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George Boole's Deductive System

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Abstract The deductive system in Boole's *Laws of Thought* (**LT**) involves both an algebra, which we call *proto-Boolean*, and a "general method in Logic" making use of that algebra. Our object is to elucidate these two components of Boole's system, to prove his principal results, and to draw some conclusions not explicit in **LT**. We also discuss some examples of incoherence in **LT**; these mask the genius of Boole's design and account for much of the puzzled and disparaging commentary **LT** has received. Our evaluation of Boole's logical system does not differ substantially from that advanced in Hailperin's exhaustive study, *Boole's Logic and Probability*. Unlike the latter work, however, we make direct use of the polynomials native to **LT** rather than appealing to formalisms such as multisets and rings.

1 Introduction

The system of inference developed in Chapters V–X of Boole's *Laws of Thought* (**LT**)[1] is an achievement of surpassing genius. It has virtually no antecedent in logic. As Heath [24] observes, "Few major innovators in any science can have had so little to learn from their predecessors as Boole."[2] Franklin summarizes Boole's "great advance" thus:

> The task which Boole accomplished was the complete solution of the problem:—given any number of statements, involving any number of terms, mixed up indiscriminately in the subjects and the predicates, to eliminate certain of those terms, that is, to see exactly what the statements amount to irrespective of them, and then to manipulate the remaining statements so that they shall read as a description of a certain other chosen term (or terms) standing by itself in a subject or a predicate. . . . This problem of Logic was completely solved by Boole. ([31, p. 543])

Not all commentators, however, have been so benign. Jevons [26, p. 65] speaks of Boole's "dark and symbolic processes." Lotze [34] calls Boole's system a "rash and

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misty analogy from the province of mathematics" (p. 278) which involves "working in the dark" (p. 277); he is consoled, however, that "these chimeras have not found their way to Germany" (p. 283). The Kneales [29, p. 421] shrink from his "fearsome apparatus of numerical coefficients." Corcoran [13] writes that "Boole has a semi-formal method of derivation that is neither sound nor complete" (p. 261) and that his work "is marred by what appear to be confusions, incoherencies, fallacies, and glaring omissions" (p. 279). Dummett [17, p. 205] warns that "anyone unacquainted with Boole's works will receive an unpleasant surprise when he discovers how ill-constructed his theory actually was and how confused his explanations of it." Wood [56] agrees, declaring that Boole was a "hack mathematician" (p. 145) whose "treatment of his own logic is so trivial and so incompetent that it constitutes a step backward from Aristotle" (p. 67).

Undoubtedly the most comprehensive analysis of **LT** is that offered by Hailperin [21] who writes, "Boole was a thorough and careful worker and the mathematical system which he elaborated for doing logic was not shown to be wrong by the historical simplification to Boolean algebra but merely replaced by it" (p. 2). Seeking in his monograph to provide "an intensive and extensive study of Boole's mathematical theories" (p. 3), Hailperin examines Boole's system through the lens of modern algebra and mathematical logic: "We show not only how to justify Boole's procedure here but to make sense of it in all respects. We do this by going over to rings of quotients of Boolean elements, elements not from the original Boolean algebra but from a certain factor algebra" (p. 4).[3] A chapter entitled "Requisites from Algebra, Logic and Probability" touches, for example, on the Gödel Incompleteness Theorem, McKinsey theories, and Lindenbaum algebras.

The key to understanding Boole, in Hailperin's view, is the *signed multiset*—a set in which multiple occurrences of elements, including negative occurrences, are allowed: "Our basic contention is: *To obtain a meaningful interpretation of Boole's system we have to use not the notion of a class (class = set) but that of a multiset*" (emphasis in the original) [21, p. 136].

1.1 Objectives and approach Hailperin's exhaustive monograph treats **LT** in its entirety, including Boole's chapters on probability. We focus, however, on the deductive system developed in Chapters V–X of **LT**. That system comprises two components: a polynomial algebra, which we call *proto-Boolean,* and a "general method in Logic." Our objects are to elucidate both components and to prove Boole's principal results. Doing so requires that we introduce some elements (notation, definitions, and propositions) not explicit in **LT** but nevertheless within his algebraic system.

The multiplicities in a Hailperin multiset, as applied to **LT**, are the coefficients of a developed polynomial in Boole's algebra (cf. Section 2.12); thus a signed multiset is equivalent to such a polynomial. We believe the polynomials native to **LT** to be simpler and more flexible than their corresponding multisets and that an abstraction from Boole's algebra such as multiset-theory diverts attention from that algebra without providing a compensating advantage.

Boole's algebra can be explicated without multisets in terms of congruences with respect to a polynomial ideal (cf. Beth [1, Section 25]). We attempt a less formal and more self-contained treatment, however, assuming only common algebra and basic logic, justifying Boole's algebra on its own terms.

1.2 Boole's general method Boole's *general method in Logic*[4] is the first formulation—amazingly complete, if less than coherently stated—of the concepts underlying the modern theory of Boolean equations.[5] The central features of that theory—development, reduction, elimination, and the construction of parametric and inclusive general solutions—are those developed in Chapters V–X of **LT**. Having the later theory in view enables the parts of Boole's system to emerge in a clear and familiar pattern.

The general method determines the consequents of a set of universal premises by solving an equation in a numerical algebra. In particular, Boole solves the

General Problem.

> *Given any equation connecting the symbols $x, y..w, z..$*
> *Required to determine the logical expression of any class expressed in any way by the symbols $x, y..$ in terms of the remaining symbols, w, x, &c.* (**LT**, p. 140)

Given a symbol, x, of interest and a set of premises, Boole reduces the premises to a single equivalent proto-Boolean equation,

$$f(x) = 0\,. \tag{1}$$

The general method eventuates in an *inclusive general solution* (cf. Section 3.3.2) of (1), representing the set of consequents of that equation that involve x.

The general method is *generative, local, sound,* and *complete.* It is *generative* in that it produces a representation of all consequents of the premises rather than verifying a given consequent, that is, proving a theorem. It is *local,* in that it seeks to relate a selected symbol (Boole's term for a variable) to the remaining symbols. All of the principal nineteenth-century writers on the algebra of logic propound local systems. The method is also *sound* because the general solution specifies nothing but consequents of (1) involving x, and *complete* because it specifies all such consequents. (Among the principal systems in the algebra of logic, only that of Jevons [26] is not complete.)

We discuss the general method in Section 4, using Example 5 in Chapter IX of **LT** for illustration. Example 5 was used as their acid test by Frege[6] [19, p. 40], Ladd [30, p. 57], Lotze [34, p. 356], Macfarlane [37], McColl [39, p. 23], Peirce [43, p. 39], Schröder [47, p. 522], Venn [54, p. 351], and Wundt[7] [57, p. 356]. Schröder called Example 5 his "touchstone." (Jevons avoided Example 5, wisely choosing simpler examples from **LT** to assert the superiority of his own methods.)

1.3 The two algebras Boole's general method (**LT**, Chapters V–X) involves operations in a polynomial algebra. He chooses in Chapters II–IV, however, to base his definitions on an incomplete "Algebra of Logic." Each symbol in that algebra represents a *class,* that is, "a collection of individuals…extended so as to include the case in which but a single individual exists…as well as the cases denoted by the terms 'nothing' and 'universe' " (**LT**, p. 28). The universal class is denoted by 1, the empty class by 0. The product, xy, of two classes represents their intersection. The sum, $x + y$, is defined only if x and y are disjoint, in which case it represents their union. At the end of Chapter II, Boole announces a companion-algebra, employing the same notation, which is numerical rather than logical:

> Let us conceive, then, of an Algebra in which the symbols, x, y, z, &c. admit indifferently of the values 0 and 1, and these values alone. The laws, the

> axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Differences of interpretation alone will divide them. Upon this principle the method of the following work is established. (**LT**, p. 37)

Boole's alternative algebra, which we call *proto-Boolean* and discuss in Section 2, is the basis for his deductive system. It consists of polynomials, of the first degree in each symbol, having integer coefficients. Polynomials are combined using the rules of common algebra, save that the rule $x^2 = x$ (called the *fundamental law of thought* on p. 49 of **LT**) is applied to symbols as needed to reduce their degree to unity. In Boole's class-algebra the sum $x + x$ does not exist (i.e., is not a class); its proto-Boolean value, however, is $2x$.

If a proto-Boolean polynomial f has the property that $f^2 = f$ (thus extending the fundamental law of thought from symbols to a polynomial), Boole calls f *interpretable*. The connection between interpretable polynomials and Boole's logical algebra of classes is discussed in Section 2.8.4.

1.4 Translation Boole summarizes the translation of premises into corresponding proto-Boolean equations (**LT**, p. 124) by the following

> Rule.—*The equations being so expressed as that the terms X and Y in the following typical forms obey the law of duality* [i.e., are interpretable], *change the equations*

$$\begin{array}{rcllrcl} X & = & vY & \text{into} & X(1-Y) & = & 0. \\ X & = & Y & \text{into} & X(1-Y)+Y(1-X) & = & 0. \\ vX & = & vY & \text{into} & vX(1-Y)+vY(1-X) & = & 0. \end{array} \tag{2}$$

The equations on the left of (2) are "the great leading types of propositions symbolically expressed" (**LT**, p. 64). X and Y are premise-terms; v is an "indefinite class symbol" (**LT**, p. 62), that is, an arbitrary parameter; all three are interpretable. The first and third equations represent, respectively, the propositions "All Xs are Ys" and "Some Xs are Ys."

Boole here makes the Aristotelian assumption (abandoned in his general method) that a term may not denote an empty class. Thus Boole "quantifies the predicate," reading the first equation in (2) as "All Xs are some Ys" and the third as "Some Xs are some Ys." To enforce his prohibition of empty classes, Boole requires that v be "the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed" (**LT**, p. 63). For "v is the representative of *some,* which, though it may include in its meaning *all,* does not include *none*" (**LT**, p. 124).

