or $k+1 \leq k' < n$) $\{\mathbf{B}_{k'}, \mathbf{B}_{k'+1}\} \in C$. We have that C connects $T\mathbf{a}_h$ with $\mathbf{B}_k$, as well as $\mathbf{B}_{k+1}$ with $T\mathbf{a}_{h'}$. Let $\mathbf{C}_k$ be $\mathbf{A}_i$, $\mathbf{A}_j$, $\neg\mathbf{A}_i$, or $\neg\mathbf{A}_j$ depending on whether $\mathbf{B}_k$ is $T\mathbf{a}_i$, $T\mathbf{a}_j$, $\neg T\mathbf{a}_i$, or $\neg T\mathbf{a}_j$, respectively, and let $\mathbf{C}_{k+1}$ be $\mathbf{A}_i$, $\mathbf{A}_j$, $\neg\mathbf{A}_i$, or $\neg\mathbf{A}_j$ depending on whether $\mathbf{B}_{k+1}$ is $T\mathbf{a}_i$, $T\mathbf{a}_j$, $\neg T\mathbf{a}_i$, or $\neg T\mathbf{a}_j$, respectively. Then $\mathbf{A}_h$ is C-equivalent to $\mathbf{C}_k$, and $\mathbf{C}_{k+1}$ is C-equivalent to $\mathbf{A}_{h'}$. But, of course, $\mathbf{C}_k$ is C-equivalent to $\mathbf{C}_{k+1}$. Hence $\mathbf{A}_h$ and $\mathbf{A}_{h'}$ are C-equivalent, and so they are C^+-equivalent.

Thus we can take the T-attributions $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$, the corresponding wffs from among $\mathbf{A}_1, \ldots, \mathbf{A}_{l+1}$, the assignment Q, and the class C^+ in the roles played in the formulation of Theorem 3 by $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$, $\mathbf{A}_1, \ldots, \mathbf{A}_l$, Q, and C, respectively. Then we know by the inductive hypothesis that there are an extension C' of C^+ and a C'-valuation V' where if all T-attributions among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$ that get $1/2$ in V' are $T\mathbf{a}_{k_1}, \ldots, T\mathbf{a}_{k_r}$, then: $C' - C^+$ involves only T-attributions from among $T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}$ other than $T\mathbf{a}_j$; for any k_h and $k_{h'}$ where $h, h' \in \{1, \ldots, r\}$ and $\mathbf{a}_{k_h}$ is other than $\mathbf{a}_{k_{h'}}$, C' connects $T\mathbf{a}_{k_h}$ and $T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_{k_{h'}}$ are C'-equivalent, and C' connects $T\mathbf{a}_{k_h}$ and $\neg T\mathbf{a}_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_{k_{h'}}$ are C'-equivalent; V' incorporates Q; V' assigns $1/2$ to all atomic wffs that are not in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$, receive no value in Q, and are not C'-associates of any T-attribution in $\{T\mathbf{a}_1, \ldots, T\mathbf{a}_{l+1}\} - \{T\mathbf{a}_j\}$ and any T-attribution receiving a value in Q; $T\mathbf{a}_1 \leftrightarrow \mathbf{A}_1, \ldots, T\mathbf{a}_{l+1} \leftrightarrow \mathbf{A}_{l+1}$ other than $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$ get 1 in V'; V' follows (Con) with respect to C'; and if V' assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ ($\varnothing$ if $r = 0$) is or is not C'-insertable between $\mathbf{C}$ and $\mathbf{D}$.

It is basic to note that C' connects $T\mathbf{a}_i$ with $T\mathbf{a}_j$ and that V' (as well as any other C'-valuation) assigns the same value to $T\mathbf{a}_i$ and $T\mathbf{a}_j$. Then, C' and V' have the properties we must show, as becomes clear mainly through the following considerations.

(a) If $T\mathbf{a}_j$ gets $1/2$ in V', then we must show that, for any $h \in \{1, \ldots, r\}$, C' connects $T\mathbf{a}_{k_h}$ and $T\mathbf{a}_j$ just in case $\mathbf{A}_{k_h}$ and $\mathbf{A}_j$ are C'-equivalent, and C' connects $T\mathbf{a}_{k_h}$ and $\neg T\mathbf{a}_j$ just in case $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_j$ are C'-equivalent. Indeed, since $T\mathbf{a}_j$ gets $1/2$ in V', $T\mathbf{a}_i$ also gets $1/2$ there, so $\mathbf{a}_i$ is one of $\mathbf{a}_{k_1}, \ldots, \mathbf{a}_{k_r}$. Say that $\mathbf{a}_{k_h}$ is other than $\mathbf{a}_i$. C' connects $T\mathbf{a}_{k_h}$ with $T\mathbf{a}_j$ if and only if it connects $T\mathbf{a}_{k_h}$ with $T\mathbf{a}_i$ if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_i$ are C'-equivalent. And, as $\mathbf{A}_i$ and $\mathbf{A}_j$ are C'-equivalent, $\mathbf{A}_{k_h}$ and $\mathbf{A}_i$ are C'-equivalent if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_j$ are. We likewise see that C' connects $T\mathbf{a}_{k_h}$ with $\neg T\mathbf{a}_j$ if and only if $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_j$ are C'-equivalent. On the other hand, in case $\mathbf{a}_{k_h}$ is $\mathbf{a}_i$, then of course C' connects $T\mathbf{a}_{k_h}$ with $T\mathbf{a}_j$, $\mathbf{A}_{k_h}$ and $\mathbf{A}_j$ are C'-equivalent, C' does not connect $T\mathbf{a}_{k_h}$ with $\neg T\mathbf{a}_j$, and $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_j$ are not C'-equivalent.

