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Thin Ultrafilters

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Abstract A free ultrafilter $\mathcal{U}$ on ω is called a T-point if, for every countable group G of permutations of ω, there exists $U \in \mathcal{U}$ such that, for each $g \in G$, the set $\{x \in U : gx \neq x, gx \in U\}$ is finite. We show that each P-point and each Q-point in ω^* is a T-point, and, under CH, construct a T-point, which is neither a P-point, nor a Q-point. A question whether T-points exist in ZFC is open.

1 Introduction

Let G be a group with the identity e. A subset T of G is called thin if the intersection $gT \cap T$ is finite for every $g \in G$, $g \neq e$. For thin subsets, its modifications, applications, and references, see Lutsenko and Protasov [6], Protasov [14]. We begin with the following generalization of thin subsets.

Let X be a set, $s : X \to X$ be an arbitrary mapping. We say that a subset T of X is *s-thin* if the set

$$\{x \in T : sx \neq x, sx \in T\}$$

is finite. Clearly, each finite subset of X is s-thin.

Given a subset S of the set X^X of all selfmappings of X, we say that T is *S-thin* if T is s-thin for each $s \in S$. We say that an ultrafilter $\mathcal{U}$ on X is *S-thin* if there is an S-thin subset of X which is a member of $\mathcal{U}$.

Let $\mathcal{S}$ be a family of subsets of X^X. We say that an ultrafilter $\mathcal{U}$ on X is *thin with respect to $\mathcal{S}$* if $\mathcal{U}$ is S-thin for each $S \in \mathcal{S}$. Every ultrafilter on X is thin with respect to the family of all finite subsets of X^X (see Proposition 2.5).

A free ultrafilter $\mathcal{U}$ on $\omega = \{0, 1, \dots\}$ is called a *T-point* if $\mathcal{U}$ is thin with respect to the family of all countable groups of permutations of ω (in words, if for every countable group G of permutations of ω there exists a G-thin subset $T \subseteq \omega$ such that $T \in \mathcal{U}$). We show that T-points generalize the classical ultrafilters on ω: P-points and Q-points.

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Recall that a free ultrafilter $\mathcal{U}$ on ω is

1. *selective* if, for every partition $\mathcal{P}$ of ω, either some block of $\mathcal{P}$ is a member of $\mathcal{U}$, or there is $U \in \mathcal{U}$ such that $|U \cap P| \leqslant 1$ for each $P \in \mathcal{P}$;
2. *P-point* if, for every partition $\mathcal{P}$ of ω, either some block of $\mathcal{P}$ is a member of $\mathcal{U}$, or there is $U \in \mathcal{U}$ such that $U \cap P$ is finite for each $P \in \mathcal{P}$;
3. *Q-point* if, for every partition $\mathcal{P}$ of ω into finite subsets, there is $U \in \mathcal{U}$ such that $|U \cap P| \leqslant 1$ for each $P \in \mathcal{P}$.

In Sections 2 and 3, we prove that each P-point and each Q-point is a T-point, and, under the Continuum Hypothesis (CH), construct a T-point which is neither P-point nor Q-point. We do not know if T-points exist in ZFC without additional set-theoretical assumptions. In Section 4, we use thin subsets to show that each nonempty open subset of the corona of a countable G-space contains a homeomorphic copy of ω^*. In Section 5, we give a "thin" characterization of selective ultrafilters.

For a discrete space X, we identify the Stone-Čech compactification βX of X with the set of all ultrafilters of X, and denote $X^* = \beta X \setminus X$. We use the universal property of βX stating that each mapping $f : X \to K$, where K is a compact Hausdorff space, extends to the continuous mapping $f^\beta : \beta X \to K$.

2 Thin Subsets and Ultrafilters

Example 2.1 Let G be an infinite group of cardinality κ. By Chou's lemma [1], there exists a subset $X \subset G$ such that $|X| = \kappa$ and $|gX \cap X| \leqslant 3$ for each $g \in G$, $g \neq e$. Moreover, by [7], there exists $Y \subset G$ such that $|Y| = \kappa$, $G = YY^{-1} \cup Y^{-1}Y$ and $|gY \cap Y| \leqslant 2$ for each $g \in G$, $g \neq e$. Since every subset of a G-thin subset is G-thin, we conclude that there are 2^{2^κ} ultrafilters on G having a G-thin subset among its members. On the other hand, if $\mathcal{U}$ is an idempotent in the semigroup G^* of all free ultrafilters on G (see [3, Chapter 5]) then, for every $U \in \mathcal{U}$ there exist $g \in U$ and $V \in \mathcal{U}$ such that $V \subseteq U$ and $gV \subseteq U$. It follows that $\mathcal{U}$ has no G-thin members.

