Notre Dame Journal of Formal Logic
Volume 59, Number 3, 2018

A Partition Theorem of ω^{ω^α}

Claribet Piña

Abstract We consider finite partitions of the closure $\overline{\mathcal{F}}$ of an ω^α-uniform barrier $\mathcal{F}$. For each partition, we get a homogeneous set having both the same combinatorial and topological structure as $\overline{\mathcal{F}}$, seen as a subspace of the Cantor space $2^{\mathbb{N}}$.

1 Introduction

Given two topological spaces X and Y and an integer $l > 1$, we denote by $X \to (\text{top}\, Y)^1_l$ the fact that, given any map $f : X \longrightarrow l$, there is H a subspace of X which is homeomorphic to Y and f-monochromatic. A very well-known result on partitions of topological spaces, due to Baumgartner [2] and Weiss [5], establishes that a countable ordinal space γ—that is, γ endowed with its order topology—satisfies that $\forall l > 1, \gamma \to (\text{top}\, \gamma)^1_l$ if and only if $\gamma = \omega^{\omega^\alpha}$ for some $\alpha < \omega_1$. Translating this statement to the realm of partitions of families in FIN, we get that given $l > 1$ and $0 < \alpha < \omega_1$, if $\mathcal{F} \subseteq \text{FIN}$ is a family having the same topological type as ω^{ω^α}, and $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{l-1}$ is a partition of $\mathcal{F}$, then there exists a subfamily $\mathcal{H} \subseteq \mathcal{F}$ with the same topological type as $\mathcal{F}$ such that $\mathcal{H} \subseteq \mathcal{F}_i$ for some $i < l$. We wondered if it is possible to find a homogeneous set $\mathcal{H}$ which also preserves the combinatorial structure of $\mathcal{F}$.

It is known that if $\mathcal{F}$ is an ω^α-uniform family endowed with the lexicographic order topology $\tau_{<_{\text{lex}}}$, then $(\mathcal{F}, \tau_{<_{\text{lex}}})$ and $(\mathcal{F} \restriction M, \tau_{<_{\text{lex}}})$ are both homeomorphic to ω^{ω^α}, for every $M \in \mathbb{N}^{[\infty]}$, where $\mathcal{F} \restriction M = \{s \in \mathcal{F} : s \subseteq M\}$. Then, at first glance, the Ramsey property for uniform families (see Theorem 2.1) seems to provide a homogeneous set $\mathcal{H} = \mathcal{F} \restriction M$ having the same combinatorial behavior as $\mathcal{F}$ (i.e., $\mathcal{F} \restriction M$ is as well an ω^α-uniform family, on M). However, this is not the case, since for our partition problem the homogeneous set $\mathcal{H}$ must be considered with the subspace topology inherited from $(\mathcal{F}, \tau_{<_{\text{lex}}})$. Piña and Uzcátegui [3] showed

Received April 2, 2015; accepted March 4, 2016
First published online June 26, 2018
2010 Mathematics Subject Classification: Primary 03E02; Secondary 05D10
Keywords: partition of topological spaces, uniform barriers, U-trees
 10.1215/00294527-2018-0001

that the topological type of a restriction $\mathcal{F} \upharpoonright M$ may change so radically, according to M, that it could be a discrete subspace of $(\mathcal{F}, \tau_{<_{\text{lex}}})$. Furthermore, they showed that given any uniform family $\mathcal{F}$, there is a partition of $\mathcal{F}$ such that $\mathcal{F} \upharpoonright M$ is a discrete subspace of $(\mathcal{F}, \tau_{<_{\text{lex}}})$ for every $M \in \mathbb{N}^{[<\infty]}$ given by the Ramsey property of $\mathcal{F}$ applied to such a partition. Therefore, in this sense $\mathcal{F} \upharpoonright M$ fails to be a homogeneous set.

A more subtle way of dealing with this problem is considering a different representation of the ordinal space ω^{ω^α} by using the closure $\overline{\mathcal{F}}$ of an ω^α-uniform family $\mathcal{F}$ endowed with the subspace topology inherited from the Cantor space $2^{\mathbb{N}}$. In this article, we study the combinatorial characteristics of a homogeneous family for a partition relation $\forall l > 1, \overline{\mathcal{F}} \to (\text{top}\,\overline{\mathcal{F}})^1_l$, where $\mathcal{F}$ is an ω^α-uniform barrier. In this sense, we cannot get a homogeneous set with exactly the same combinatorial behavior as $\overline{\mathcal{F}}$. Say, for example, we cannot get as a homogeneous set a family $\overline{\mathcal{F} \upharpoonright M}$, or the closure $\overline{\mathcal{B}}$ of another ω^α-uniform family $\mathcal{B}$, since a coloring defined on $\overline{\mathcal{F}}$ using the sizes of the elements in $\overline{\mathcal{F}}$ clearly forbids any of these families to be monochromatic. However, in Theorem 3.4, we will obtain a homogeneous set which is nothing else than the image, under an $\sqsubseteq$-order-preserving embedding, of the closure of another ω^α-uniform barrier. As a consequence of Theorem 3.4, we clearly obtain an alternative proof for the Baumgartner–Weiss result (Corollary 3.5).

In most of our proofs, we will assume the existence of a nonprincipal selective ultrafilter. It is well known that the existence of selective ultrafilters requires additional set-theoretic assumptions. However, it has been proved that they exist under the assumption of hypothesis as the continuum hypothesis (CH) or as Martin's axiom (MA). Therefore, by absoluteness, all the results presented here are in fact ZFC-results.

The article is organized as follows. In Section 2 we present some basic facts, notation, and well-known notions from Ramsey theory. In that section, we also prove some technical lemmas that we will use in our induction proofs. In Section 3 we prove our main result (Theorem 3.4) making use of Theorem 3.1, and we give an alternative proof for the Baumgartner–Weiss theorem (Corollary 3.5). Finally, Section 4 is devoted to giving an inductive proof of our more technical result (Theorem 3.1).

2 Preliminaries

We will start this section by introducing some notation, definitions, and well-known facts that we will use throughout the article. We will only include in this section the proofs of results that, as far as we know, are new. (For all the facts stated without a proof, see Argyros and Todorcevic [1], Baumgartner [2], and Todorcevic [4].)

2.1 Basic notions We denote by FIN the set of all *nonempty* finite subsets of $\mathbb{N}$. If $M \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we denote by $M^{[k]}$ the set of subsets of M with size k, by $M^{[\leq k]}$ the set of subsets of M with size less than or equal to k, and by $M^{[<\infty]}$ the set of all finite subsets of M. If, moreover, $M \subseteq \mathbb{N}$ is infinite, then we denote by $M^{[\infty]}$ the set of all infinite subsets of M. Given $s \in$ FIN and $M \subseteq \mathbb{N}$, we denote by M/s the set $\{x \in M : \max(s) < x\}$. If $s = \{n\}$, we write instead M/n. We put $M > s$ if $\min(M) > \max(s)$. We write $s \sqsubset M$ if there is $m \in M$ such that $\{0, 1, \ldots, m\} \cap M = s$, we write $s \sqsubseteq M$ if $s \sqsubset M$ or $s = M$, and we denote by

$s \not\sqsubseteq M$ the negation of $s \sqsubseteq M$. We say that a mapping φ, with domain $\mathcal{F} \subseteq \mathrm{FIN}$ and values in FIN, is $\sqsubseteq$-*order-preserving* if $s \sqsubseteq t$ implies $\varphi(s) \sqsubseteq \varphi(t)$ whenever $s, t \in \mathcal{F}$. We say that a family $\mathcal{F} \subseteq \mathrm{FIN}$ is *thin* if for every different $s, t \in \mathcal{F}$ we have $s \not\sqsubseteq t$.

Given a family $\mathcal{F} \subseteq \mathrm{FIN}$ and $M \in \mathbb{N}^{[\infty]}$, we say that $\mathcal{F}$ is a *front on* M if it is thin, and for every $N \in M^{[\infty]}$ there is $t \in \mathcal{F}$ such that $t \sqsubset N$. If, moreover, we have that $s \not\subseteq t$ for every $s, t \in \mathcal{F}$, we say that $\mathcal{F}$ is a *barrier on* M. Clearly, every barrier on M is a front on M. When $M = \mathbb{N}$, we simply say that $\mathcal{F}$ is a front (resp., a barrier). The *Schreier barrier*, which we will often use, is defined by

$$\mathcal{S} = \{s \in \mathrm{FIN} : |s| = \min(s) + 1\}.$$

If $\mathcal{F} \subseteq \mathrm{FIN}$ and $M \in \mathbb{N}^{[\infty]}$, the sets $\mathcal{F} \restriction M$ and $\mathcal{F}[M]$ are defined as follows:

$$\mathcal{F} \restriction M = \{t \in \mathcal{F} : t \subseteq M\}, \qquad \mathcal{F}[M] = \{t \cap M : t \in \mathcal{F}\}.$$

If $\mathcal{F}$ is a front (resp., a barrier) on M, and $N \in M^{[\infty]}$, then $\mathcal{F} \restriction N$ is a front (resp., a barrier) on N. We say that a family $\mathcal{F} \subseteq \mathrm{FIN}$ is *Ramsey* (or that it has the *Ramsey property*) if for every finite partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ and every $M \in \mathbb{N}^{[\infty]}$, there is $N \in M^{[\infty]}$ such that at most one of the restrictions $\mathcal{F}_0 \restriction N$, $\mathcal{F}_1 \restriction N, \dots,$ $\mathcal{F}_k \restriction N$ is nonempty. The following classic result of Ramsey theory ensures the existence of Ramsey families.

Theorem 2.1 (Nash-Williams) *Every thin family is Ramsey.*

We consider $\mathcal{F} \subseteq \mathrm{FIN}$ as a topological subspace of $\wp(\mathbb{N})$ by endowing this last one with the topology of pointwise convergence from the Cantor space $2^{\mathbb{N}}$, that is, identifying sets with their characteristic functions. We say that a sequence $(s_n)_n$ in FIN is a *block sequence* if $s_n < s_m$ for every $n < m$. We denote by $\overline{\mathcal{F}}$ the topological closure of $\mathcal{F}$. Moreover, we say that $\mathcal{F}$ is *precompact* if $\overline{\mathcal{F}} \subseteq \mathrm{FIN}$. We will also consider the downward closures of $\mathcal{F} \subseteq \mathrm{FIN}$ defined by

$$\overline{\mathcal{F}}^{\sqsubseteq} = \{s : s \sqsubseteq t \text{ for some } t \in \mathcal{F}\}, \qquad \overline{\mathcal{F}}^{\subseteq} = \{s : s \subseteq t \text{ for some } t \in \mathcal{F}\}.$$

It is a fact that if $\mathcal{F}$ is a front on M, then $\overline{\mathcal{F}}^{\sqsubseteq} = \overline{\mathcal{F}}$, and that if $\mathcal{F}$ is a barrier on M, then $\overline{\mathcal{F}}^{\subseteq} = \overline{\mathcal{F}}^{\sqsubseteq} = \overline{\mathcal{F}}$. Moreover, a family $\mathcal{F} \subseteq \mathrm{FIN}$ is precompact if and only if $\overline{\mathcal{F}}^{\sqsubseteq}$ is compact.