(b) Since $T\mathbf{a}_i \leftrightarrow \mathbf{A}_i$ gets 1 in V', $T\mathbf{a}_i$ and $\mathbf{A}_i$ have the same value there. But, also, $T\mathbf{a}_j$ has the same value there as $T\mathbf{a}_i$, and $\mathbf{A}_i$ has the same as $\mathbf{A}_j$. Thus $T\mathbf{a}_j$ and $\mathbf{A}_j$ have the same value in V'. If that value is 1 or 0, then $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$ gets 1 in V'. But if the common value of $T\mathbf{a}_j$, $T\mathbf{a}_i$, $\mathbf{A}_i$, and $\mathbf{A}_j$ is $1/2$ (in which case $\mathcal{A}_i$ is one of $\mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_r}$), then again $T\mathbf{a}_j \leftrightarrow \mathbf{A}_j$ gets 1 in V'. The reason is that $\mathcal{A}_i$, and so $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ too, is C'-insertable between $T\mathbf{a}_j$ and $\mathbf{A}_j$, as well as between $\mathbf{A}_j$ and $T\mathbf{a}_j$; for $T\mathbf{a}_j$ C'-implies $T\mathbf{a}_i$ and conversely, and $\mathbf{A}_i$ C'-implies $\mathbf{A}_j$ and conversely.

(c) Assuming that $T\mathbf{a}_j$ gets $1/2$ in V', we must show that if V' assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r} \cup \mathcal{A}_j$ is or is not C'-insertable between $\mathbf{C}$ and $\mathbf{D}$. So what we should demonstrate is that if $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r} \cup \mathcal{A}_j$ is

C'-insertable between $\mathbf{C}$ and $\mathbf{D}$, then so is $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$. Indeed, provided that $T\mathbf{a}_j$ has $1/2$ in V', $T\mathbf{a}_i$ will also be getting $1/2$ there, so $\mathcal{A}_i$ will be one of $\mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_r}$. Say there are $\mathbf{E}_1, \ldots, \mathbf{E}_n$ ($n \geq 2$) such that $\mathbf{E}_1$ is $\mathbf{C}$, $\mathbf{E}_n$ is $\mathbf{D}$, and for every h ($1 \leq h < n$) either $\mathbf{E}_h$ C'-implies $\mathbf{E}_{h+1}$ or $\mathbf{E}_h \to \mathbf{E}_{h+1}$ belongs to $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r} \cup \mathcal{A}_j$. For every h such that $\mathbf{E}_h \to \mathbf{E}_{h+1}$ belongs to $\mathcal{A}_j$, make the following insertions between $\mathbf{E}_h$ and $\mathbf{E}_{h+1}$: if $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $\mathbf{A}_j \to T\mathbf{a}_j$ and not $T\mathbf{a}_j \to \mathbf{A}_j$, insert $\mathbf{A}_i$ and $T\mathbf{a}_i$ (in that order); if $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $T\mathbf{a}_j \to \mathbf{A}_j$, insert $T\mathbf{a}_i$ and $\mathbf{A}_i$; if $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $\neg T\mathbf{a}_j \to \neg\mathbf{A}_j$, insert $\neg T\mathbf{a}_i$ and $\neg\mathbf{A}_i$; and if $\mathbf{E}_h \to \mathbf{E}_{h+1}$ is $\neg\mathbf{A}_j \to \neg T\mathbf{a}_j$ and not $\neg T\mathbf{a}_j \to \neg\mathbf{A}_j$, insert $\neg\mathbf{A}_i$ and $\neg T\mathbf{a}_i$. The resulting (longer) sequence of wffs shows that $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ is C'-insertable between $\mathbf{C}$ and $\mathbf{D}$. For, in each insertion, what is inserted right after $\mathbf{E}_h$ is C'-implied by $\mathbf{E}_h$, and what is inserted right before $\mathbf{E}_{h+1}$ C'-implies $\mathbf{E}_{h+1}$.

Finally, the last, fifth case (i.e., there are i and j among $1, \ldots, l+1$ such that $i \neq j$ and $\mathbf{A}_i$ is C-equivalent to $\neg\mathbf{A}_j$) is similar to the case we have just discussed, but we should consider the set $C \cup \{\{T\mathbf{a}_i, \neg T\mathbf{a}_j\}, \{\neg T\mathbf{a}_i, T\mathbf{a}_j\}\}$ instead of the set $C \cup \{\{T\mathbf{a}_i, T\mathbf{a}_j\}, \{\neg T\mathbf{a}_i, \neg T\mathbf{a}_j\}\}$. □

4.6 If W is a finite and nonempty set of wffs, the classes of subsets of W can be enumerated. The subsets of W will be finitely many, as will their classes. Say we already have an enumeration $\mathcal{E}(W)$ of the members of W. Then, the subsets of W can be ordered in such a way that if S_1 and S_2 are any two subsets, S_1 precedes S_2 just in case S_2 contains $\mathbf{B}$, but S_1 does not, where $\mathbf{B}$ is the first wff in $\mathcal{E}(W)$ with respect to which S_1 and S_2 differ (i.e., the one contains it, but the other does not). Let $\mathcal{E}^s(W)$ be the resulting enumeration of the subsets of W. Then, the classes of subsets of W can be ordered in such a way that if C_1 and C_2 are any two classes, C_1 precedes C_2 just in case C_2 contains S, but C_1 does not, where S is the first set in $\mathcal{E}^s(W)$ with respect to which C_1 and C_2 differ. Let $\mathcal{E}^c(W)$ be the resulting enumeration of the classes of subsets of W.

Again, suppose that $\mathcal{V}$ is a finite and nonempty set of valuations, and $\mathbf{B}_1, \ldots, \mathbf{B}_r$ ($r \geq 1$) are distinct atomic wffs such that if we take any two valuations in $\mathcal{V}$, there will be at least one wff among $\mathbf{B}_1, \ldots, \mathbf{B}_r$ which receives different values in those valuations. In that case, the valuations in $\mathcal{V}$ can be ordered in such a way that if V_1 and V_2 are any two valuations, and $\mathbf{B}_i$ is the first wff in $\mathbf{B}_1, \ldots, \mathbf{B}_r$ with respect to which they differ, then V_1 precedes V_2 just in case $|\mathbf{B}_i|_{V_1} < |\mathbf{B}_i|_{V_2}$. Let the resulting enumeration of the valuations in $\mathcal{V}$ be $\mathcal{E}(\mathcal{V}, \mathbf{B}_1, \ldots, \mathbf{B}_r)$.

It is time to return to the wffs $\mathbf{A}_1, \mathbf{A}_2, \ldots$ as these were defined in Section 3, that is, to $\mathcal{R}(a_1), \mathcal{R}(a_2), \ldots$. We now have enough tools to establish the Central Theorem, repeated here.