Example 2.2 For infinite cardinals $\kappa, \mu, \mu \leqslant \kappa$, we denote by S_κ the group of all permutations of κ and put

$$S_{\kappa,\mu} = \{g \in S_\kappa : |\operatorname{supp} g| < \mu\},$$

where $\operatorname{supp} g = \{x \in \kappa : gx \neq x\}$. Clearly, each subset of κ is $S_{\kappa,\aleph_0}$-thin, and each $S_{\kappa,\aleph_1}$-thin subset of κ is finite.

Example 2.3 Let $S \subset \omega^\omega$ be a countable family of finite-to-one mappings. To construct a countable S-thin subset of ω, we enumerate $S = \{s_n : n \in \omega\}$, put $F_n = \{s_i : i \leqslant n\}$, $n \in \omega$, choose an arbitrary element $x_0 \in \omega$ and suppose that we have chosen the elements $\{x_i : i \leqslant n\}$ such that the subsets $\{F_i x_i : i \leqslant n\}$ are pairwise disjoint. Since each mapping from F_{n+1} is finite-to-one, we can choose $x_{n+1} \in \omega$ such that $F_{n+1}x_{n+1} \cap F_i x_i = \varnothing$ for every $i \leqslant n$. After ω steps, we get an S-thin subset $X = \{x_n : n \in \omega\}$.

Example 2.4 We point out a countable subset $S \subset \omega^\omega$ such that each S-thin subset of ω is finite. For each $n \in \omega$, we define $\sigma_n \in \omega^\omega$ by $\sigma_n(x) = n$ for each $x \in \omega$, and put $S = \{\sigma_n : n \in \omega\}$. Let X be a nonempty S-thin subset of ω, $n \in X$. Since X is σ_n-thin, we see that X is finite.

In what follows we use the 4-set lemma [3, Lemma 3.33]. Let X be a set, $f : X \to X$. Then there exists a partition

$$X = X_0 \cup X_1 \cup X_2 \cup X_3$$

such that $X_0 = \{x \in X : f(x) = x\}$ and $f(X_i) \cap X_i = \varnothing$ for every $i \in \{1, 2, 3\}$.

Proposition 2.5 *Let κ be a cardinal, $\mathcal{U} \in \beta\kappa$. For every finite subset $F \subset \kappa^\kappa$, there exists an F-thin subset $T \in \mathcal{U}$.*

Proof Using the 4-set lemma, for every $f \in F$, we choose $T_f \in \mathcal{U}$ such that either $f|T_f \equiv id$ or $f(T_f) \cap T_f = \varnothing$. Put $T = \bigcap_{f \in F} T_f$. □

Let X be a topological space, μ be an infinite cardinal. A point $x \in X$ is called a *P_μ-point* if the intersection of any μ neighborhood of x is a neighborhood of x. In the case $X = \omega^*$ and $\mu = \aleph_0$, we get a P-point in ω^*.

Proposition 2.6 *Let κ, μ be infinite cardinals, $\mathcal{U}$ be a P_μ-point in κ^*, $S \subset \kappa^\kappa$ be a family of finite-to-one mappings, $|S| = \mu$. Then there exists an S-thin subset $T \in \mathcal{U}$.*

Proof Using the 4-set lemma, for each $s \in S$, we pick a subset $T_s \in \mathcal{U}$ such that either $s|T_s \equiv id$ or $s(T_s) \cap T_s = \varnothing$. Since $\mathcal{U}$ is a P_μ-point in κ^* and $|S| = \mu$, we can choose a subset $T \in \mathcal{U}$ such that $T \setminus T_s$ is finite for each $s \in S$. Since the set $s^{-1}(T \setminus T_s)$ is finite, T is s-thin for each $s \in S$. □

Proposition 2.7 *Let G be a countable group of permutations of ω, $\mathcal{U} \in \omega^*$. If $\mathcal{U}$ is a Q-point then there exists a G-thin subset $T \in \mathcal{U}$.*

Proof Enlarging G, we may suppose that G acts transitively on ω. We enumerate $G = \{g_n : n \in \omega\}$, $g_0 = e$, put $F_n = \{g_0^{\pm 1}, \dots, g_n^{\pm 1}\}$ and denote by F_n^m the product of m copies of F_n.