Recall that if X is a topological space and $\alpha < \omega_1$, then the *αth Cantor–Bendixson derivative* of X is defined recursively as follows:

$$\begin{aligned} X' &= \{x \in X : x \text{ is not isolated in } X\}, \\ X^{(\beta+1)} &= (X^{(\beta)})', \\ X^{(\lambda)} &= \bigcap_{\beta<\lambda} X^{(\beta)}, \quad \text{if } \lambda > 0 \text{ is a limit ordinal.} \end{aligned}$$

It is known that if $\mathcal{A} \subseteq \mathrm{FIN}$ is compact, then there is $\alpha < \omega_1$ such that $\mathcal{A}^{(\alpha)} = \mathcal{A}^{(\alpha+1)} = \emptyset$. The minimal ordinal $\alpha < \omega_1$ with this property is called the *Cantor–Bendixson rank* of $\mathcal{A}$, and we denote it by $\mathrm{rk}_{\mathrm{CB}}(\mathcal{A})$.

We recall that the *lexicographic order* $<_{\mathrm{lex}}$ is defined on FIN by $s <_{\mathrm{lex}} t$ if and only if $\min(s \triangle t) \in s$. An important fact about the lexicographic order is that every precompact family $\mathcal{F} \subseteq \mathrm{FIN}$ is lexicographically well-ordered. With this in mind, we adopt the concept of an order-homeomorphism. We say that $\mathcal{A} \subseteq \mathrm{FIN}$ is *order-homeomorphic* to a countable ordinal space γ—that is, γ endowed with its order topology—if there is an order isomorphism between $(\mathcal{A}, <_{\mathrm{lex}})$ and γ which is also a topological homeomorphism.

Proposition 2.2 *If $\mathcal{F}$ is an (infinite) precompact family on $M \in \mathbb{N}^{[\infty]}$ with $\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{F}}^{\sqsubseteq}) > 0$, then $(\overline{\mathcal{F}}^{\sqsubseteq}, \sqsubseteq)$ is order-homeomorphic to $\omega^{\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{F}}^{\sqsubseteq})} + 1$. Therefore, $\overline{\mathcal{F}}^{\sqsubseteq} \setminus \{\emptyset\}$ is homeomorphic to $\omega^{\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{F}}^{\sqsubseteq})}$.*

2.2 α-uniform families We now introduce the notion of uniform families, which will give us a way of representing countable ordinal spaces as families on FIN.

Given $\mathcal{F} \subseteq \mathrm{FIN}$, $M \in \mathbb{N}^{[\infty]}$, and $\alpha < \omega_1$, the concept of an α-uniform family on M is defined recursively as follows. *$\mathcal{F}$ is α-uniform on M* if one of the following conditions holds:

1. $\alpha = 0$ and $\mathcal{F} = \{\emptyset\}$,
2. $\alpha = \beta + 1$ and $\mathcal{F}_{\{n\}} := \{s \in \mathrm{FIN} : n < s \text{ and } \{n\} \cup s \in \mathcal{F}\}$ is β-uniform on M/n for every $n \in M$,
3. α is a limit ordinal and there is a strictly increasing sequence of ordinals $(\alpha_n)_n$ converging to α such that $\mathcal{F}_{\{n\}}$ is α_n-uniform on M/n for every $n \in M$.

We say that a family is *uniform on some $M \in \mathbb{N}^{[\infty]}$* if it is α-uniform on M for some $\alpha < \omega_1$. If a family $\mathcal{F}$ is uniform on $\mathbb{N}$, we simply say that $\mathcal{F}$ is *uniform*. It follows by induction on k that given $k > 0$, a family $\mathcal{F} \subseteq \mathrm{FIN}$ is k-uniform on $M \in \mathbb{N}^{[\infty]}$ if and only if $\mathcal{F} = M^{[k]}$. Therefore, $\mathcal{F}$ is ω-uniform on M if and only if there exists a strictly increasing sequence of positive integers $(m_n)_{n \in M}$ such that $\mathcal{F}_{\{n\}} = M^{[m_n]}$ for every $n \in M$. Particularly, the Schreier barrier $\mathcal{S}$ is an ω-uniform family. On the other hand, it follows from the definition that for every $\alpha < \omega_1$, there exists an α-uniform family.

Proposition 2.3 *Given $\alpha < \omega_1$ and $\mathcal{F}$ an α-uniform family on $M \in \mathbb{N}^{[\infty]}$, we have the following.*

1. *$\mathcal{F}$ is a front on M. Therefore, $\mathcal{F}$ is Ramsey.*
2. *$\mathcal{F} \upharpoonright N$ is an α-uniform family on N for every $N \in M^{[\infty]}$.*
3. *The order type of $(\mathcal{F}, <_{\mathrm{lex}})$ is ω^α. Moreover, $\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{F}}) = \alpha$.*

Theorem 2.4 *Given $\mathcal{F} \subseteq \mathrm{FIN}$ a nonempty precompact family on $M \subseteq \mathbb{N}$ with Cantor–Bendixson rank $\alpha < \omega_1$, there is $N \in M^{[\infty]}$ such that $\mathcal{F}[N]$ is the closure of an α-uniform barrier on N.*

Theorem 2.5 *Given $\mathcal{F} \subseteq \mathrm{FIN}$ a barrier on $M \in \mathbb{N}^{[\infty]}$ with $\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{F}}) = \alpha$, there is an infinite set $N \subseteq M$ such that $\mathcal{F} \upharpoonright N$ is an α-uniform barrier on N.*

Lemma 2.6 *Given $\alpha < \beta < \omega_1$, $M \in \mathbb{N}^{[\infty]}$, $\mathcal{A}$ an α-uniform family on M, and $\mathcal{B}$ a β-uniform family on M, there is an infinite set $N \subseteq M$ such that $\mathcal{A} \upharpoonright N \subseteq \overline{\mathcal{B}}$.*

We recall that an *ultrafilter on* $\mathbb{N}$ is a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ with the following properties:

1. $\emptyset \notin \mathcal{U}$ and $\mathbb{N} \in \mathcal{U}$,
2. $M \subseteq N$ and $M \in \mathcal{U}$ imply $N \in \mathcal{U}$,

3. $N = N_0 \cup N_1$ and $N \in \mathcal{U}$ imply $N_0 \in \mathcal{U}$ or $N_1 \in \mathcal{U}$, and
4. $M \in \mathcal{U}$ and $N \in \mathcal{U}$ imply $M \cap N \in \mathcal{U}$.

We say that an ultrafilter $\mathcal{U}$ is *nonprincipal* if $\{n\} \notin \mathcal{U}$ for all $n \in \mathbb{N}$. We call an ultrafilter $\mathcal{U}$ *selective* if given a barrier $\mathcal{F}$ and any finite coloring of it, there is $M \in \mathcal{U}$ such that $\mathcal{F} \upharpoonright M$ is monochromatic; or, equivalently, $\mathcal{U}$ is selective if for every sequence $(A_t)_{t\in\mathrm{FIN}}$ of elements of $\mathcal{U}$, there is $A \in \mathcal{U}$ such that $A/t \subseteq A_t$ for every $t \in A^{[<\infty]}$. Given a tree $T \subseteq \mathbb{N}^{[<\infty]}$ and $\overrightarrow{\mathcal{U}} = \{\mathcal{U}_s : s \in \mathbb{N}^{[<\infty]}\}$ a family of nonprincipal ultrafilters on $\mathbb{N}$, we call T a $\overrightarrow{\mathcal{U}}$*-tree* if $\{n \in \mathbb{N} : t \cup \{n\} \in T\} \in \mathcal{U}_t$ for every $t \in T$. If the sequence $\overrightarrow{\mathcal{U}}$ is constant, that is, if $\mathcal{U}_s = \mathcal{U}$ for every $s \in \mathbb{N}^{[<\infty]}$, then we will suppress the arrow. For our proofs, we will always consider $\mathcal{U}$-trees with stem $\mathrm{st}(T) = \emptyset$; that is, $\mathrm{st}(T)$ is the maximal node comparable with every other node of T. Also, we adopt the convention that, from now on, whenever we consider a selective ultrafilter we mean a nonprincipal selective ultrafilter.

If T is a $\mathcal{U}$-tree and $\mathcal{B}$ is a family on FIN, then we will denote by $T \upharpoonright \mathcal{B}$ the set of elements of T up to $\mathcal{B}$. That is, the terminal nodes of $T \upharpoonright \mathcal{B}$ are in $\mathcal{B}$.

Lemma 2.7 *Let $\mathcal{U}$ be a selective ultrafilter, and let $0 < \alpha < \omega_1$. If T is a $\mathcal{U}$-tree and $\mathcal{B}$ is an α-uniform barrier on $M \in \mathcal{U}$, then there is a set $N \subseteq M$ in $\mathcal{U}$ such that $T \cap (\mathcal{B} \upharpoonright N)$ is an α-uniform barrier on N.*

Proof (By induction on α) Fix $\mathcal{U}$ a selective ultrafilter, T a $\mathcal{U}$-tree, and $\mathcal{B}$ an α-uniform family on $M \in \mathcal{U}$. If $\alpha = 1$, then take $M_1 = \{n : \{n\} \in T\} \in \mathcal{U}$. Since $M \in \mathcal{U}$, the set $N = M \cap M_1 \subseteq M$ belongs to $\mathcal{U}$. Moreover, $T \cap (\mathcal{B} \upharpoonright N) = N^{[1]}$ is a 1-uniform barrier on N.

Suppose that the lemma is true for every $0 < \beta < \alpha$. Let $(\alpha_i)_i$ be an increasing sequence of ordinals converging to α if α is limit, and let $\alpha_i = \beta$ for all $i \in M$ if $\alpha = \beta + 1$ such that $\mathcal{B}_{\{n\}}$ is an α_n-uniform barrier on M/n for every $n \in M$. Note that $T_{\{n\}}$ is a $\mathcal{U}$-tree for every $n \in M_1$ (for M_1 as in the base case). Then, by the inductive hypothesis, for every $n \in M \cap M_1$ there is $M_n \subseteq M/n$ in $\mathcal{U}$ such that $T_{\{n\}} \cap (\mathcal{B}_{\{n\}} \upharpoonright M_n)$ is an α_n-uniform barrier on M_n. Finally, consider $N \subseteq M \cap M_1$ in $\mathcal{U}$ such that $N/n \subseteq M_n$ for every $n \in N$, and note that $T \cap (\mathcal{B} \upharpoonright N)_{\{n\}} = T_{\{n\}} \cap (\mathcal{B}_{\{n\}} \upharpoonright N/n)$ is an α_n-uniform barrier on N/n for every $n \in N$. $\square$

2.3 Schreier families We now present some combinatorial properties of the ω^α-uniform families. Mainly, we will introduce the families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$, which are obtained in a very canonical way from the Schreier barrier, and which are contained in the closure of any ω^α-uniform family (see Lemma 2.11). In view of this, later on, these families will be an important tool in our proofs considering partitions of the closure of ω^α-uniform barriers.

