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Comparing Borel Reducibility and Depth of an ω-Stable Theory

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Abstract In "A proof of Vaught's conjecture for ω-stable theories," the notions of ENI-NDOP and eni-depth have been introduced, which are variants of the notions of NDOP and depth known from Shelah's classification theory. First, we show that for an ω-stable first-order complete theory, ENI-NDOP allows tree decompositions of *countable* models. Then we discuss the relationship between eni-depth and the complexity of the isomorphism relation for countable models of such a theory in terms of Borel reducibility as introduced by Friedman and Stanley and construct, in particular, a sequence of complete first-order ω-stable theories $(T_\alpha)_{\alpha<\omega_1}$ with increasing and cofinal eni-depth and isomorphism relations which are strictly increasing with respect to Borel reducibility.

1 Introduction

Friedman and Stanley [5] introduced a model theoretic context for the descriptive set theoretic notion of Borel reducibility. If L is a countable language, the set $\mathrm{Mod}(\sigma)$ of models with underlying set ω of an $L_{\omega_1\omega}$-sentence σ form a standard Borel space and Borel reducibility applies to the isomorphism relation for those countable models, which we denote by $\cong_\sigma$: If L and L' are countable and $\sigma \in L_{\omega_1\omega}$, $\sigma' \in L'_{\omega_1\omega}$, we say that $\cong_\sigma$ *reduces to* $\cong_{\sigma'}$ (notation: $\cong_\sigma \ \leq_B \ \cong_{\sigma'}$) if there is a Borel map $f : \mathrm{Mod}(\sigma) \to \mathrm{Mod}(\sigma')$ such that for all $M, N \in \mathrm{Mod}(\sigma)$, $M \cong N$ if and only if $f(M) \cong f(N)$. For example, if R is a binary relation symbol and $L = \{R\}$, σ the empty L-theory, it is a well-known fact that every $\cong_\tau$ reduces to $\cong_\sigma$; that is, the theory of graphs is maximal with respect to the partial preordering $\leq_B$.

The investigation of the ordering $\leq_B$ is closely related to the classification problem of countable models: if $\cong_\sigma \ \leq_B \ \cong_{\sigma'}$ then complete isomorphism invariants for countable models of σ' give rise to complete invariants for countable models of σ; that is, the classification problem for σ is at most as complicated as the one for σ'.

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The ordering $\leq_B$ has been extensively investigated and special attention has been payed to so-called essentially countable isomorphism relations (see, e.g., [6], [8], [10], [7], [9], [11], [16]).

However, not much has been said yet about the case of first-order theories or more specifically *complete* first-order theories (for some first results see [14], [15], and [17]). We focus on the context of ω-stable theories which allows us to use strong tools; in particular, (in well-behaved cases) we have tree decompositions for models. References for ω-stability and Shelah's classification theory are [20] and [1]. If T is an ω-stable theory, the notion of NDOP (the negation of DOP) can be defined, which implies that each model $M \models T$ is prime over an independent tree of "finitely generated" countable submodels (see [20] or [1]). The supremum of ordinal depths of such decomposition trees is called the *depth* of T, which is a countable ordinal or ∞. In the latter case we call T *deep*; otherwise it is *shallow*. Intuitively, ω-stable theories with DOP are more complicated than those with NDOP, and among those, deep theories are more complicated than shallow ones. In the shallow case, the depth can be considered as a measure of complexity of that theory.

There does not seem to be any relationship between depth and Borel-reducibility, since theories of any depth and even deep or DOP theories can be $\aleph_0$-categorical. However, [19] introduces a variation of the notion of depth which is more pertinent to the case of countable models, the *eni-depth*. This is done by essentially focusing on so-called ENI types, that is, strongly regular types based on finite sets that can have finite dimensions in models. We will add one more natural notion of depth which we denote by ENI-depth and we will see that ENI-NDOP allows us to decompose *countable* models by trees (a result established independently in [14]) and that the eni-depth is to some extent related to the Borel reducibility notion of complexity, whereas the ENI-depth is not an appropriate notion to measure complexity of countable models.

A third notion of complexity is the *Scott height* of a theory which roughly speaking measures how long back-and-forths must be to distinguish nonisomorphic models of that theory. There exist various versions of that notion in literature, some of which are mentioned in [13]. Our definition here will be the following.

Definition 1.1 Let M and N be two L-structures and for some $n < \omega$, $\bar{a} \in M^n$, $\bar{b} \in N^n$. Then

1. $(M, \bar{a}) \equiv_0 (N, \bar{b})$ if $\bar{a}, \bar{b}$ have the same quantifier free type over $\varnothing$;
2. $(M, \bar{a}) \equiv_\alpha (N, \bar{b})$ (α limit) if $(M, \bar{a}) \equiv_\beta (N, \bar{b})$ for all $\beta < \alpha$;
3. $(M, \bar{a}) \equiv_{\alpha+1} (N, \bar{b})$ if
   (a) $\forall a \in M \exists b \in N \ (M, \bar{a}, a) \equiv_\alpha (N, \bar{b}, b)$ and
   (b) $\forall b \in N \exists a \in M \ (M, \bar{a}, a) \equiv_\alpha (N, \bar{b}, b)$;
4. $(M, \bar{a}) \equiv_\infty (N, \bar{b})$ if $(M, \bar{a}) \equiv_\alpha (N, \bar{b})$ for all α.

If T is a theory and M a countable model of T, by Scott's Isomorphism Theorem, there exists a countable α such that for all countable $N \models T$, $(M, \varnothing) \equiv_\alpha (N, \varnothing)$ implies $(M, \varnothing) \equiv_\infty (N, \varnothing)$ (and thus actually $M \cong N$). Call a minimal such α the *Scott height* $\mathrm{SH}(M)$ of M and let $\mathrm{SH}(T) = \sup\{\mathrm{SH}(M) | M \models T, M \text{ countable } \}$.

By definition, $\mathrm{SH}(T)$ is either countable or ω_1. Becker and Kechris prove in [2] that it is ω_1 if and only if the isomorphism for countable models of T is not Borel (although it does have to be analytic). [5] shows that the theory of Abelian p-groups is not Borel, but not very complicated with respect to $\leq_B$. This example is far from being first-order axiomatizable. On the other hand, an isomorphism relation of maximal complexity (bi-reducible with graphs) must be non-Borel. We do not know any first-order example which has non-Borel isomorphism but is not of maximal complexity, and much less a complete or ω-stable such.

Our motivation for considering Scott heights is that countable trees of ordinal rank at most α have Scott height $\omega \cdot \alpha$ and for ω-stable ENI-NDOP theories T, isomorphism types of countable models are described by trees of rank at most the ENI-depth of T. This suggests that such theories of low ENI-depth might have bounds on their Scott height. We will see here that this intuition is valid for the eni-depth 1 case, but as will be exposed in another paper (in preparation, presenting an example of [13]), it fails drastically in general.