We fix an arbitrary $a \in \omega$, put

$$X_0 = \{a\}, X_{n+1} = F_{n+1}^{n+1} a \setminus F_n^n a,\ n \in \omega,$$

and claim that, for each $n > 0$,

$$(*) \qquad F_{n-1} X_n \subseteq X_{n-1} \cup X_n \cup X_{n+1}.$$

For $n = 1$, (*) is evident so let $n \geqslant 2$. Since

$$X_{n-1} \cup X_n \cup X_{n+1} = (F_{n-1}^{n-1} a \setminus F_{n-2}^{n-2} a) \cup (F_n^n a \setminus F_{n-1}^{n-1} a) \cup (F_{n+1}^{n+1} a \setminus F_n^n a) = F_{n+1}^{n+1} a \setminus F_{n-2}^{n-2} a,$$

it suffices to verify that

$$F_{n-1}(F_n^n a \setminus F_{n-1}^{n-1} a) \subseteq F_{n+1}^{n+1} a \setminus F_{n-2}^{n-2} a.$$

Clearly, $F_{n-1}(F_n^n a \setminus F_{n-1}^{n-1} a) \subseteq F_{n-1} F_n^n a \subseteq F_{n+1}^{n+1} a$. If $F_{n-1}(F_n^n a \setminus F_{n-1}^{n-1} a) \cap F_{n-2}^{n-2} a \neq \varnothing$, then $F_n^n a \setminus F_{n-1}^{n-1} a \cap F_{n-1} F_{n-2}^{n-2} a \neq \varnothing$, contradicting $F_{n-1} F_{n-2}^{n-2} \subseteq F_{n-1}^{n-1}$.

Then we put

$$Y_0 = \bigcup_{n \in \omega} X_{3n},\ Y_1 = \bigcup_{n \in \omega} X_{3n+1},\ Y_2 = \bigcup_{n \in \omega} X_{3n+2}.$$

Since G acts transitively on ω, we have $\omega = \bigcup_{n \in \omega} X_n$ so $\omega = Y_0 \cup Y_1 \cup Y_2$. Since $X_n \cap X_m = \varnothing$ for all distinct m, n and $\mathcal{U}$ is a Q-point, there exist $T \in \mathcal{U}$ and

$i \in \{1,2,3\}$ such that $T \subseteq Y_i$ and $|T \cap X_n| \leqslant 1$ for each $n \in \omega$. We take an arbitrary $g_m \in G$. If $t \in T \cap X_n$, $n \geqslant m+1$ and $g_m t \in T$ then, by $(*)$, $g_m t \in X_n$ so $g_m t = t$. Hence, T is G-thin. □

3 T-Points

Theorem 3.1 *Every P-point and every Q-point in ω^* are T-points.*

Proof Apply Propositions 2.6 and 2.7. □

Shelah produced a ZFC-model in which there are no P-points in ω^* [16]. On the other hand, there is also a model in which there are no Q-points [10]. But it is unknown [2, Question 25] if there is a model in which there are no P-points and Q-points. By [4] and [8], if $\mathfrak{c} \leqslant \aleph_2$ there is either a P-point or a Q-point (and, by Theorem 3.1, a T-point).

Recall that the ultrafilters $\mathcal{U}, \mathcal{V}$ on ω are of the same type if there is a bijection $f : \omega \to \omega$ such that, for any $X \subseteq \omega$, $X \in \mathcal{U}$ if and only if $f(X) \in \mathcal{V}$. If $\mathcal{V}$ is an ultrafilter, and $(\mathcal{U}_n)_{n\in\omega}$ is a sequence of ultrafilters on ω, a subset $A \subseteq \omega$ is a member of the ultrafilter $\mathcal{V}\text{-}\lim \mathcal{U}_n$ if and only if $\{n \in \omega : A \in \mathcal{U}_n\} \in \mathcal{V}$.