Central Theorem *For each assignment K of values to one or more sentential letters, there is a valuation that gives* 1 *to all of $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \ldots$, incorporates K, and has (Prop).*

Proof We will inductively define appropriate classes $C_1, C_2, \ldots$ and valuations $V_1, V_2, \ldots$ such that, for each n, V_n is a C_n-valuation.

We begin with C_1 and V_1. If we take $l = 1$, Ta_1 as $T\mathbf{a}_l$, $\mathbf{A}_1$ as $\mathbf{A}_l$, K as Q, and $\varnothing$ as C, then the conditions for applying Theorem 3 are met. So there are an extension C' of $\varnothing$ and a C'-valuation V such that C' involves no T-attribution other than Ta_1; V incorporates K; V assigns $1/2$ to all atomic wffs that are other than Ta_1 and receive no value in K; $Ta_1 \leftrightarrow \mathbf{A}_1$ gets 1 in V; V follows (Con) with respect to C'; and if V assigns $1/2$ to Ta_1 and also $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_1$ is or is not C'-insertable between $\mathbf{C}$ and $\mathbf{D}$, whereas if V assigns 1 or 0 to Ta_1, but $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether or not $\mathbf{C}$ C'-implies $\mathbf{D}$. But there is only one extension of $\varnothing$ that involves no T-attribution other than Ta_1, namely, $\varnothing$ itself. C_1 will just be that extension. So there is a valuation V such that V incorporates K and assigns $1/2$ to all atomic wffs that are other than Ta_1 and receive no value in K; $Ta_1 \leftrightarrow \mathbf{A}_1$ gets 1 in V; V follows (Con) with respect to $\varnothing$; and if V assigns $1/2$ to Ta_1 and also $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_1$ is or is not $\varnothing$-insertable between $\mathbf{C}$ and $\mathbf{D}$, whereas if V assigns 1 or 0 to Ta_1, but $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether or not $\mathbf{C}$ $\varnothing$-implies $\mathbf{D}$. If $\mathcal{V}$ is the set of such valuations, then $\mathcal{V}$ cannot have more than three members. V_1 will be the first valuation in $\mathcal{E}(\mathcal{V}, Ta_1)$.

Suppose that the appropriate class C_n and the C_n-valuation V_n have been defined in such a way that if all T-attributions among $Ta_1, \ldots, Ta_n$ that get $1/2$ in V_n are $Ta_{j_1}, \ldots, Ta_{j_l}$, where $j_1, \ldots, j_l$ are distinct from one another and arranged in increasing order, then: C_n involves no T-attribution other than $Ta_1, \ldots, Ta_n$; if C_n connects Ta_{j_h} with $Ta_{j_{h'}}$ ($1 \leq h \leq l$, $1 \leq h' \leq l$), then $\mathbf{A}_{j_h}$ and $\mathbf{A}_{j_{h'}}$ are C_n-equivalent; and if C_n connects Ta_{j_h} with $\neg Ta_{j_{h'}}$, then $\mathbf{A}_{j_h}$ and $\neg\mathbf{A}_{j_{h'}}$ are C_n-equivalent.

Let $Ta_{i_1}, \ldots, Ta_{i_m}$ be all T-attributions among $Ta_1, \ldots, Ta_n$ that get 1 or 0 in V_n, and let I be the fragment of V_n which concerns all and only the T-attributions $Ta_{i_1}, \ldots, Ta_{i_m}$. Then, we can take $Ta_{j_1}, \ldots, Ta_{j_l}, Ta_{n+1}$, $\mathbf{A}_{j_1}, \ldots, \mathbf{A}_{j_l}, \mathbf{A}_{n+1}$, $I \cup K$, and C_n in the roles played in the formulation of Theorem 3 by $T\mathbf{a}_1, \ldots, T\mathbf{a}_l$, $\mathbf{A}_1, \ldots, \mathbf{A}_l$, Q, and C, respectively. For C_n does not combine any one of $Ta_{j_1}, \ldots, Ta_{j_l}$ with a T-attribution receiving a value in $I \cup K$. If it combined Ta_{j_1} with Ta_{i_1}, for example, then Ta_{j_1} and Ta_{i_1} would have either the same value or opposite values in V_n. And C_n does not combine Ta_{n+1} with any T-attribution, not even itself. Moreover, $I \cup K$ conforms with C_n, since V_n does so.

Thus, as we know from Theorem 3, there are an extension C' of C_n and a C'-valuation V such that if all T-attributions among $Ta_{j_1}, \ldots, Ta_{j_l}, Ta_{n+1}$ that get $1/2$ in V are $Ta_{k_1}, \ldots, Ta_{k_r}$, then: C' involves no T-attribution other than $Ta_1, \ldots, Ta_{n+1}$; for any k_h and $k_{h'}$ where $h, h' \in \{1, \ldots, r\}$ and a_{k_h} is other than $a_{k_{h'}}$, C' connects Ta_{k_h} with $Ta_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\mathbf{A}_{k_{h'}}$ are C'-equivalent, and C' connects Ta_{k_h} with $\neg Ta_{k_{h'}}$ if and only if $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_{k_{h'}}$ are C'-equivalent; V incorporates I and K; V assigns $1/2$ to all atomic wffs that are not in $\{Ta_1, \ldots, Ta_{n+1}\}$ and receive no value in K; $Ta_{j_1} \leftrightarrow \mathbf{A}_{j_1}, \ldots, Ta_{j_l} \leftrightarrow \mathbf{A}_{j_l}$, $Ta_{n+1} \leftrightarrow \mathbf{A}_{n+1}$ get 1 in V; V follows (Con) with respect to C'; and if V assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{k_1} \cup \cdots \cup \mathcal{A}_{k_r}$ ($\varnothing$ if $r = 0$) is or is not C'-insertable between $\mathbf{C}$ and $\mathbf{D}$.