The results of this paper are the following:

1. ENI-NDOP allows tree decompositions of countable models (we understand Laskowski and Shelah prove the same result in [14] independently). Moreover, those decompositions can be chosen to realize only ENI types.
2. If a theory has only $\kappa \leq \aleph_0$ many countable models, its isomorphism reduces to equality on a set of κ elements.
3. Since Laskowski and Shelah already deal with the case of ENI-DOP and eni-deep theories in [14] by showing that their isomorphism has maximal complexity, we can focus on eni-shallow theories. We show that eni-depth 1 theories have isomorphisms which reduce to equality on the real numbers (i.e., they are "smooth").
4. As our main result, we construct an ω_1-sequence of complete first-order ω-stable theories with strictly increasing eni-depth and also isomorphism relations which increase strictly with respect to $\leq_B$. Similar theories have been used in [5] (trees as subsets of $[\omega]^{<\omega}$) and in [10] (hereditarily countable sets of bounded rank), but these are not complete first-order axiomatizable and thus do not fit in our context.

2 ENI Types and Tree Decompositions

We assume throughout that our theories eliminate quantifiers (otherwise we can replace it with its Morleyization). The main consequences of our general assumption of ω-stability are the following.

1. There is a well-behaved notion of (forking-)independence of sets in models (notation for "A is independent from B over C" is $A \underset{C}{\downarrow} B$).
2. This leads to the notion of orthogonality of types (denoted by $\perp$): Two types p over A and q over B are orthogonal, if for all $C \supset A \cup B$ and a, b realizing nonforking extensions of p and q to C, respectively, $a \underset{C}{\downarrow} b$.
3. We recall that a type p over a set A is strongly regular if there is a formula $\varphi \in p$ such that for all $B \supset A$ and $q \in S(B)$ containing φ, q is either a nonforking extension of p or orthogonal to p. Strongly regular types have a well-defined dimension in models and there are "enough" strongly regular types: for all models $M \subsetneq N$, there is an $a \in N \setminus M$ such that $t(a/M)$ is strongly regular.
4. All types are based on (i.e., do not fork over) *finite* sets of parameters. We can also always find finite sets of parameters which make types *stationary*,

that is, such that there is a unique nonforking extension of the type to any superset of parameters.

5. There are prime models over any set of parameters (which are unique up to isomorphism fixing pointwise those parameters).

We will also use the notion of a type p being *orthogonal to a set* A ($p \perp A$) meaning that $p \perp q$ for all $q \in S(A)$, and the notion of a set A being *independent over* B, meaning that $a \underset{B}{\downarrow} A \setminus \{a\}$ for all $a \in A$. If $p \in S(A)$ is stationary and $A \subset B$, let $p|B$ denote the unique nonforking extension of p to B.

Notation From now on, throughout this paper, let T be an ω-stable theory.

The notion of an *ENI type* was first defined in [19]. For another exposition, see [1]. For A finite, a strongly regular type $p \in S(A)$ is ENI if there is a finite $B \supset A$ such that the nonforking extension of p to B is nonisolated (strongly regular types are stationary by definition). Equivalently, p is ENI if its dimension is finite in a prime model over A. If p is strongly regular and not ENI we say it is NENI. Types over infinite sets are ENI if they are nonforking extensions of ENI-types over finite sets. It can be shown that a strongly regular type nonorthogonal to an ENI type must also be ENI (see [19]).

We now define trees of models and decomposition trees.

Definition 2.1 A *tree of models* is an application

$$\mu : A \to \{N|N \models T\}$$

with

(i) $A \subset \kappa^{<\omega}$ is a tree (for some cardinal κ);
(ii) $\mu(\varnothing)$ is prime (over $\varnothing$);
(iii) for $s \in A$ nonempty, let s^- denote its unique predecessor; then $\mu(s)$ is prime over $\mu(s^-) \cup \{a\}$ for some a realizing a strongly regular type over $\mu(s^-)$;
(iv) for all $s \in A$, the family $\{\mu(t)|t^- = s\}$ is independent over $\mu(s)$;
(v) for $s \in A$ with $|s| \geq 2$, $t(\mu(s)/\mu(s^-)) \perp \mu(s^{--})$.

Later, we will use the following straightforward fact about trees of models.

Lemma 2.2 *Let μ be a tree of models, $A \subset$ dom(μ) a subtree of* dom(μ) *and $B \subset$* dom$(\mu) \setminus A$ *the set of successors of elements of A in* dom$(\mu) \setminus A$. *Then, if M is prime over $\bigcup$* im$(\mu \restriction A)$,

(i) *if $a \in A$, $\{\mu(b)|b \in B, b \supset a\} \underset{\mu(a)}{\downarrow} M$,*
(ii) *$\{\mu(b)|b \in B\}$ is independent over M.*

Definition 2.3 A tree of models $\mu : A \to \{N|N \preceq M\}$ is called a *decomposition* of a model M if M is prime over $\bigcup \mathrm{im}(\mu)$.

Definition 2.4 An ω-stable theory T has ENI-NDOP if for all models

$$M_0, M_1, M_2, N \text{ with } M_0 \subset M_1 \cap M_2,\ M_1 \underset{M_0}{\downarrow} M_2,\ N \text{ prime over } M_1 \cup M_2,$$
$$\text{if } p \in S(N) \text{ is ENI, then } p \not\perp M_1 \text{ or } p \not\perp M_2.$$

We have an "ENI-version" (with the same proof) of Lemma XVII, 2.2 in [1].

Lemma 2.5 *Let μ be a decomposition of M and p be an* ENI *type nonorthogonal to M. Then* ENI-NDOP *implies that there is an $s \in$* dom(μ) *with $p \not\perp \mu(s)$.*

3 Maximal Atomic Models

Definition 3.1 A submodel M of N is *maximal atomic* over a set $A \subset M$ in N if

(i) it is atomic over A,

(ii) for all $M' \subset N$ containing M, if M' is atomic over A, then $M' = M$.

This notion is already used in [19] and that article also essentially contains the ingredients for the following two lemmas.

Lemma 3.2 *If M is maximal atomic in N (over $\varnothing$), then all strongly regular types $p \in S(M)$ realized in N are* ENI.

Proof Otherwise, if $a \models p$ and p is based on a finite $B \subset M$, then for all finite $C \supset B$, $t(a/C)$ would be isolated, and hence also $t(aC/\varnothing)$, which shows that $M \cup \{a\}$ is atomic, contradicting maximality of M. □

Lemma 3.3 *If $M, A \subset N$ with A independent over M and $M' \subset N$ is maximal atomic in N over $M \cup A$, then every strongly regular type $p \in S(M')$ orthogonal to M realized in N is* ENI.