Given $\mathcal{A}$ and $\mathcal{B}$ two families on FIN, we define $\mathcal{A} \oplus \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{B}$ by

$$\mathcal{A} \oplus \mathcal{B} = \{s \cup t : s < t, s \in \mathcal{B} \text{ and } t \in \mathcal{A}\},$$

$$\mathcal{A} \otimes \mathcal{B} = \big\{s_1 \cup s_2 \cup \cdots \cup s_n : s_1 < s_2 < \cdots < s_n \text{ are in } \mathcal{A} \text{ and } \big\{\min(s_i) : 1 \le i \le n\big\} \in \mathcal{B}\big\}.$$

Lemma 2.8 *If $\mathcal{A}, \mathcal{B} \subseteq$ FIN are barriers on $M \in \mathbb{N}^{[\infty]}$, then the following hold:*

1. *$\mathcal{A} \oplus \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{B}$ are barriers on M,*

2. $\overline{\mathcal{A} \oplus \mathcal{B}} = \overline{\mathcal{A}} \oplus \overline{\mathcal{B}}$,
3. $\overline{\mathcal{A} \otimes \mathcal{B}} = \overline{\mathcal{A}} \otimes \overline{\mathcal{B}}$.

Lemma 2.9 *Let $\alpha, \beta < \omega_1$, let $M \in \mathbb{N}^{[\infty]}$, let $\mathcal{A}$ be an α-uniform family on M, and let $\mathcal{B}$ be a β-uniform family on M. Then, $\mathcal{A} \oplus \mathcal{B}$ is $(\alpha + \beta)$-uniform on M, and $\mathcal{A} \otimes \mathcal{B}$ is $(\alpha \cdot \beta)$-uniform on M.*

Given $0 < \alpha < \omega_1$, we define $\mathcal{S}_\alpha$ as follows:

$$\begin{aligned} \mathcal{S}_\alpha &= \mathcal{S}, \quad \text{if } \alpha = 1, \\ \mathcal{S}_\alpha &= \mathcal{S}_\beta \otimes \mathcal{S}, \quad \text{if } \alpha = \beta + 1, \\ \mathcal{S}_\alpha &= \bigcup_{n<\omega} \mathcal{S}_{\alpha_n} \oplus \big\{\{n\}\big\}, \quad \text{if } \alpha = \sup_n \alpha_n. \end{aligned}$$

Remark 2.10 Note that for every $0 < \alpha < \omega_1$, the barrier $\mathcal{S}_\alpha$ is ω^α-uniform.

Lemma 2.11 shows how the families $\mathcal{S}_\alpha$ are in some sense minimal ω^α-uniform barriers. In the following, we will implicitly make use of Lemmas 2.6, 2.8, and 2.9.

Lemma 2.11 *Given $\mathcal{U}$ a selective ultrafilter and $\mathcal{F}$ an ω^α-uniform family on $M \in \mathcal{U}$, there is $N \subseteq M$ in $\mathcal{U}$ such that*

1. $(\mathcal{S}_\beta \otimes \mathbb{N}^{[i]}) \restriction N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ *for every $i \in N$, if $\alpha = \beta + 1$; and*
2. $\mathcal{S}_{\alpha_i} \restriction N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ *for every $i \in N$, if $\alpha = \sup_i \alpha_i$.*

Proof Consider $\mathcal{F}$ an ω^α-uniform family on $M \in \mathcal{U}$ and $(\gamma_i)_i$ an increasing sequence of ordinals converging to ω^α such that $\mathcal{F}_{\{i\}}$ is γ_i-uniform on M/i for every $i \in M$.

If $\alpha = \beta + 1$, then for every $i \in M$ consider $n_i \in M$ such that $\omega^\alpha \cdot n_i < \gamma_i \le \omega^\alpha \cdot (n_i + 1)$. Since $i \le n_i$, for every $i \in M$ there is $M_i \subseteq M/i$ in $\mathcal{U}$ such that $(\mathcal{S}_\alpha \otimes \mathbb{N}^{[i]}) \restriction M_i \subseteq \overline{\mathcal{F}_{\{i\}}}$. We consider now $N \subseteq M$ in $\mathcal{U}$ such that $N/i \subseteq M_i$ for every $i \in N$. Then, we get that $(\mathcal{S}_\beta \otimes \mathbb{N}^{[i]}) \restriction N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ for every $i \in N$.

If α is a limit ordinal, then consider $(\alpha_i)_i$ a strictly increasing sequence of ordinals converging to α. For every $i \in M$, consider $n_i \in M$ such that $\omega^{\alpha_i} \le \omega^{\alpha_{n_i}} < \gamma_i \le \omega^{\alpha_{n_i}+1}$. Then, for every $i \in M$, we obtain $M_i \subseteq M/i$ such that $\mathcal{S}_{\alpha_i} \restriction M_i \subseteq \overline{\mathcal{F}_{\{i\}}}$. Finally, consider $N \subseteq M$ in $\mathcal{U}$ such that $N/i \subseteq M_i$ for all $i \in N$, to get that $\mathcal{S}_{\alpha_i} \restriction N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ for every $i \in N$. □

3 Main Results

In this section we will use Theorem 3.1 to prove our main theorem (Theorem 3.4), which, besides giving an alternative proof for the Baumgartner–Weiss theorem (Corollary 3.5), also provides a combinatorial description of a homogeneous set for the partition relation $\forall l > 1, \overline{\mathcal{F}} \to (\text{top}\,\overline{\mathcal{F}})^1_l$, where $\mathcal{F}$ is an ω^α-uniform barrier.

In order to simplify the reading, the proof of Theorem 3.1 will be left to the next section.

Theorem 3.1 *Given $l > 1$, $0 < \alpha < \omega_1$, $\mathcal{U}$ a selective ultrafilter, $\mathcal{F}$ an ω^α-uniform barrier on some set $M \in \mathcal{U}$, and a coloring $f : \overline{\mathcal{F}} \longrightarrow l$, there exist a $\mathcal{U}$-tree T, an ω^α-uniform barrier $\mathcal{B}$, on some set $N \subseteq M$ of $\mathcal{U}$, and a mapping $\varphi : (T \restriction \mathcal{B}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$ such that*

1. *f is constant on $rg(\varphi)$;*
2. *$s \subseteq \varphi(s)$ for every $\emptyset \neq s \in T \restriction \mathcal{B}$;*

3. $\min(\varphi(s)) = \min(s)$ *for every* $\emptyset \neq s \in T \restriction \mathcal{B}$*;*
4. *if* $u \cup \{n\} \in T \restriction \mathcal{B}$*, then* $\varphi(u \cup \{n\}) = \varphi(u) \cup u_n$ *for some finite set* $u_n > \varphi(u)$*;*
5. *if* $u \cup \{n\}, u \cup \{m\} \in T \restriction \mathcal{B}$*,* $u \neq \emptyset$ *and* $m > n$*, then there is a finite set* v *such that* $\varphi(u) \sqsubseteq v \sqsubset \varphi(u \cup \{m\}), \varphi(u \cup \{n\})$ *and* $\varphi(u \cup \{m\}) \setminus v > \varphi(u \cup \{n\}) \setminus v$*; moreover, if* $\{n\}, \{m\} \in T \restriction \mathcal{B}$ *and* $m > n$*, then* $\varphi(\{m\}) > \varphi(\{n\})$*.*

Remark 3.2 As we will see in Section 4, the fact that Theorem 3.1 is stated in terms of selective ultrafilters and $\mathcal{U}$-trees will allow us to develop a shorter proof for it, since we will be able to restrict ourselves to infinite sets inside our original set M without having to carry out diagonalization procedures. However, notice that, given $\mathcal{F}$ an ω^α-uniform barrier, and a coloring $f : \overline{\mathcal{F}} \longrightarrow l$, for any selective ultrafilter $\mathcal{U}$ and any $M \in \mathcal{U}$, the barrier $\mathcal{F} \restriction M$ is ω^α-uniform on M. Hence, if T, $\mathcal{B}$, and φ are as in Theorem 3.1, then, by Lemma 2.7, there will be $N \in M^{[\infty]}$ such that $T \cap (\mathcal{B} \restriction N) \subseteq T \restriction \mathcal{B}$ is an ω^α-uniform barrier on N. In this way, we obtain the following simplified version of Theorem 3.1 which is the one that we will essentially use.

Corollary 3.3 *Given* $l > 1$*,* $0 < \alpha < \omega_1$*,* $\mathcal{F}$ *an* ω^α*-uniform barrier, and a coloring* $f : \overline{\mathcal{F}} \longrightarrow l$*, there exist an* ω^α*-uniform barrier* $\mathcal{B}$*, on some set* $M \in \mathbb{N}^{[\infty]}$*, and a map* $\varphi : \overline{\mathcal{B}} \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$ *such that*

1. f *is constant on* $rg(\varphi)$*;*
2. $s \subseteq \varphi(s)$ *for every* $\emptyset \neq s \in \overline{\mathcal{B}}$*;*
3. $\min(\varphi(s)) = \min(s)$ *for every* $\emptyset \neq s \in \overline{\mathcal{B}}$*;*
4. *if* $u \cup \{n\} \in \overline{\mathcal{B}}$*, then* $\varphi(u \cup \{n\}) = \varphi(u) \cup u_n$ *for some finite set* $u_n > \varphi(u)$*;*
5. *if* $u \cup \{n\}, u \cup \{m\} \in \overline{\mathcal{B}}$*,* $u \neq \emptyset$ *and* $m > n$*, then there is a finite set* v *such that* $\varphi(u) \sqsubseteq v \sqsubset \varphi(u \cup \{m\}), \varphi(u \cup \{n\})$ *and* $\varphi(u \cup \{m\}) \setminus v > \varphi(u \cup \{n\}) \setminus v$*; moreover, if* $\{n\}, \{m\} \in \overline{\mathcal{B}}$ *and* $m > n$*, then* $\varphi(\{m\}) > \varphi(\{n\})$*.*

Theorem 3.4 (Main theorem) *Given* $l > 1$*,* $0 < \alpha < \omega_1$*,* $\mathcal{F}$ *an* ω^α*-uniform barrier, and a coloring* $f : \overline{\mathcal{F}} \longrightarrow l$*, there exist an* ω^α*-uniform barrier* $\mathcal{B}$*, on some set* $M \in \mathbb{N}^{[\infty]}$*, and an* $\sqsubseteq$*-order-preserving embedding* $\varphi : \overline{\mathcal{B}} \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$ *such that* f *is constant on* $rg(\varphi)$*.*

Proof We just need to prove that the map φ given by Corollary 3.3 is indeed an $\sqsubseteq$-order-preserving embedding. Notice that from property (4), φ is $\sqsubseteq$-order-preserving and, moreover, that $\varphi''[s] \subseteq [\varphi(s)] \cap rg(\varphi)$ for every $s \in \overline{\mathcal{B}} \setminus \{\emptyset\}$, where for a given $a \in \mathrm{FIN}$, $[a]$ denotes the basic open $[a] = \{t \in \mathrm{FIN} : a \sqsubseteq t\}$. Thus, it follows that φ is continuous. For the injectivity, consider $a, b \in \overline{\mathcal{B}} \setminus \{\emptyset\}$ with $a \neq b$, and suppose $a \not\sqsubseteq b$. Let $u \sqsubset a, b$ be such that $\min(a \setminus u) \neq \min(b \setminus u)$, say, $\min(b \setminus u) = m > n = \min(a \setminus u)$. Then, by properties (4) and (5), there is a finite set v such that $v \sqsubset \varphi(u \cup \{m\}), \varphi(u \cup \{n\})$ and $\varphi(u \cup \{n\}) \setminus v < \varphi(u \cup \{m\}) \setminus v \sqsubseteq \varphi(b) \setminus v$. Then, $\min(\varphi(u \cup \{n\}) \setminus v) \in \varphi(a) \setminus \varphi(b)$. $\square$

Corollary 3.5 (Baumgartner–Weiss) $\omega^{\omega^\alpha} \rightarrow (\mathrm{top}\, \omega^{\omega^\alpha})^1_l$*, for every* $l > 1$ *and every* $0 < \alpha < \omega_1$*.*

Proof Given $l > 1$ and $0 < \alpha < \omega_1$, we can always consider $\mathcal{F}$ an ω^α-uniform barrier, whose closure (by Propositions 2.2 and 2.3(3)) is order-homeomorphic to $\omega^{\omega^\alpha} + 1$. Then, given an l-coloring of ω^{ω^α}, we naturally get an l-coloring of $\overline{\mathcal{F}}$. Then by Theorem 3.4, we get the conclusion. $\square$

4 Proof of Theorem 3.1

The aim of this section is to prove Theorem 3.1, which will be proved by induction on α. In Section 4.1 we deal with the base case $\alpha = 1$, and then in Section 4.2 we carry out the inductive step.