Theorem 3.2 *Let $\{\mathcal{U}_n : n \in \omega\}$ be a family of P-points in ω^* of distinct types, $\mathcal{V}$ be an arbitrary ultrafilter from ω^*. Then $\mathcal{V}\text{-}\lim \mathcal{U}_n$ is a T-point.*

Proof Let $G = \{g_m : m \in \omega\}$ be a countable group of permutations of ω. Since the ultrafilters $\{\mathcal{U}_n : n \in \omega\}$ are of distinct types, for each $m, n, k \in \omega$, $n \neq k$, we can choose $U_{m,n,k} \in \mathcal{U}_n$ and $V_{m,n,k} \in \mathcal{U}_k$ such that

$$g_m U_{m,n,k} \cap V_{m,n,k} = \varnothing.$$

Since $\mathcal{U}_n$ is a P-point, there exists $U_n \in \mathcal{U}_n$ such that $U_n \setminus U_{m,n,k}$ and $U_n \setminus V_{m,k,n}$ are finite for all $m, n, k, k \neq n$. Thus, $g_m U_n \cap U_k$ is finite for all $m, n, k, n \neq k$.

Since $g_m^\beta(\mathcal{U}_n)$ is a P-point, $g_m^\beta(\mathcal{U}_n)$ is not in the closure of the set $\{\mathcal{U}_k : k \in \omega, k \neq n\}$. Hence, we can choose inductively the sets $\{W_n : n \in \omega\}$ such that $W_n \subseteq U_n$, $W_n \in \mathcal{U}_n$ and

$$g_m W_n \cap W_k = \varnothing$$

for all $m \leqslant n < k < \omega$.

Using the 4-set lemma, for every $n \in \omega$, we choose a decreasing family $\{U_{n,m} \in \mathcal{U}_n : m \in \omega\}$ such that $U_{n,m} \subseteq W_n$ and either $g_m|U_{n,m} \equiv id$ or $g_m U_{n,m} \cap U_{n,m} = \varnothing$. Since $\mathcal{U}_n$ is a P-point, we can choose $T_n \in \mathcal{U}_n$ such that $T_n \subseteq W_n$ and $T_n \setminus U_{n,m}$ is finite for every $m \in \omega$.

At last, we put

$$T = \bigcup_{n\in\omega} (T_n \cap U_{n,n})$$

and note that $T \in \mathcal{V}\text{-}\lim \mathcal{U}_n$. By the construction, T is G-thin. □

Let X be a topological space. A point $p \in X$ is called a *weak P-point* if $p \notin cl_X Y$ for any countable subset $Y \subseteq X \setminus \{p\}$. In contrast to P-points, the weak P-points in ω^* exist in ZFC (see [5], [9]). To prove this statement, Kunen introduced the following delicate notion.

A point $p \in X$ is called an OK-*point* if, for any countable family $\{U_n : n \in \omega\}$ of neighborhoods of p, there exists an uncountable family $\mathcal{F}$ of neighborhoods of p

such that, for each $n \geqslant 1$ and each subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size n, $\bigcap \mathcal{F}' \subseteq U_n$. Every OK-point is a weak P-point and OK-points in ω^* exist in ZFC.

An ultrafilter $\mathcal{U} \in \omega^*$ is called an NWD-point if, for every injective mapping $f : \omega \to \mathbb{R}$, there exists $U \in \mathcal{U}$ such that $f(U)$ is nowhere dense in $\mathbb{R}$. To see that every P-point is an NWD-point, we can use the following simple topological characterization: an ultrafilter $\mathcal{U} \in \omega^*$ is a P-point if and only if, for every Hausdorff topology τ on ω, there exists $U \in \mathcal{U}$ such that U has at most one limit point in (X, τ).

Proposition 3.3 *Under* CH*, there exists a T-point in ω^* which is neither a weak P-point nor an* NWD*-point nor a Q-point. For every ultrafilter $\mathcal{V} \in \omega^*$, there exists a T-point $\mathcal{U} \in \omega^*$ and a mapping $f : \omega \to \omega$ such that $\mathcal{V} = f^\beta(\mathcal{U})$.*

Proof Using CH, we can construct a family $\{\mathcal{U}_n : n \in \omega\}$ of P-points of distinct types such that each $\mathcal{U}_n$ is not a Q-point. Let $\mathcal{V}$ be an arbitrary ultrafilter from ω^*. By Theorem 3.2, $\mathcal{W} = \mathcal{V}\text{-}\lim \mathcal{U}_n$ is a T-point. Clearly, $\mathcal{W}$ is neither a weak P-point nor a Q-point.