Proof Assume p is NENI and realized by $a \in N$ and let p be based on a finite $B \subset M'$. We contradict the maximality of M' by showing that for all finite $C \subset M'$ containing B, $t(aC/MA)$ is isolated. It is enough to show that $t(a/MAC)$ is isolated.

Let $A_0 \subset A$ be finite with $C \underset{MA_0}{\downarrow} MA$. The set A is independent over M; hence $A \setminus A_0 \underset{M}{\downarrow} A_0$ and thus $A \setminus A_0 \underset{M}{\downarrow} A_0C$ which implies that $A \setminus A_0$ is independent over MA_0C. Now $p \perp M$ implies $p \perp t(b/M)$ for all $b \in A$. In particular, $p \perp t(b/MA_0C)$ for all $b \in A \setminus A_0$ and thus $p \perp t(A \setminus A_0/MA_0C)$. Noting $q|X$ the nonforking extension of a stationary type q, we now have $p|MA_0C \vdash p|MAC$ since p was based on C. Since we want to show that $t(a/MAC) = p|MAC$ is isolated, it now suffices to show that $p|MA_0C$ is isolated.

Let $D \subset M$ be finite with $CA_0 \underset{D}{\downarrow} M$. $p \perp M$ implies $p \perp D$, whence $p \perp t(M/D)$. This and $CA_0 \underset{D}{\downarrow} M$ imply $p \perp t(M/DA_0C)$ and since p was based on C, $p|DA_0C \vdash p|MA_0C$. Now, DA_0C is finite and p was supposed to be NENI; hence $p|DA_0C$ is isolated as well as $p|MA_0C$. □

4 Construction of ENI-Decompositions

Let M be any *countable* model of T. Assuming ENI-NDOP and in particular the conclusion of Lemma 2.5, we will construct a decomposition of M where all types used in that construction will be ENI. The main idea is to choose certain prime models *maximal atomic* and then use Lemmas 3.2 and 3.3.

We will inductively construct the following objects:

1. trees $S_n \subset \omega^{<n+1}$ (for each $n < \omega$) such that S_{n+1} extends S_n;
2. for each $n < \omega$ and $s \in S_{n+1} \setminus S_n$ a cardinal $\kappa_s \leq \aleph_0$ with
$$\{t \in S_{n+2} | t^- = s\} = \{s^\frown i | i < \kappa_s\};$$
3. for each $n < \omega$ and $s \in S_n$ a model $M_s \subset M$;
4. for each $n < \omega$, $s \in S_n$, $i < \kappa_s$ an element $a_i^s \in M$.
(Notation: $A_s = \{a_i^s | i < \kappa_s\}$)

5. An increasing sequence of submodels N_n of M ($n < \omega$).

These objects will have the following properties throughout the induction:

1. all a_i^s realize ENI types over M_s;
2. for s the empty sequence, M_s is maximal atomic in M (over $\varnothing$) and A_s is maximal independent in M over M_s;
3. if $s = t^\frown i$, M_s is maximal atomic in M over $M_t a_i^t$ and A_s is independent over M_s, maximal in M with the property that $t(a_j^s/M_s) \perp M_t$ for all $j < \kappa_s$;
4. $N_0 = M_\varnothing$ and N_{n+1} is both prime over $N_n \cup \bigcup_{s \in S_{n+1} \setminus S_n} M_s$ and maximal atomic in M over $N_n \cup \bigcup_{s \in S_n,\ |s|=n} A_s$.

Our construction is as follows:

1. Let $M_\varnothing$ be maximal atomic in M (over $\varnothing$), $N_0 = M_\varnothing$ and $S_0 = \{\varnothing\}$. Choose a maximal $M_\varnothing$-independent set $A_\varnothing$ of realizations of strongly regular types over $M_\varnothing$ in M. Let $\kappa_\varnothing = |A_\varnothing|$ and $\{a_i^\varnothing | i < \kappa_\varnothing\}$ be an enumeration of $A_\varnothing$. By Lemma 3.2, all $a_i^\varnothing$ realize ENI-types over $M_\varnothing$.
2. We now assume the following objects are already constructed for some n (and have the properties mentioned above): S_n, N_n, M_s, $A_s = \{a_i^s | i < \kappa_s\}$ for all $s \in S_n$. Let $S = \{s \in S_n |\ |s| = n\}$ and $A_n = \bigcup_{s \in S} A_s$. If $A_n = \varnothing$ we can finish the construction by setting $N_k = N_n$ and $S_k = S_n$ for all $k > n$. Otherwise,
   (a) let $S_{n+1} = S_n \cup \{s^\frown i | s \in S, i < \kappa_s\}$ and let N_{n+1} be maximal atomic in M over $N_n A_n$;
   (b) for all $s \in S$ and $i < \kappa_s$, let $M'_{s^\frown i} \subset N_{n+1}$ be prime over $M_s a_i^s$;
   (c) let $N'_{n+1} \subset N_{n+1}$ be prime over $N_n \cup \bigcup_{s \in S_{n+1} \setminus S_n} M'_s$;
   (d) N'_{n+1} is also prime over $N_n A_n$ and thus $N_n A_n$-isomorphic to N_{n+1}; for all $s \in S$ and $i < \kappa_s$, let $M_{s^\frown i}$ be the image of $M'_{s^\frown i}$ under such an isomorphism;
   (e) then we define for each $s \in S_{n+1} \setminus S_n$ the sets A_s as a maximal M_s-independent set of realizations of strongly regular types over M_s, orthogonal to M_{s^-};
   (f) enumerate A_s as $A_s = \{a_i^s | i < \kappa_s\}$ for some $\kappa_s \leq \aleph_0$; by Lemma 2.2, A_s is actually independent over N_n and by Lemma 3.3, all a_i^s must realize ENI types over M_s.

Let $S = \bigcup_{n<\omega} S_n$ and define μ as $\mu(s) = M_s$ for all $s \in S$. Clearly, μ is a tree of models realizing only ENI types. We will show that M is prime over $\bigcup \mathrm{dom}(\mu)$. Set $\bar{M} = \bigcup_{n<\omega} N_n$. We first show that $\bar{M}$ is prime over $\bigcup \mathrm{dom}(\mu)$. Let $\bar{M}' \subset \bar{M}$ be prime over $\bigcup \mathrm{dom}(\mu)$. We will construct inductively an embedding of $\bar{M}$ into $\bar{M}'$ which fixes $\bigcup \mathrm{dom}(\mu)$. Let f_0 be the identity on N_0. For an arbitrary $n < \omega$, suppose f_n is an embedding of N_n into $\bar{M}'$ such that

1. $f_n \restriction \mathrm{dom}(\mu \restriction S_n)$ is the identity on $\mathrm{dom}(\mu \restriction S_n)$,
2. $\mathrm{im}(f_n)$ is prime over $\mathrm{dom}(\mu \restriction S_n)$.