4.1 The case $\alpha = 1$ Let us fix $l > 1$, $\mathcal{U}$ a selective ultrafilter, $M \in \mathcal{U}$, $\mathcal{F}$ an ω-uniform barrier on M, and a coloring $f : \overline{\mathcal{F}} \longrightarrow l$. Then we have the following.

Claim 4.1.1 *There are $B \in \mathcal{U}$ and $l^* < l$ such that for every $s \in \mathcal{F} \restriction B$, there is $\psi(s) \subseteq s$ such that if $n = \min(s)$, then*

(i) $\forall x \in \psi(s)\ (f(\{0, 1, \ldots, x\} \cap s) = l^*)$, *and*
(ii) $x \in \psi(s)$ *iff* $|\{0, 1, \ldots, x\} \cap s| = m_i^n$ *for some* $0 < i \leq k_n$,

where $k_n = |\psi(s)|$ *and* $0 < m_1^n < m_2^n < \cdots < m_{k_n}^n \leq |s|$ *depend only on* $n \in B$.

Proof We start by choosing $\psi(s) \subseteq s$ for each $s \in \mathcal{F}$ as the biggest of the sets $\{x \in s : f(\{0, 1, \ldots, x\} \cap s) = 0\}$, $\{x \in s : f(\{0, 1, \ldots, x\} \cap s) = 1\}$, $\ldots, \{x \in s : (\{0, 1, \ldots, x\} \cap s) = l - 1\}$ (if there are several of these sets with maximal size, pick and fix any of them). Define $\varphi_0 : \mathcal{F} \longrightarrow l$ by

$$\varphi_0(s) = i \quad \text{iff} \quad \forall x \in \psi(s) \quad \big(f\big(\{0, 1, \ldots, x\} \cap s\big) = i\big).$$

Since $\mathcal{U}$ is selective and $\mathcal{F}$ is a barrier, we can pick $A \subseteq M$ in $\mathcal{U}$ and $l^* < l$ such that $\varphi_0(s) = l^*$ for all $s \in \mathcal{F} \restriction A$. That is,

$$\forall s \in \mathcal{F} \restriction A\ \forall x \in \psi(s) \quad \big(f\big(\{0, 1, \ldots, x\} \cap s\big) = l^*\big). \tag{4.1}$$

Since $\mathcal{F}$ is an ω-uniform family on M, there exits $(m_n)_n$ a strictly increasing sequence of integers such that $\mathcal{F}_{\{n\}} = M^{[m_n]}$ for all $n \in M$. Therefore, if $t \in \mathcal{F}$ and $\min(t) = n$, then $|\psi(t)| \leq |t| = m_n + 1$. Define $\varphi_n : \mathcal{F}_{\{n\}} \restriction A \longrightarrow \{1, 2, \ldots, m_n + 1\}$, for every $n \in A$, by

$$\varphi_n(s) = i \quad \text{iff} \quad \big|\psi\big(\{n\} \cup s\big)\big| = i.$$

Pick $L_n \subseteq A/n$ in $\mathcal{U}$ such that φ_n is constant on $\mathcal{F}_{\{n\}} \restriction L_n$. That is,

$$\exists k_n \in \{1, 2, \ldots, m_n + 1\} \quad \text{such that } \big|\psi\big(\{n\} \cup s\big)\big| = k_n \quad \forall s \in \mathcal{F}_{\{n\}} \restriction L_n. \tag{4.2}$$

We now define $\psi_n : \mathcal{F}_{\{n\}} \restriction L_n \longrightarrow \wp(m_n + 2)$ by

$$\psi_n(s) = \Big\{\big|\{0, 1, \ldots, x\} \cap \big(\{n\} \cup s\big)\big| : x \in \psi\big(\{n\} \cup s\big)\Big\}.$$

Then, pick $a_n \in \wp(m_n + 2)$ and $M_n \subseteq L_n$ in $\mathcal{U}$ such that $\psi_n(s) = a_n$ for all $s \in \mathcal{F}_{\{n\}} \restriction M_n$. That is, there are $0 < m_1^n < m_2^n < \cdots < m_{k_n}^n \leq m_n + 1$ such that for all $s \in \mathcal{F}_{\{n\}} \restriction M_n$,

$$\begin{aligned} &x \in \psi\big(\{n\} \cup s\big) \\ &\quad \text{iff} \quad \big|\{0, 1, \ldots, x\} \cap \big(\{n\} \cup s\big)\big| = m_i^n \quad \text{for some } 0 < i \leq k_n. \end{aligned} \tag{4.3}$$

With this procedure we obtain $(M_n)_{n \in A}$ as a sequence of sets in $\mathcal{U}$ and $(k_n)_{n \in A}$ and $\{(m_i^n)_{0 < i \leq k_n} : n \in A\}$ as sequences of integers such that if $s \in \mathcal{F} \restriction A$, $\min(s) = n$ and $s/n \in M_n$, then (by (4.1), (4.2), and (4.3)) we have the following:

1. $\forall x \in \psi(s)\ (f(\{0, 1, \ldots, x\} \cap s)) = l^*$;
2. $|\psi(s)| = k_n$;
3. $x \in \psi(s)$ if and only if $|\{0, 1, \ldots, x\} \cap s| = m_i^n$ for some $0 < i \leq k_n$.

Note that $m > k_n \cdot l$ implies $k_m > k_n$. Then we can assume, by shrinking A if necessary, that $(k_n)_{n\in A}$ is strictly increasing. On the other hand, by $\mathcal{U}$ being selective, we can pick $B \subseteq A$ in $\mathcal{U}$ such that $B/n \subseteq M_n$ for all $n \in B$. This implies that for every $s \in \mathcal{F} \upharpoonright B$, if $\min(s) = n$, then

$$\begin{aligned}&\text{(i) } \forall x \in \psi(s) \quad \big(f\big(\{0,1,\ldots,x\} \cap s\big) = l^*\big), \qquad \text{and}\\ &\text{(ii) } x \in \psi(s) \quad \text{iff} \quad \big|\{0,1,\ldots,x\} \cap s\big| = m_i^n \quad \text{for some } 0 < i \leq k_n. \end{aligned} \tag{4.4}$$

□

Notice that by $B \in \mathcal{U}$ and $\mathcal{U}$ being nonprincipal, $B^{[<\infty]}$ is a $\mathcal{U}$-tree. Thus, if for $n \in B$ we put $r_n = \max\{m^i_{k_i} : i \leq n\}$ and we take $v_n \in B^{[r_n]}$ with $\min(v_n) > n$, then

$$A_n := \big\{m \in \mathbb{N} : \exists v \in B^{[r_n]}(m > v > n)\big\} \supseteq \big\{m \in \mathbb{N} : v_n \cup \{m\} \in B^{[<\infty]}\big\} \in \mathcal{U}.$$

Therefore, $A_n \in \mathcal{U}$ for every $n \in B$, and we can choose $M_* \subseteq B$ in $\mathcal{U}$ such that $M_*/n \subseteq A_n$ for every $n \in M_*$. Consider

$$\mathcal{G} = \big\{\psi(s) : s \in \mathcal{F} \upharpoonright B\big\} \subseteq \overline{\mathcal{F}}.$$

Claim 4.1.2 *There exist $N \subseteq M_*$ in $\mathcal{U}$ and $\mathcal{B}$ an ω-uniform barrier on N such that $\mathcal{G}[N] = \overline{\mathcal{B}}$.*

Proof Since $\mathcal{G}[M_*]$ is a precompact family on M_* and $\mathcal{G}[M_*][N] = \mathcal{G}[N]$ for every $N \in M_*^{[<\infty]}$, by Theorem 2.4, it is enough to show that $\mathcal{G}[M_*]$ has Cantor–Bendixson rank ω.

To see this, we fix $p > 1$ and $q \in M_*$ such that $k_q > p$. Then, it will be enough to show that $(M_*/k_q)^{[\leq p]} \subseteq \mathcal{G}[M_*]$. In the following, we will repeatedly use the fact that $B \setminus M_*$ is infinite, which follows from the choice of M_*. Let $x_1 < x_2 < \cdots < x_p$ be in M_*/k_q, and choose $u_1, u_2, \ldots, u_p \subseteq B$ with

$$\begin{aligned}
|u_1| &= m_1^q - 2 && \text{and} && q < u_1 < x_1,\\
|u_2| &= m_2^q - m_1^q - 1 && \text{and} && x_1 < u_2 < x_2,\\
|u_3| &= m_3^q - m_2^q - 1 && \text{and} && x_2 < u_3 < x_3,\\
&\vdots\\
|u_p| &= m_p^q - m_{p-1}^q - 1 && \text{and} && x_{p-1} < u_p < x_p.
\end{aligned}$$

By (4.4) we can choose $s_1, s_2, \ldots, s_p \in \mathcal{F} \upharpoonright B$ such that $\{q\} \cup u_1 \cup \{x_1\} \sqsubset s_1$, $\{q\} \cup u_1 \cup \{x_1\} \cup u_2 \cup \{x_2\} \sqsubset s_2$, $\{q\} \cup u_1 \cup \{x_1\} \cup u_2 \cup \{x_2\} \cup u_3 \cup \{x_3\} \sqsubset s_3, \ldots$, $\{q\} \cup u_1 \cup \{x_1\} \cup u_2 \cup \{x_2\} \cup \cdots \cup u_p \cup \{x_p\} \sqsubset s_p$ and $s_1/x_1, s_2/x_2, s_3/x_3, \ldots$, $s_p/x_p \subseteq B \setminus M_*$. Then, $\{x_1\} = \psi(s_1) \cap M_*$, $\{x_1, x_2\} = \psi(s_2) \cap M_*$, $\{x_1, x_2, x_3\} = \psi(s_3) \cap M_*, \ldots, \{x_1, x_2, x_3, \ldots, x_p\} = \psi(s_p) \cap M_*$. Then, it follows that $(M_*/k_q)^{[\leq p]} \subseteq \mathcal{G}[M_*]$. □

Let T be the $\mathcal{U}$-tree $M_*^{[<\infty]}$. Then we define $\varphi : (T \upharpoonright \mathcal{B}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$, satisfying the conclusions of the theorem as follows. If $\emptyset \neq t \in T \upharpoonright \mathcal{B}$, then there is $s \in \mathcal{F} \upharpoonright B$ such that $t = \psi(s) \cap N$. Let us put $\min(s) = p$ and $t = \{n_1, n_2, \ldots, n_k\}_<$; then we should have $k \leq k_p \leq k_{n_1}$ and $n_2 \in A_{n_1}$,