C_{n+1} will be the first class C' in $\mathcal{E}^c(\{Ta_1, \neg Ta_1, \ldots, Ta_{n+1}, \neg Ta_{n+1}\})$ such that, for some valuation V, C' and V have the properties set out in the preceding paragraph—where $\mathcal{E}(\{Ta_1, \neg Ta_1, \ldots, Ta_{n+1}, \neg Ta_{n+1}\})$ is the enumeration being displayed. The valuations which together with C_{n+1} have those properties are finitely many. For if we take any two such valuations, there will be at least one wff among $Ta_{j_1}, \ldots, Ta_{j_l}, Ta_{n+1}$ which receives different values in them. Let $\mathcal{V}$ be the set of those finitely many valuations. V_{n+1} will be the first valuation in $\mathcal{E}(\mathcal{V}, Ta_{j_1}, \ldots, Ta_{j_l}, Ta_{n+1})$.

All T-attributions among $Ta_1, \ldots, Ta_{n+1}$ that get $1/2$ in V_{n+1} are $Ta_{k_1}, \ldots, Ta_{k_r}$ (i.e., all T-attributions among $Ta_{j_1}, \ldots, Ta_{j_l}, Ta_{n+1}$ that get $1/2$ in V_{n+1}). The appropriate class C_{n+1} and the C_{n+1}-valuation V_{n+1} have been defined in such a way that C_{n+1} involves no T-attribution other than $Ta_1, \ldots, Ta_{n+1}$; if C_{n+1} connects Ta_{k_h} with $Ta_{k_{h'}}$ ($1 \leq h \leq r$, $1 \leq h' \leq r$), then $\mathbf{A}_{k_h}$ and $\mathbf{A}_{k_{h'}}$ are C_{n+1}-equivalent; and if C_{n+1} connects Ta_{k_h} with $\neg Ta_{k_{h'}}$, then $\mathbf{A}_{k_h}$ and $\neg\mathbf{A}_{k_{h'}}$ are C_{n+1}-equivalent.

It is clear that for every n, if all T-attributions among $Ta_1, \ldots, Ta_n$ that get $1/2$ in V_n are $Ta_{j_1}, \ldots, Ta_{j_l}$, then: V_n incorporates K; it assigns $1/2$ to all atomic wffs that are other than $Ta_1, \ldots, Ta_n$ and receive no value in K; it follows (Con) with respect to C_n; if V_n assigns $1/2$ to $\mathbf{C}$ and to $\mathbf{D}$, then it gives 1 or $1/2$ to $\mathbf{C} \to \mathbf{D}$ depending on whether $\mathcal{A}_{j_1} \cup \cdots \cup \mathcal{A}_{j_l}$ ($\varnothing$ if $l = 0$) is or is not C_n-insertable between $\mathbf{C}$ and $\mathbf{D}$; and for any h and h' where $h, h' \in \{1, \ldots, l\}$ and a_{j_h} is other than $a_{j_{h'}}$, C_n connects Ta_{j_h} with $Ta_{j_{h'}}$ if and only if $\mathbf{A}_{j_h}$ and $\mathbf{A}_{j_{h'}}$ are C_n-equivalent, and C_n connects Ta_{j_h} with $\neg Ta_{j_{h'}}$ if and only if $\mathbf{A}_{j_h}$ and $\neg\mathbf{A}_{j_{h'}}$ are C_n-equivalent. Of course C_{n+1} is an extension of C_n, and $Ta_n \leftrightarrow \mathbf{A}_n$ gets 1 in V_n.

We can now demonstrate that, for any wff $\mathbf{B}$ and any n, if $\mathbf{B}$ has 1 in V_n, then it also has 1 in V_{n+1}, and if $\mathbf{B}$ has 0 in V_n, then it also has 0 in V_{n+1}. The demonstration proceeds by induction on the number of occurrences of connectives in $\mathbf{B}$. In the base clause of the induction, we appeal to the fact that any atomic wff that gets 1 or 0 in V_n either is one of $Ta_{i_1}, \ldots, Ta_{i_m}$ or receives a value in K, and so it keeps its value in V_{n+1}. In the inductive clause, the interesting cases are two. (a) Say that $\mathbf{B}$ is $\mathbf{C} \wedge \mathbf{D}$ and gets 0 in V_n while $\mathbf{C}$ and $\mathbf{D}$ have $1/2$ or the one has $1/2$ and the other 1. Since V_n follows (Con) with respect to C_n, the deep conjuncts of $\mathbf{B}$ do not all have 1 in any C_n-valuation and do not all have 0 in any C_n-valuation. Thus $\mathbf{C}$ and $\mathbf{D}$ do not both have 1 in V_{n+1}. If one of them has 0 there, then of course $\mathbf{B}$ gets 0, too. But also if $\mathbf{C}$ and $\mathbf{D}$ have $1/2$ in V_{n+1}, or one has $1/2$ and the other 1, then $\mathbf{B}$ gets 0 there. For V_{n+1} follows (Con) with respect to C_{n+1}, and the deep conjuncts of $\mathbf{B}$ will not all have 1 in any C_{n+1}-valuation and will not all have 0 in any C_{n+1}-valuation. (b) Say that $\mathbf{B}$ is $\mathbf{C} \to \mathbf{D}$ and gets 1 in V_n while $\mathbf{C}$ and $\mathbf{D}$ have $1/2$. Then $\mathcal{A}_{j_1} \cup \cdots \cup \mathcal{A}_{j_l}$ is C_n-insertable between $\mathbf{C}$ and $\mathbf{D}$. But for each h ($1 \leq h \leq l$) all of $Ta_{j_h} \to \mathbf{A}_{j_h}$, $\mathbf{A}_{j_h} \to Ta_{j_h}$, $\neg\mathbf{A}_{j_h} \to \neg Ta_{j_h}$, and $\neg Ta_{j_h} \to \neg\mathbf{A}_{j_h}$ get 1 in V_{n+1}. Moreover, if $\mathbf{E}$ C_n-implies $\mathbf{E}'$, then $\mathbf{E} \to \mathbf{E}'$ gets 1 in V_{n+1}. For V_{n+1} is a C_n-valuation, so it cannot be that $|\mathbf{E}|_{V_{n+1}} > |\mathbf{E}'|_{V_{n+1}}$; and if V_{n+1} assigns $1/2$ to both $\mathbf{E}$ and $\mathbf{E}'$, then it will give 1 to $\mathbf{E} \to \mathbf{E}'$ because $\mathbf{E}$ C_{n+1}-implies $\mathbf{E}'$. Hence, by transitivity, $\mathbf{C} \to \mathbf{D}$ gets 1 in V_{n+1}.

