

On the Elementary Theory of Restricted Real and Imaginary Parts of Holomorphic Functions

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Abstract We show that the ordered field of real numbers with restricted $\mathbb{R}_{\mathcal{H}}$ -definable analytic functions admits quantifier elimination if we add a function symbol $^{-1}$ for the function $x \mapsto \frac{1}{x}$ (with $0^{-1} = 0$ by convention), where $\mathbb{R}_{\mathcal{H}}$ is the real field augmented by the functions in the family \mathcal{H} of restricted parts (real and imaginary) of holomorphic functions which satisfies certain conditions. Further, with another condition on \mathcal{H} we show that the structure $(\mathbb{R}_{\mathcal{H}}, \text{constants})$ is strongly model complete.

1 Introduction

The quantifier elimination property is one of the central properties in model theory. It was successfully used to demonstrate that various theories possess certain model-theoretic properties like decidability and completeness. In 1948, Tarski showed (see [5]) a quantifier elimination method for the elementary theory of real closed fields. As noted by Tarski, any quantifier elimination method for this theory provides also a decision method, which enables one to decide whether any sentence of the theory is true or false. Subsequently, Denef and van den Dries proved (see [1]) the strong result that the ordered field with restricted analytic functions admits quantifier elimination if we add a function symbol $^{-1}$ for $x \mapsto \frac{1}{x}$, where $0^{-1} = 0$ by convention. However, this result is not known in general for subclasses of the class of restricted analytic functions. In this paper, we will give some classes where we have a positive answer. More precisely, we will prove a quantifier elimination result for the real field augmented by all restricted $\mathbb{R}_{\mathcal{H}}$ -definable analytic functions. Here \mathcal{H} is an arbitrary family of restricted real and imaginary parts of holomorphic functions with certain conditions (see conditions (P1) and (P2) in Section 2) and $\mathbb{R}_{\mathcal{H}}$ is the real field augmented by all functions of \mathcal{H} . Moreover, under a new condition (see condition (P3) in Section 2) on \mathcal{H} we show that the structure $(\mathbb{R}_{\mathcal{H}}, \text{constants})$ is

Received October 8, 2010; accepted April 18, 2011; printed April 5, 2012

2010 Mathematics Subject Classification: Primary 03C10; Secondary 14P15

Keywords: quantifier elimination, Weierstrass systems

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strongly model complete. Throughout this paper, “definable” means “definable with parameters.”

In Section 3, we will give some examples of the family \mathcal{H} . In Section 4, we shall recall the definition of Weierstrass systems and give some examples. In Section 5, we prove the first main result. Finally, in Section 6, we will prove the model completeness result mentioned above. Now we state the main results of this work.

2 Statement of the Main Results

Define $I := [-1, 1]$ and $J := (-1, 1)$. For each nonnegative integer n and each analytic function $f : U \rightarrow \mathbb{R}$, where U is some open neighborhood of the closed box I^n in \mathbb{R}^n , write $\dim(f) = n$, and let $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$\tilde{f}(x) := \begin{cases} f(x), & x \in I^n; \\ 0, & \text{otherwise.} \end{cases}$$

We denote by \mathcal{F} the collection of all \tilde{f} . From now on, we fix a subset \mathcal{H} of \mathcal{F} such that the following two conditions are satisfied:

- (P1) If $\tilde{h} \in \mathcal{H}$, then there is $\tilde{g} \in \mathcal{H}$ such that $\dim(h) = \dim(g) = 2n$, for some $n \in \mathbb{N}$, and either $h + ig$ or $g + ih$ is holomorphic on $J^{2n} \subset \mathbb{C}^n$, where we identify \mathbb{C}^n with $\mathbb{R}^2 \times \dots \times \mathbb{R}^2 = \mathbb{R}^{2n}$ in the usual way.
- (P2) If $\tilde{h} \in \mathcal{H}$, then there are $0 < \varepsilon < 1$ and $\tilde{g} \in \mathcal{H}$ such that $\dim(h) = \dim(g) = 2n$, for some $n \in \mathbb{N}$, and $h(x) = g(\varepsilon x)$ for all $x \in I^{2n}$.