$n_3 \in A_{n_2}, \dots, n_k \in A_{n_{k-1}}$. From this, we can define the following sets:

$$\begin{aligned}
u_1 &= \{n_1\},\\
u_2 &= v_2 \cup \{n_2\}, \quad \text{for } v_2 \text{ the } <_{\text{lex}} - \max v \subseteq B \text{ with}\\
&\quad |v| = m_1^{n_1} - 2 \text{ and } n_2 > v > n_1,\\
u_3 &= v_3 \cup \{n_3\}, \quad \text{for } v_3 \text{ the } <_{\text{lex}} - \max v \subseteq B \text{ with}\\
&\quad |v| = m_2^{n_1} - m_1^{n_1} - 1 \text{ and } n_3 > v > n_2,\\
&\quad \vdots\\
u_k &= v_k \cup \{n_k\}, \quad \text{for } v_k \text{ the } <_{\text{lex}} - \max v \subseteq B \text{ with}\\
&\quad |v| = m_{k-1}^{n_1} - m_{k-2}^{n_1} - 1 \text{ and } n_k > v > n_{k-1}.
\end{aligned}$$

It is clear that the choice of such sets is uniquely determined by t. Take $\varphi(t) = \bigcup_{0<j\leq k} u_j$. Then as $u_1 < u_2 < \cdots < u_k$, we have $\min(\varphi(t)) = n_1$ and $|\varphi(t)| = m_{k-1}^{n_1}$. Moreover, by (4.4) we can choose $v \in \mathcal{F} \restriction B$ such that $\varphi(t) \sqsubset v$ and $\max(\varphi(t)) \in \psi(v)$, thus $f(\varphi(t)) = l^*$. Then it is clear that φ defined in this way satisfies conditions (1)–(4) in the statement of Theorem 3.1. To see that φ also satisfies condition (5), note that if $t \cup \{m\}, t \cup \{n\} \in T \restriction \mathcal{B}$ and $m > n$, then $\varphi(t \cup \{m\}) = \bigcup_{0<j\leq k} u_j \cup u_m$ and $\varphi(t \cup \{n\}) = \bigcup_{0<j\leq k} u_j \cup u_n$ with $u_m = v_m \cup \{m\}$ and $u_n = v_n \cup \{n\}$, for some $v_m, v_n \subseteq B$, such that $n_k < v_m < m$, $n_k < v_n < n$ and $|v_m| = |v_n| = m_k^{n_1} - m_{k-1}^{n_1} - 1$. Moreover, $m \in A_n$. Therefore, there is $v \subseteq B$ such that $n < v < m$ and $|v| \geq m_{k_{n_1}}^{n_1} \geq m_k^{n_1}$. This, together with the fact that v_m should be taken as the $<_{\text{lex}}$-maximal with such properties, implies $u_m > u_n$. □

4.2 The case $\alpha > 1$ Consider $1 < \alpha < \omega_1$, $\mathcal{U}$ a selective ultrafilter, $\mathcal{F}$ an ω^α-uniform barrier on $M \in \mathcal{U}$, and $f : \overline{\mathcal{F}} \longrightarrow l$ a finite coloring. Let us assume that Theorem 3.1 holds for every $0 < \beta < \alpha$, and let us prove it for α.

Case α limit. By Lemma 2.11, we can consider $N \subseteq M$ in $\mathcal{U}$ such that $\mathcal{S}_{\alpha_i} \restriction N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ for every $i \in N$, where $\alpha = \sup_i(\alpha_i)$. Define the map $f_k : \overline{\mathcal{S}_{\alpha_k}} \restriction N/k \longrightarrow l$ by $f_k(t) = f(\{k\} \cup t)$, for every $k \in N$. Then, apply the inductive hypothesis to find a $\mathcal{U}$-tree T_k, an ω^{α_k}-uniform barrier $\mathcal{B}_k$ on some set $M_k \subseteq N/k$ of $\mathcal{U}$, and a map $\varphi_k : (T_k \restriction \mathcal{B}_k) \setminus \{\emptyset\} \longrightarrow (\overline{\mathcal{S}_{\alpha_k}} \restriction N/k) \setminus \{\emptyset\}$ satisfying conditions (1)–(5). Let us put $f(\{k\} \cup t) = l_k$ for every $t \in rg(\varphi_k)$.

Next, consider $A \subseteq N$ in $\mathcal{U}$ and $l^* < l$ such that $l_k = l^*$ and $A/k \subseteq M_k$ for all $k \in A$. Define a coloring $\theta : A^{[2]} \longrightarrow \{0, 1\}$ by $\theta(\{k, m\}) = 1$ if there is $\{n\} \in T_k \restriction \mathcal{B}_k$ such that $k < \varphi_k(\{n\}) < m$, and $\theta(\{k, m\}) = 0$ if not. Then there is $M_* \subseteq A$ in $\mathcal{U}$ such that θ is constant on $M_*^{[2]}$. Notice that θ should be constant 1 on $M_*^{[2]}$. If $k \in M_*$, then by property (5) of φ_k, there is $\{n\} \in T_k \restriction \mathcal{B}_k$ such that $k < \varphi_k(\{n\})$. Therefore, for any $m \in M_*$ bigger than $\varphi_k(\{n\})$, we have $\theta(\{k, m\}) = 1$.

For each $k \in M_*$ let m_k be the minimal integer n such that $\{n\} \in T_k \restriction \mathcal{B}_k$ and $k < \varphi_k(\{n\}) < k^+$, where k^+ denotes the successor of k in M_*. Since $\mathcal{B}_k$ is an ω^{α_k}-uniform barrier on M_k, we can choose $(\gamma_i(\omega^{\alpha_k}))_i$ a sequence of ordinals converging to ω^{α_k} such that $(\mathcal{B}_k)_{\{i\}}$ is a $\gamma_i(\omega^{\alpha_k})$-uniform barrier on M_k/i for each $i \in M_k$. Moreover, we can assume that $\omega^{\alpha_{k-1}} < \gamma_i(\omega^{\alpha_k}) < \omega^{\alpha_k}$ for all $i < \omega$. Then, it is clear that $\sup_k\{\gamma_{m_k}(\omega^{\alpha_k})\} = \omega^\alpha$. On the other hand, since $m_k \in \varphi_k(\{m_k\})$, we get $x \geq k^+ > m_k$ for every $x \in M_*/k$. Therefore, $M_*/k \subseteq A/m_k \subseteq M_k/m_k$. From this, $(\mathcal{B}_k)_{\{m_k\}}$ is a $\gamma_{m_k}(\omega^{\alpha_k})$-uniform barrier on M_*/k for every $k \in M_*$. Thus, $\mathcal{B}$ defined by $\mathcal{B} = \bigcup_{k \in M_*} (\mathcal{B}_k)_{\{m_k\}} \oplus \{\{k\}\}$ is an ω^α-uniform barrier on M_*.

Let T be the $\mathcal{U}$-tree $T = (\bigcup_{k \in M_*} (T_k)_{\{m_k\}} \oplus \{\{k\}\}) \cup \{\emptyset\}$, and define the mapping $\varphi : (T \upharpoonright \mathcal{B}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$ by

$$\varphi(s) = \{k\} \cup \varphi_k\big(\{m_k\} \cup s/k\big), \quad \text{if } \min(s) = k.$$

Then it is clear that f is constant l^* on $rg(\varphi)$ and that it satisfies conditions (2) and (3) of the statement. On the other hand, if $u \cup \{n\} \in T \upharpoonright \mathcal{B}$ and $\min(u) = k$, then $\varphi(u \cup \{n\}) = \{k\} \cup \varphi_k(\{m_k\} \cup u/k \cup \{n\})$, and since φ_k satisfies condition (4), there is a finite set $u_n > \varphi_k(\{m_k\} \cup u/k)$ such that $\varphi_k(\{m_k\} \cup u/k \cup \{n\}) = \varphi_k(\{m_k\} \cup u/k) \cup u_n$, and then $\varphi(u \cup \{n\}) = \{k\} \cup \varphi_k(\{m_k\} \cup u/k) \cup u_n = \varphi(u) \cup u_n$ with $u_n > \varphi(u)$. Moreover, if also $u \cup \{m\} \in T \upharpoonright \mathcal{B}$ and $m > n$, then $\{m_k\} \cup u/k \cup \{n\}, \{m_k\} \cup u/k \cup \{m\}$ are in $T_k \upharpoonright \mathcal{B}_k$, and by the inductive hypothesis, there is a finite set v such that $\varphi_k(\{m_k\} \cup u/k) \sqsubseteq v \sqsubset \varphi_k(\{m_k\} \cup u/k \cup \{m\}), \varphi_k(\{m_k\} \cup u/k \cup \{n\})$ and $\varphi_k(\{m_k\} \cup u/k \cup \{m\}) \setminus v > \varphi_k(\{m_k\} \cup u/k \cup \{n\}) \setminus v$, and then $\varphi(u) = \{k\} \cup \varphi_k(\{m_k\} \cup u/k) \sqsubseteq \{k\} \cup v \sqsubset \varphi(u \cup \{m\}), \varphi(u \cup \{n\})$ and $\varphi(u \cup \{m\}) \setminus (\{k\} \cup v) = \varphi_k(\{m_k\} \cup u/k \cup \{m\}) \setminus v > \varphi_k(\{m_k\} \cup u/k \cup \{n\}) \setminus v = \varphi(u \cup \{n\}) \setminus (\{k\} \cup v)$. Finally, note that if $\{k\}, \{k'\} \in T \upharpoonright \mathcal{B}$ and $k' > k$, then $k' \geq k^+ > \varphi_k(\{m_k\}) > k$, which implies $\varphi(\{k'\}) > \varphi(\{k\})$.

Case $\alpha = \beta + 1$. By Lemma 2.11, we can consider $N \subseteq M$ in $\mathcal{U}$ such that $(\mathcal{S}_\beta \otimes \mathbb{N}^{[i]}) \upharpoonright N/i \subseteq \overline{\mathcal{F}_{\{i\}}}$ for every $i \in N$. Let us fix $k \in N$, and, in order to simplify notation, let us put $\mathcal{C} = \mathcal{S}_\beta \upharpoonright N/k$. Notice that if $a_1 < a_2 < \cdots < a_i$ are in $\overline{\mathcal{C}}$ and $i \leq k$, then $\{k\} \cup a_1 \cup a_2 \cup \cdots \cup a_i \in \overline{\mathcal{F}}$.