Thus, for any wff $\mathbf{B}$ there are three possibilities: it gets $1/2$ in all of $V_1, V_2, \ldots$, or there is an n such that $\mathbf{B}$ has $1/2$ in all of $V_1, \ldots, V_{n-1}$ (if $n > 1$) and gets 1 in all of $V_n, V_{n+1}, \ldots$, or there is an n such that $\mathbf{B}$ has $1/2$ in all of $V_1, \ldots, V_{n-1}$ (if $n > 1$) and gets 0 in all of $V_n, V_{n+1}, \ldots$.

We now define the assignment V_ω of values to wffs. If an atomic wff has 1 in some one of $V_1, V_2, \ldots$, then it gets 1 in V_ω too. If it has 0 in some one of $V_1, V_2, \ldots$, then it gets 0 in V_ω. And if it has $1/2$ in all of $V_1, V_2, \ldots$, then it gets $1/2$ in V_ω. Values for compound wffs are calculated as follows. If V_ω assigns $1/2$ to $\mathbf{A}$ and to $\mathbf{B}$, or assigns $1/2$ to the one and 1 to the other, then it gives 0 or $1/2$ to $\mathbf{A} \wedge \mathbf{B}$ depending on whether it is or it is not (respectively) the case that $\mathbf{A} \wedge \mathbf{B}$ has more than one deep conjunct and, for some n, has 0 in V_n. If V_ω assigns $1/2$ to $\mathbf{A}$ and to $\mathbf{B}$, then it gives 1 or $1/2$ to $\mathbf{A} \rightarrow \mathbf{B}$ depending on whether there is or there is not an n such that $\mathbf{A} \rightarrow \mathbf{B}$ has 1 in V_n. In all other cases, V_ω just follows the tables for the connectives.

It can be shown that V_ω accords with the rules that accompany the tables for $\rightarrow$ and $\wedge$ in valuations, and so it is a valuation. In the case of $\rightarrow$ first, if $\mathbf{A}$ implies $\mathbf{B}$, then $\mathbf{A} \rightarrow \mathbf{B}$ gets 1 in V_n for every n, whether or not V_n assigns $1/2$ to $\mathbf{A}$ and $\mathbf{B}$. Moreover, if $\mathbf{A} \rightarrow \mathbf{B}$ has 1 in V_n for some n, and $\mathbf{B} \rightarrow \mathbf{C}$ has 1 in V_m for some m, then both $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{C}$ have 1 in V_l where $l = \max\{n, m\}$, so $\mathbf{A} \rightarrow \mathbf{C}$ has 1 in V_l.

In the case of $\wedge$ now, if the deep conjuncts of $\mathbf{A} \wedge \mathbf{B}$ do not all have 1 in any valuation$_*$ and do not all have 0 in any valuation$_*$, then $\mathbf{A} \wedge \mathbf{B}$ has more than one deep conjunct (as we know) and gets 0 in V_n for every n, since it cannot get either 1 or $1/2$. Finally, suppose that $\mathbf{C}$ and $\mathbf{D}$ are conjunctions such that every deep conjunct of $\mathbf{C}$ is a deep conjunct of $\mathbf{D}$, $\mathbf{C}$ has more than one deep conjunct, and $\mathbf{C}$ gets 0 in V_n for some n. Then, $\mathbf{D}$ of course has more than one deep conjunct. But it also gets 0 in V_n. This is obvious if a deep conjunct of $\mathbf{C}$ has 0 in V_n. If, on the other hand, no deep conjunct of $\mathbf{C}$ has 0 in V_n, then a conjunction $\mathbf{C}'$, possibly $\mathbf{C}$ itself, is a well-formed part of $\mathbf{C}$ such that every deep conjunct of $\mathbf{C}'$ is a deep conjunct of $\mathbf{C}$, and $\mathbf{C}'$ gets 0 in V_n not because it has a conjunct that gets 0 there, but because V_n follows (Con) with respect to C_n. Thus, the deep conjuncts of $\mathbf{C}'$ do not all have 1 in any C_n-valuation and do not all have 0 in any C_n-valuation. But then the deep conjuncts of $\mathbf{D}$ share that feature. So the conjuncts of $\mathbf{D}$ cannot both have 1 in V_n. If one of them gets 0 in V_n, then so does $\mathbf{D}$. But also if they both have $1/2$ in V_n, or the one has $1/2$ and the other 1, then once again $\mathbf{D}$ gets 0 in V_n because of (Con).

It should next be demonstrated that, for any wff $\mathbf{B}$, if it has 1 in some one of $V_1, V_2, \ldots$, it gets 1 in V_ω too; if it has 0 in some one of $V_1, V_2, \ldots$, it gets 0 in V_ω; and if it has $1/2$ in all of $V_1, V_2, \ldots$, it gets $1/2$ in V_ω. The demonstration is inductive, and I will consider only the nontrivial cases, which are the following five:

(i) $\mathbf{B}$ is $\mathbf{C} \vee \mathbf{D}$ and has $1/2$ in all of $V_1, V_2, \ldots$. Then, there are only two possibilities for $\mathbf{C}$ and $\mathbf{D}$: either both have $1/2$ in all of $V_1, V_2, \ldots$ or the one has $1/2$ in all of $V_1, V_2, \ldots$ while the other has 0 in some one of $V_1, V_2, \ldots$ (and in all subsequent ones of course). For if, say, $\mathbf{C}$ had 0 in V_n for some n, and $\mathbf{D}$ had 0 in V_m for some m, then both would have 0 in V_l where $l = \max\{n, m\}$, and so $\mathbf{C} \vee \mathbf{D}$ would also have 0 in V_l. In the first of the two possibilities, $\mathbf{C}$ and $\mathbf{D}$, by the inductive hypothesis, get $1/2$ in V_ω. In the second, again by the inductive hypothesis, one of $\mathbf{C}$ and $\mathbf{D}$ gets $1/2$ in V_ω while the other gets 0 there. At any rate, $\mathbf{C} \vee \mathbf{D}$ gets $1/2$ in V_ω.