As a consequence of Lemma 2.2, we can extend f_n to an application g defined on $N_n \cup \mathrm{dom}(\mu \restriction S_{n+1})$. Now, since N_{n+1} is prime over that set, we can extend g to an embedding of N_{n+1} into $\bar{M}'$ which defines f_{n+1}.

Finally, we show that actually $\bar{M} = M$. Suppose not and let $c \in M \setminus \bar{M}$ be such that $p = t(c/\bar{M})$ is strongly regular. There is a minimal $n < \omega$ such that $p \not\perp N_n$ and thus a strongly regular $q \in S(N_n)$ such that $q \not\perp p$. We must have $n > 0$; otherwise, the existence of $c \notin \bar{M}$ contradicts the maximality of $A_\varnothing$. We thus have $p \perp N_{n-1}$ and also $q \perp N_{n-1}$. p must be ENI; otherwise, q is NENI and has positive dimension in M contradicting the maximality of N_n (Lemma 3.3).

We can now apply Lemma 2.5 and find an $s \in S$ such that $p \not\perp M_s$. If we take the *minimal* such s, $p \perp M_{s^-}$ and we find a strongly regular $r \in S(M_s)$ nonorthogonal to p which must also satisfy $r \perp M_{s^-}$. But now, the existence of $c \notin \bar{M}$ contradicts the maximality of A_s.

5 Depth

Now that we have decomposition trees for countable models, we can define corresponding notions of depth. There are two natural choices: first ENI-depth which counts only ENI types and thus measures the depth of decomposition trees which we obtain in Section 4. It will turn out that ENI-depth does not reflect correctly the complexity of countable models; another notion, denoted by eni-depth, seems more accurate for that purpose. It measures the lengths of paths in decomposition trees which have an ENI type on top. Here are the formal definitions.

Definition 5.1 A stationary type $p \in S(A)$ is said to *support* another type q, if there is a model $M \supset A$ and $a \models p|M$ such that $q \perp M$ and $q \not\perp M[a]$ (where $M[a]$ is a model prime over $M \cup \{a\}$).

Let p be a strongly regular type. We now define its ENI-depth and eni-depth.

Definition 5.2

1. $\mathrm{ENI} - \mathrm{dp}(p) \geq 0$ for all p.
2. For limit α, $\mathrm{ENI} - \mathrm{dp}(p) \geq \alpha$ if $\forall \beta < \alpha$ $\mathrm{ENI} - \mathrm{dp}(p) \geq \beta$.
3. $\mathrm{ENI} - \mathrm{dp}(p) \geq \alpha + 1$ if p is ENI and supports an ENI type q with $\mathrm{ENI} - \mathrm{dp}(q) \geq \alpha$.

Then we set $\mathrm{ENI} - \mathrm{dp}(p) = \infty$ if $\mathrm{ENI} - \mathrm{dp}(p) \geq \alpha$ for all α; otherwise, $\mathrm{ENI} - \mathrm{dp}(p) = \min\{\alpha | \mathrm{ENI} - \mathrm{dp}(p) \geq \alpha \text{ and } \mathrm{ENI} - \mathrm{dp}(p) \not\geq \alpha + 1\}$.

Note in particular that NENI types have ENI-depth 0 by definition. Also, the original definition of *depth* of a type (denoted by $\mathrm{dp}(p)$) is the same as that for ENI-depth, where the requirements of types being ENI are dropped.

Definition 5.3

1. $\mathrm{eni} - \mathrm{dp}(p) \geq 0$ for all p.
2. $\mathrm{eni} - \mathrm{dp}(p) \geq 1$ if p supports an ENI type.
3. For limit α, $\mathrm{eni} - \mathrm{dp}(p) \geq \alpha$ if $\forall \beta < \alpha$ $\mathrm{eni} - \mathrm{dp}(p) \geq \beta$.
4. $\mathrm{eni} - \mathrm{dp}(p) \geq \alpha + 1$ if p supports a q with $\mathrm{eni} - \mathrm{dp}(q) \geq \alpha$.

Again we set $\mathrm{eni} - \mathrm{dp}(p) = \infty$ if $\mathrm{eni} - \mathrm{dp}(p) \geq \alpha$ for all α; otherwise, $\mathrm{eni} - \mathrm{dp}(p) = \min\{\alpha | \mathrm{eni} - \mathrm{dp}(p) \geq \alpha \text{ and } \mathrm{eni} - \mathrm{dp}(p) \not\geq \alpha + 1\}$

If $\mathrm{eni} - \mathrm{dp}(T) = \infty$ (respectively, $\mathrm{ENI} - \mathrm{dp}(T) = \infty$), we call T *eni-deep* (*ENI-deep*); otherwise, *eni-shallow* (*ENI-shallow*).

It is easy to verify that $\mathrm{ENI} - \mathrm{dp}(p) \leq \mathrm{eni} - \mathrm{dp}(p) \leq \mathrm{dp}(p)$ for all p and that all these notions of depth are invariant under nonorthogonality (see [13] for details). Also, all depths must be *countable* ordinals or ∞.

We then define the ENI-depth and eni-depth of T as follows.

Definition 5.4 Let A be the set of all types realized in tree decompositions of models of T. Then

1. $\mathrm{ENI} - \mathrm{dp}(T) = \sup\{\mathrm{ENI} - \mathrm{dp}(p) + 1 | p \in A\}$,
2. $\mathrm{eni} - \mathrm{dp}(T) = \sup\{\mathrm{eni} - \mathrm{dp}(p) + 1 | p \in A\}$.

In [19], Shelah, Harrington, and Makkai show the following.

Proposition 5.5 *If T has* ENI-NDOP *and* $\mathrm{eni} - \mathrm{dp}(T) \geq 3$*, then T must have $2^{\aleph_0}$ countable models (up to isomorphism).*

The condition $\mathrm{eni} - \mathrm{dp}(T) \geq 3$ is optimal since there exist theories with eni-depth 2 with countably many countable models (see [1] or [13]). This and the following result by Laskowski and Shelah suggest that eni-depth is a good candidate to measure the complexity of the class of countable models of T.

Theorem 5.6 ([15],[14]) *If T either has* ENI-DOP *or has* ENI-NDOP *and is eni-deep, then the isomorphism relation for countable models (denoted by $\cong_T$) of T is* Borel-complete*, meaning that for all L and $\sigma \in L_{\omega_1\omega}$, $\cong_\sigma \leq_B \cong_T$.*

We know examples of ENI-NDOP theories which are eni-deep but have low ENI-depth. Since they are of maximal complexity by the last theorem, the notion of ENI-depth is not appropriate to measure complexity for countable models.