Define $f_0 : \overline{\mathcal{C}} \longrightarrow l$ by $f_0(u) = f(\{k\} \cup u)$, and apply the inductive hypothesis to find a $\mathcal{U}$-tree T_0, an ω^β-uniform barrier $\mathcal{B}_0$, on some set $M_0 \subseteq N/k$ of $\mathcal{U}$, and a mapping $\varphi_0 : (T_0 \upharpoonright \mathcal{B}_0) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{C}} \setminus \{\emptyset\}$ satisfying conditions (1)–(5). Let us put

$$f\big(\{k\} \cup \varphi_0(v)\big) = l_0 \quad \forall v \in T_0 \upharpoonright \mathcal{B}_0, v \neq \emptyset.$$

For each $v \in T_0 \cap \mathcal{B}_0$, define $f_1^v : \overline{\mathcal{C}}/\varphi_0(v) \longrightarrow l$ by $f_1^v(u) = f(\{k\} \cup \varphi_0(v) \cup u)$, and consider T_1^v a $\mathcal{U}$-tree, $\mathcal{B}_1^v$ an ω^β-uniform barrier on some set of $\mathcal{U}$, and a mapping $\varphi_1^v : (T_1^v \upharpoonright \mathcal{B}_1^v) \setminus \{\emptyset\} \longrightarrow (\overline{\mathcal{C}}/\varphi_0(v)) \setminus \{\emptyset\}$ satisfying conditions (1)–(5). Let us put

$$f\big(\{k\} \cup \varphi_0(v) \cup u\big) = l_v \quad \forall u \in rg(\varphi_1^v).$$

By Lemma 2.7, there is $M_1 \subseteq M_0$ in $\mathcal{U}$ such that $T_0 \cap (\mathcal{B}_0 \upharpoonright M_1)$ is an ω^β-uniform barrier on M_1. Moreover, by the selectivity of $\mathcal{U}$, we can assume that M_1 is such that $l_v = l_{v'}$ for every $v, v' \in T_0 \cap (\mathcal{B}_0 \upharpoonright M_1)$. That is, there exits $l_1 < l$ such that

$$f\big(\{k\} \cup \varphi_0(v) \cup \varphi_1^v(w)\big) = l_1 \quad \forall v \in T_0 \cap (\mathcal{B}_0 \upharpoonright M_1)\ \forall w \in T_1^v \upharpoonright \mathcal{B}_1^v, w \neq \emptyset.$$

In general, by repeatedly applying the inductive hypothesis, we obtain for each $i < k$ a collection of $\mathcal{U}$-trees $T_0, T_1^{v_0}, \ldots, T_i^{v_0, v_1, \ldots, v_{i-1}}$; a collection of ω^β-uniform barriers $\mathcal{B}_0$, $\mathcal{B}_1^{v_0}, \ldots, \mathcal{B}_i^{v_0, v_1, \ldots, v_{i-1}}$ on some set of $\mathcal{U}$; a collection of mappings

$$\varphi_0 : (T_0 \upharpoonright \mathcal{B}_0) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{C}} \setminus \{\emptyset\},$$
$$\varphi_1^{v_0} : (T_1^{v_0} \upharpoonright \mathcal{B}_1^{v_0}) \setminus \{\emptyset\} \longrightarrow \big(\overline{\mathcal{C}}/\varphi_0(v_0)\big) \setminus \{\emptyset\},$$

$$\vdots$$

$$\varphi_i^{v_0,v_1,\ldots,v_{i-1}}:$$
$$(T_i^{v_0,v_1,\ldots,v_{i-1}} \restriction \mathcal{B}_i^{v_0,v_1,\ldots,v_{i-1}}) \setminus \{\emptyset\} \longrightarrow \big(\overline{\mathcal{C}}/\varphi_{i-1}^{v_0,v_1,\ldots,v_{i-2}}(v_{i-1})\big) \setminus \{\emptyset\},$$

satisfying conditions (2)–(5) of the statement; a color $l_i < l$ and a set $M_i \in \mathcal{U}$ with $M_i \subseteq M_{i-1}$, such that if $v_0 \in T_0 \cap (\mathcal{B}_0 \restriction M_i)$, $v_1 \in T_1^{v_0} \cap (\mathcal{B}_1^{v_0} \restriction M_i), \ldots,$ $v_{i-1} \in T_{i-1}^{v_0,v_1,\ldots,v_{i-2}} \cap (\mathcal{B}_{i-1}^{v_0,v_1,\ldots,v_{i-2}} \restriction M_i), \emptyset \neq v_i \in T_i^{v_0,v_1,\ldots,v_{i-1}} \restriction \mathcal{B}_i^{v_0,v_1,\ldots,v_{i-1}}$, then

$$f\big(\{k\} \cup \varphi_0(v_0) \cup \varphi_1^{v_0}(v_1) \cup \varphi_2^{v_0,v_1}(v_2) \cup \cdots \cup \varphi_i^{v_0,v_1,\ldots,v_{i-1}}(v_i)\big) = l_i, \quad (4.5)$$

where the superscripts in $T_j^{v_0,v_1,\ldots,v_{j-1}}$, $\mathcal{B}_j^{v_0,v_1,\ldots,v_{j-1}}$, and $\varphi_j^{v_0,v_1,\ldots,v_{j-1}}$ indicate that the choice of the trees, the uniform barriers, and the mappings depends on the sets $v_0, v_1, \ldots, v_{j-1}$ previously fixed, for every $0 < j \leq i$.

Later on, we find it convenient to regroup the superscripts if we have many of them. For example, if we put $\overline{v} = \{v_0, v_1, \ldots, v_{j-1}\}$, then $\varphi_j^{\overline{v}} = \varphi_j^{v_0,v_1,\ldots,v_{j-1}}$ indicates the dependence of the mapping on all the sets in $\overline{v}$, and we will do the same for the trees and the uniform barriers. Moreover, we will have a collection of such trees, barriers, and mappings as above for every different choice of $v_0, v_1, \ldots, v_{i-1}$. On the other hand, notice that by the choice of $\varphi_j^{v_0,v_1,\ldots,v_{j-1}}$, for every $j \leq i$, each of these maps satisfies conditions (2)–(5) of the statement of Theorem 3.1.

Until now, we have chosen a color $l_i < l$ for each $i < k$. Then, for k big enough, some of these colors must repeat, and then should exist $a_k \subseteq k$ and $l^k < l$ such that $l_i = l^k$ for every $i \in a_k$. Suppose $a_k = \{i_1, i_2, \ldots, i_p\}$; then $l_{i_1} = l_{i_2} = \cdots = l_{i_p} = l^k$. For future reference, we will put $p_k := |a_k| = p$.

The following claim will allow us to define (below) a sequence of barriers (by varying $k \in N$) with uniformities converging to $\omega^{\beta+1}$, so that we may then proceed as in the α-limit case.

Claim 4.2.1 *There exist an $(\omega^\beta \cdot p_k)$-uniform barrier $\mathcal{B}_k$ on some set of $\mathcal{U}$, a $\mathcal{U}$-tree T_k, and a mapping $\varphi_k : (T_k \restriction \mathcal{B}_k) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ satisfying conditions (2) and (4) and the following:*

1′. *$f(\{k\} \cup \varphi_k(u)) = l^k$ for every $\emptyset \neq u \in T_k \restriction \mathcal{B}_k$,*

5′. *if $u \cup \{n\}, u \cup \{m\} \in T_k \restriction \mathcal{B}_k$ and $m > n$, then there is a finite set v such that $\varphi_k(u) \sqsubseteq v \sqsubset \varphi_k(u \cup \{m\})$, $\varphi_k(u \cup \{n\})$ and $\varphi_k(u \cup \{m\}) \setminus v > \varphi_k(u \cup \{n\}) \setminus v$; moreover, there is a finite set $a > k$ such that $a \sqsubset \varphi_k(\{n\})$ for every $\{n\} \in T_k \restriction \mathcal{B}_k$, and $(\varphi_k(\{n\}) \setminus a)_n$ is a block sequence.*

Proof In the following, we will recursively define a $\mathcal{U}$-tree T_{i_j}, an $(\omega^\beta \cdot j)$-uniform barrier $\mathcal{B}_{i_j}$ on some set of $\mathcal{U}$, and a map $\varphi_{i_j} : (T_{i_j} \restriction \mathcal{B}_{i_j}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ satisfying conditions (2), (4), (1′), and (5′) (for φ_{i_j} instead of φ_k) for each $0 < j \leq p$, where $i_i, i_2, \ldots, i_p$ are the elements of a_k.

We start by fixing $v_0 \in T_0 \cap (\mathcal{B}_0 \restriction M_{i_1})$, $v_1 \in T_1^{v_0} \cap (\mathcal{B}_1^{v_0} \restriction M_{i_1}), \ldots, v_{i_1-1} \in T_{i_1-1}^{v_0,\ldots,v_{i_1-2}} \cap (\mathcal{B}_{i_1-1}^{v_0,\ldots,v_{i_1-2}} \restriction M_{i_1})$, where all these structures are as in the argument above. Put $\overline{v} = \{v_0, v_1, \ldots, v_{i_1-1}\}$, $\mathcal{B}_{i_1} = \mathcal{B}_{i_1}^{\overline{v}}$, and $T_{i_1} = T_{i_1}^{\overline{v}}$, and define $\varphi_{i_1} : (T_{i_1} \restriction \mathcal{B}_{i_1}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ by

$$\varphi_{i_1}(u) = \varphi_0(v_0) \cup \varphi_1^{v_0}(v_1) \cup \cdots \cup \varphi_{i_1-1}^{v_0,v_1,\ldots,v_{i_1-2}}(v_{i_1-1}) \cup \varphi_{i_1}^{\overline{v}}(u).$$

Then, by (4.5) we have

$$f\big(\{k\} \cup \varphi_{i_1}(u)\big) = l_{i_1} = l^k.$$

Notice that φ_{i_1} satisfies conditions (2) and (4) because $\varphi_{i_1}^{\overline{v}}$ does. Moreover, φ_{i_1} satisfies (5′) since $\varphi_{i_1}^{\overline{v}}$ satisfies (5), where we put $a = \varphi_0(v_0) \cup \varphi_1^{v_0}(v_1) \cup \cdots \cup \varphi_{i_1-1}^{v_0,v_1,\ldots,v_{i_1-2}}(v_{i_1-1})$ for the second part of statement (5′). Let $N_{i_1} \in \mathcal{U}$ be the set on which $\mathcal{B}_{i_1}$ is defined.

In order to define φ_{i_2}, we will consider for each $u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction M_{i_2})$, sets $u_{i_1+1} \in T_{i_1+1}^{\overline{v},u} \cap (\mathcal{B}_{i_1+1}^{\overline{v},u} \restriction M_{i_2})$, $u_{i_1+2} \in T_{i_1+2}^{\overline{v},u,u_{i_1+1}} \cap (\mathcal{B}_{i_1+2}^{\overline{v},u,u_{i_1+1}} \restriction M_{i_2})$, $\ldots$, $u_{i_2-1} \in T_{i_2-1}^{\overline{v},u,u_{i_1+1},\ldots,u_{i_2-2}} \cap (\mathcal{B}_{i_2-1}^{\overline{v},u,u_{i_1+1},\ldots,u_{i_2-2}} \restriction M_{i_2})$. Then we will put $\overline{u} = \{u, u_{i_1+1}, \ldots, u_{i_2-1}\}$, and we will fix the uniform barrier $\mathcal{B}_{i_2}^{\overline{v},\overline{u}}$. Let $M_{i_2}^{\overline{v},\overline{u}} \in \mathcal{U}$ be the set where $\mathcal{B}_{i_2}^{\overline{v},\overline{u}}$ is defined. We choose, by selectivity, a set $N_{i_2} \subseteq M_{i_2} \cap N_{i_1}$ in $\mathcal{U}$ such that $N_{i_2}/u \subseteq M_{i_2}^{\overline{v},\overline{u}}$ for every $u \in N_{i_2}^{[<\infty]}$. Let us now define $\mathcal{B}_{i_2}$ and T_{i_2} by

$$\mathcal{B}_{i_2} = \bigcup_{u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})} \mathcal{B}_{i_2}^{\overline{v},\overline{u}} \oplus \{u\},$$

$$T_{i_2} = \Big(\bigcup_{u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})} T_{i_2}^{\overline{v},\overline{u}} \oplus \{u\}\Big) \cup \big(T_{i_1} \cap \overline{(\mathcal{B}_{i_1} \restriction N_{i_2})}\big).$$

Clearly, T_{i_2} is a $\mathcal{U}$-tree. Moreover, by Lemma 2.7, we may assume that N_{i_2} is such that $T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$ is an ω^β-uniform barrier on N_{i_2}. Therefore, $\mathcal{B}_{i_2}$ is a barrier on N_{i_2} with $\mathrm{rk}_{\mathrm{CB}}(\overline{\mathcal{B}_{i_2}}) = \omega^\beta \cdot 2$. Thus, by Theorem 2.5, we may assume, moreover, that N_{i_2} was taken such that $\mathcal{B}_{i_2}$ is in fact an $(\omega^\beta \cdot 2)$-uniform barrier on N_{i_2}.