None of Boole's strictures concerning v is enforced, however, in his general method. For equations of the first type, Rule (2) removes v and with it Aristotle's restriction on the size of classes. (Equation $X(1-Y) = 0$ includes the non-Aristotelian value $X = 0$ among its solutions.) Equations of the third type do not appear in any of the examples in Chapters V–X; Boole excludes them without comment. Particular propositions cannot, in fact, be represented in Boole's equational system.[8]

The parameter v, removed at the outset of Boole's general method, reappears (without Aristotelian trammels and written sometimes as $\frac{0}{0}$) in his parametric general solution, discussed below.

1.5 Deduction via solution Boole infers the consequents of (1) involving x by solving (1) for x in terms of the remaining symbols. This approach is "vitiated," according to Corcoran and Wood [14], "by the fallacy of supposing that a solution to an equation is necessarily a logical consequence of the equation." This *Solutions Fallacy* is discussed subsequently by Carnielli [12], Corcoran [13], and Nambiar [41]. As Corcoran observes, a solution—by which he means a *particular* solution, defined in Section 3.2—is not necessarily one of the consequents of an equation. (A particular solution of an equation is, in fact, one of its antecedents.) Boole does not, however, construct particular solutions of (1). Instead, he constructs a *general solution*—a representation of all particular solutions—in two forms.

1.5.1 Parametric form A *parametric* general solution of (1),

$$x = r + vs \tag{3}$$

$$0 = t, \tag{4}$$

involves interpretable functions, r, s, and t ($rs = 0$), and an arbitrary interpretable parameter, v. Equation (4) expresses the most general "independent relation" (**LT**, p. 108) deducible from (1); it is also the necessary and sufficient condition that (1) be consistent, that is, that it possess a solution.

1.5.2 Inclusive form Boole augments the parametric general solution (3, 4) with "modes of expression more agreeable to those of common discourse" (**LT**, p. 112), using the ingredients r and s of (3) to form an alternative general solution comprising two consequents of (1), namely,

$$\text{"all } r \text{ is } x\text{"} \quad \text{and} \quad \text{"all } x \text{ is } r + s.\text{"} \tag{5}$$

These consequents, stated various ways in **LT**, are called by Boole the "reverse interpretation" and "direct interpretation," respectively. The system (5, 4) constitutes an *inclusive general solution* of (1), for whose attainment the parametric form is a way station (Boole seems unable to construct the inclusive form directly). System (5, 4) is equivalent to (1), and is therefore both a consequent and an antecedent of (1). Hence Boole's general method is not afflicted with the Solutions Fallacy.

1.6 Sources of misunderstanding Commentators typically assume one or more of the following concerning Boole's general method:

1. It includes and extends traditional Aristotelian logic.
2. Addition is defined only for disjoint summands.
3. Division is one of its operations.
4. Because the only copula in Boole's system is $=$, he does not express an inclusive consequent.
5. Intermediate equations cannot be interpreted logically.
6. The parameter v is part of particular propositions and cannot be zero.

None of these statements is true of Boole's general method. Every one of them, however, is found—explicitly or by implication—in **LT**. This conundrum results because the parts of **LT** do not cohere. Boole seems, in the early chapters of **LT**, to accept the canons of Aristotelian logic and to lay the groundwork for a system of inference based on an algebra of logic. However, the deductive system that emerges, beginning in Chapter V, is not Aristotelian and relies on a numerical algebra.

There is serious disagreement between the logical calculus of Chapters II–IV of **LT** and the algebra employed in Chapters V–X (the chapters of interest for this paper). There are also contradictions within these parts of **LT**. Among the more contradictory are his treatments of logical "or," Aristotelian logic, and the uninterpretable intermediate steps in his general method.

1.6.1 Logical "or" In Chapter II of **LT** (p. 32), Boole states that a phrase aggregating two or more classes implies that the classes are disjoint:

> In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another.

He revokes this linguistic constraint, however, in Chapter IV. His "rule of expression" specifies interpretable formulas for the term, "Either x's or y's" in both its exclusive and nonexclusive senses (**LT**, p. 57).

1.6.2 Aristotelian logic Although Aristotelian logic per se is exiled in **LT** to Chapter XV (the last chapter on logic), Boole implies in the preceding chapters that he is treating particular as well as universal propositions in traditional logic. Particular propositions and Aristotle's restrictions on class-extent are abandoned, however, in Boole's general method.

1.6.3 Uninterpretable equations The intermediate steps in Boole's general method have evoked particular censure. Typical comments are the following:

> It is entirely paradoxical to say that... we can start from equations having a meaning and arrive at equations having a meaning by passing through equations having no meaning. ([36], p. 352)

> It is thus, properly speaking, only the premises and the final results of treatment which with [sic] Boole directly represent[s] logical facts, whereas the road by which he proceeds from premises to results, is, logically speaking, meaningless nonsense. ([27], p. 115)

The first five pages of Chapter V of **LT** constitute an extended apology for his uninterpretable intermediate equations, including an appeal to the analogous "uninterpretable symbol $\sqrt{-1}$, in the intermediate processes of trigonometry" (**LT**, p. 69). This appeal is puzzling because, fewer than ten pages after invoking $\sqrt{-1}$, Boole states that "*equations* are always reducible... to interpretable forms" (**LT**, p. 78). In Chapter X, Boole shows how to make every step of a solution interpretable, remarking that doing so "would involve in some instances no slight labour of preliminary reduction. But it is still interesting to know that this can be done" (**LT**, p. 151).

2 Proto-Boolean Algebras

2.1 Proto-Boolean Polynomials and Forms Let $x_1, \ldots, x_n$ be indeterminates, that is, abstract signs or tokens that commute with integers. Following Boole (**LT**, p. 27), we call these signs *symbols* and assume that they satisfy his "fundamental law," $x_i^2 = x_i$ (**LT**, p. 49).

Setting $x_i^{-1} = 1$ and $x_i^1 = x_i$ ($i = 1, 2, \ldots, n$), we define a *proto-Boolean polynomial* (henceforth simply *polynomial*) on $x_1, \ldots, x_n$ to be an expression,

$$\sum_{k_1,\ldots,k_n \in \{-1,1\}} a_{k_1,\ldots,k_n} x_1^{k_1} \cdots x_n^{k_n}, \tag{6}$$

in which the $a_{k_1,\ldots,k_n}$ are integers. Examples are 5 and $-2 + 3x_1 - 4x_1x_3$.

Let f and g be polynomials, and let $f+g$, $f-g$, and $f \dot{\times} g$ be the common (high school) sum, difference, and product of polynomials—except that a computed product is made linear in each symbol, x_i, by applications of Boole's law, $x_i \dot{\times} x_i = x_i$. (In most of what follows, we write fg for $f \dot{\times} g$.) A polynomial or a formula comprising sums, differences, and products of polynomials will be called a *proto-Boolean form* (or simply *p-form*).

Let P_n be the set of n-symbol p-forms (P if n is not specified). It is clear that $P_0 \subset P_1 \subset P_2 \cdots$. We call the system $(P_n, +, \dot{\times}, 0, 1)$ an n-symbol *proto-Boolean algebra*. Every element of P is equivalent to a unique polynomial; thus the p-form $(x-2y)(x+xy)+3x$ is equivalent to $4x-3xy$.

The linkage between logic and polynomials of this type was remarked in 1933 by Whitney [55]. Ten years later, Hoff-Hansen [25] showed that expressions in the propositional calculus may be interpreted as such polynomials. Skolem [49] pointed out that Hoff-Hansen's system is the polynomial ideal generated by the basis $x_1^2 - x_1, x_2^2 - x_2, \ldots$.[9] Laita et al. [32, p. 426] write that Boole was working implicitly in the same polynomial ring but seem unaware of the earlier work of Hoff-Hansen and Skolem. Polynomials of this kind have also been applied in operations research and engineering.[10]

2.2 Interpretability Boole's concept of *interpretability* extends his fundamental law from symbols to certain p-forms. Thus Boole calls $f \in P$ interpretable if $f^2 = f$ (**LT**, p. 93) and calls an equation interpretable if both its members are interpretable. We denote the set of n-symbol interpretable p-forms by I_n (by I, if n is not specified). Clearly, $I_n \subset P_n$.

2.3 Development A polynomial, $f(x)$, in P_1 has the form $a+bx$, where a and b are integers. Thus $f(x) = a(1-x) + (a+b)x$; that is,

$$f(x) = f(0)(1-x) + f(1)x\,. \tag{7}$$

Generalizing to P_2,

$$f(x_1, x_2) = f(0,0)\bar{x}_1\bar{x}_2 + f(0,1)\bar{x}_1 x_2 + f(1,0)x_1\bar{x}_2 + f(1,1)x_1x_2\,, \tag{8}$$

where $\bar{x}_i$ is Boole's shorthand for $1-x_i$ (**LT**, p. 119). We extend this shorthand to elements of P, that is, $(\forall f \in P)[\,\bar{f} = 1 - f\,]$.

Boole appeals (**MAL**, p. 60; **LT**, p. 72) to the Taylor-Maclaurin theorem to justify the extension of this development to any number of symbols.

Example 2.1 Let f be expressed by the polynomial form

$$f(x_1, x_2) = 3 - 8x_1 - 3x_2 + 9x_1x_2\,. \tag{9}$$

Computing $f(0,0) = 3$, $f(0,1) = 0$, $f(1,0) = -5$, and $f(1,1) = 1$, we arrive at the development

$$f(x_1, x_2) = 3\bar{x}_1\bar{x}_2 - 5x_1\bar{x}_2 + x_1x_2\,. \tag{10}$$

An alternative form for f is a polynomial in $\bar{x}_1, \bar{x}_2, \ldots$. Thus,

$$f(x_1, x_2) = 1 - \bar{x}_1 - 6\bar{x}_2 + 9\bar{x}_1\bar{x}_2\,. \tag{11}$$

Expressions (9), (10), and (11) are Whitney's three *normal forms* for f [55]. Each is a unique representation of f, up to the order of sums and products.