We identify ω with $\mathbb{Q}$ and, for each $n \in \omega$, choose an injective sequence $(a_{nm})_{m\in\omega}$ converging to n. Then we take a family $\{\mathcal{U}_n : n \in \omega\}$ of P-points of distinct types such that $\{a_{nm} : m \in \omega\} \in \mathcal{U}_n$ and each $\mathcal{U}_n$ is not a Q-point. At last, we take an ultrafilter $\mathcal{V}$ on $\mathbb{Q}$ such that every member of $\mathcal{V}$ is not nowhere dense in $\mathbb{Q}$. Then $\mathcal{W}$ is not an NWD-point.

To prove the second statement, we choose an arbitrary family $\{\mathcal{U}_n : n \in \omega\}$ of P-points of distinct types. Since the set $\{\mathcal{U}_n : n \in \omega\}$ is discrete in ω^*, we can choose a disjoint family $\{U_n \in \mathcal{U}_n : n \in \omega\}$. We define a mapping $f : \bigcup_{n\in\omega} U_n \to \omega$ by $f(x) = n$ if and only if $x \in U_n$, and extend f on ω arbitrarily. Then $\mathcal{V} = f^\beta(\mathcal{W})$. □

Question 3.4 *Let $\mathcal{U}$ be a T-point in ω^*, $S \subset \omega^\omega$ be a countable family of finite-to-one mappings. Does there exist an S-thin subset $T \in \mathcal{U}$?*

Question 3.5 *Let $\mathcal{U}$ be a T-point in ω^*, $f : \omega \to \omega$ be a finite-to-one mapping. Is $f^\beta(\mathcal{U})$ a T-point?*

Question 3.6 *Is every weak P-point (*OK*-point,* NWD*-point) a T-point?*

Let G be a countable group of permutations of ω. We say that an ultrafilter $\mathcal{U} \in \omega^*$ is a *T_G-point* if, for every bijection $f : \omega \to \omega$, there exists $U \in \mathcal{U}$ such that $f(U)$ is G-thin. It is easy to see that a T-point is a T_G-point for every countable group of permutations of ω. On the other hand, if G is a group of all permutations of ω with finite support then each $\mathcal{U} \in \omega^*$ is a T_G-point.

Question 3.7 *Given a countable subgroup G of S_ω, does there exist a T_G-point which is not a T-point? Does there exist a T_G-point in* ZFC*?*

4 Corona

All G-spaces in this section are assumed to be discrete and transitive. Let G be a group and let X be a G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. By the universal property of the Stone-Čech compactification βX of X, the action of G on X extends to the continuous action of G on βX. Since the subspace $X^* = \beta X \setminus X$ of all free ultrafilters is G-invariant, it also has a natural structure of G-space.

We denote by E the orbit equivalence on X^* defined by

$$(x, y) \in E \Leftrightarrow Gx = Gy,$$

and following [13], consider the smallest by inclusion, closed in $X^* \times X^*$ equivalence $\check{E}$ on X^* such that $E \subseteq \check{E}$. The factor-space $\check{X} = X^*/\check{E}$ is called a corona of X.

For every $\mathcal{U} \in X^*$, we denote by $\check{\mathcal{U}}$ the class of $\check{E}$-equivalence containing $\mathcal{U}$, and say that two ultrafilters $\mathcal{U}, \mathcal{V} \in X^*$ are *corona equivalent* if $\check{\mathcal{U}} = \check{\mathcal{V}}$. To detect whether two ultrafilters are corona equivalent, we use the G-slowly oscillating functions on X.

A function $h : X \to [0, 1]$ is called G*-slowly oscillating* if, for any $g \in G$ and $\varepsilon > 0$, there exists a finite subset K of X such that

$$|h(gx) - h(x)| < \varepsilon$$

for every $x \in X \setminus K$.

By [12, Proposition 1], the ultrafilters $\mathcal{U}, \mathcal{V} \in X^*$ are corona equivalent if and only if $h^\beta(\mathcal{U}) = h^\beta(\mathcal{V})$ for every G-slowly oscillating function h on X.

Given a subset A of X and a filter Φ on X, we put

$$\overline{A} = \{\mathcal{U} \in X^* : A \in \mathcal{U}\}, \quad \overline{\Phi} = \cap\{\overline{A} : A \in \Phi\},$$

and note that, for every nonempty closed subset Y of X^*, there exists a filter Φ on X such that $Y = \overline{\Phi}$. For $\mathcal{U} \in X^*$, we denote by $\psi_{\mathcal{U}}$ the filter on X such that $\overline{\psi_{\mathcal{U}}} = \check{\mathcal{U}}$.

