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A Note on Majkić's Systems

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Abstract The present note offers a proof that systems developed by Majkić are actually extensions of intuitionistic logic, and therefore not paraconsistent.

1 Introduction

In [3], Majkić developed two hierarchies of "paraconsistent" logic called Z_n and CZ_n ($1 \leq n < \omega$), which are variations of da Costa's hierarchy C_n (cf. da Costa [2]). As is mentioned in [3], this was motivated by the lack of "a kind of (relative) compositional model-theoretic semantics" (cf. [3, p. 404]) for da Costa's systems.

Now, the aim of the present note is to prove the following two facts.

Fact 1.1 Two hierarchies Z_n and CZ_n are *not* actually a hierarchy in the sense that for any $i \neq j$, $\mathrm{Th}(Z_i) = \mathrm{Th}(Z_j)$ and $\mathrm{Th}(CZ_i) = \mathrm{Th}(CZ_j)$ hold, where $\mathrm{Th}(S)$ stands for the set of theorems in a system S.

Fact 1.2 Systems Z_n and CZ_n are *not* paraconsistent, but instead they are extended systems of intuitionistic propositional calculus.

These will be proved by giving a simple axiomatization for Z_n and CZ_n which is different from the original one.

2 Formulation of Z_n and CZ_n

We shall first revisit the systems Z_n and CZ_n. First, the positive part of these systems is intuitionistic; that is, it consists of the following axiom schemata and a rule of inference (we shall refer to this system as IPC^+):

(1) $A \supset (B \supset A)$
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
(3) $(A \wedge B) \supset A$
(4) $(A \wedge B) \supset B$

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(5) $A \supset (B \supset (A \wedge B))$
(6) $A \supset (A \vee B)$
(7) $B \supset (A \vee B)$
(8) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

(MP) $\dfrac{A \qquad A \supset B}{B}$.

In addition to the system IPC$^+$, Z_n has some axiom schemata, which are related to negation, but before stating them we need the following definition as it is done in da Costa's systems.

Definition 2.1 Let A be a formula and $1 \leq n < \omega$. Then we define A°, A^{n}, and $A^{(\mathrm{n})}$ as follows:

$$A^\circ =_{\mathrm{def}} \neg(A \wedge \neg A)$$
$$A^{\mathrm{n}} =_{\mathrm{def}} A^{\overbrace{\circ\circ\cdots\circ}^{n}}$$
$$A^{(\mathrm{n})} =_{\mathrm{def}} A^1 \wedge A^2 \wedge \cdots \wedge A^{\mathrm{n}}\,.$$

Remark 2.2 Note that the definition given by Majkić in [3, p. 403] is inaccurate. Here we have adopted the original definition given by da Costa in [2, p. 500].

With the help of the above definition, we obtain the system Z_n for each n by adding the following schemata to the system IPC$^+$.

(11) $B^{(\mathrm{n})} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$
(12) $(A^{(\mathrm{n})} \wedge B^{(\mathrm{n})}) \supset ((A \wedge B)^{(\mathrm{n})} \wedge (A \vee B)^{(\mathrm{n})} \wedge (A \supset B)^{(\mathrm{n})})$
(9b) $(A \supset B) \supset (\neg B \supset \neg A)$
(10b) $1 \supset \neg 0, \neg 1 \supset 0$
(11b) $A \supset 1, 0 \supset A$
(12b) $(\neg A \wedge \neg B) \supset \neg(A \vee B)$

Finally, the hierarchy CZ_n can be obtained by adding the following formula:

(13b) $\neg(A \wedge B) \supset (\neg A \vee \neg B)$.

Remark 2.3 Note here that 0 and 1 are considered as contradiction and tautology nullary logic operators (constants), respectively, in the present system (cf. [3, p. 412]).

In the following section, we shall give another formulation of Z_n and CZ_n.

3 Another Formulation of Z_n and CZ_n

We now consider systems which are inferentially equivalent to Z_n and CZ_n.

Definition 3.1 Let Ω be a system which consists of the following axiom schemata in addition to IPC$^+$:

(9b) $(A \supset B) \supset (\neg B \supset \neg A)$
(10b) $1 \supset \neg 0, \neg 1 \supset 0$
(11b) $A \supset 1, 0 \supset A$.

Also, we shall refer to the extended system of Ω enriched with the following axiom scheme as $C\Omega$:

(13b) $\neg(A \wedge B) \supset (\neg A \vee \neg B)$.