Boole calls $f(0,0)$, $f(0,1), \ldots$ the *coefficients* of the development (8), and $\bar{x}_1\bar{x}_2$, $\bar{x}_1 x_2, \ldots$ its *constituents*. Some additional terminology will be useful. We represent n-tuples by uppercase letters; thus $A = (a_1, a_2, \ldots)$, $X = (x_1, x_2, \ldots)$, and $(a, X) = (a, x_1, x_2, \ldots)$. We adopt Boole's convention (**LT**, p. 100) of writing $f(x)$ in place of $f(x, Y)$, if Y is understood. For $i \in \{0, 1\}$, we write $f_i(Y)$ and f_i in place of $f(i, Y)$ and $f(i)$, respectively.

To each $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$ and $X = (x_1, \ldots, x_n)$ there corresponds an n-symbol constituent, $X^A = \prod_{i=1}^{n} x_i^{a_i}$, where $x_i^0 = 1 - x_i$ and $x_i^1 = x_i$. Thus $(x_1, x_2, x_3)^{(0,1,0)} = x_1^0 x_2^1 x_3^0 = (1 - x_1)x_2(1 - x_3) = \bar{x}_1 x_2 \bar{x}_3$.

Computations involving constituents are facilitated by the following readily-verified properties, where $A, B \in \{0, 1\}^n$ and $X = (x_1, x_2, \ldots, x_n)$:

$$A^A = 1 \tag{12}$$

$$A^B = 0 \qquad \text{if } A \neq B \tag{13}$$

$$X^A \cdot X^A = X^A \tag{14}$$

$$X^A \cdot X^B = 0 \qquad \text{if } A \neq B \tag{15}$$

$$\sum_{A \in \{0,1\}^n} X^A = 1\,. \tag{16}$$

Proposition 2.1 *If $f \in P_n$, then*

$$f(X) = \sum_{A \in \{0,1\}^n} f(A) X^A\,. \tag{17}$$

Proof To any polynomial, f, in P_1 there correspond integers a and b such that $f(x) = a + bx = a(1 - x) + (a + b)x$. Hence $f(x) = f(0)\bar{x} + f(1)x$. Suppose the proposition to be true for $n = k > 0$. Then to any polynomial, f, in P_{k+1} there correspond polynomials g and h in P_k such that $f(x, Y) = g(Y) + xh(Y) = (1 - x)g(Y) + x(g(Y) + h(Y))$. Thus,

$$\begin{aligned} f(x, Y) &= x^0 \sum_{B \in \{0,1\}^k} g(B) Y^B + x^1 \sum_{B \in \{0,1\}^k} (g(B) + h(B)) Y^B \\ &= \sum_{B \in \{0,1\}^k} f(0, B) x^0 Y^B + \sum_{B \in \{0,1\}^k} f(1, B) x^1 Y^B \\ &= \sum_{A \in \{0,1\}^{k+1}} f(A)(x, Y)^A\,, \end{aligned}$$

verifying (17). □

Given $f, g \in P_n$ and $X = (x_1, \ldots, x_n)$, the following properties derive directly from Proposition 2.1 and the combining properties (14), (15), and (16) of constituents:

$$\overline{f(X)} = \sum_{A \in \{0,1\}^n} \overline{f(A)} X^A \tag{18}$$

$$f(X)g(X) = \sum_{A \in \{0,1\}^n} f(A)g(A) X^A \tag{19}$$

$$f(X) + g(X) = \sum_{A \in \{0,1\}^n} (f(A) + g(A)) X^A \tag{20}$$

$$f(X) - g(X) = \sum_{A \in \{0,1\}^n} (f(A) - g(A)) X^A\,. \tag{21}$$

2.4 P-Functions The symbols $x_1, x_2, \dots x_n$ in a p-form $f \in P_n$ are indeterminates satisfying Boole's law, $x_i^2 = x_i$. If, however, we view these symbols as variables, then f generates a *proto-Boolean polynomial function* (or simply *p-function*) $\widehat{f}$, mapping n p-forms into a p-form.[11] Specifically, $\widehat{f} : P^n \to P$ is defined as follows:

1. For integer a and for $p_1, \dots, p_n \in P$,
$$\widehat{a}(p_1, \dots, p_n) = a\,.$$
2. For $i = 1, 2, \dots,$ and for $p_1, \dots, p_n \in P$,
$$\widehat{x_i}(p_1, \dots, p_n) = p_i\,.$$
3. For $f, g \in P_n$ and $p_1, \dots, p_n \in P$,
$$\begin{aligned}\widehat{f+g}(p_1, \dots, p_n) &= \widehat{f}(p_1, \dots, p_n) + \widehat{g}(p_1, \dots, p_n)\\ \widehat{f-g}(p_1, \dots, p_n) &= \widehat{f}(p_1, \dots, p_n) - \widehat{g}(p_1, \dots, p_n)\\ \widehat{fg}(p_1, \dots, p_n) &= \widehat{f}(p_1, \dots, p_n) \dot{\times} \widehat{g}(p_1, \dots, p_n)\,.\end{aligned}$$

Every p-form specifies a unique p-function; every p-function is specified by any one of an equivalence-class of p-forms. We denote by $\widehat{P_n}$ $(\widehat{I_n})$ the set of p-functions generated by p-forms in P_n (I_n). If stated to be in $\widehat{P_n}$ or $\widehat{I_n}$, a letter $f, g, h, \dots$ will denote a p-function; otherwise, $\widehat{f}, \widehat{g}, \widehat{h}, \dots$ will denote the p-functions generated by p-forms $f, g, h, \dots$.

2.5 Development of p-functions Development (17), which holds for a p-form $f(x_1, \dots, x_n) \in P_n$ $(x_1, \dots, x_n$ indeterminate), may be extended to a p-function $f : P^n \to P$, where $f \in \widehat{P_n}$.

Proposition 2.2

$$(\forall f \in \widehat{P_n})(\forall X \in P^n)\left[\, f(X) = \sum_{A \in \{0,1\}^n} f(A) X^A \,\right]. \tag{22}$$

Proof Consider first the development of $f \in \widehat{P}$ with respect to a single variable, x. Let $g \in P$. Then $f(x) = px + q$ and $g = rx + s$, where $p, q \in \widehat{P}$ and $r, s \in P$, none of p, q, r, s involving x. Thus $f(g) = p \dot{\times} (rx + s) + q$ $= (1 - (rx + s))q + (rx + s)(p + q) = \overline{g} f(0) + g f(1)$. Suppose (22) to hold for $n = k \geq 1$. Let $(x, Y) = (x, y_1, \dots, y_k) \in P^{k+1}$ and consider $f \in \widehat{P}_{k+1}$: $f(x, Y) = \bar{x} f(0, Y) + x f(1, Y) = \bar{x} \sum_{B \in \{0,1\}^k} f(0, B) Y^B + x \sum_{B \in \{0,1\}^k} f(1, B) Y^B$ $= \sum_{(i,B) \in \{0,1\}^{k+1}} f(i, B) x^i Y^B$. Thus $f(x, Y) = \sum_{A \in \{0,1\}^{k+1}} f(A)(x, Y)^A$. $\square$

Development (22) should be applied with caution if $X \notin I^n$, in which case the products X^A in (22) satisfy neither of properties (14) and (15) of constituents.[12] Boole develops the system in Chapters V–X of **LT** in terms of functions in $\widehat{P_n}$ whose domains comprise only p-forms in I. Accordingly, we consider functions only of the form $f : I^n \to P$ henceforth, where $f \in \widehat{P_n}$.

2.6 Equations For $f \in \widehat{P_n}$, we say the equation

$$f(X) = 0, \tag{23}$$

is *consistent* if $\exists (p_1, \dots, p_n) \in I^n$ such that $f(p_1, \dots, p_n) = 0$ is an identity.

Proposition 2.3 *For all $f \in \widehat{P}_n$ and $X \in I^n$, the following are equivalent:*

$$f(X) = 0 \tag{24}$$

$$(\forall A \in \{0,1\}^n)\,[\,f(A) = 0 \text{ or } X^A = 0\,]\,. \tag{25}$$

Proof (25) $\Longrightarrow$ (24) (Proposition 2.1). Conversely, (24) $\Longrightarrow \sum_{B \in \{0,1\}^n} f(B) X^B = 0$. Multiplying the latter by X^A, for arbitrary $A \in \{0,1\}^n$ and recalling (14) and (15), we infer $f(A)X^A = 0$. If $f(A) = 0$, then (25) follows. If $f(A) \neq 0$, then $f(A)X^A = 0 \Longrightarrow X^A = 0 \Longrightarrow$ (25). □

Proposition 2.4 $(\forall f \in \widehat{P}, x \in I)\big[[f(x) = 0] \Longleftrightarrow [\bar{x} f(0) = 0 \textit{ and } x f(1) = 0]\big]$.

Proof $f(x) = 0 \Longleftrightarrow \bar{x} f(0) + x f(1) = 0$ (Proposition 2.1). Multiplying by $\bar{x}$ (x), we infer $\bar{x} f(0) = 0$ $(x f(1) = 0)$. The converse follows from Proposition 2.1. □

2.7 Verification

A proto-Boolean identity on I may be verified by considering only 0 and 1 as symbol-values.