Now we suppose that G and X are countable and fix some numerations $G = \{g_i : i \in \omega\}$, $X = \{x_i : i \in \omega\}$. For $A \subseteq X$ and $f : \omega \to \omega$, we put

$$\Psi_{A,f} = \bigcup_{i \in \omega} g_i(A \setminus \{x_0, \dots, x_{f(i)}\}).$$

Proposition 4.1 *Let G be a countable group, X be a countable G-space, $\mathcal{U} \in X^*$. Then the family $\{\Psi_{U,f} : U \in \mathcal{U}, f \in \omega^\omega\}$ forms a base for $\psi_{\mathcal{U}}$.*

Proof We take an arbitrary $\mathcal{V} \notin \check{\mathcal{U}}$. By [12, Proposition 1], there exists a G-slowly oscillating function $h : X \to [0, 1]$ such that $h^\beta(\mathcal{U}) = 0, h^\beta(\mathcal{V}) = 1$. We choose $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $h(x) < \frac{1}{4}$ for every $x \in U$, and $h(x) > \frac{3}{4}$ for every $x \in V$. Since h is G-slowly oscillating, for every $i \in \omega$, there exists $f(i) \in \omega$ such that

$$|h(g_i x) - h(x)| < \frac{1}{4}$$

for each $x \in X \setminus \{x_0, \dots, x_{f(i)}\}$. Then $g_i(U \setminus \{x_0, \dots, x_{f(i)}\}) \cap V = \varnothing$ so $\Psi_{U,f} \cap V = \varnothing$ and $\mathcal{V} \notin \psi_{\mathcal{U}}$.

On the other hand, assume that $\Psi_{U,f} \notin \mathcal{V}$ for some $U \in \mathcal{U}$, $f \in \omega^\omega$ and $\mathcal{V} \in \check{\mathcal{U}}$. Applying Theorem 2.1 from [11], we get a G-slowly oscillating function $h : X \to [0, 1]$ such that $h^\beta(\mathcal{U}) = 0, h^\beta(\mathcal{V}) = 1$. By [12, Proposition 1], $\mathcal{V} \notin \check{\mathcal{U}}$ and we get a contradiction. □

Proposition 4.2 *Let G be a countable group, X be a countable G-space. For an infinite subset T of X, the following statements are equivalent:*

(1) T is G-thin;

(2) the restriction $\check{}\,|T^*$ of the mapping $\check{}\,: X^* \to \check{X}$ is injective.

Proof (1) $\Rightarrow$ (2) We choose distinct $\mathcal{U}, \mathcal{V} \in X^*$ such that $T \in \mathcal{U}$, $T \in \mathcal{V}$ and show that $\check{\mathcal{U}} \neq \check{\mathcal{V}}$. Let $F_n = \{g_0, \dots, g_n\}, n \in \omega$. Since T is G-thin, there is an increasing function $h \in \omega^\omega$ such that

$$F_n x \cap F_n y = \varnothing$$

for all distinct $x, y \in T \setminus \{x_0, \dots, x_{h(n)}\}$. Put $t(0) = h(0)$ and define inductively a function $t \in \omega^\omega$ such that, for each $n \in \omega$, $t(n) \geqslant h(n)$ and

$$(F_{n+1}^{-1} F_n \{x_0, \dots, x_{h(n+1)}\}) \cap (T \setminus \{x_0, \dots, x_{t(n+1)}\}) = \varnothing.$$

Then, for any $n, m \in \omega$ and $x \in T \setminus \{x_0, \dots, x_{t(n)}\}$, $y \in T \setminus \{x_0, \dots, x_{t(m)}\}$,

$$g_n x = g_m y \Rightarrow x = y.$$

We choose disjoint $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that $U \subseteq T$, $V \subseteq T$. By the construction of t,

$$\Psi_{U,t} \cap \Psi_{V,t} = \varnothing,$$

and, applying Proposition 4.1, we get $\check{\mathcal{U}} \neq \check{\mathcal{V}}$.