Define $\varphi_{i_2} : (T_{i_2} \restriction \mathcal{B}_{i_2}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ by $\varphi_{i_2}(s) = \varphi_{i_1}(s)$, if $s \in T_{i_1} \cap \overline{(\mathcal{B}_{i_1} \restriction N_{i_2})}$; and $\varphi_{i_2}(s) = \varphi_{i_1}(u) \cup \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \cup \varphi_{i_2-1}^{\overline{v},u,\ldots,u_{i_2-2}}(u_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u}}(w)$, if $s = u \cup w$ with $u < w$, $u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$ and $w \neq \emptyset$.

Clearly, φ_{i_2} satisfies condition (2) since φ_{i_1} and $\varphi_{i_2}^{\overline{v},\overline{u}}$ do. Notice that by (4.5), we have $f(\{k\} \cup \varphi_{i_2}(s)) \in \{l_{i_1}, l_{i_2}\}$ for all $s \in T_{i_2} \restriction \mathcal{B}_{i_2} \setminus \{\emptyset\}$, but $l_{i_1} = l_{i_2} = l^k$. Therefore,

$$f\big(\{k\} \cup \varphi_{i_2}(s)\big) = l^k \quad \text{for all } \emptyset \neq s \in T_{i_2} \restriction \mathcal{B}_{i_2}.$$

Moreover, φ_{i_2} satisfies conditions (4) and (5′). Indeed, if $u \cup \{n\} \in T_{i_2} \restriction \mathcal{B}_{i_2}$ and $u \cup \{n\} \in T_{i_1} \cap \overline{(\mathcal{B}_{i_1} \restriction N_{i_2})}$, then condition (4) holds since φ_{i_1} satisfies it. If $u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$, then by definition of φ_{i_2}, we have $\varphi_{i_2}(u) = \varphi_{i_1}(u)$ and $\varphi_{i_2}(u \cup \{n\}) = \varphi_{i_2}(u) \cup \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \cup \varphi_{i_2-1}^{\overline{v},u,\ldots,u_{i_2-2}}(u_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u}}(\{n\})$. Then take $u_n = \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \cup \varphi_{i_2-1}^{\overline{v},u,\ldots,u_{i_2-2}}(u_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u}}(\{n\})$ to get (4). Finally, if $u \notin T_{i_1} \cap \overline{(\mathcal{B}_{i_1} \restriction N_{i_2})}$, then (4) holds for φ_{i_2} since $\varphi_{i_2}^{\overline{v},\overline{u}}$ satisfies it.

To see that φ_{i_2} satisfies condition (5′), consider $u \cup \{m\} \in T_{i_2} \restriction \mathcal{B}_{i_2}$ with $m > n$. Then, if $u \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$, we have

$$\begin{aligned}\varphi_{i_2}\big(u \cup \{m\}\big) &= \varphi_{i_2}(u) \cup \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \\ &\quad \cup \varphi_{i_2-1}^{\overline{v},u,\dots,u_{i_2-2}}(u_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u}}\big(\{m\}\big),\\ \varphi_{i_2}\big(u \cup \{n\}\big) &= \varphi_{i_2}(u) \cup \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \\ &\quad \cup \varphi_{i_2-1}^{\overline{v},u,\dots,u_{i_2-2}}(u_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u}}\big(\{n\}\big).\end{aligned}$$

Thus, since $\varphi_{i_2}^{\overline{v},\overline{u}}$ satisfies condition (5), we have $\varphi_{i_2}^{\overline{v},\overline{u}}(\{m\}) > \varphi_{i_2}^{\overline{v},\overline{u}}(\{n\})$. Therefore, if we put

$$w = \varphi_{i_2}(u) \cup \varphi_{i_1+1}^{\overline{v},u}(u_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u,u_{i_1+1}}(u_{i_1+2}) \cup \cdots \cup \varphi_{i_2-1}^{\overline{v},u,\dots,u_{i_2-2}}(u_{i_2-1}),$$

then $\varphi_{i_2}(u) \sqsubset w \sqsubset \varphi_{i_2}(u \cup \{m\}), \varphi_{i_2}(u \cup \{n\})$ and $\varphi_{i_2}(u \cup \{m\}) \setminus w > \varphi_{i_2}(u \cup \{n\}) \setminus w$.

If $u = u' \cup x$ with $u' \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$ and $x \neq \emptyset$, then

$$\begin{aligned}\varphi_{i_2}(u) &= \varphi_{i_1}(u') \cup \varphi_{i_1+1}^{\overline{v},u'}(u'_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u',u'_{i_1+1}}(u'_{i_1+2}) \cup \cdots \\ &\quad \cup \varphi_{i_2-1}^{\overline{v},u',\dots,u'_{i_2-2}}(u'_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u'}}(x),\\ \varphi_{i_2}\big(u \cup \{m\}\big) &= \varphi_{i_1}(u') \cup \varphi_{i_1+1}^{\overline{v},u'}(u'_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u',u'_{i_1+1}}(u'_{i_1+2}) \cup \cdots \\ &\quad \cup \varphi_{i_2-1}^{\overline{v},u',\dots,u'_{i_2-2}}(u'_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u'}}\big(x \cup \{m\}\big),\\ \varphi_{i_2}\big(u \cup \{n\}\big) &= \varphi_{i_1}(u') \cup \varphi_{i_1+1}^{\overline{v},u'}(u'_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u',u'_{i_1+1}}(u'_{i_1+2}) \cup \cdots \\ &\quad \cup \varphi_{i_2-1}^{\overline{v},u',\dots,u'_{i_2-2}}(u'_{i_2-1}) \cup \varphi_{i_2}^{\overline{v},\overline{u'}}\big(x \cup \{n\}\big).\end{aligned}$$

Thus, using property (5) of $\varphi_{i_2}^{\overline{v},\overline{u'}}$, we get a finite set w such that $\varphi_{i_2}^{\overline{v},\overline{u'}}(x) \sqsubseteq w \sqsubset \varphi_{i_2}^{\overline{v},\overline{u'}}(x \cup \{m\}), \varphi_{i_2}^{\overline{v},\overline{u'}}(x \cup \{n\})$ and $\varphi_{i_2}^{\overline{v},\overline{u'}}(x \cup \{m\}) \setminus w > \varphi_{i_2}^{\overline{v},\overline{u'}}(x \cup \{n\}) \setminus w$. Then, if we put $w' = \varphi_{i_1}(u') \cup \varphi_{i_1+1}^{\overline{v},u'}(u'_{i_1+1}) \cup \varphi_{i_1+2}^{\overline{v},u',u'_{i_1+1}}(u'_{i_1+2}) \cup \cdots \cup \varphi_{i_2-1}^{\overline{v},u',\dots,u'_{i_2-2}}(u'_{i_2-1})$, we get $\varphi_{i_2}(u) \sqsubseteq w' \cup w \sqsubset \varphi_{i_2}(u \cup \{m\}), \varphi_{i_2}(u \cup \{n\})$ and $\varphi_{i_2}(u \cup \{m\}) \setminus (w' \cup w) > \varphi_{i_2}(u \cup \{n\}) \setminus (w' \cup w)$.

Finally, notice that φ_{i_2} satisfies the second part of (5′) since $\varphi_{i_2}(\{n\}) = \varphi_{i_1}(\{n\})$, for every $\{n\} \in T_{i_2} \restriction \mathcal{B}_{i_2}$. Then, we conclude that φ_{i_2} satisfies conditions (2), (4), (1′), and (5′).

Take $1 < r < p$, and suppose that for every $1 < j \leq r$, we have already defined an infinite set $N_{i_j} \subseteq M_{i_j}$ in $\mathcal{U}$, an $(\omega^\beta \cdot j)$-uniform barrier $\mathcal{B}_{i_j}$ on N_{i_j}, a $\mathcal{U}$-tree T_{i_j}, and a mapping $\varphi_{i_j} : (T_{i_j} \restriction \mathcal{B}_{i_j}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ satisfying the following conditions:

$$\mathcal{B}_{i_j} = \bigcup_{u \in T_{i_{j-1}} \cap (\mathcal{B}_{i_{j-1}} \restriction N_{i_j})} \mathcal{B}_{i_j}^{\overline{v},\overline{u^1},\overline{u^2},\dots,\overline{u^{j-1}}} \oplus \{u\},$$

where

- $u = u^1 \cup u^2 \cup \cdots \cup u^{j-1}$ with $u^1 \in T_{i_1} \cap (\mathcal{B}_{i_1} \restriction N_{i_2})$, $u^1 \cup u^2 \in T_{i_2} \cap (\mathcal{B}_{i_2} \restriction N_{i_3}), \dots, u^1 \cup u^2 \cup \cdots \cup u^{j-2} \in T_{i_{j-2}} \cap (\mathcal{B}_{i_{j-2}} \restriction N_{i_{j-1}})$,

- $u^{j-1} \in T_{i_{j-1}}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}}} \cap (B_{i_{j-1}}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}}} \upharpoonright M_{i_j})$,
- $u^{j-1}_{i_{j-1}+1} \in T_{i_{j-1}+1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1}} \cap (B_{i_{j-1}+1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1}} \upharpoonright M_{i_j})$, $u^{j-1}_{i_{j-1}+2} \in T_{i_{j-1}+2}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1},u^{j-1}_{i_{j-1}+1}} \cap (B_{i_{j-1}+2}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1},u^{j-1}_{i_{j-1}+1}} \upharpoonright M_{i_j}), \ldots,$ $u^{j-1}_{i_j-1} \in T_{i_j-1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1},u^{j-1}_{j-1+1},\ldots,u^{j-1}_{i_j-2}} \cap$ $(B_{i_j-1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1},u^{j-1}_{j-1+1},\ldots,u^{j-1}_{i_j-2}} \upharpoonright M_{i_j})$,
- $\overline{u^{j-1}} = \{u^{j-1}, u^{j-1}_{i_{j-1}+1}, \ldots, u^{j-1}_{i_j-1}\}$,
- $f(\{k\} \cup \varphi_{i_j}(u)) = l^k$ for every $\emptyset \neq u \in T_{i_j} \upharpoonright \mathcal{B}_{i_j}$,
- φ_{i_j} satisfies conditions (2), (4), and (5′), and
- if $s = u \cup x$ with $x \in \mathcal{B}_{i_j}^{\overline{v},\overline{u^1},\overline{u^2},\ldots,\overline{u^{j-1}}}$, then

$$\begin{aligned}\varphi_{i_j}(s) = \varphi_{i_{j-1}}(u) &\cup \varphi_{i_{j-1}+1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1}}(u^{j-1}_{i_{j-1}+1})\\ &\cup \varphi_{i_{j-1}+2}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},u^{j-1},u^{j-1}_{i_{j-1}+1}}(u^{j-1}_{i_{j-1}+2})\\ &\cup \cdots \cup \varphi_{i_j}^{\overline{v},\overline{u^1},\ldots,\overline{u^{j-2}},\overline{u^{j-1}}}(x).\end{aligned}$$