Proposition 2.5 *The following are equivalent for all $f, g \in \widehat{P}_n$:*

$$(\forall X \in I^n) \quad \big[\, f(X) = 0 \;\Longrightarrow\; g(X) = 0 \,\big] \tag{26}$$

$$(\forall A \in \{0,1\}^n) \; \big[\, f(A) = 0 \;\Longrightarrow\; g(A) = 0 \,\big]\,. \tag{27}$$

Proof Suppose (24) to be consistent. Clearly (26) $\Longrightarrow$ (27). If (27) holds, on the other hand, then applying Proposition 2.3 twice,

$$(\forall X \in I^n)\left[\begin{array}{lll} f(X) = 0 & \Longrightarrow & (\forall A \in \{0,1\}^n)[f(A) = 0 \text{ or } X^A = 0] \\ & \Longrightarrow & (\forall A \in \{0,1\}^n)[g(A) = 0 \text{ or } X^A = 0] \\ & \Longrightarrow & g(X) = 0. \end{array}\right].$$

Thus (27) $\Longrightarrow$ (26). If (24) is inconsistent, both (26) and (27) are true. □

Proposition 2.6 *The following are equivalent for all $f, g \in \widehat{P}_n$:*

$$(\forall X \in I^n) \quad \big[\, f(X) = 0 \;\Longleftrightarrow\; g(X) = 0 \,\big]$$

$$(\forall A \in \{0,1\}^n) \; \big[\, f(A) = 0 \;\Longleftrightarrow\; g(A) = 0 \,\big]\,.$$

Proof Follows from Proposition 2.5. □

In Boolean algebra, Propositions 2.5 and 2.6 are forms of the *verification theorem*.[13]

2.8 Interpretable forms and functions

2.8.1 Characterization of I_n

Proposition 2.7 (LT, Chap. V, Prop. IV, p. 79)

$$f \in I_n \Longleftrightarrow (\forall A \in \{0,1\}^n)[f(A) \in \{0,1\}].$$

Proof $f \in I_n \Longleftrightarrow (\forall X \in I^n)\;[f(X)^2 = f(X)] \Longleftrightarrow (\forall A \in \{0,1\}^n)$ $[f(A)^2 = f(A)]$ (Proposition 2.6) $\Longleftrightarrow (\forall A \in \{0,1\}^n)[f(A) \in \{0,1\}]$. □

2.8.2 Formally interpretable p-forms We define the set of *formally interpretable* p-forms as follows:

1. 0 is formally interpretable.
2. 1 is formally interpretable.
3. If x is a symbol, then x is formally interpretable.
4. If f and g are formally interpretable, then so are
   (a) $(f)(g)$
   (b) $1-(f)$
   (c) $f+g$ if $(f)(g)=0$.

(A parenthesis-pair may be removed if doing so does not introduce ambiguity.) Boole consistently expresses interpretable p-forms so they are formally interpretable. Thus he writes $x+y-xy$ as $x+(1-x)y$.

2.8.3 Interpretable p-forms as classes We define operator $\dot{+}$ on P by

$$f\dot{+}g = f+g-f\dot{\times}g,$$

where $f\dot{\times}g$, which we normally write as fg, is defined in Section 2.1. (Whitney [55] writes $\dot{+}$ for the same operation.)

If $f \in I_n$, each of the 2^n coefficients, $f(A)$, in development (17) has value 0 or 1 (Proposition 2.7); thus I_n comprises 2^{2^n} elements. $\mathcal{I}_n = (I_n, \dot{+}, \dot{\times}, {}^{-}, 0, 1)$ is a Boolean algebra [21, Theorem 2.32] and is thus isomorphic to an algebra of classes [51]. Let $\xi_1, \ldots, \xi_n$ denote distinct subsets of a set, U, and let S_n be the 2^{2^n}-element field built up from these subsets, using the operators $\cup$, $\cap$, and $'$. Then $\mathcal{S}_n = (S_n, \cup, \cap, {}', \varnothing, U)$ is a Boolean algebra of classes (sets).

Let $x_1, x_2, \ldots, x_n$ be proto-Boolean symbols, let f and g be elements of I_n, and let $\varphi : I_n \to S_n$ be defined by

1. $\varphi(0) = \varnothing$
2. $\varphi(1) = U$
3. $\varphi(x_i) = \xi_i$
4. $\varphi(fg) = \varphi(f) \cap \varphi(g)$
5. $\varphi(\bar{f}) = (\varphi(f))'$.

Then φ is an isomorphism of $\mathcal{I}_n$ onto $\mathcal{S}_n$.[14]

2.8.4 Boole's algebra of logic and I_n Boole's "rule of expression" (**LT**, Chap. IV, p. 57) states,

> Let the expression, "Either x's or y's," be expressed by $x(1-y)+y(1-x)$, when the classes denoted by x and y are exclusive, by $x+y(1-x)$ when they are not exclusive [Emphasis in the original].

In terms of $\dot{+}$, however,

$$\begin{aligned} x(1-y)+y(1-x) &= x\bar{y} \dot{+} y\bar{x} \\ x+y(1-x) &= x \dot{+} y \\ x+y &= x \dot{+} y \quad \text{if } xy=0. \end{aligned}$$

Terms aggregating classes are thus translated by Boole to elements of I_n, albeit expressed using $+$, $-$, and $\dot{\times}$. (Boole's copious examples bear this out.) The intersection of classes x and y is translated as xy, and the "contrary class" (**LT**, p. 48) of x is translated as $1-x$: both xy and $1-x$ are members of I_n. Thus the "algebra of Logic" of **LT**, Chapters II–IV, is $\mathcal{I}_n$, which may be interpreted as the class-algebra, $\mathcal{S}_n$.

2.9 The arithmetic order-relation We define the relation $\leq$ on I_n as follows: If $f, g \in \widehat{I_n}$, then $f \leq g$ provided $f(A) \leq g(A)$ for all $A \in \{0,1\}^n$. This relation does not appear in **LT**.[15] Its use, however, is natural in proto-Boolean algebra and clarifies the discussion in Section 3.3.2 of inclusive general solutions.

Proposition 2.8 *If $f, g \in \widehat{I_n}$, then the following are equivalent:*

$$\begin{aligned}&(\forall X \in I^n)\,[\, f(X) = 0 \implies g(X) = 0\,]\\ &(\forall X \in I^n)\,[\,(1 - f(X))\, g(X) = 0\,]\\ &(\forall X \in I^n)\,[\, g(X) \leq f(X)\,].\end{aligned}$$

Proof It suffices for each of the three statements to replace X by A and I^n by $\{0,1\}^n$ (Proposition 2.6), in which case $f(A), g(A) \in \{0,1\}$ (Proposition 2.7). Thus all three statements are false if $f(A) = 0$ and $g(A) = 1$, and all three are true if $f(A) = 1$ or $g(A) = 0$. □

2.10 Composability and reduction A p-function, $f \in \widehat{P_n}$, as well as the equation $f(X) = 0$, will be called *composable* if $(\forall A \in \{0,1\}^n)\,[\, f(A) \geq 0\,]$.

Proposition 2.9 *If $f \in \widehat{P_n}$, then f^2 is composable.*

Proof $(\forall X \in I^n)\left[\, f^2(X) = \sum_{B\in\{0,1\}^n} f(B)^2 X^B\,\right]$, whence $(\forall A \in \{0,1\}^n)$ $\left[\, f^2(A) = \sum_{B\in\{0,1\}^n} f(B)^2 A^B = f(A)^2 \geq 0\,\right]$ (cf. (14) and (15)). □

Proposition 2.10 (LT, Chap. VIII, Prop. II, p. 120) *Let $f_1, f_2, \ldots, f_m$ be composable p-functions. Then*

$$(\forall X \in I^n)\left[\,(\forall i \in \{1,\ldots,m\})\,[\, f_i(X) = 0\,] \iff \sum_{i=1}^{m} f_i(X) = 0. \right. \tag{28}$$

Proof By Proposition 2.6, (28) $\iff$ $(\forall A \in \{0,1\}^n)\big[(\forall i \in \{1,\ldots,m\})[f_i(A) = 0]$ $\iff \sum_{i=1}^{m} f_i(A) = 0\big]$, which clearly holds for composable $f_1, f_2, \ldots, f_m$. □

Proposition 2.11 *If $f \in \widehat{P_n}$, then*

$$(\forall X \in I^n)\,[\,(f(X))^2 = 0 \iff f(X) = 0\,]. \tag{29}$$

Proof (29) $\iff$ $(\forall A \in \{0,1\}^n)[\,(f(A))^2 = 0 \iff f(A) = 0\,]$ (Proposition 2.6). □

Proposition 2.12 (LT, Chap. VIII, Prop. III, p. 121) *If $f_1, f_2, \ldots, f_m \in \widehat{P_n}$, then for all $X \in I^n$ the system $f_1(X) = 0,\ f_2(X) = 0, \ldots, f_m(X) = 0$ is equivalent to the single composable equation $\sum_{i=1}^{m} f_i^2(X) = 0$.*

Proof Follows from Propositions 2.9, 2.10, and 2.11. □

2.11 Elimination

Proposition 2.13 (LT, Chap. VII, Prop. I)

$$(\forall f \in \widehat{P_n})(\forall x \in I)\,[\, f(x) = 0 \implies f(0) f(1) = 0\,].$$

Proof $f(x) = 0 \implies \bar{x} f(0) + x f(1) = 0 \implies [\bar{x} f(0) f(1) = 0 \text{ and } x f(0) f(1) = 0]$ $\implies (\bar{x} + x) f(0) f(1) = 0 \implies f(0) f(1) = 0$. □

Boole calls $f(0)f(1) = 0$ "the complete result of the elimination of x" from $f(x) = 0$ (**LT**, p. 101).[16] Hailperin questions the term "complete," noting that Boole "has only shown that $f(1)f(0) = 0$ is an algebraic consequence of $f(x) = 0$" [21, p. 100]. We think it likely, however, that Boole means the following: among the consequents of $f(x) = 0$ not involving x, $f(1)f(0) = 0$ is the most general in the sense that it implies every other such consequent.