(2) $\Rightarrow$ (1) Suppose that T is not G-thin and choose $g \in G$ such that T is not g-thin. Then the set $A = \{x \in T : gx \neq x, gx \in T\}$ is infinite. We take an arbitrary ultrafilter $\mathcal{U} \in X^*$ such that $A \in \mathcal{U}$. Since $gx \neq x$ for all $x \in A$, $g\mathcal{U} \neq \mathcal{U}$. Clearly, $T \in g\mathcal{U}$ but $\check{\mathcal{U}} = (g\check{\mathcal{U}})$ so $\check{}|T^*$ is not injective. $\square$

Corollary 4.3 *Let G be a countable group, X be a countable G-space. Every nonempty open subset of $\check{X}$ contains a homeomorphic copy of ω^*.*

Corollary 4.4 *Let G be a countable group, X be a countable G-space, T be an infinite G-thin subset of X. If $\mathcal{U} \in X^*$ and $T \in \mathcal{U}$ then $\mathcal{U}$ is an isolated point in $\check{\mathcal{U}}$.*

Question 4.5 *Let G be a countable group, X be a countable G-space, $\mathcal{U} \in X^*$. Has $\mathcal{U}$ a G-thin member provided that $\mathcal{U}$ is an isolated point in $\check{\mathcal{U}}$? If not then characterize $\mathcal{U} \in X^*$ which are isolated in $\check{\mathcal{U}}$.*

Remark 4.6 Let μ be a left invariant Banach measure on an infinite group G, A be a subset of G such that $\mu(A) > 0$. We consider G as a left regular G-space and show that A is not G-thin. Let $\mu(A) > \frac{1}{n}$, $g_0, \dots, g_n$ be distinct elements of G. Since $\mu(G) = 1$ and μ is additive, there exist distinct $i, j \in \{0, \dots, n\}$ such that $\mu(g_i A \cap g_j A) > 0$, so $\mu(g_j^{-1} g_i A \cap A) > 0$ and A is not $g_j^{-1} g_i$-thin. By [13, Lemma 4.3], there exists a Banach measure μ on $\mathbb{Z}$ such that if $\mathcal{U} \in \mathbb{Z}^*$ and $\mu(U) = 0$ for some $U \in \mathcal{U}$ then, for every $\mathcal{V} \in \check{\mathcal{U}}$, there exists $V \in \mathcal{V}$ such that $\mu(V) = 0$. Let $\mathcal{W}$ be an ultrafilter on $\mathbb{Z}$ such that $\mu(W) > 0$ for each $W \in \mathcal{W}$. Then the corona class $\check{\mathcal{W}}$ has no G-thin ultrafilters.

Remark 4.7 Let G be a countable group, $\mathcal{U}, \mathcal{V}$ be right cancelable ultrafilters from G^* (see [8, Chapter 3]). Then $\mathcal{W} = \mathcal{U}\mathcal{V}$ is right cancelable and, by [3, Theorem 8.11], G-orbit of $\mathcal{W}$ is discrete, so $\mathcal{W}$ is isolated in its orbit. On the other hand, $\mathcal{W}$ is a limit point of the G-orbit of $\mathcal{V}$, so $\mathcal{V} \in \check{\mathcal{W}}$ and $\mathcal{W}$ is not isolated in $\check{\mathcal{W}}$.

Remark 4.8 We can use G-thin subsets also to extend onto G-spaces the Chou's theorem on the number of invariant means (Banach measures) on an amenable group [1].

Let X be a discrete G-space, $C(X)$ be the set of all bounded functions from X to $\mathbb{R}$ with sup norm. A continuous linear functional $m : C(X) \to \mathbb{R}$ is called an *invariant mean* if

1. $m(f) \geqslant 0$ for all $f \geqslant 0$ and $m(1) = 1$;
2. $m(f_g) = m(f)$ for each $g \in G$, where $f_g(x) = f(gx)$.

We assume that X is an infinite G-space, $|X| = |G|$, and there exists a G-thin subset T of X such that $|X| = |T|$. Let $\mathcal{U}, \mathcal{V}$ be distinct free ultrafilters on X such that $T \in \mathcal{U}$, $T \in \mathcal{V}$ and $|U| = |V| = |T|$ for all $U \in \mathcal{U}$, $V \in \mathcal{V}$. We enumerate $G = \{g_\alpha : \alpha < \kappa\}$. Since T is G-thin and $T \in \mathcal{U}$, $T \in \mathcal{V}$, we can choose inductively the families $\{U_\alpha : \alpha < \kappa\}$, $\{V_\alpha : \alpha < \kappa\}$ of members of $\mathcal{U}$ and $\mathcal{V}$ such that $g_\alpha U_\alpha \cap g_\gamma V_\gamma = \varnothing$ for all $\alpha, \gamma < \kappa$. Since $(\bigcup_{\alpha<\kappa} g_\alpha U_\alpha) \cap (\bigcup_{\alpha<\kappa} g_\alpha V_\alpha) = \varnothing$, we have