(ii) $\mathbf{B}$ is $\mathbf{C} \wedge \mathbf{D}$ and has 0 in some one of $V_1, V_2, \ldots$, but in none of $V_1, V_2, \ldots$ does either $\mathbf{C}$ or $\mathbf{D}$ have 0. Then, there are two possibilities for $\mathbf{C}$ and $\mathbf{D}$: either both have $1/2$ in all of $V_1, V_2, \ldots$ or the one has $1/2$ in all of $V_1, V_2, \ldots$ and the other has 1 in some one of $V_1, V_2, \ldots$ (and in all subsequent ones). Thus, in V_ω, either both $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ or the one gets $1/2$ and the other 1. Moreover, $\mathbf{C} \wedge \mathbf{D}$ possesses more than one deep conjunct. For if it possessed only one, it could not have 0 in a valuation without $\mathbf{C}$ or $\mathbf{D}$ also having 0 there. So $\mathbf{C} \wedge \mathbf{D}$ gets 0 in V_ω.

(iii) $\mathbf{B}$ is $\mathbf{C} \wedge \mathbf{D}$ and has $1/2$ in all of $V_1, V_2, \ldots$. The case is similar to the preceding one, and we can see that $\mathbf{C} \wedge \mathbf{D}$ gets $1/2$ in V_ω.

(iv) $\mathbf{B}$ is $\mathbf{C} \to \mathbf{D}$ and has 1 in some one of $V_1, V_2, \dots$, but in none of $V_1, V_2, \dots$ does $\mathbf{C}$ have 0, and in none of $V_1, V_2, \dots$ does $\mathbf{D}$ have 1. Then there is only one possibility for $\mathbf{C}$ and $\mathbf{D}$: both have $1/2$ in all of $V_1, V_2, \dots$. For if $\mathbf{C}$ had 1 in some one of $V_1, V_2, \dots$ and in all subsequent ones, and $\mathbf{D}$ had $1/2$ in all of $V_1, V_2, \dots$, then, for some n, $\mathbf{C} \to \mathbf{D}$ would have $1/2$ in all of $V_n, V_{n+1}, V_{n+2}, \dots$, and so it would have 1 in some one of $V_1, V_2, \dots$ and $1/2$ in some subsequent ones. If $\mathbf{C}$ had 1 in some one of $V_1, V_2, \dots$ and in all subsequent ones, and $\mathbf{D}$ had 0 in some one of $V_1, V_2, \dots$ and in all subsequent ones, then $\mathbf{C} \to \mathbf{D}$ would have 1 in some one of $V_1, V_2, \dots$ and 0 in some subsequent ones. And if $\mathbf{C}$ had $1/2$ in all of $V_1, V_2, \dots$, and $\mathbf{D}$ had 0 in some one of $V_1, V_2, \dots$ and in all subsequent ones, then $\mathbf{C} \to \mathbf{D}$ would have 1 in some one of $V_1, V_2, \dots$ and $1/2$ in some subsequent ones. Thus, by the inductive hypothesis, $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V_ω. So $\mathbf{C} \to \mathbf{D}$ gets 1 there.

(v) $\mathbf{B}$ is $\mathbf{C} \to \mathbf{D}$ and has $1/2$ in all of $V_1, V_2, \dots$. Then there are three possibilities for $\mathbf{C}$ and $\mathbf{D}$: $\mathbf{C}$ has 1 in some one of $V_1, V_2, \dots$ and $\mathbf{D}$ has $1/2$ in all of $V_1, V_2, \dots$, or $\mathbf{C}$ has $1/2$ in all of $V_1, V_2, \dots$ and $\mathbf{D}$ has 0 in some one of $V_1, V_2, \dots$, or both $\mathbf{C}$ and $\mathbf{D}$ have $1/2$ in all of $V_1, V_2, \dots$. By the inductive hypothesis, in the first possibility $\mathbf{C}$ and $\mathbf{D}$ get 1 and $1/2$, respectively, in V_ω, and in the second possibility they get $1/2$ and 0, respectively, in that valuation. In both cases, $\mathbf{C} \to \mathbf{D}$ gets $1/2$ in V_ω. In the third possibility, both $\mathbf{C}$ and $\mathbf{D}$ get $1/2$ in V_ω, so $\mathbf{C} \to \mathbf{D}$ also gets $1/2$ there.

Thus V_ω gives 1 to all the biconditionals $Ta_1 \leftrightarrow \mathbf{A}_1, Ta_2 \leftrightarrow \mathbf{A}_2, \dots$ and incorporates K. It remains to show that it has the property (Prop).

Say there are sequences $\mathbf{E}_1, \dots, \mathbf{E}_m$ and $\mathbf{E}'_1, \dots, \mathbf{E}'_m$ of wffs as described in (Prop). Since, for every k, Ta_k has the same value in V_ω as $\mathbf{A}_k$, V_ω gives the same value to all of $\mathbf{E}_1, \dots, \mathbf{E}_m$. Likewise, it gives the same value to all of $\mathbf{E}'_1, \dots, \mathbf{E}'_m$. If $\mathbf{E}_1$ has 0 in V_ω, then of course $\mathbf{C} \wedge \mathbf{D}$ gets 0 there. If $\mathbf{E}_1$ has 1 in V_ω, then $\mathbf{E}_m$ gets 1 there, so $\mathbf{E}'_m$ and $\mathbf{E}'_1$ have 0, and thus again $\mathbf{C} \wedge \mathbf{D}$ gets 0 in V_ω.