Proposition 2.14 $(\forall f \in \widehat{P}_n,\ g \in \widehat{P}_{n-1},\ n \geq 1)$, *the following are equivalent:*

$$\begin{array}{lrcl}
\text{(i)} & (\forall (x,Y) \in I^n)\ [\quad f(x,Y) = 0 & \Longrightarrow & g(Y) = 0\,] \\
\text{(ii)} & (\forall Y \in I^{n-1})\ [\ f(0,Y)f(1,Y) = 0 & \Longrightarrow & g(Y) = 0\,].
\end{array}$$

Proof Invoking Proposition 2.6 twice,

$$\begin{aligned}
\text{(i)} \Longleftrightarrow\ & (\forall (i,A) \in \{0,1\}^n)\ \big[\ f(i,A) = 0 \Longrightarrow g(A) = 0\big] \\
\Longleftrightarrow\ & (\forall A \in \{0,1\}^{n-1})\ \big[\,[f(0,A) = 0 \text{ or } f(1,A) = 0] \Longrightarrow g(A) = 0\big] \\
\Longleftrightarrow\ & (\forall A \in \{0,1\}^{n-1})\ \big[\ f(0,A)f(1,A) = 0 \Longrightarrow g(A) = 0\big] \\
\Longleftrightarrow\ & \text{(ii)}.
\end{aligned}$$

See [8] for a proof that (i) $\Longleftrightarrow$ (ii) in standard Boolean algebra. □

Proposition 2.15 (LT, Chap. IX, Prop. III, p. 133) *Let $f(x, y, z, \ldots) \in P$ be expressed in the partially developed form,*

$$f(x, y, z, \ldots) = g(y, z, \ldots)(1 - x) + h(y, z, \ldots)x.$$

Then the resultant of elimination of y from $f(x, y, z, \ldots) = 0$ is

$$g(0, z, \ldots)g(1, z, \ldots)(1 - x) + h(0, z, \ldots)h(1, z, \ldots)x = 0\,. \tag{30}$$

Proof Form (30) is achieved if each factor of $f(x, 0, y, \ldots)f(x, 1, z, \ldots) = 0$ (the desired resultant) is developed with respect to x. □

2.12 An alternative formulation: Multisets Hailperin [21] presents a model of Boole's algebra, equivalent to the polynomial formulation, based on *signed multisets*. A multiset representing an element f in Boole's algebra displays the same objects (constituents and their coefficients) as the development (17). To each constituent, X^A, having coefficient $f(A)$, there corresponds an element, $(f(A))X^A$, in the associated multiset. Thus the multiset-version of the function in Example 2.1 repackages (10) as $\{(3)\bar{x}_1\bar{x}_2, (0)\bar{x}_1 x_2, (-5)x_1\bar{x}_2, (1)x_1 x_2\}$.

A multiset-formulation equivalent to Hailperin's was proposed by Whitney [55] in 1933. Whitney associates with each element of his *generalized sets* "any integer, positive, negative or zero, instead of merely one or zero." Whitney represents a subset F of a set U by its *characteristic function*, call it φ, mapping each element of U to its multiplicity in F. If generalized set F represents function f having development (17) (Whitney's first normal form for f), then φ maps the constituents in (17) to their coefficients. In the case of Example 2.1, $\varphi(\bar{x}_1\bar{x}_2) = 3$, $\varphi(\bar{x}_1 x_2) = 0$, $\varphi(x_1\bar{x}_2) = -5$, and $\varphi(x_1 x_2) = 1$.

Although Hailperin's multisets and Whitney's generalized sets are equivalent, their applications are essentially opposite: Whitney uses polynomials to represent sets; Hailperin uses multisets to represent Boole's polynomials.

3 Solution of Proto-Boolean Equations

Boole carries out logical inference by solving equations of the form

$$f(x, Y) = 0 \tag{31}$$

for x in terms of $Y = (y_1, \ldots, y_{n-1})$. As before, we follow Boole in writing (31) as $f(x) = 0$ if Y is not specified. Unless otherwise noted, f is a function in $\widehat{P}_n$ mapping I^n to P.

3.1 The interpretable image We associate with $f \in \widehat{P}_n$ an *interpretable image*, $f^* \in \widehat{I}_n$, as follows:

$$f^*(X) = \sum_{\substack{A \in \{0,1\}^n \\ f(A) \neq 0}} X^A.$$

Thus $(\forall A \in \{0, 1\}^n)[\, f^*(A) = 0 \Longleftrightarrow f(A) = 0 \text{ and } f^*(A) = 1 \Longleftrightarrow f(A) \neq 0\,]$.

3.2 Particular solutions A *particular solution* of (31) is an equation, $x = g(Y)$, where $g \in P_{n-1}$ such that $f(g(Y), Y) = 0$ is an identity.

Proposition 3.1 (LT, Chap. X, Prop. I; [21], Theorem 2.35) *$f(x) = 0$ and $f^*(x) = 0$ possess the same set of interpretable solutions; that is,*

$$(\forall f \in \widehat{P}_n)(\forall x \in I_{n-1})\,[\, f(x) = 0 \Longleftrightarrow f^*(x) = 0\,].$$

Proof For $f \in \widehat{P}_n$ and $g \in I_{n-1}$,

$$\begin{aligned}
&(\forall Y \in I^{n-1})[\, f(g(Y), Y) = 0\,] \\
\Longleftrightarrow\ &(\forall A \in \{0, 1\}^{n-1})[\, \bar{g}(A) f(0, A) = 0 \text{ and } g(A) f(1, A) = 0\,] \\
\Longleftrightarrow\ &(\forall A \in \{0, 1\}^{n-1})[\, \bar{g}(A) f^*(0, A) = 0 \text{ and } g(A) f^*(1, A) = 0\,] \\
\Longleftrightarrow\ &(\forall Y \in I^{n-1})[\, f^*(g(Y), Y) = 0\,],
\end{aligned}$$

where we apply Propositions 2.4 and 2.6 twice and note that $g(A) \in \{0, 1\}$. □

Proposition 3.2 $(\forall f \in \widehat{P}_n)\big[\, (\exists x \in I\,)[\, f(x) = 0\,] \Longleftrightarrow f(0) f(1) = 0\,\big]$.

Proof Let $g \in I$. Then $[f(g) = 0] \Longrightarrow [\bar{g} f_0 + g f_1 = 0] \Longrightarrow [\bar{g} f_0 f_1 = 0$ and $g f_0 f_1 = 0\,] \Longrightarrow [\,(\bar{g} + g) f_0 f_1 = 0\,] \Longrightarrow [\, f_0 f_1 = 0\,]$. Conversely, $(\forall Y \in I^{n-1})\,[f(0, Y) f(1, Y) = 0] \Longrightarrow (\forall A \in \{0, 1\}^{n-1})[f(0, A) f(1, A) = 0]$. Choosing $x = f^*(0, Y)$,

$$f(f^*(0, Y), Y) = \sum_{A \in \{0,1\}^{n-1}} [\overline{f^*(0, A)} f(0, A) + f^*(0, A) f(1, A)]\, Y^A. \tag{32}$$

If $f(0, A) = 0$, then $f^*(0, A) = 0$. If $f(0, A) \neq 0$, then $f(1, A) = 0$ and $\overline{f^*(0, A)} = 0$. Thus each term of (32) vanishes; that is, $x = f^*(0, Y)$ is an interpretable solution of $f(x, Y) = 0$. (cf. [21, Theorem 2.34] for a different proof.) □

The condition $f(0) f(1) = 0$ is thus not only "the complete result of the elimination of x from $[f(x) = 0]$" (**LT**, p. 101), but it is also the necessary and sufficient condition for the existence of solutions in I of that equation.

3.3 General solutions A *general solution* of the proto-Boolean equation $f(x) = 0$ is a representation of all, and nothing but, its particular solutions. Boole considers only interpretable solutions; therefore it suffices, in view of Proposition 3.1, to solve $f^*(x) = 0$.

There are two methods, other than by enumeration, to express a general solution of a proto-Boolean equation: (a) by an equation involving an *arbitrary parameter* or (b) by a pair of *inclusions*. These correspond to the two basic forms in the theory of standard Boolean equations [7; 45] differing only in the underlying algebra and Boole's focus on a single dependent variable. Boole includes both forms in his "general method in Logic" (cf. Section 4).

3.3.1 Parametric form Löwenheim [35, p. 190] defines a general solution, in parametric form, of a standard Boolean equation as follows:

> [The system]
>
> $$\begin{aligned} x &= \varphi(u, v, \ldots)\,, \\ y &= \psi(u, v, \ldots)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$
>
> is a "general solution" of [a Boolean equation] if
>
> 1) it is a solution for any values of the arbitrary parameters $u, v, \ldots$, and
>
> 2) it is capable of representing any solution of [that equation]; that is, if a certain solution $x_0, y_0, \ldots$ of the equation is given, then it must be possible to find certain values of $u, v, \ldots$ for which
>
> $$\begin{aligned} x_0 &= \varphi(u, v, \ldots)\,, \\ y_0 &= \psi(u, v, \ldots)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

We formalize this definition in proto-Boolean terms, following the approach of Deschamps [16] and Rudeanu [45, p. 56]: a *parametric general solution* of $f^*(x) = 0$ is a system

$$x = \varphi(v) \tag{33}$$

$$0 = f^*(0) f^*(1) \tag{34}$$

such that

$$(\forall x \in I)\,(\forall v \in I)\big[\, x = \varphi(v) \implies f^*(x) = 0\,\big] \tag{35}$$

$$(\forall x \in I)\big[\, f^*(x) = 0 \implies (\exists v \in I)\,[\,x = \varphi(v)\,]\,\big]. \tag{36}$$

We believe that Löwenheim's definition, formalized in (35) and 36), expresses Boole's intent; namely, that as v is assigned values on I, (33) generates (a) all solutions (condition (36)) and (b) nothing but solutions (condition (35)) of $f^*(x) = 0$. A seemingly different interpretation is advanced by Hailperin: "We take Boole's $w = A + vC$ to be $\exists v(w = A + vC)$" [21, p. 156]. This purely existential view has been expressed by other commentators, for example, [6, p. 92], [14, p. 623], and [56, p. 130]. In this view, a parametric general solution of $f^*(x) = 0$ is a system, (33, 34), satisfying the single condition

$$(\forall x \in I)\big[\, f^*(x) = 0 \iff (\exists v \in I)\,[\,x = \varphi(v)\,]\,\big]. \tag{37}$$

The two definitions, as we now show, are equivalent.