$$cl_{\beta X}\{g^\beta(\mathcal{U}) : g \in G\} \cap cl_{\beta X}\{g^\beta(\mathcal{V}) : g \in G\} = \varnothing,$$

and repeating the Chou's argument, we conclude that if X admits an invariant mean then there are 2^{2^κ} distinct invariant means on X. In particular (see Example 2.2), if X, G are countable and X is amenable then X admits $2^{\mathfrak{c}}$ distinct invariant means.

5 Ballean Context

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the set of radii. Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

Following [15], we say that a ball structure $\mathcal{B} = (X, P, B)$ is a *ballean* if

1. for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$
2. for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

A subset $Y \subseteq X$ is called *bounded* if $Y \subseteq B(x, \alpha)$ for some $x \in X$ and $\alpha \in P$. We say that a subset $T \subseteq X$ is *thin* (or pseudodiscrete in terminology from [15]) if, for every $\alpha \in P$, there exists a bounded subset $Y \subseteq X$ such that

$$B(x, \alpha) \cap B(x', \alpha) = \varnothing$$

for all distinct $x, x' \in T \setminus Y$.

Every G-space X determines the ballean $(X, \mathcal{F}_G, B)$, where $\mathcal{F}_G$ is the family of all finite subsets of G containing the identity e and $B(x, F) = Fx$ for all $x \in X, F \in \mathcal{F}_G$. It is easy to verify that a subset T of a transitive G-space X is G-thin if and only if T is thin in the ballean $(X, \mathcal{F}_G, B)$. A metric space (X, d) can be considered as the ballean $(X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{y \in X : d(x, y) \leqslant r\}$ for all $x \in X$, $r \in \mathbb{R}^+$. Recall that a metric d on X is an ultrametric if $d(x, y) \leqslant \max d(x, z), d(z, y)$ for all $x, y, z \in X$.

Theorem 5.1 *For an ultrafilter $\mathcal{U} \in \omega^*$, the following statements are equivalent:*

(1) $\mathcal{U}$ is selective;

(2) for every metric d on ω, there exists a thin subset T of (ω, d) such that $T \in \mathcal{U}$;

(3) for every ultrametric d on ω, there exists a thin subset T of (ω, d) such that $T \in \mathcal{U}$.

Proof (1) $\Rightarrow$ (2) We fix $x_0 \in \omega$, $r > 0$ and put

$$X_0 = \{x_0\}, X_{n+1} = B_d(x_0, (n+1)r) \setminus B_d(x_0, nr), n \in \omega,$$

$$Y_0 = \bigcup_{n\in\omega} X_{2n}, Y_1 = \bigcup_{n\in\omega} X_{2n+1}.$$

Since $\mathcal{U}$ is selective, there exist $T \in \mathcal{U}$ and $i \in \{0, 1\}$ such that $T \subseteq Y_i$ and either $|T \cap X_n| \leqslant 1$ for each $n \in \omega$ or $T = X_m$ for some $m \in \omega$. In the first case, $B_d(x, r) \cap B_d(x', r) = \varnothing$ for all distinct $x, x' \in T$. In the second case, T is bounded. Thus, in both cases T is thin in (ω, d).

(2) $\Rightarrow$ (3) Evident.

(3) $\Rightarrow$ (1) Let $\{P_n : n \in \omega\}$ be a partition of ω. We define an ultrametric d on ω by the rule:

$$d(x, y) = \begin{cases} 0, \text{ if } x = y; \\ 1, \text{ if } x \neq y,\ x, y \in P_n; \\ \max\{n, m\}, \text{ if } x \in P_n, y \in P_m, n \neq m. \end{cases}$$

Then we choose a thin subset T of (ω, d) such that $T \in \mathcal{U}$. By the definition of thin subset, there exists a bounded subset Y of (ω, d) such that $B_d(x, 1) \cap B_d(x', 1) = \varnothing$ for all distinct $x, x' \in T \setminus Y$. If T is bounded then some block P_n is a member of $\mathcal{U}$. Otherwise, $T \setminus Y \in \mathcal{U}$ and $|(T \setminus Y) \cap P_n| \leqslant 1$ for each $n \in \omega$. Hence, $\mathcal{U}$ is selective. □

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