Now suppose that $\mathbf{E}_1, \dots, \mathbf{E}_m$ and $\mathbf{E}'_1, \dots, \mathbf{E}'_m$ all have $1/2$ in V_ω, and consider a valuation V_n such that $j_1, \dots, j_{m-1}, j'_1, \dots, j'_{m-1} \leq n$, the numbers j_1, and so on, being as described in (Prop). Clearly, $\mathbf{E}_1, \dots, \mathbf{E}_m, \mathbf{E}'_1, \dots, \mathbf{E}'_m$ have $1/2$ in V_n, and so $Ta_{j_1}, \dots, Ta_{j_{m-1}}, Ta_{j'_1}, \dots, Ta_{j'_{m-1}}$ have $1/2$ there. Assuming that $\mathbf{E}_{i+1}$ and $\mathbf{E}'_{i+1}$ get opposite values in each C_n-valuation, we can see that $\mathbf{E}_i$ and $\mathbf{E}'_i$ also get opposite values in each C_n-valuation. For, as we know,

$$\mathbf{E}_i = \overbrace{\neg \cdots \neg}^{l \text{ times}} Ta_{j_i}, \qquad \mathbf{E}_{i+1} = \overbrace{\neg \cdots \neg}^{l \text{ times}} \mathbf{A}_{j_i},$$

$$\mathbf{E}'_i = \overbrace{\neg \cdots \neg}^{r \text{ times}} Ta_{j'_i}, \qquad \mathbf{E}'_{i+1} = \overbrace{\neg \cdots \neg}^{r \text{ times}} \mathbf{A}_{j'_i}$$

for some l and r.

If l and r are both even or both odd, then $\mathbf{A}_{j_i}$ and $\mathbf{A}_{j'_i}$ have opposite values in every C_n-valuation. In other words, $\mathbf{A}_{j_i}$ and $\neg\mathbf{A}_{j'_i}$ are C_n-equivalent. Moreover, a_{j_i} is other than $a_{j'_i}$, for if $\mathbf{A}_{j_i}$ is $\mathbf{A}_{j'_i}$, then $\mathbf{A}_{j_i}$ will get $1/2$ in all C_n-valuations, which is impossible. Hence, C_n connects Ta_{j_i} with $\neg Ta_{j'_i}$. Thus $\mathbf{E}_i$ and $\mathbf{E}'_i$ get opposite values in every C_n-valuation.

On the other hand, if one of l and r is even and the other is odd, then, in each C_n-valuation, $\mathbf{A}_{j_i}$ and $\mathbf{A}_{j'_i}$ have the same value. In other words, $\mathbf{A}_{j_i}$ and $\mathbf{A}_{j'_i}$ are

C_n-equivalent. So if a_{j_i} is other than $a_{j_i'}$, then C_n connects Ta_{j_i} with $Ta_{j_i'}$. So, in each C_n-valuation, Ta_{j_i} and $Ta_{j_i'}$ get the same value, as is of course also the case if a_{j_i} is $a_{j_i'}$. Thus, once more, $\mathbf{E}_i$ and $\mathbf{E}_i'$ get opposite values in every C_n-valuation.

Therefore, since $\mathbf{E}_m$ and $\mathbf{E}_m'$ get opposite values in every C_n-valuation, so do $\mathbf{E}_1$ and $\mathbf{E}_1'$. Thus $\mathbf{C} \wedge \mathbf{D}$ gets 0 in V_n. For not all its deep conjuncts have 1 in V_n: $\mathbf{E}_1$ and $\mathbf{E}_1'$ do not. And if $\mathbf{C}$ and $\mathbf{D}$ have $1/2$ in V_n, or the one has $1/2$ and the other 1, then $\mathbf{C} \wedge \mathbf{D}$ gets 0 there because V_n follows (Con) with respect to C_n. Moreover, $\mathbf{C} \wedge \mathbf{D}$ must have more than one deep conjunct. For if it had only one, $\mathbf{E}_1$ would be identical with $\mathbf{E}_1'$ and so get $1/2$ in every C_n-valuation.

Thus $\mathbf{C} \wedge \mathbf{D}$ gets 0 in V_ω, too, if $\mathbf{C}$ and $\mathbf{D}$ have $1/2$ there or the one has $1/2$ and the other 1. Clearly, $\mathbf{C} \wedge \mathbf{D}$ also gets 0 in V_ω if one of $\mathbf{C}$ and $\mathbf{D}$ has 0 there. And it cannot be that both $\mathbf{C}$ and $\mathbf{D}$ have 1 in V_ω, since some deep conjuncts of $\mathbf{C} \wedge \mathbf{D}$ (namely, $\mathbf{E}_1$ and $\mathbf{E}_1'$) do not have 1 there. □

4.7 Finally, taking up a few issues that we left over in Section 3, we should demonstrate that some conjunctions of the form $Ta_k \wedge Ta_h$ where, for some $\mathbf{B}$ and $\mathbf{C}$, $\mathbf{A}_k$ is $\mathbf{B} \vee \mathbf{C}$ while $\mathbf{A}_h$ is $\neg\mathbf{B} \wedge \neg\mathbf{C}$ get a value other than 0 in a valuation in $\mathcal{S}$. The proof of the Central Theorem shows how we can define a valuation, V_ω, which assigns $1/2$ to p_1 and p_2 and meets the conditions for membership in $\mathcal{S}$. Let $\mathbf{A}_k$ and $\mathbf{A}_h$ be $p_1 \vee p_2$ and $\neg p_1 \wedge \neg p_2$, respectively. Then, p_1 and p_2 have $1/2$ in all of $V_1, V_2, \ldots,$ and $p_1 \vee p_2$ gets $1/2$ in V_ω. As for $\neg p_1 \wedge \neg p_2$, it has $1/2$ in V_n for every n. For V_n follows (Con) with respect to C_n, and it is easy to construct a C_n-valuation in which $\neg p_1$ and $\neg p_2$ both have 1 (or have any combination of values, for that matter). Thus $\neg p_1 \wedge \neg p_2$ gets $1/2$ in V_ω too. Hence Ta_k and Ta_h get $1/2$ in V_ω, and so they have $1/2$ in all of $V_1, V_2, \ldots$. Consider any n such that $k \leq n$ and $h \leq n$. If $Ta_k \wedge Ta_h$ has 0 and not $1/2$ in V_n, then Ta_k and Ta_h will not both have 1 in any C_n-valuation. This cannot be so if C_n does not combine Ta_k and Ta_h or if it connects Ta_k with Ta_h. Thus C_n will connect Ta_k with $\neg Ta_h$. But then $\mathbf{A}_k$ and $\neg\mathbf{A}_h$ will be C_n-equivalent, so $p_1 \vee p_2$ and $\neg p_1 \wedge \neg p_2$ will have opposite values in all C_n-valuations. In fact, however, it is easy to construct a C_n-valuation in which $p_1 \vee p_2$ gets $1/2$, but $\neg p_1 \wedge \neg p_2$ gets 0. (Consider a C_n-valuation that assigns $1/2$ to p_1 and p_2, but gives 0 to every conjunction $\mathbf{C} \wedge \mathbf{D}$ in case (i) of Table 3 unless $\mathbf{C} \wedge \mathbf{D}$ possesses only one deep conjunct.) Thus, $Ta_k \wedge Ta_h$ has $1/2$ in V_n for every n such that $k \leq n$ and $h \leq n$. So $Ta_k \wedge Ta_h$ gets $1/2$ in V_ω too.