Here is a brief description of such an example (complete, ω-stable, eliminating quantifiers) having ENI-depth 2 (for more details, see [13]): Let $L = \{U, V, R, S, \pi\}$. U and V are unary predicates which partition the universe and π is a surjection of V onto U. Let S be a successor function on V (to create ENI types) such that π-fibers are unions of connected S-components. Finally, let R define a directed graph without cycles on U such that each element has infinitely many R-successors and R-predecessors.

R itself defines a deep theory with arbitrary long supportive chains of NENI-types, and since for each $a \in U$, the type $p(x) = \{\pi(x) = a\}$ is ENI, we get eni-depth ∞. On the other hand, the only chain of successive ENI-types has length 2: for $a \in U$ the type which says "x is not in the connected R-component of a" which supports the type $p(x)$ mentioned above.

6 Low Complexity

Since Theorem 5.6 deals already with the most complicated theories, we are left with the case of ENI-NDOP, eni-shallow theories. In this section, we will investigate theories of minimal eni-depth. But first, we can make an easy remark on theories with few countable models

In [19], Vaught's conjecture for ω-stable theories is proved. We now assume that T has less than $2^{\aleph_0}$ many countable models; thus T has only countably many countable models.

Proposition 6.1 *If T has $\kappa \leq \aleph_0$ many countable models, $\cong_T$ bi-reduces to the equality relation on a set S with κ elements.*

Proof Let X_L be the Polish space of L-structures with universe ω and $\mathrm{Mod}(T) \subset X_L$ the subspace of models of T. For each $M \in \mathrm{Mod}(T)$, its isomorphism type $\{N \in \mathrm{Mod}(T) | N \cong M\}$ is a Borel set (see, e.g., [12], Theorem 15.14), so the obvious reduction maps are Borel maps: in one direction, map each isomorphism type to a different element of S. In the other direction, map different elements of S to elements of different isomorphism types. $\square$

Bouscaren proved in [3] Martin's conjecture for ω-stable theories, which implies that such a theory with few countable models has Scott height at most $\omega \cdot 2$. This is an optimal bound since the eni-depth 2 example with $\aleph_0$ many countable models due to Shelah, exposed in [1] in XVIII,4 has Scott height $\omega \cdot 2$.

Now we can assume that T has $2^{\aleph_0}$ countable models. We can find a strong bound on complexity for theories with minimal eni-depth.

Theorem 6.2 *If T has* ENI-NDOP *and eni-depth 1, then $\cong_T$ reduces to equality on the Cantor space (it is called* smooth *or* tame*).*

Proof In [4], Bouscaren and Lascar show that if all strongly regular types (over finite sets of parameters) which are orthogonal to $\varnothing$ have infinite dimension in every model (which in our terminology means that supported types must be NENI; that is, T must have eni-depth 1), then all countable models are *almost homogeneous*, meaning that for all M and $\bar{a}, \bar{b} \in M^n$ having same strong type, there exists a strong automorphism f of M such that $f(\bar{a}) = \bar{b}$.

Pillay shows in [18] that under that condition, countable models are isomorphic if and only if they realize the same types over $\varnothing$. Let $(p_i)_{i<\omega}$ be a list of all types over $\varnothing$ and define $\chi : \mathrm{Mod}(T) \to 2^\omega$ by $\chi(M)(n) = 1$ if and only if p_n is realized in M. This application is clearly Borel and, by Pillay's result, we have $M \cong N$ if and only if $\chi(M) = \chi(N)$. $\square$

Pillay's characterization of countable models also implies that under the assumptions of Theorem 6.2, we must have $\mathrm{SH}(T) \leq \omega$.

7 A Sequence of Increasing Complexity

We now present a procedure for inductively constructing more and more complicated theories. Basically, in successor stages, the theory consists of an equivalence relation with infinitely many classes each of which contains a model of the preceding theory. In limit stages, we have for each preceding theory a unary predicate which contains a model of that theory. For technical reasons, our languages have to be relational. We will obtain a sequence of theories with strictly increasing complexity with respect to eni-depth, Borel-reducibility, and Scott height. In the following, whenever we do not give the proof of a lemma or proposition, this means that it is straightforward.

Let L be a language and E a new binary relation symbol. We inductively define for each L-formula φ and variable x an $L \cup \{E\}$-formula $\varphi^{E(x,-)}$ by relativizing quantifiers: $\forall y \psi$ becomes $\forall y (E(x, y) \to \psi)$ and $\exists y \psi$ becomes $\exists y (E(x, y) \wedge \psi)$.

Definition 7.1 If L is relational and T an L-theory, define the $L \cup \{E\}$-theory T' by the following axioms:

(1) E is an equivalence relation with infinitely many equivalence classes;
(2) for $n < \omega$ and $R \in L$ an n-ary relation,
$$\forall x_1, \ldots, x_n (R(x_1, \ldots, x_n) \to \bigwedge_{1 \leq i < j \leq n} E(x_i, x_j));$$
(3) for all $\varphi \in T$, $\forall x\, \varphi^{E(x,-)}$ (with x not already contained in φ).

It is straightforward to see that if T is complete, ω-stable, or eliminates quantifiers, T' still has these properties.

Notation From now on we assume that T eliminates quantifiers.

If $M \models T'$ and $a \in M$, let $[a]$ denote the E-class containing a which is actually a model of T by axiom (2) and the fact (proven by induction over the complexity of the formula) that for $\varphi \in L$, $[a] \models \varphi$ if and only if $M \models \varphi^{E(a,-)}$. We now describe what the 1-types in T' are.

Proposition 7.2 *Let $A \subset M \models T'$ and $p(x) \in S_1(A)$.*

1. *If $\neg E(x,a) \in p(x)$ for all $a \in A$, p is completely characterized by the set of formulas of the form $R(x,\ldots,x)$ it contains (for $R \in L$).*
2. *If $E(x,a) \in p(x)$ for some $a \in A$, let $\hat{p}$ be the T-type obtained from p by restricting it to L and the parameters $A \cap [a]$. Then p is equivalent (modulo T') to $\hat{p} \cup \{E(x,a)\}$.*

Now supposing that T (and thus T') is ω-stable, we can show the following.

Proposition 7.3 *Let $M \models T'$ and $p(x) \in S_1(M)$.*

1. *If $E(x,a) \in p$ for some $a \in M$, p is strongly regular if and only if $\hat{p} \in S_1([a])$ is strongly regular. In that case, p is* ENI *if and only if $\hat{p}$ is* ENI.
2. *If $\{\neg E(x,a) | a \in M\} \subset p$ the p is strongly regular if and only if $p \restriction \varnothing$ is isolated. In that case p is* NENI.