Let us define $\mathcal{B}_{i_{r+1}}$, $T_{i_{r+1}}$, and $\varphi_{i_{r+1}}$ satisfying all those conditions. Notice that for each $w \in T_{i_r} \cap (\mathcal{B}_{i_r} \upharpoonright M_{i_{r+1}})$, there is $u \in T_{i_{r-1}} \cap (\mathcal{B}_{i_{r-1}} \upharpoonright N_{i_r})$ as in the list above (we fix $u^1, u^2, \ldots, u^{r-1}$ verifying those conditions) and $x \in T_{i_r}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}}} \cap (\mathcal{B}_{i_r}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}}} \upharpoonright M_{i_{r+1}})$ such that $u < x$ and $w = u \cup x$. Then, fix

$$\begin{aligned}&x_{i_r+1} \in T_{i_r+1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x} \cap (\mathcal{B}_{i_r+1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x} \upharpoonright M_{i_{r+1}}),\\ &x_{i_r+2} \in T_{i_r+2}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x,x_{i_r+1}} \cap (\mathcal{B}_{i_r+2}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x,x_{i_r+1}} \upharpoonright M_{i_{r+1}}),\\ &\vdots\\ &x_{i_{r+1}-1} \in T_{i_{r+1}-1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x,x_{i_r+1},\ldots,x_{i_{r+1}-2}}\\ &\qquad\qquad \cap (\mathcal{B}_{i_{r+1}-1}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},x,x_{i_r+1},\ldots,x_{i_{r+1}-2}} \upharpoonright M_{i_{r+1}}),\end{aligned}$$

and put $\overline{x} = \{x, x_{i_r+1}, \ldots, x_{i_{r+1}-1}\}$. Arguing as we did for $j = 2$, we can take $N_{i_{r+1}} \subseteq M_{i_{r+1}}$ in $\mathcal{U}$ such that $\mathcal{B}_{i_{r+1}}$ defined as below is a uniform barrier on $N_{i_{r+1}}$,

$$\mathcal{B}_{i_{r+1}} = \bigcup_{w \in T_{i_r} \cap (\mathcal{B}_{i_r} \upharpoonright N_{i_{r+1}})} \mathcal{B}_{i_r}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},\overline{x}} \oplus \{w\},$$

where $w = u^1 \cup u^2 \cup \cdots \cup u^{r-1} \cup x$ and all the sets $u^1, u^2, \ldots, u^{r-1}, x$ are determined as before for every $w \in T_{i_r} \cap (\mathcal{B}_{i_r} \upharpoonright M_{i_{r+1}})$. Define the $\mathcal{U}$-tree $T_{i_{r+1}}$ by

$$T_{i_{r+1}} = \Big(\bigcup_{w \in T_{i_r} \cap (\mathcal{B}_{i_r} \upharpoonright N_{i_{r+1}})} T_{i_{r+1}}^{\overline{v},\overline{u^1},\ldots,\overline{u^{r-1}},\overline{x}} \oplus \{w\}\Big) \cup \big(T_{i_r} \cap \overline{(\mathcal{B}_{i_r} \upharpoonright N_{i_{r+1}})}\big),$$

and the mapping $\varphi_{i_{r+1}} : (T_{i_{r+1}} \restriction \mathcal{B}_{i_{r+1}}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ by $\varphi_{i_{r+1}}(s) = \varphi_{i_r}(s)$, if $s \in T_{i_r} \cap \overline{(\mathcal{B}_{i_r} \restriction N_{i_{r+1}})}$; and $\varphi_{i_{r+1}}(s) = \varphi_{i_r}(w) \cup \varphi_{i_r+1}^{\overline{v},\overline{u^1},\dots,\overline{u^{r-1}},x}(x_{i_r+1}) \cup \cdots \cup \varphi_{i_{r+1}-1}^{\overline{v},\overline{u^1},\dots,\overline{u^{r-1}},x,\dots,x_{i_{r+1}-2}}(x_{i_{r+1}-1}) \cup \varphi_{i_{r+1}}^{\overline{v},\overline{u^1},\dots,\overline{u^{r-1}},\overline{x}}(y)$, if $s = w \cup y$, $w = u^1 \cup u^2 \cup \cdots \cup u^{r-1} \cup x$ with $x < y \neq \emptyset$ and $u^1, u^2, \dots, u^{r-1}, x$ are defined as above for $w \in T_{i_r} \cap (\mathcal{B}_{i_r} \restriction N_{i_{r+1}})$.

Then, given $s \in (T_{i_{r+1}} \restriction \mathcal{B}_{i_{r+1}}) \setminus \{\emptyset\}$, if $s \in T_{i_r} \cap \overline{(\mathcal{B}_{i_r} \restriction N_{i_{r+1}})}$, we get by hypothesis that $f(\{k\} \cup \varphi_{i_{r+1}}(s)) = f(\{k\} \cup \varphi_{i_r}(s)) = l^k$. Moreover, by the last condition on the list and (4.5), we have that $f(\{k\} \cup \varphi_{i_{r+1}}(s)) = l_{i_{r+1}} = l^k$ if $s \notin T_{i_r} \cap \overline{(\mathcal{B}_{i_r} \restriction N_{i_{r+1}})}$. In order to verify that $\varphi_{i_{r+1}}$ satisfies conditions (4) and (5′), we just need to proceed as we did for $j = 2$. Note, moreover, that $\mathcal{B}_{i_{r+1}}$ is $(\omega^\beta \cdot (r+1))$-uniform on $N_{i_{r+1}}$ because, by Lemma 2.7, we can choose $N_{i_{r+1}}$ such that $T_{i_r} \cap (\mathcal{B}_{i_r} \restriction N_{i_{r+1}})$ is $(\omega^\beta \cdot r)$-uniform on $N_{i_{r+1}}$ and $\mathcal{B}_{i_r}^{\overline{v},\overline{u^1},\dots,\overline{u^{r-1}},\overline{x}}$ is ω^β-uniform on $N_{i_{r+1}}/w$ for every $w = u^1 \cup \cdots \cup u^{r-1} \cup x \in T_{i_r} \cap (\mathcal{B}_{i_r} \restriction N_{i_{r+1}})$. This finishes the recursion.

Then, we have proved Claim 4.2.1 once we take $\mathcal{B}_k = \mathcal{B}_{i_p}$, $T_k = T_{i_p}$, and $\varphi_k = \varphi_{i_p}$. □

We are almost ready now to define φ satisfying the conditions in Theorem 3.1. First, we apply Claim 4.2.1 for every $k \in N$ in order to get an $(\omega^\beta \cdot p_k)$-uniform barrier $\mathcal{B}_k$ on some set $N_k \in \mathcal{U}$, a $\mathcal{U}$-tree T_k, and a mapping $\varphi_k : (T_k \restriction \mathcal{B}_k) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}_{\{k\}}} \setminus \{\emptyset\}$ satisfying (1′), (2), (4), and (5′), where $p_k = |a_k|$. Moreover, we can choose $A \subseteq N$ in $\mathcal{U}$ and $l^* < l$ such that $A/k \subseteq N_k$ and $l^k = l^*$ for every $k \in A$. Arguing as in the α-limit case, we may also assume that for every $\{k, m\} \in A^{[2]}$ there is $\{n\} \in T_k \restriction \mathcal{B}_k$ such that $k < \varphi_k(\{n\}) < m$, and take m_k the smallest integer satisfying such a property when $m = k^+$ is the successor of k in A. For every $k \in A$, consider $(\gamma_i^k(\omega^\beta \cdot p_k))_i$ an increasing sequence of ordinals converging to $\omega^\beta \cdot p_k$ such that $(\mathcal{B}_k)_{\{i\}}$ is a $\gamma_i^k(\omega^\beta \cdot p_k)$-uniform barrier on N_k/i and $\omega^\beta \cdot p_{k-1} < \gamma_i^k(\omega^\beta \cdot p_k) < \omega^\beta \cdot p_k$ for every $i \in N_k$. Then, as $p_k \to \infty$ when $k \to \infty$, we get that $\sup_k\{\gamma_{m_k}^k(\omega^\beta \cdot p_k)\} = \omega^\beta \cdot \omega = \omega^\alpha$. Therefore,

$$\mathcal{B} = \bigcup_{k \in A} (\mathcal{B}_k)_{\{m_k\}} \oplus \{\{k\}\}$$

is an ω^α-uniform barrier on A. Finally, consider the $\mathcal{U}$-tree

$$T = \Big(\bigcup_{k \in A} (T_k)_{\{m_k\}} \oplus \{\{k\}\}\Big) \cup \{\emptyset\},$$

and define $\varphi : (T \restriction \mathcal{B}) \setminus \{\emptyset\} \longrightarrow \overline{\mathcal{F}} \setminus \{\emptyset\}$ by

$$\varphi(s) = \{k\} \cup \varphi_k\big(\{m_k\} \cup s/k\big), \quad \text{if } \min(s) = k.$$ □

References

[1] Argyros, S. A., and S. Todorcevic, *Ramsey Methods in Analysis*, Birkhäuser, Basel, 2005. Zbl 1092.46002. MR 2145246. 388

[2] Baumgartner, J. E., "Partition relations for countable topological spaces," *Journal of Combinatorial Theory, Series A*, vol. 43 (1986), pp. 178–95. Zbl 0647.05005. MR 0867644. DOI 10.1016/0097-3165(86)90059-2. 387, 388

[3] Piña, C., and C. Uzcátegui, "On topological properties of families of finite sets," *Journal of Combinatorial Theory, Series A*, vol. 119 (2012), pp. 1066–77. Zbl 1294.54004. MR 2891382. DOI 10.1016/j.jcta.2012.02.003. 387

[4] Todorcevic, S., *Introduction to Ramsey Spaces*, vol. 174 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, 2010. Zbl 1205.05001. MR 2603812. 388

[5] Weiss, W., "Partitioning topological spaces," pp. 154–71 in *Mathematics of Ramsey Theory*, edited by J. Nešetřil and V. Rödl, vol. 5 of *Algorithms and Combinatorics*, Springer, Berlin, 1990. Zbl 0755.54002. MR 1083599. 387

Acknowledgment

The author would like to thank the anonymous referee for a careful review of this article and for contributing valuable comments and suggestions which have improved the quality of the work.

Department of Mathematics
University of Paris 7—Denis Diderot
Paris
France
and
Department of Mathematics
University of the Andes
La Hechicera Campus
Mérida
Venezuela
claribet@math.univ-paris-diderot.fr
claribet@ula.ve