We should also see that $p_1 \wedge \neg Ta_m$, where $\mathbf{A}_m$ is p_1, gets a value other than 0 in a valuation in $\mathcal{S}$, so that $\neg[p_1 \wedge \neg Ta_m]$ does not belong to the theory of truth. Indeed, $p_1 \wedge \neg Ta_m$ gets $1/2$ in a valuation V_ω that assigns $1/2$ to p_1 and is defined as described in the proof of the Central Theorem. For if p_1 gets $1/2$ in V_ω, then Ta_m also gets $1/2$ there, and so p_1 and Ta_m have $1/2$ in all of $V_1, V_2, \ldots$. Thus, $p_1 \wedge \neg Ta_m$ has $1/2$ in V_n for every n, since there will be a C_n-valuation in which p_1 and $\neg Ta_m$ both have 1 (or have any combination of values, for that matter). We can similarly see that $Ta_m \wedge \neg p_1$ gets $1/2$ in V_ω.

5 Concluding Remarks

I will end the article by mentioning some of the ways in which I have extended the work presented here. For one thing, the language has been enriched with a primitive determinacy operator, Δ. The meaning of $\Delta\mathbf{A}$ is “It is determinate whether $\mathbf{A}$.” The idea is that it may not be a determinate matter whether or not the sentence (L) is true. Reality itself may not have an answer to the question, so to speak. If, however, we accept that it is not determinate whether (L) is true, we commit ourselves to the view that it is determinate whether (it is determinate whether (L) is true). I would say that either it is not a determinate matter whether (L) is true or it is not a determinate matter whether (it is a determinate matter whether (L) is true) or indeterminacy may lie even deeper. The theory of truth in the enriched language allows us to prove an analogue of the Central Theorem and, in fact, has both models (assignments of the values 1, $1/2$, and 0) which assign 1 to the claim "It is not determinate whether the liar sentence is true" and models which do not assign it 1 (but $1/2$).

More importantly, the work has been extended to a first-order setting. The language now includes the universal and the existential quantifier. In each model, instead of assigning values to wffs, we assign values to pairs $\langle s, \mathbf{A}\rangle$ where s is a sequence of objects from the domain. The language also possesses function symbols or other means that enable us to describe wffs syntactically. There result a nonclassical first-order logic and some theories of truth embedded in the framework of that logic. Each theory allows us to prove an analogue of the Central Theorem and incorporates various generalizations about truth. The proofs are, however, more complicated than those in the present article. If the language possesses a function symbol for concatenation, then we can define a falsity predicate F such that to call a wff F is to say that the concatenation of $\neg$ and that wff is true. The theories include generalizations to the effect that no sentence is both true and false and no sentence is both not true and not false.

Notes

1. Zardini [12], who validates both the law of noncontradiction and the substitution of equivalents, does not accept that every sentence $\mathbf{A}$ entails $\mathbf{A} \wedge \mathbf{A}$. The main difference between his logic and that presented here is that the former is substructural in that it rejects the principle that if $\mathbf{A}$ considered twice entails $\mathbf{B}$, then $\mathbf{A}$ considered once entails $\mathbf{B}$.

2. The logic to be presented validates the inferences of the form "$\mathbf{S} \vee \mathbf{S}', \neg\mathbf{S}$; hence $\mathbf{S}'$." This is consistent with claiming that we should not freely make such inferences when we have assumed that, contrary to what the logic teaches, a contradictory state of affairs obtains.

3. The point goes back at least to Fine [5, pp. 142–43, 146–50].

4. Outermost brackets are omitted. The rule of the association to the left applies; for example, $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$ abbreviates $[[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}]$. The brackets that enclose a disjunction or conjunction are omitted when the disjunction or conjunction is an argument of an occurrence of $\rightarrow$ or $\leftrightarrow$; so $\mathbf{A} \rightarrow \mathbf{B} \vee \mathbf{C}$ abbreviates $[\mathbf{A} \rightarrow [\mathbf{B} \vee \mathbf{C}]]$. No other brackets are omitted.

5. As far as I know, in Avron's work on propositional logic for nondeterministic semantics, there are no rules that are additional to the truth tables and constrain the valuations, such as the rules that will be employed here. A rule of that kind appears in his work on first-order logic, ensuring the substitutivity of identicals (see [2, pp. 285–87]).

6. On the other hand, taking a doxastic or epistemic attitude toward each one of two statements is not always the same as taking it toward their conjunction. One can attach a high probability to $\mathbf{S}$ and to $\mathbf{S}'$ without attaching a high probability to $\mathbf{S} \wedge \mathbf{S}'$. But in the case of such attitudes, too, it is difficult to treat $\mathbf{S}$ and $\mathbf{S} \wedge \mathbf{S}$ differently.

7. For example, Aristotle calls it, or a variant of it, "the most certain of all principles" [1, p. 66].

8. For the concept of an inference whose premises are other inferences, see, for example, Prawitz [8, p. 228], [9, pp. 69–70].

9. So in the theory, as opposed to the logic, the letter T and the individual constants are not schematic letters.

10. Here, and up to the end of the proof of Theorem 3, the subscripted letters $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_n$, and so on, are just variables ranging over all wffs. The letter $\mathbf{A}_1$, for example, does not specifically denote the wff $\mathcal{R}(a_1)$.

11. Appropriate classes enter as follows: When we move to a version of Theorem 2 that involves many T-attributions, a new disjunct appears in the theorem to the effect that the wffs corresponding to two of those T-attributions are equivalent to each other. This adds a case to the induction proving Theorem 3. In order to deal with that case, we need to ensure that the T-attributions get the same value, and we do that by means of an appropriate class. We also use those classes in order to show that the valuation we construct in the proof of the Central Theorem has (Prop).

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