Using Lemma 7.4, we can show Proposition 7.5.

Lemma 7.4 *Let $M \models T'$ and $p(x), q(x) \in S_1(M)$.*

1. *If both p and q contain a formula $E(x,a)$ for some $a \in M$, then $p \perp q$ if and only if $\hat{p} \perp \hat{q}$.*
2. *Let N be a submodel of M. If p contains $E(x,a)$ for some $a \in M$ then $p \perp N$ if and only if $\hat{p} \perp N \cap [a]$.*

Proposition 7.5 *If T has the* (ENI-)NDOP, *then T' has the* (ENI-)NDOP.

Supposing from now on that T, and thus T', has ENI-NDOP, we will show that $\mathrm{eni}-\mathrm{dp}(T') = \mathrm{eni}-\mathrm{dp}(T) + 1$ and that the ENI-depth does not change. This follows from the following proposition.

Proposition 7.6 *Let $M \models T'$ and $p(x) \in S_1(M)$ strongly regular.*

1. *If $E(x,a) \in p$ for some $a \in M$, then $\mathrm{eni}-\mathrm{dp}(p) = \mathrm{eni}-\mathrm{dp}(\hat{p})$ and $\mathrm{ENI}-\mathrm{dp}(p) = \mathrm{ENI}-\mathrm{dp}(\hat{p})$.*
2. *If $\{\neg E(x,a) | a \in M\} \subset p$, then $\mathrm{eni}-\mathrm{dp}(p) = \mathrm{eni}-\mathrm{dp}(T)$ and $\mathrm{ENI}-\mathrm{dp}(p) = 0$ (since p is* NENI*).*

Having shown that by our construction, the eni-depth increases by one, we now come to the notion of Scott height and will see that it increases by ω.

Definition 7.7 For a limit ordinal α, a sequence $(M_i)_{i \leq \omega}$ of pairwise nonisomorphic models is an *α-chain* if there is an increasing sequence $(\alpha_i)_{i<\omega}$ cofinal in α such that for all $i < j \leq \omega$, $M_i \equiv_{\alpha_i} M_j$.

Now given a sequence $(M_i)_{i \leq \omega}$ of models of T, we construct a sequence $(M'_i)_{i \leq \omega}$ of models of T' by choosing for each M'_i exactly one E-class containing M_j for each $j < \omega$ and exactly i E-classes containing M_ω.

Proposition 7.8 *If $(M_i)_{i\leq\omega}$ is an α-chain, $(M'_i)_{i\leq\omega}$ is an $(\alpha+\omega)$-chain.*

Proof We show that for $k < l \leq \omega$, $M'_k \equiv_{\alpha+k} M'_l$ by constructing a back-and-forth procedure of length k resulting in k-tuples $\bar{a} = (a_1, \ldots, a_k) \in (M'_k)^k$, $\bar{b} = (b_1, \ldots, b_k) \in (M'_l)^k$ such that $(M'_k, \bar{a}) \equiv_\alpha (M'_l, \bar{b})$.

Whenever an a_i is chosen in an E-class containing a copy of M_n ($n < \omega$), we choose the corresponding b_i using an isomorphism between the copies of M_n in M'_k and M'_l. If a_i is in an M_ω component, we choose any M_ω-component in M'_l and an isomorphism between these components to assign a corresponding b_i. We use the symmetric procedure to make an element b_i in M'_l correspond to an a_i in M'_k.

To see that we finally have $(M'_k, \bar{a}) \equiv_\alpha (M'_l, \bar{b})$, we use the following observation (proved by induction on β). If $(N_i)_{i<\omega}$ and $(\tilde{N}_i)_{i<\omega}$ are enumerations of the E-classes of models $N, \tilde{N} \models T'$ and $\bar{c} \in N^m$, $\bar{d} \in \tilde{N}^m$ and if for all $i < \omega$, $(N_i, \bar{c}|i) \equiv_\beta (\tilde{N}_i, \bar{d}|i)$ (where $\bar{c}|i$ is the subtuple of $\bar{c}$ of elements in N_i), then $(N, \bar{c}) \equiv_\beta (\tilde{N}, \bar{d})$.

Using the fact that $(M_i)_{i\leq\omega}$ is an α-chain and the particular form of our back-and-forth described above, we can find such enumerations of the E-classes of M'_k and M'_l, respectively. □

This implies that if $\mathrm{SH}(T)$ is a limit ordinal and T admits an $\mathrm{SH}(T)$-chain, T' admits an $(\mathrm{SH}(T)+\omega)$-chain, and thus $\mathrm{SH}(T') \geq \mathrm{SH}(T)+\omega$.

To see that $\mathrm{SH}(T') \leq \mathrm{SH}(T)+\omega$, we show that for $M, N \models T'$ with $M \equiv_{\mathrm{SH}(T)+\omega} N$, and $\tilde{M} \models T$, M and N must contain the same number of copies of $\tilde{M}$. This follows from $M \equiv_{\mathrm{SH}(T)+\omega} N$ and the following lemma.

Lemma 7.9 *If $M, N \models T'$ and $\bar{a} = (a_1, \ldots, a_k) \in M^k$, $\bar{b} = (b_1, \ldots, b_k) \in N^k$ satisfy $(M, \bar{a}) \equiv_\beta (N, \bar{b})$, then for each $i \in \{1, \ldots, k\}$, $([a_i], \bar{a}|i) \equiv_\beta ([b_i], \bar{b}|i)$, where $\bar{a}|i$ is the subtuple of $\bar{a}$ of the a_j belonging to $[a_i]$.*

This finishes our investigation of the construction T' for the moment. Next, we will introduce a construction which can be seen as the "disjoint sum" of theories.

Definition 7.10 For some α and $\beta < \alpha$, let L_β be relational, disjoint languages and let T_β be an L_β-theory. Let A_β be new unary predicates and let

1. $\sum L_\beta$ be the language $\bigcup_{\beta<\alpha} (L_\beta \cup \{A_\beta\})$;
2. $\sum T_\beta$ be the $\sum L_\beta$-theory axiomatized by
   (a) the A_β are pairwise disjoint,
   (b) for all $i < \alpha$, $n < \omega$ and n-ary $R \in L_i$: $\forall x_1, \ldots, x_n (R(x_1, \ldots, x_n) \rightarrow \bigwedge_{1\leq j\leq n} A_i(x_j))$,
   (c) for all $i < \beta$ and $\varphi \in T_i$: the relativization of φ to A_i which we denote by φ^{A_i} (i.e., all quantifiers of φ^{A_i} are relativized to A_i, similar to our definition of $\varphi^{E(x,-)}$ above).