Let $\mathbb{R}_{\mathcal{H}} := (\mathbb{R}, <, 0, 1, +, -, \dots, \mathcal{H})$ be the expansion of the ordered field of real numbers by the functions in \mathcal{H} , and let $\mathbb{R}_{\mathcal{D}}$ be the expansion of the ordered field of real numbers by the functions in

$$\mathcal{D} := \left\{ \tilde{f} \in \mathcal{F} : f \text{ is definable in } \mathbb{R}_{\mathcal{H}} \right\}.$$

Finally, let $(\mathbb{R}_{\mathcal{D}},^{-1})$ be the expansion of $\mathbb{R}_{\mathcal{D}}$ by the function

$$x^{-1} := \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\}; \\ 0, & x = 0. \end{cases}$$

We can now state the first main result, proved in Section 5.

Theorem A *The structure $(\mathbb{R}_{\mathcal{D}},^{-1})$ admits quantifier elimination.*

Let $\mathcal{R} := (\mathbb{R}_{\mathcal{H}}, \text{constants})$ be the expansion of $\mathbb{R}_{\mathcal{H}}$ obtained by adding a name for each real number to the language. Let \mathcal{L} be the language of \mathcal{R} . A subset $A \subseteq \mathbb{R}^n$ is strongly definable in \mathcal{R} if there is a quantifier-free \mathcal{L} -formula $\psi(X, Y)$, $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_p)$ such that A is defined in \mathcal{R} by the existential formula $\exists Y \psi(X, Y)$ and there is for each $a \in A$ exactly one $b \in \mathbb{R}^p$ such that $\mathcal{R} \models \psi(a, b)$. The structure \mathcal{R} is strongly model complete if its definable sets are strongly definable in \mathcal{R} . Given $D \subset \mathbb{R}^m$, we call a map $f : D \rightarrow \mathbb{R}^n$ strongly definable in \mathcal{R} if its graph and the complement $\mathbb{R}^m \setminus D$ of its domain are strongly definable in \mathcal{R} .

We consider the following condition:

- (P3) If $\tilde{f} \in \mathcal{H}$ and $\dim(f) = n$, then $\frac{\partial f}{\partial x_k} \in \mathcal{H}$, for $k = 1, \dots, n$.

We can now state the other main result of this paper, proved in Section 5.

Theorem B *If \mathcal{H} also satisfies (P3), then the structure \mathcal{R} is strongly model complete.*

The reader familiar with early work on o-minimal expansions of the real field may have noticed that \mathcal{R} , with (P3), is a structure to which Gabrielov's theorem on reducts of \mathbb{R}_{an} applies (see [3]), and hence is model complete, but my main point here is to prove that, with (P3), \mathcal{R} is strongly model complete. As noted by van den Dries (in [2]) strong model-completeness is the next best thing after quantifier elimination and is stronger than model-completeness.

3 Examples of the Family \mathcal{H}

Let $h : V \rightarrow \mathbb{R}$ be a harmonic function, where V is a simply connected open neighborhood of I^2 in \mathbb{R}^2 . Let $\langle \varepsilon_k : k \in \mathbb{N} \rangle$ be a sequence of positive real numbers such that $\varepsilon_k < \varepsilon_{k+1} < 1$ for all $k \in \mathbb{N}$. Put $V_k = \left\{ \frac{z}{\varepsilon_k} : z \in V \right\}$, and let $f_k, g_k : V_k \rightarrow \mathbb{R}$ be the analytic functions given by $f_k(z) = \frac{\partial h}{\partial y}(\varepsilon_k z)$ and $g_k(z) = \frac{\partial h}{\partial x}(\varepsilon_k z)$. Note that V_k is an open neighborhood of I^n . Now put

$$\mathcal{H}_h := \left\{ \widetilde{f}_k, \widetilde{g}_k : k \in \mathbb{N} \right\} \subset \mathcal{F}.$$

Proposition 3.1 *The conditions (P1) and (P2) hold for the family \mathcal{H}_h .*

Proof The function h is harmonic and V is simply connected. By [4], §7.1.4, there is a harmonic function $g : V \rightarrow \mathbb{R}$ such that $H := h + ig$ is holomorphic on $V \subseteq \mathbb{C}$. We have

$$\frac{\partial H}{\partial y} = \frac{\partial h}{\partial y} + i \frac{\partial g}{\partial y} = iH',$$

where H' denotes the derivative of H , and by Cauchy-Riemann,

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y}.$$

Then

$$\frac{\partial h}{\partial y} + i \frac{\partial h}{\partial x} = iH'.$$

Now it is easy to see that, for each $k \in \mathbb{N}$, the function $f_k + ig_k$ is holomorphic on J^2 . Thus (P1) holds for \mathcal{H}_h . On the other hand, $0 < \frac{\varepsilon_k}{\varepsilon_{k+1}} < 1$, $f_k(z) = f_{k+1}\left(\frac{\varepsilon_k}{\varepsilon_{k+1}}z\right)$, and $g_k(z) = g_{k+1}\left(\frac{\varepsilon_k}{\varepsilon_{k+1}}z\right)$, for all $k \in \mathbb{N}$ and all $z \in I^2$. Hence, (P2) holds for \mathcal{H}_h . \square