Proposition 3.3 (37) $\iff$ (35, 36).

Proof (37) is equivalent to the system

$$(\forall x \in I)\left[\, f^*(x) = 0 \implies (\exists v \in I)\,[\, x = \varphi(v)\,]\,\right] \tag{38}$$
$$(\forall x \in I)\left[(\exists v \in I)\,[\, x = \varphi(v)\,] \implies f^*(x) = 0\,\right]. \tag{39}$$

(38) is identical to (36). Further,

$$\begin{aligned}
(39) \iff\ & (\forall x \in I)\left[\sim [\,(\exists v \in I)\,[\, x = \varphi(v)\,]\,] \vee [\, f^*(x) = 0\,]\right] \\
\iff\ & (\forall x \in I)\left[[\,(\forall v \in I) \sim [\, x = \varphi(v)\,]\,] \vee [\, f^*(x) = 0\,]\right] \\
\iff\ & (\forall x \in I)\,(\forall v \in I)\left[\,\sim [\, x = \varphi(v)\,] \vee [\, f^*(x) = 0\,]\right] \\
\iff\ & (35).
\end{aligned}$$

□

3.3.2 Inclusive form An *inclusive general solution*[17] of $f^*(x) = 0$ is a system,

$$g \le x \le h \tag{40}$$
$$f^*(0)\, f^*(1) = 0, \tag{41}$$

where $g, h \in I$, such that (40) $\iff f^*(x) = 0$.

Proposition 3.4 *The system*

$$f^*(0) \le x \le \overline{f^*(1)} \tag{42}$$
$$f^*(0)\, f^*(1) = 0 \tag{43}$$

is an inclusive general solution of $f^*(x) = 0$.

Proof (42) $\iff [\, f^*(0)\bar{x} = 0$ and $f^*(1)x = 0]$ (Proposition 2.8) $\iff f^*(x) = 0$.
□

Proposition 3.5 *If* $g, h \in I$, *then* $g \le x \le h$ *is related to* $f^*(x) = 0$ *as follows:*

$$(\forall x \in I)\left[f^*(x) = 0 \implies g \le x \le h\right] \iff \left[g \le f^*(0) \text{ and } \overline{f^*(1)} \le h\right] \tag{44}$$
$$(\forall x \in I)\left[g \le x \le h \implies f^*(x) = 0\right] \iff \left[f^*(0) \le g \text{ and } h \le \overline{f^*(1)}\right]. \tag{45}$$

Proof Applying Proposition 2.8, (44) becomes

$$(\forall x \in I)\,\left[\overline{f^*(0)} g \bar{x} + \overline{f^*(1)} \bar{h} x = 0\right] \iff \left[\overline{f^*(0)} g = 0 \ \text{ and } \ \overline{f^*(1)}\, \bar{h} = 0\right].$$

The two statements are clearly equivalent. (45) is proved analogously. □

The set of antecedents (consequents) of $f^*(x) = 0$ is thus the set of subintervals (superintervals) of (42). Hence, given $f^*(0) f^*(1) = 0$ (equivalently, $f^*(0) \le \overline{f^*(1)}$), (42) is both an antecedent and a consequent of $f^*(x) = 0$.

The *completeness* (and *soundness*) of the inclusive general solution (42, 43) of $f^*(x) = 0$ follows from the fact that the superintervals of (42) comprise *all of* (and *nothing but*) the consequents of that equation.

3.4 Boole's general solutions

3.4.1 Parametric form Boole's parameter-based solution of (31) has the form

$$x = r(Y) + v(Y)s(Y) \tag{46}$$
$$0 = t(Y). \tag{47}$$

(Boole writes (46) as $w = A + vC$ and (47) as $D = 0$ (**LT**, p. 92) and calls (47) the "independent relation.") The functions r, s, t, and v are defined by

$$\left.\begin{array}{lcl} r(Y) & = & \sum (Y\text{-constituents mandatory in the solution}) \\ s(Y) & = & \sum (Y\text{-constituents optional in the solution}) \\ t(Y) & = & \sum (Y\text{-constituents to be set to zero}) \\ v(Y) & = & \sum (\text{arbitrary subset of the } Y\text{-constituents}). \end{array}\right\} \tag{48}$$

Each of these sums is interpretable, and the sums defining $r(Y)$ and $s(Y)$ are disjoint; hence (46) is formally interpretable. To determine the allocation of each constituent, Y^A, to one of the sums (48), Boole expresses (31) as $(1 - x) f_0(Y) + x f_1(Y) = 0$ and assumes a solution, $x = g(Y)$. Thus

$$(f_0(Y) - f_1(Y))\, g(Y) = f_0(Y)\,, \tag{49}$$

whence

$$g(Y) = \frac{f_0(Y)}{f_0(Y) - f_1(Y)} = \sum_{A \in \{0,1\}^n} \frac{f_0(A)}{f_0(A) - f_1(A)} Y^A\,. \tag{50}$$

Boole's use of such indicated quotients has exercised his critics more, probably, than any other of his apparent faults. He does not intend $\frac{f_0(Y)}{f_0(Y) - f_1(Y)}$, however, to signify an algebraic fraction. Rather, "the operation of division cannot be *performed* with the symbols with which we are now engaged. Our resource, then, is to *express* the operation, and develop the result" (**LT**, p. 89). It is more convenient, that is, to develop an ordered pair, expressed as a quotient as shown in (50), than to develop the two sides of (49) separately.

Boole bases the allocation of constituents on four "canons" (**LT**, p. 92), shown below. These assign each constituent, Y^A, to one of the summations in (48), or to none, based on the value of $\frac{f_0(A)}{f_0(A) - f_1(A)}$.

> 1st. The symbol 1 [i.e., $\frac{n}{n}$, $n \neq 0$], as the coefficient of a term in a development, indicates that the whole of the class which that constituent represents, is to be taken.
>
> 2nd. The coefficient 0 [i.e., $\frac{0}{n}$, $n \neq 0$], indicates that none of the class are to be taken.
>
> 3rd. The symbol $\frac{0}{0}$ indicates that a perfectly *indefinite* portion of the class, that is, *some, none,* or *all* of its members are to be taken.
>
> 4th. Any other symbol as a coefficient indicates that the constituent to which it is prefixed must be equated to 0.[18]

Table 1 shows the dependence of $\frac{f_0(A)}{f_0(A) - f_1(A)}$ on f_0 and f_1 for Boole's four cases (m and n in the table are distinct nonzero integers). The case numbers correspond to Boole's, except that the table lists two possibilities for Case 4. Boole seeks only interpretable solutions; thus, noting Proposition 3.2, only constituents for which

$f_0(A)f_1(A) = 0$ (i.e., Cases 1, 2, or 3) may contribute to (46) in the general solution. Case 2 constituents, however, are discarded. Case 4 constituents are assigned to (47), the consistency-condition.

Case 4b shows that solutions of $f(x, Y) = 0$ may exist if $f_0(A)f_1(A) \neq 0$ for one or more $A \in \{0, 1\}^n$. By Proposition 3.2, such solutions are not interpretable.[19]

Case	$f_0(A)$	$f_1(A)$	$(f_0(A) - f_1(A))\, g(A) = f_0(A)$	$g(A)$
1.	n	0	$ng(A) = n$	1
2.	0	n	$-ng(A) = 0$	0
3.	0	0	$0 = 0$	arbitrary
4a.	n	n	$0 = n$	impossible
4b.	n	m	$(n - m)g(A) = n$	$n/(n - m)$

Table 1 Dependence of $g(A) = \frac{f_0(A)}{f_0(A) - f_1(A)}$ on $f_0(A)$ and $f_1(A)$.

Proposition 3.6 *Boole's parameter-based solution,* (46, 47)*, of* (31) *is expressed in terms of f^* (the interpretable image of f) as follows:*[20]

$$x = f^*(0, Y)\ \overline{f^*(1, Y)} + v(Y)\ \overline{f^*(0, Y)}\ \overline{f^*(1, Y)} \tag{51}$$

$$0 = f^*(0, Y)\ f^*(1, Y). \tag{52}$$

Proof Comparing Boole's canons with Table 1,

$$\text{Case 1:} \quad r(Y) = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) \neq 0 \\ f_1(A) = 0}} Y^A = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) \neq 0}} Y^A \sum_{\substack{B \in \{0,1\}^n \\ f_1(B) = 0}} Y^B$$

$$\text{Case 3:} \quad s(Y) = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) = 0 \\ f_1(A) = 0}} Y^A = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) = 0}} Y^A \sum_{\substack{B \in \{0,1\}^n \\ f_1(B) = 0}} Y^B$$

$$\text{Case 4:} \quad t(Y) = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) \neq 0 \\ f_1(A) \neq 0}} Y^A = \sum_{\substack{A \in \{0,1\}^n \\ f_0(A) \neq 0}} Y^A \sum_{\substack{B \in \{0,1\}^n \\ f_1(B) \neq 0}} Y^B$$

$$\begin{aligned} \text{Thus} \quad r(Y) &= f^*(0, Y)\ \overline{f^*(1, Y)} \\ s(Y) &= \overline{f^*(0, Y)}\ \overline{f^*(1, Y)} \\ t(Y) &= f^*(0, Y)\ f^*(1, Y). \end{aligned}$$

□

It remains to show that Boole's solution, (51, 52), of $f(x) = 0$ is a general solution of that equation, that is, that it satisfies condition (37).

Proposition 3.7 (51, 52) *is a parametric general solution of* $f^*(x) = 0$.