Remark 3.2 It might be curious why we refrain from referring to the system introduced above simply as Z, without the subscript n. The reason is that since there already is a system of paraconsistent logic called Z studied in Béziau [1], we wanted to avoid any misunderstanding.

Now we shall prove some theses of Ω. Note that several theses, listed below, can be proved in IPC$^+$:

(T1) $A \supset A$
(T2) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$
(T3) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
(T4) $(A \supset (B \supset C)) \supset ((C \supset D) \supset (A \supset (B \supset D)))$
(T5) $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$
(T6) $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$
(T7) $(A \wedge B) \supset (B \wedge A)$.

We shall make use of these in the proof of the following proposition.

Proposition 3.3 *The following theses are provable in* Ω*:*

(T8) $A \supset (\neg A \supset B)$
(T9) $(A \supset 0) \supset \neg A$
(T10) $\neg(A \wedge \neg A)$
(T11) $(A \supset (B \wedge \neg B)) \supset \neg A$
(T12) $(\neg A \wedge \neg B) \supset \neg(A \vee B)$.

Proof We can prove the proposition as follows:

For (T8):

1.	$(1 \supset A) \supset (\neg A \supset \neg 1)$	(9b)
2.	$A \supset (1 \supset A)$	(1)
3.	$A \supset (\neg A \supset \neg 1)$	1, 2, (T3), (MP)
4.	$\neg 1 \supset B$	(10b), (11b), (T3), (MP)
5.	$A \supset (\neg A \supset B)$	3, 4, (T4), (MP)

For (T9):

1.	$(A \supset A) \supset 1$	(11b)
2.	$1 \supset \neg 0$	(10b)
3.	$(A \supset A) \supset \neg 0$	1, 2, (T3), (MP)
4.	$\neg 0$	3, (T1), (MP)
5.	$(A \supset 0) \supset (\neg 0 \supset \neg A)$	(9b)
6.	$\neg 0 \supset ((A \supset 0) \supset \neg A)$	5, (T2), (MP)
7.	$(A \supset 0) \supset \neg A$	4, 6, (MP)

For (T10):

1.	$((A \wedge \neg A) \supset 0) \supset \neg(A \wedge \neg A)$	(T9)
2.	$(A \wedge \neg A) \supset 0$	(T8), (T6), (MP)
3.	$\neg(A \wedge \neg A)$	1, 2, (MP)

For (T11):

1.	$(A \supset (B \wedge \neg B)) \supset (\neg(B \wedge \neg B) \supset \neg A)$	(9b)
2.	$\neg(B \wedge \neg B) \supset ((A \supset (B \wedge \neg B)) \supset \neg A)$	1, (T2), (MP)
3.	$(A \supset (B \wedge \neg B)) \supset \neg A$	2, (T10), (MP)

For (T12):

1.	$((A \vee B) \wedge (\neg A \wedge \neg B)) \supset 0$	(T8), (T6), (8), (MP)
2.	$((\neg A \wedge \neg B) \wedge (A \vee B)) \supset 0$	1, (T7), (T3), (MP)
3.	$(\neg A \wedge \neg B) \supset ((A \vee B) \supset 0)$	2, (T5), (MP)
4.	$(\neg A \wedge \neg B) \supset \neg(A \vee B)$	3, (T9), (T3), (MP)

□

Remark 3.4 It should be noted that IPC^+ together with (T8) and (T11) give a formulation of intuitionistic propositional calculus. Therefore, $\mathbf{\Omega}$ contains intuitionistic propositional calculus as its subsystem. In other words, $\mathbf{\Omega}$ is an extension of intuitionistic propositional calculus.

Making use of this proposition, we can prove the following theorem.

Theorem 3.5 *For each n, Z_n and CZ_n are inferentially equivalent to $\mathbf{\Omega}$ and $C\mathbf{\Omega}$, respectively.*

Proof We shall first consider the systems Z_n and $\mathbf{\Omega}$. It is obvious that $\mathbf{\Omega}$ is a subsystem of Z_n, so it would be sufficient to show that Z_n is a subsystem of $\mathbf{\Omega}$. For this purpose, we need to prove that axioms (11), (12), and (12b) are theses of $\mathbf{\Omega}$. But this is an immediate consequence of the previous proposition. As for the inferential equivalence of CZ_n and $C\mathbf{\Omega}$, just add (13b) to both Z_n and $\mathbf{\Omega}$. □

Remark 3.6 Therefore, combining Remark 3.4 and Theorem 3.5, we conclude that systems Z_n and CZ_n are *not* paraconsistent but they are extensions of intuitionistic propositional calculus.

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