Now put $\mathcal{H}_{\text{har}} := \left\{ \widetilde{f} \in \mathcal{F} : \dim(f) = 2 \text{ and } f \text{ is harmonic} \right\}$.

Proposition 3.2 *The family \mathcal{H}_{har} satisfy (P1), (P2), and (P3).*

Proof Let $\widetilde{f} \in \mathcal{H}_{\text{har}}$. We can assume that $f : U \rightarrow \mathbb{R}$ with U is a simply connected open neighborhood of I^2 in \mathbb{R}^2 . Hence, by [4], §7.1.4, there is a harmonic function $g : U \rightarrow \mathbb{R}$ such that $f + ig$ is holomorphic on $U \subseteq \mathbb{C}$ and $\widetilde{g} \in \mathcal{H}_{\text{har}}$. Thus (P1) holds for \mathcal{H}_{har} . Let $r > 1$ be such that $r^2 \cdot J^2 \subset U$. We consider the function $u : r \cdot J^2 \rightarrow \mathbb{R}$ defined by $u(z) = f(rz)$. It is clear that $r \cdot J^2$ is an open neighborhood of I^2 and u is harmonic. Then \widetilde{u} belongs to \mathcal{H}_{har} . Put $\varepsilon = \frac{1}{r} < 1$. We have $f(z) = u(\varepsilon z)$ for all $z \in r \cdot J^2$, so that $I^2 \subset r \cdot J^2$. Then (P2) holds for \mathcal{H}_{har} . The functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are harmonic, then $\widetilde{\frac{\partial f}{\partial x}}$ and $\widetilde{\frac{\partial f}{\partial y}}$ belong to \mathcal{H}_{har} . Therefore, (P3) holds for \mathcal{H}_{har} . \square

4 Weierstrass Systems

We shall use the following notations. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , and let $X = (X_1, \dots, X_n)$ be a tuple of distinct indeterminates with $n \geq 0$. We denote by $\mathbb{K}[[X]]$ the ring of formal power series in X_1, \dots, X_n over \mathbb{K} , and by $\mathbb{K}\langle X \rangle$ the ring of power series over \mathbb{K} in X_1, \dots, X_n which converge in some neighborhood of the origin. The subring of polynomials over \mathbb{K} in X_1, \dots, X_n will be denoted by $\mathbb{K}[X]$. Recall that a polydisc in \mathbb{K}^n centered at a point $a \in \mathbb{K}^n$ is an open set of the form

$$\{x \in \mathbb{K}^n : |x_1 - a_1| < r_1, \dots, |x_n - a_n| < r_n\},$$

where $0 < r_1, \dots, r_n \in \mathbb{R}$.

Following Denef and Lipshitz, we define a Weierstrass system over \mathbb{K} to be a family of rings $\langle \mathcal{P}_n : n \geq 0 \rangle$ such that for all n the following three conditions hold:

- (W1) $\mathbb{K}[X] \subset \mathcal{P}_n \subset \mathbb{K}[[X]]$, and if χ is a permutation of $\{1, \dots, n\}$ and $f(X_1, \dots, X_n) \in \mathcal{P}_n$, then $f(X_{\chi(1)}, \dots, X_{\chi(n)}) \in \mathcal{P}_n$. Moreover, for each $m > 0$, $\mathcal{P}_{n+m} \cap \mathbb{K}[[X]] = \mathcal{P}_n$.
- (W2) If $f \in \mathcal{P}_n$ is a unit in $\mathbb{K}[[X]]$, then f is a unit in \mathcal{P}_n .
- (W3) (Weierstrass division) If $f \in \mathcal{P}_{n+1}$ and $f(0, X_{n+1}) \in \mathbb{K}[[X_{n+1}]]$ is nonzero of order d , then for every $g \in \mathcal{P}_{n+1}$ there is $q \in \mathcal{P}_{n+1}$ and there are $r_j \in \mathcal{P}_n$, $j = 0, 1, \dots, d-1$ such that