Proof $(\exists v \in I)[x = \varphi(v)] \iff (\varphi(0)\bar{x} + \overline{\varphi}(0)x)(\varphi(1)\bar{x} + \overline{\varphi}(1)x) = 0$ (Proposition 3.2) $\iff \varphi(0)\varphi(1)\bar{x} + \overline{\varphi}(0)\overline{\varphi}(1)x = 0$. Let $\varphi(v) = f_0^*\overline{f_1^*} + v\overline{f_0^*}\ \overline{f_1^*}$. Then $\varphi(0) = f_0^*(1 - f_1^*) = f_0^* - f_0^* f_1^* = f_0^*$ (invoking (52)) and $\varphi(1) = f_0^*(1 - f_1^*) + (1 - f_0^*)(1 - f_1^*) = (f_0^* + 1 - f_0^*)(1 - f_1^*) = (1 - f_1^*)$. Thus $(\exists v)[x = \varphi(v)] \iff f_0^*(1 - f_1^*)\bar{x} + (1 - f_0^*)f_1^* x = 0 \iff f_0^*\bar{x} + f_1^* x = 0 \iff f^*(x) = 0$. □

3.4.2 Inclusive form The final step of Boole's inferential process is to transform his parametric general solution to inclusive form. To gain insight into Boole's approach, we consider two instances in **LT** of such transformation. The first concerns class- (primary) logic:

> The equation [$w = stp + \frac{0}{0}st(1-p)$, where $\frac{0}{0}$ is an arbitrary interpretable parameter] may be interpreted in the following manner: *Wealth is either limited in supply, transferrable, and productive of pleasure, or limited in supply, transferrable, and not productive of pleasure. And reversely, whatever is limited in supply, transferrable, and productive of pleasure is wealth.* Reverse interpretations, similar to the above, are always furnished when the final development introduces terms having unity as a coefficient. (**LT**, p. 112)

Thus $w = stp + \frac{0}{0}st\overline{p}$ is transformed into $stp \subseteq w \subseteq (stp \cup st\overline{p})$.

The second instance concerns propositional (secondary) logic:

> **Principle**.—*Any constituent term or terms in a particular member of an equation which have for their coefficient unity, may be taken as the antecedent of a proposition, of which all the terms in the other member form the consequent.*
>
> Thus the equation
>
> $$y = xz + vx(1-z) + (1-x)(1-z)$$
>
> would have the following interpretations:
>
> **Direct Interpretation**.—*If the proposition Y is true, then either X and Z are true, or X is true and Z false, or X and Z are both false.*
>
> **Reverse Interpretation**.—*If either X and Z are true, or X and Z are false, Y is true.*
>
> The aggregate of these partial interpretations will express the whole significance of the equation given. (**LT**, p. 173)

Thus $y = xz + \bar{x}\bar{z} + vx\bar{z}$ is transformed into $xz \vee \bar{x}\bar{z} \rightarrow y \rightarrow xz \vee \bar{x}\bar{z} \vee x\bar{z}$. We conclude from the foregoing illustrations that Boole transforms (51, 52) into verbal statements corresponding to the inclusive form,

$$f^*(0)\,\overline{f^*(1)} \leq x \leq f^*(0)\,\overline{f^*(1)} + \overline{f^*(0)}\;\overline{f^*(1)} \tag{53}$$

$$f^*(0)f^*(1) = 0\,. \tag{54}$$

Boole does not include a symbol for the arithmetic order-relation in **LT**. That omission, together with his being unaware, apparently, of the closed form (51, 52), forces Boole to express (53, 54) verbally, and by means of examples.

Proposition 3.8 *System* (53, 54) *is an inclusive general solution of* $f^*(x) = 0$.

Proof (54) $\Longrightarrow \big[\,[\,f^*(0)\,\overline{f^*(1)} = f^*(0)\,]$ and $[\,f^*(0)\,\overline{f^*(1)} + \overline{f^*(0)}\;\overline{f^*(1)} = \overline{f^*(1)}\,]\big]$. Thus (53) $\Longleftrightarrow [\,f^*(0) \leq x \leq \overline{f^*(1)}\,] \Longleftrightarrow \big[\,[\,\bar{x}f^*(0) = 0\,]$ and $[\,xf^*(1) = 0\,]\big] \Longleftrightarrow [\,f^*(x) = 0\,]$ (Propositions 2.4 and 2.8). □

4 The General Method

The steps in Boole's general method are summarized in Figure 1. To illustrate the method, we follow Boole's solution of one part of Example 5, Chapter IX (**LT**, p. 146). As noted in Section 1.2, this example was used by nineteenth-century logicians as the acid test for their methods.

1.	**Translation**	Express the premises, involving logical symbols $x, y_1, \ldots, y_m, z_1, \ldots, z_n$, by a system $$\begin{array}{rcl} g_1(x, y_1, \ldots, y_m, z_1, \ldots, z_n) &=& 0 \\ &\vdots& \\ g_p(x, y_1, \ldots, y_m, z_1, \ldots, z_n) &=& 0 \end{array}$$ of proto-Boolean equations.
2.	**Reduction**	Condense to a single equivalent equation: $g(x, Y, Z) = 0$
3.	**Elimination**	Deduce a general consequent not involving Z: $f(x, Y) = 0$
4.	**Parametric General Solution**	Construct a formally-interpretable parametric general solution for x, $$\begin{array}{rcl} x &=& r(Y) + v(Y)\, s(Y) \\ 0 &=& t(Y), \end{array}$$ of the equation in Step 3, where v is an arbitrary interpretable p-function and $rs = 0$.
5.	**Inclusive General Solution**	Using the functions r, s, and t derived in Step 4, express the general solution as $$\begin{array}{l} r(Y) \le x \le r(Y) + s(Y) \\ t(Y) = 0 \end{array}$$ (Boole states this result verbally.)

Figure 1 Steps in Boole's general method.

Example 5 (Venn's statement [54], p. 351, of Boole's Example 5, changing Venn's u to Boole's v) Let the observation of a class of natural productions be supposed to have led to the following general results.

1. Wherever x and z are missing, v is found, with one (but not both) of y and w.
2. Wherever x and w are found while v is missing, y and z will both be present or both absent.
3. Wherever x is found with either or both of y and v there will z or w (but not both) be found, and conversely.

Boole specifies that v (an ordinary symbol, not a parameter, in this example) is to be eliminated and poses two problems based on the result: first, that x be concluded in terms of w, y, and z; second, that y be concluded in terms of w, x, and z. We study the second problem.

Boole expresses the premises by

$$\begin{array}{rcl} \bar{x}\bar{z} &=& qv(\bar{w}y + w\bar{y}) \\ \bar{v}wx &=& q(yz + \bar{y}\bar{z}) \\ xy + vx\bar{y} &=& w\bar{z} + \bar{w}z \end{array} \tag{55}$$

where q is an arbitrary parameter and $\bar{v}, \bar{w}, \ldots,$ stand for $1 - v, 1 - w, \ldots$.

4.1 Translation Following Rule (2), Boole translates (55) to the system

$$\begin{aligned} \bar{x}\bar{z}(1 - v(\bar{w}y + w\bar{y})) &= 0 \\ \bar{v}wx(\bar{y}z + y\bar{z}) &= 0 \\ (xy + vx\bar{y})(\bar{w}\bar{z} + wz) + (1 - xy - vx\bar{y})(w\bar{z} + \bar{w}z) &= 0\,. \end{aligned} \tag{56}$$

(Boole writes $x\bar{y} + \bar{x}y$ and $\bar{x}\bar{y} + xy$, rather than $1 - (\bar{x}\bar{y} + xy)$ and $1 - (x\bar{y} + \bar{x}y)$, respectively, assuming the simpler forms to be familiar to the reader.)

4.2 Reduction The equations in (56) are formally interpretable; hence they are composable. They may therefore be combined into an equivalent single equation by simple addition (Proposition 2.10). (Noncomposable equations can be made composable using Boole's method of squaring, cf. Proposition 2.12.) System (56) is therefore equivalent to $g = 0$, where

$$g = \bar{x}\bar{z}(1 - v(\bar{w}y + w\bar{y})) + \bar{v}wx(\bar{y}z + y\bar{z}) + (xy + vx\bar{y})(\bar{w}\bar{z} + wz) + (1 - xy - vx\bar{y})(w\bar{z} + \bar{w}z). \tag{57}$$

4.3 Elimination The resultant of elimination of v from $g(v, w, x, y, z) = 0$ is the most general consequent of that equation not involving v (Proposition 2.14). Boole develops (57) partially with respect to y; namely, $g = (1 - y)g_0(v, w, \ldots) + yg_1(v, w, \ldots)$, where $g_0 = g(v, w, x, 0, z)$ and $g_1 = g(v, w, x, 1, z)$; that is,

$$\begin{aligned} g_0(v, w, \ldots) &= \bar{x}\bar{z}(1 - vw) + \bar{v}wxz + vx(\bar{w}\bar{z} + wz) + (1 - vx)(w\bar{z} + \bar{w}z) \\ g_1(v, w, \ldots) &= \bar{x}\bar{z}(1 - v\bar{w}) + \bar{v}wx\bar{z} + x(\bar{w}\bar{z} + wz) + \bar{x}(w\bar{z} + \bar{w}z)\,. \end{aligned}$$

Applying Proposition 2.15 (**LT**, Chap. IX, Prop. III), Boole expresses the resultant of elimination of v from $g = 0$ as $f = 0$, where f is given by

$$\begin{aligned} f &= (1 - y)g_0(0, w, \ldots)g_0(1, w, \ldots) + yg_1(0, w, \ldots)g_1(1, w, \ldots) \\ &= (1 - y)\left[(\bar{x}\bar{z} + wxz + w\bar{z} + \bar{w}z)(\bar{w}\bar{x}\bar{z} + x(\bar{w}\bar{z} + wz) + \bar{x}(w\bar{z} + \bar{w}z))\right] \\ &\quad + y\ \left[(\bar{x}\bar{z} + wx\bar{z} + x(\bar{w}\bar{z} + wz) + \bar{x}(w\bar{z} + \bar{w}z))\right. \\ &\qquad \left.(w\bar{x}\bar{z} + x(\bar{w}\bar{z} + wz) + \bar{x}(w\bar{z} + \bar{w}z))\right]\,. \end{aligned}$$