Again, if the T_β are all complete, ω-stable, or eliminate quantifiers, $\sum T_\beta$ will have the same properties. If $B \subset M \models \sum T_\beta$, let $A_\beta(B)$ be the set of elements of B satisfying $A_\beta(x)$. It is easy to see that $A_\beta(M)$ (with the induced structure) is a model of T_β.

Proposition 7.11 *Let $B \subset M \models \sum T_\beta$ and $p \in S_1(B)$. There are two possibilities:*

1. *p is the unique type containing $\{\neg A_\beta(x) | \beta < \alpha\}$;*
2. *$A_\beta(x) \in p$ for some $\beta < \alpha$; in that case, p is equivalent (modulo $\sum T_\beta$) to the type $\tilde{p} \cup \{A_\beta(x)\}$, where $\tilde{p} \in S_1(A_\beta(B))$ is the L_β-type obtained from p by restricting that type to L_β and to the parameters $A_\beta(B)$.*

Similar to the construction of T' above, we obtain the following results.

Proposition 7.12 *If $M \models \sum T_\beta$ and $p \in S_1(M)$ contains $\{\neg A_\beta(x) | \beta < \alpha\}$, then p is strongly regular and* ENI.

Proposition 7.13 *If $M \models \sum T_\beta$ and $p \in S_1(M)$ contains $A_\beta(x)$ for some $\beta < \alpha$, then*

1. *p is strongly regular if and only if $\tilde{p}$ is,*
2. *p is* ENI *if and only if $\tilde{p}$ is,*
3. *if $q \in S_1(M)$ is another type containing $A_\beta(x)$, then $p \perp q$ if and only if $\tilde{p} \perp \tilde{q}$,*
4. *if N is a submodel of M, $p \perp N$ if and only if $\tilde{p} \perp A_\beta(N)$.*

This implies that if *all* T_β have the (ENI-)NDOP, then also $\sum T_\beta$.

Proposition 7.14 *If $M \models \sum T_\beta$ and $p \in S_1(M)$ contains $A_\beta(x)$ for some $\beta < \alpha$, and is strongly regular,* $\mathrm{eni}-\mathrm{dp}(p) = \mathrm{eni}-\mathrm{dp}(\tilde{p})$ *and* $\mathrm{ENI}-\mathrm{dp}(p) = \mathrm{ENI}-\mathrm{dp}(\tilde{p})$.

And since a type containing $\{\neg A_\beta(x) | \beta < \alpha\}$ does not support any other type, we have the following theorem.

Theorem 7.15

$$\mathrm{eni}-\mathrm{dp}\left(\sum T_\beta\right) = \sup\{\mathrm{eni}-\mathrm{dp}(T_\beta) | \beta < \alpha\}$$

$$\textit{and } \mathrm{ENI}-\mathrm{dp}\left(\sum T_\beta\right) = \sup\{\mathrm{ENI}-\mathrm{dp}(T_\beta) | \beta < \alpha\}.$$

Concerning Scott heights, we get the following result.

Theorem 7.16 $\mathrm{SH}(\sum T_\beta) = \max(\omega, \sup\{\mathrm{SH}(T_\beta) | \beta < \alpha\})$ *and moreover, if* $\mathrm{SH}(\sum T_\beta)$ *is a limit ordinal and each T_β admits a $\gamma_\beta = \mathrm{SH}(T_\beta)$-chain with $(\gamma_\beta)_{\beta<\alpha}$ strictly increasing, then $\sum T_\beta$ admits a $\mathrm{SH}(\sum T_\beta)$-chain.*

This finishes our investigation of the sum of countably many theories and we can now proceed to the inductive definition of an ω_1-sequence of theories $(T_\alpha)_{1 \leq \alpha < \omega_1}$:

1. Let $L_1 = \{C_i\}_{i<\omega}$ where the C_i are unary predicates and let T_1 be the L_1-theory stating that all C_i are infinite and pairwise disjoint.
2. Let $T_{\alpha+1} = T'$.
3. For a limit ordinal $\alpha < \omega_1$, let $T_\alpha = \sum T_\beta$ where the β range over all ordinals smaller than α.

First of all, T_1 is complete, ω-stable, and eliminates quantifiers and thus all T_α have these properties. Also $\mathrm{eni}-\mathrm{dp}(T_1) = \mathrm{ENI}-\mathrm{dp}(T_1) = 1$ and thus $\mathrm{eni}-\mathrm{dp}(T_\alpha) = \mathrm{ENI}-\mathrm{dp}(T_\alpha) = \alpha$ for all α.

T_1 has Scott height ω and admits an ω-chain $(M_i)_{i\leq\omega}$ (where M_i realizes the type $p(x) = \{\neg C_j(x) | j < \omega\}$ exactly i times). Consequently, all T_α admit $\mathrm{SH}(T_\alpha)$-chains and we have $\mathrm{SH}(T_\alpha) = \omega \cdot \alpha$ for all α.

There remains to show that the isomorphisms $\cong_{T_\alpha}$ for countable models of the theories T_α form a strictly increasing chain with respect to the Borel reducibility ordering.

We will prove a slightly weaker result, namely, that the "hereditarily countable sets of rank α" formalized in [10] reduce to $\cong_{T_{1+\alpha}}$ which implies that there must be a strictly increasing (with respect to Borel reducibility) subsequence of $(T_\alpha)_{1\leq\alpha<\omega}$ of length ω_1, since [10] show that hereditarily countable sets of rank α form a strictly increasing ω_1-sequence which, moreover, is "Borel-cofinal" (meaning that all *Borel* isomorphism relations (for countable models of an $L_{\omega_1\omega}$ axiomatizable theory) reduce to a member of that sequence).

Since our proof uses standard coding techniques and the details are not very deep but long to write, we will only give a somewhat informal description of it. For more details, the reader may refer to [13]. We begin with the definition of κ-uniform models of T_α.

Definition 7.17 For $\alpha < \omega_1$ and $\kappa \leq \aleph_0$, let $M^\kappa_\alpha \in X_{T_\alpha}$ be inductively defined as follows.

1. If $\alpha = 1$, let M^κ_α be the countable model of T_1 whose E-classes have all exactly κ elements realizing $\{\neg C_j(x) | j < \omega\}$.
2. Given M^κ_α, let $M^\kappa_{\alpha+1}$ be the model whose E-classes contain copies of M^κ_α.
3. For α a limit ordinal, let M^κ_α be the model omitting the type $\{\neg A_\beta(x) | \beta < \alpha\}$ which has a copy of M^κ_β in each predicate A_β $(\beta < \alpha)$.