$$g = qf + (r_{d-1}X_{n+1}^{d-1} + \dots + r_0).$$

We say that a Weierstrass system over \mathbb{K} as above is convergent if each $f \in \mathcal{P}_n$ converges on a polydisc Δ in \mathbb{K}^n centered at $0 \in \mathbb{K}^n$, and is also such that for each $a \in \Delta$ the series

$$f(X+a) := \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^{|\alpha|} f}{\partial X^\alpha}(a) \frac{X^\alpha}{\alpha!}$$

belongs to \mathcal{P}_n . (Here α ranges over \mathbb{N}^n and we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial X^\alpha = \partial X_1^{\alpha_1} \dots \partial X_n^{\alpha_n}$, $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$, and $\alpha! = \alpha_1! \dots \alpha_n!$.)

For each polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin let \mathcal{C}_Δ be the collection of all power series $h \in \mathbb{C}[[X]]$ which converge on Δ to a holomorphic function $H : \Delta \rightarrow \mathbb{C}$ definable in $\mathbb{R}_{\mathcal{H}}$. Now, put $\mathcal{C}_n = \bigcup_{\Delta} \mathcal{C}_\Delta$, the union over all polydiscs $\Delta \subset \mathbb{C}^n$ centered at the origin. Clearly, the elements of \mathcal{C}_n can be added and multiplied in the usual way and are easily seen to form a local ring.

Proposition 4.1 $\langle \mathcal{C}_n : n \geq 0 \rangle$ is a convergent Weierstrass system over \mathbb{C} .

For the proof we require two lemmas.

Lemma 4.2 Let $f, g \in \mathcal{C}_n$ and $h \in \mathbb{C}\langle X \rangle$ be such that $f \neq 0$ and $g = fh$. Then $h \in \mathcal{C}_n$.

Proof We can assume that f, g , and h are functions defined on a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin. We suppose also that f and g are $\mathbb{R}_{\mathcal{H}}$ -definable. We have to prove that the graph of h , Γ_h , is $\mathbb{R}_{\mathcal{H}}$ -definable. Let

$$Y = \{x \in \Delta : f(x) = 0\}.$$

Since Δ is connected, the interior of Y is empty; hence $\overline{\Delta \setminus Y} \cap \Delta = \Delta$. The set $\Gamma_h \cap (\Delta \setminus Y) \times \mathbb{C}$ is definable in $\mathbb{R}_{\mathcal{H}}$. Hence so is $\overline{\Gamma_h \cap (\Delta \setminus Y) \times \mathbb{C}} \cap \Delta \times \mathbb{C}$. Since h is continuous, we have $\Gamma_h = \overline{\Gamma_h \cap (\Delta \setminus Y) \times \mathbb{C}} \cap \Delta \times \mathbb{C}$. Hence the graph of h is definable in $\mathbb{R}_{\mathcal{H}}$. \square

Lemma 4.3 *Let $f \in \mathcal{C}_{n+1}$ be such that $f(0, X_{n+1}) \in \mathbb{C}[[X_{n+1}]]$ is nonzero of order d , and let $g \in \mathcal{C}_{n+1}$. Then there is $q \in \mathcal{C}_{n+1}$ and there are $r_j \in \mathcal{C}_n$, $j = 0, 1, \dots, d - 1$ such that $g = qf + (r_{d-1}X_{n+1}^{d-1} + \dots + r_0)$.*

The proof of Lemma 4.3 is, modulo obvious changes, the same as that given for [2], Lemma 3.6. (Use “ $\mathbb{R}_{\mathcal{H}}$ -definable” in place of “strongly definable” and Lemma 4.2 in place of [2], Lemma 3.4).

Proof of Proposition 4.1 Clearly, (W1) and (W2) hold for the system \mathcal{C} . Surely, (W3) is essentially Lemma 4.3. \square

Put $\mathcal{A}_n = \mathcal{C}_n \cap \mathbb{R}[[X]]$. Then $\mathcal{A} := \langle \mathcal{A}_n : n \geq 0 \rangle$ is a convergent Weierstrass system over \mathbb{R} . Let $L_{\mathcal{A}}$ be the language of ordered rings $\{<, 0, 1, +, -, \cdot\}$ augmented by a new function symbol for each $f \in \mathcal{F}$ such that all translated series $f(X + a)$ with $a \in I^n$ belong to \mathcal{A}_n , and denote by $\mathbb{R}_{\mathcal{A}}$ the reals with its natural $L_{\mathcal{A}}$ -structure. Then we have the following theorem.