Thus the simplified resultant is

$$(1 - y)(\bar{w}\bar{x}\bar{z} + \bar{w}\bar{x}z + 2w\bar{x}\bar{z} + wxz) + y(\bar{w}\bar{x}z + \bar{w}x\bar{z} + 4w\bar{x}\bar{z} + wxz) = 0. \tag{58}$$

4.4 Solutions

4.4.1 Boole's parametric general solution Boole converts coefficients 2 and 4 in (58) to unity (cf. Prop. 3.1) and solves the resulting equation for y:

$$y = \frac{\bar{w}\bar{x}\bar{z} + w\bar{x}\bar{z} + \bar{w}\bar{x}z + wxz}{\bar{w}\bar{x}\bar{z} - \bar{w}x\bar{z}}\,.$$

In developed form,

$$\begin{aligned} y \quad = \quad & \bar{w}\bar{x}\bar{z}\left[\tfrac{1}{1}\right] + \bar{w}\bar{x}z\left[\tfrac{1}{0}\right] + \bar{w}x\bar{z}\left[\tfrac{0}{-1}\right] + \bar{w}xz\left[\tfrac{0}{0}\right] + \\ & w\bar{x}\bar{z}\left[\tfrac{1}{0}\right] + w\bar{x}z\left[\tfrac{0}{0}\right] + wx\bar{z}\left[\tfrac{0}{0}\right] \ + wxz\left[\tfrac{1}{0}\right]. \end{aligned}$$

From Boole's four canons (Section 3.4.1), he derives the solution

$$\begin{aligned} y &= \bar{w}\bar{x}\bar{z} + \tfrac{0}{0}(w\bar{x}z + wx\bar{z} + \bar{w}xz) && (7)\\ wxz &= 0 && (8)\\ w\bar{x}\bar{z} &= 0 && (9)\\ \bar{w}\bar{x}z &= 0 && (10)\end{aligned}$$

(**LT**, p. 148, Boole's numbering); $\frac{0}{0}$ is an arbitrary interpretable parameter.

4.4.2 Boole's inclusive general solution (In quoting Boole's inclusive solution, we omit his translation of w, x, y, and z into D, A, B, and C, respectively.) Boole first analyzes his condition (10): *"If property x is absent and z is present, w is present."* He then accounts for the remainder of the solution:

> 1st. *If the property y be present in one of the productions, either the properties w, x, and z are all absent, or some one alone of them is absent.* And conversely, *if they are all absent it may be concluded that the property y is present* (7).
>
> 2nd. *If x and z are both present or both absent, w will be absent, quite independently of the presence or absence of y* (8) and (9).
>
> (**LT**, p. 149)

Expressed symbolically:

$$\begin{aligned} \bar{w}\bar{x}\bar{z} \le{} & y \le \bar{w}\bar{x}\bar{z} + \bar{w}xz + w\bar{x}z + wx\bar{z} && (7)\\ \bar{x}\bar{z} + xz \le{} & \bar{w} && (8), (9)\\ \bar{x}z \le{} & w && (10)\end{aligned}$$

The inclusive general solution is the final step in Boole's general method. It is discussed in a variety of Boole's examples (**LT**, pp. 112, 120, 129, 149, 173, 222), including the widely cited Example 5, and is the form used exclusively by the successors of Boole cited earlier, beginning with Jevons [26] (who states only the direct interpretation). It is also more prominent than the parametric form in the contemporary theory of Boolean equations [7; 45]. Nevertheless, it has been little noticed by critics of **LT**, who discuss the parametric form almost exclusively—perhaps because of Boole's verbal, rather than symbolic, expression of the inclusive form.[21]

Notes

1. George Boole's published works on logic are *The Mathematical Analysis of Logic* (**MAL**) [2] in 1847, "The calculus of logic" [3] in 1848, and *An Investigation of the Laws of Thought* (**LT**) [4] in 1854. The first fifteen chapters of **LT** are devoted to logic: Chapters I–X treat class ("primary") logic; Chapters XI–XIV treat propositional ("secondary") logic, and Chapter XV discusses the syllogistic logic of Aristotle. The remaining seven chapters concern probability.

2. Cited by Wood [56, p. 67].

3. See Burris [11, p. 105] for further analysis of Boole's work in a presentation "quite close to that of Hailperin." Burris notes, "We do not know of any such scholarly evaluation of Boole's work that was available before Hailperin's book."

4. The term "general method in Logic" appears several times in **LT** (pp. 7–10, 70) but is not given a definite meaning. We follow van Evra [53, p. 366] in taking it to mean the deductive procedure presented in Chapters V ff. of **LT**.

5. Boolean-equation theory, in the modern sense, originates with McColl [38; 39] and Peirce [43] and is treated at length in Schröder [47, Vol. 1]. Later works include Couturat [15], Löwenheim [35], Lewis [33], Jørgensen [27], Rudeanu [45], and Brown [7].

6. See Schroeder-Heister [48] concerning Frege's analysis of Boole's Example 5.

7. Cited by Ladd [30, p. 58].

8. Burris [10] demonstrates that a slight modification of Boole's algebra allows particular categorical statements to be handled in an equational system.

9. It is not clear in the sources available to us (a summary by Beth [1, pp. 65 and 66], and a brief review [28]) whether Hoff-Hansen or Skolem related his system to Boole's calculus.

10. Applications in operations research are based on *pseudo-Boolean functions* [23; 22] (see [5] for a survey). Engineering applications are discussed by Papaioannou and Barrett [42] (who call a proto-Boolean polynomial the "real transform" of a Boolean function) and by Schneeweiss [46] ("the (true) polynomial form"). The latter text restricts variable-values (but not coefficients) to the integers 0 and 1, noting that such variables "allow for the use of standard algebra to write Boolean functions; George Boole did this" (p. v).

11. Hailperin's formalization of Boole's logic [21, Chap. 2] is solely in terms of functions. Thus x, y, and so on, are defined on p. 142 as variables ranging over I.

12. Hailperin's form of Prop. 2.2 [21, Theorem 2.33]) restricts X to I^n.

13. See Rudeanu [45, pp. 99 and 100] for the Boolean version of the verification theorem, first stated in 1901 as Müller's "Verifikationstheorem" [40] and discussed later in Müller's "Abriss" to Schröder [47, Vol. 3, Section 126]. Löwenheim [35] quotes it as his Proposition 14b: "We can discover whether an equation or subsumption in which $x_1, x_2, \ldots x_n$ appear is valid in general by whether it is valid for any system of values 0, 1 of the domains $x_1, x_2, \ldots, x_n$."

14. The members of $(I_n, \dot{+}, \dot{\times}, \bar{\ }, 0, 1)$ and $(S_n, \cup, \cap, {}', \varnothing, U)$ are examples of what Rudeanu [45, pp. 17 and 23] calls *simple* Boolean functions. The defining property of a simple Boolean function, $f : B^n \to B$, is that $f(A) \in \{0, 1\}$ for all $A \in \{0, 1\}^n$. Thus f is expressed by a development not involving constants, other than 0 and 1, from B. The distinction between Boolean and simple Boolean functions seems to have been made first in [44], where the latter are called "Boolean functions in the restricted sense."

15. Boole rejected a suggestion in 1848 to include $>$ in his system [50, p. 32].

16. In accord with modern usage [45, p. 62], we call $f(0)f(1) = 0$ the *resultant* (rather than Boole's "result") of elimination of x from $f(x) = 0$.

17. Called a *subsumptive* general solution in [9] in the context of Boolean algebra.

18. Styazhkin [52, p. 184] takes canon 4 to mean the constituent is "discarded."

19. Taking note of the cases listed in Table 1, it can be shown that for all $f \in \widehat{P}_n$, $f(x, Y) = 0$ possesses a solution in P^n, not just in I^n, if and only if the condition

$$(\forall A \in \{0,1\}^{n-1}) \left[\begin{array}{lr} f_0(A) f_1(A) = 0 & \text{(Cases 1,2,3)} \\ \text{or} & \\ f_0(A) f_1(A) \neq 0 \quad \text{and} \quad \left| \frac{f_0(A) - f_1(A)}{d(A)} \right| = 1 & \text{(Case 4b)} \end{array} \right]$$

is satisfied, where $d(A) = \gcd(f_0(A), f_1(A))$.

Consider $f(x, y) = 2-2x+xy$, for which $f_0(y) = 2$ and $f_1(y) = y$. Proposition 3.2 does not apply, because $f_0(y) f_1(y) \neq 0$; however, the foregoing condition is satisfied; that is, $f_0(0) f_1(0) = 0$ and $\frac{f_0(1)-f_1(1)}{d(1)} = \frac{2-1}{1} = 1$. Thus a solution of $f(x, y) = 0$ is $x = g(y) = (1-y)[\frac{f_0(0)}{f_0(0)-f_1(0)}] + y[\frac{f_0(1)}{f_0(1)-f_1(1)}] = (1-y)[\frac{2}{2-0}] + y[\frac{2}{2-1}] = 1 + y$.

20. The representation $x = f_0^* \overline{f_1^*} + v \overline{f_0^*}\, \overline{f_1^*}$ for Boole's parameter-based solution has been given (without proof and without requiring f to be interpretable) by Feys [18, p. 110]. It is surprising that Boole, who employs f_0 and f_1 extensively and with extraordinary insight, does not mention this representation.

21. In a note added to Chapter 2 of [21], Hailperin cites inclusive interpretations in an example on p. 222 of **LT**.

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