First, we show that for $\alpha \leq \beta$, $\cong_{T_\alpha} \leq_B \cong_{T_\beta}$. Let X_{T_α} and X_{T_β} be the Polish spaces of models of T_α and T_β with universe ω. We fix α and define a Borel function $g_{\alpha\beta} : X_{T_\alpha} \to X_{T_\beta}$, witnessing that reduction, by induction on β:

1. $g_{\alpha\alpha}$ is the identity function;
2. given $g_{\alpha\beta}$, $g_{\alpha(\beta+1)}$ maps an $M \in X_{T_\alpha}$ into one E-class using $g_{\alpha\beta}$ and "fills up" the remaining E-classes with copies of $M^{\aleph_0}_\beta$;
3. for limit β, $g_{\alpha\beta}$ maps an $M \in X_{T_\alpha}$ identically onto A_α and fills up the remaining A_γ $(\gamma \neq \alpha)$ with $\aleph_0$-uniform models as before.

The particular form of the models used to "fill up" classes and predicates guarantees that we have $M \cong N$ if and only if $g_{\alpha\beta}(M) \cong g_{\alpha\beta}(N)$, and the definitions can be carried out in a Borel way.

Now, we will recall what precisely the "hereditarily countable sets" used in [10] are. We start with the natural numbers ω and inductively iterate the process of taking countable subsets: set $\mathcal{P}^0(\omega) = \omega$ and $\mathcal{P}^\alpha(\omega) = [\omega \cup \bigcup_{\beta<\alpha} \mathcal{P}^\beta(\omega)]^{\leq\aleph_0}$. [10] gives an $L_{\omega_1\omega}$ axiomatization of the class of sets $\mathcal{P}^\alpha(\omega)$ for all $\alpha < \omega_1$: For $\alpha > 0$, let $L(\alpha) = \{(R_\beta)_{\beta\leq\alpha}, \epsilon, E, F, v_0, (r_i)_{i<\omega}\}$ be the language with R_β $(\beta < \alpha)$ unary predicates, ϵ, E, F binary relation symbols, and v_0 and r_i $(i < \omega)$ constant symbols. Let P^α be the class of structures in $X_{L(\alpha)}$ satisfying the following conditions:

(i) The R_β partition ω, $v_0 \notin R_0$, $R_0 = \{r_i | i < \omega\}$, the r_i are distinct.

(ii) Noting $R^+ = \bigcup_{1\leq\beta\leq\alpha} R_\beta$, $E \subset (R^+)^2$ defines a symmetric and irreflexive connected graph, without cycles so that for all elements $x \in R^+$, there is a (unique) path from x to v_0. We note $y \prec x$ if and only if there exists some $n < \omega$ and $y_1, \ldots, y_n$ such that $(v_0, y_1, \ldots, y_n, x, y)$ is the unique E-path from v_0 to y. We say that x is a *terminal* element, if there is no y with $y \prec x$.

(iii) The relation $\prec$ is well-founded.

(iv) We define the *rank* $\mathrm{rk}(x)$ of an element x by $\mathrm{rk}(x) = 1$ if x is terminal and $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 | y \prec x\}$ otherwise. Then, we postulate that $R_\beta = \{x | \mathrm{rk}(x) = \beta\}$ for $1 \leq \beta \leq \alpha$ (in particular, $R_\alpha = \{v_0\}$).

(v) $F \subset R_0 \times R_1$.

(vi) To each x, we assign sets $||x||$ by
 (a) $||x|| = n$ if $x \in R_0$ and $x = r_n$,
 (b) $||x|| = \{n | F(r_n, x)\}$ if $x \in R_1$,
 (c) $||x|| = \{||y|| \, | y \prec x\}$ otherwise,
 and we stipulate that for every $x \in R^+$ *which is not terminal* and $y, z \prec x$, we have $||y|| = ||z||$ if and only if $y = z$.

(vii) We define the relation ϵ by $x \epsilon y$ if and only if $||x|| \in ||y||$.

Let $\cong_\alpha$ denote the isomorphism relation on P^α. Then, for $M \in P^\alpha$ we set $||M|| = ||v_0||$, and [10] shows that $\mathcal{P}^\alpha(\omega) = \{||M|| \, | M \in P^\alpha\}$ and $M \cong N$ if and only if $||M|| = ||N||$ for all $M, N \in P^\alpha$.

Definition 7.18 Let $0 < \alpha < \omega_1$ and $M \in P^\alpha$. Then each $x \in \omega$ satisfying $x E v_0$ in M defines (up to isomorphism) a $P^{\mathrm{rk}(x)}$-structure $M(x)$. The universe B_x of $M(x)$ is the union of R_0 and the *E-cone*

$$\{x\} \cup \{y \in \omega | \exists n < \omega \exists y_1, \dots y_n \ (y, y_1, \dots y_n, x) \text{ is a } E\text{-path}\}$$

and the relations ϵ, E, F, R_β ($\beta \leq \mathrm{rk}(x)$) are those induced by the corresponding relations on M. We consider $M(x)$ as an element of $P^{\mathrm{rk}(x)}$ (i.e., with universe ω) using the natural bijections $B_x \leftrightarrow \omega$ respecting the ordering of ω.

Now, we can inductively define functions $f_\alpha : P^\alpha \to X_{T_{1+\alpha}}$ which witness the reductions $\cong_\alpha \leq_B \cong_{T_{1+\alpha}}$:

1. For $\alpha = 1$ and $M \models P^1$, let $f_1(M) \in X_{T_2}$ be a model which contains exactly one E-class realizing $\{\neg C_j(x) | j < \omega\}$ exactly $i + 1$ times if and only if $i \in ||M||$. If $||M||$ is finite, let the remaining E-classes omit $\{\neg C_j(x) | j < \omega\}$.
2. Given $f_\delta : P^\delta \to X_{T_{1+\delta}}$ for all $\delta \leq \alpha$ and $M \in P^{\alpha+1}$, let $B = \{b | E(b, v_0)\}$ and, reusing the functions $g_{\beta\gamma}$ defined above, let $f_{\alpha+1}(M)$ be a model whose E-classes contain the T_α-models $g_{\mathrm{rk}(a)\alpha}(f_{\mathrm{rk}(a)}(M(b)))$ for all $b \in B$ and fill up remaining E-classes using 1-uniform models.
3. If α is a limit ordinal and $M \in P^\alpha$, let $B = \{b | E(b, v_0)\}$, and for all $\beta < \alpha$, let $B_\beta \subset B$ be defined as $B_\beta = \{a \in B | \mathrm{rk}(a) = \beta\}$. Then, let $f_\alpha(M)$ be the T_α-model omitting $\{\neg A_\beta(x) | \beta < \alpha\}$, which for all $\beta < \alpha$ and $b \in B_\beta$ has the models $f_\beta(M(b))$ in the E-classes of $A_{\beta+1}$, possibly filling up remaining classes with 1-uniform models as in the successor case, as well as the A_β for limit β.

The definition of the functions f_α can be carried out in a Borel manner.

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