Theorem 4.4 ([2]) *The structure $(\mathbb{R}_{\mathcal{A}},^{-1})$ admits quantifier elimination.*

It is well known that the proof of the main results of [2] is based on some elementary properties of the functions sine and exponential (for example, $e^{a+b} = e^a e^b$) using complex power series methods. By van den Dries [2], 4.7, it seems plausible that these results hold for elliptic functions. We ask the question: which restricted real and imaginary parts of holomorphic functions? This paper gives an answer to this question. For the proof of the main theorems of this paper we use also complex power series methods and the fact that the real and imaginary parts of the Weierstrass coefficients are in the same Weierstrass system (i.e., Corollary 5.5, that plays in this paper the role of the properties of elementary functions that are used by van den Dries in [2]). This paper is a generalization of the work of van den Dries [2]. Indeed, we may create an example of the family \mathcal{H} by the elementary functions (sine and exponential) where we obtain the main results of [2]. The reader can remark that the structures considered in [6] are examples of the structures $\mathbb{R}_{\mathcal{H}}$ (without parameters). Wilkie [6] gives a characterization of functions locally definable in neighborhood of generic points in terms of complex-analytically natural closure conditions. As noted by Wilkie the structures of [6] are model complete but here we prove that (with parameters) they are strongly model complete. It is not hard to prove that if the conjecture [6], 1.8 holds, then every function locally definable (in the structures considered in [6]) is locally strongly definable, then it seems plausible that Theorem B could be deduced (for the structures considered in [6]) if the characterization of Wilkie holds in general case (not only in neighborhood of generic points). Now we prove the main theorems of this paper.

5 Proof of Theorem A

Let X_1, \dots, X_{2n} and Y_1, \dots, Y_n be distinct indeterminates, and put $X := (X_1, \dots, X_n)$ and $Y := (Y_1, \dots, Y_n)$. We also let $X' = (X_1, \dots, X_{2n})$. Given $g \in \mathbb{C}[[X, Y]]$, then

$$g = \sum_{\alpha} g_{\alpha} X^{\alpha_1} Y^{\alpha_2},$$

and put

$$g^* := \sum_{\alpha} \bar{g}_{\alpha} X^{\alpha_1} Y^{\alpha_2},$$

where the sum is over all $\alpha = (\alpha_1, \alpha_2)$ from $\mathbb{N}^n \times \mathbb{N}^n$, all g_α belong to \mathbb{C} and $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$ the usual conjugation. From now on,

$$Z = (Z_1, \dots, Z_n) := (X_1 + iY_1, \dots, X_n + iY_n).$$

If $f \in \mathbb{C}[[X]]$, $f(Z) \in \mathbb{C}[[X, Y]]$ denotes the power series given by substituting $X = Z$ in f , and put

$$\Re[f] := \frac{1}{2}(g + g^*) \quad \text{and} \quad \Im[f] := \frac{1}{2i}(g - g^*),$$

where $g = f(Z)$. Note that $\Re[f], \Im[f] \in \mathbb{R}[[X, Y]]$ and the equality $f(Z) = \Re[f] + i\Im[f]$ in this form is unique.

Theorem 5.1 *Let $f \in \mathcal{A}_n$. Then $\Re[f](X')$ and $\Im[f](X')$ belong to \mathcal{A}_{2n} .*

Lemma 5.2 *If $f \in \mathcal{A}_n$ and $g_1, \dots, g_n \in \mathcal{A}_p$ are such that $g_1(0) = \dots = g_n(0) = 0$, then $f(g_1, \dots, g_n) \in \mathcal{A}_p$.*

The proof of Lemma 5.2 is easy.

Lemma 5.3 *Let $k \geq 1$ be an integer. If $n \geq k$ and $f \in \mathcal{A}_n$, then there are $u, v \in \mathcal{A}_{n+k}$ such that*

$$f(Z_1, \dots, Z_k, X_{k+1}, \dots, X_n) = (u + iv)(X, Y_1, \dots, Y_k).$$

Proof We proceed by induction on k . Assume that $k = 1$. Let $n \geq 1$ and $f \in \mathcal{A}_n$. Put $g(X, Y_1) = f(X_1 + Y_1, X_2, \dots, X_n)$. Then by Lemma 5.2, $g(X, X_{n+1}) \in \mathcal{A}_{n+1}$. Introduce a new variable T ; by (W3), there are $q \in \mathcal{A}_{n+2}$ and $r_0, r_1 \in \mathcal{A}_{n+1}$ such that

$$g(X, T) = (Y_1^2 + T^2)q(X, Y_1, T) + r_1(X, Y_1)T + r_0(X, Y_1). \quad (4.1)$$

By substituting $T = iY_1$ in (4.1), we obtain

$$f(Z_1, X_2, \dots, X_n) = iY_1r_1(X, Y_1) + r_0(X, Y_1).$$

Then $u = r_0$ and $v = X_{n+1}r_1$.

Now suppose that the lemma holds for each p such that $1 \leq p \leq k$. Let $n \geq k+1$ and $f \in \mathcal{A}_n$. By the inductive assumption, there are $u_0, v_0 \in \mathcal{A}_{n+1}$ such that

$$f(Z_1, X_2, \dots, X_n) = u_0(X, Y_1) + iv_0(X, Y_1). \quad (4.2)$$

Moreover, by (W1) and the inductive assumption, there are $s, t, s', t' \in \mathcal{A}_{n+k+1}$ such that

$$u_0(X_1, Z_2, \dots, Z_{k+1}, X_{k+2}, \dots, X_n, Y_1) = (s + it)(X, Y_1, \dots, Y_{k+1})$$

and

$$v_0(X_1, Z_2, \dots, Z_{k+1}, X_{k+2}, \dots, X_n, Y_1) = (s' + it')(X, Y_1, \dots, Y_{k+1}).$$

By substituting $(X_2, \dots, X_{k+1}) = (Z_2, \dots, Z_{k+1})$ in (4.2), we obtain

$$f(Z_1, \dots, Z_{k+1}, X_{k+2}, \dots, X_n) = (u + iv)(X, Y_1, \dots, Y_{k+1}),$$

where $u = s - t'$ and $v = t + s'$ belong to \mathcal{A}_{n+k+1} . Thus the lemma is proved. \square

Proof of Theorem 5.1 By Lemma 5.3 for $k = n$, there are $u, v \in \mathcal{A}_{2n}$ such that $f(Z) = u(X, Y) + iv(X, Y)$. Hence by uniqueness, $u(X, Y) = \Re[f]$ and $v(X, Y) = \Im[f]$. \square

In what follows, we denote the coordinates in \mathbb{C}^n by $z = (z_1, \dots, z_n)$, with $z_i = x_i + iy_i$, $x_i = \operatorname{Re}(z_i)$ and $y_i = \operatorname{Im}(z_i)$ are, respectively, the real and the imaginary parts of z_i . Also, we consider the usual conjugation $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \mapsto (\bar{z}_1, \dots, \bar{z}_n)$. If $f \in \mathbb{C}[[X]] \subset \mathbb{C}[[X, Y]]$, then f^* is well defined.

Proposition 5.4 *If $f \in \mathcal{C}_n$, then $f^* \in \mathcal{C}_n$.*

Proof Let $f \in \mathcal{C}_n$. Then there is a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin such that f converges on Δ to an $\mathbb{R}_{\mathcal{H}}$ -definable (holomorphic) function $H : \Delta \rightarrow \mathbb{C}$. Clearly, $\sigma(\Delta) = \Delta$ and f^* converges on Δ to the holomorphic function $H^{\text{SR}} : \Delta \rightarrow \mathbb{C}$ given by $H^{\text{SR}}(z) = \overline{H(\sigma(z))}$; H^{SR} is the Schwarz Reflection of H . Clearly, H^{SR} is definable in $\mathbb{R}_{\mathcal{H}}$. Hence $f^* \in \mathcal{C}_n$. \square

Corollary 5.5 *Let $f \in \mathcal{C}_n$. Then $\Re[f](X')$ and $\Im[f](X')$ belong to \mathcal{C}_{2n} .*

Proof Put

$$r = \frac{f + f^*}{2} \quad \text{and} \quad m = \frac{f - f^*}{2i}.$$

Clearly, $f = r + im$ and $f(Z) = r(Z) + im(Z)$. We obtain then

$$\Re[f] = \Re[r] - \Im[m] \quad \text{and} \quad \Im[f] = \Im[r] + \Re[m].$$

By Proposition 5.4, $r, m \in \mathcal{A}_n$. Therefore, by Theorem 5.1, $\Re[f](X')$ and $\Im[f](X')$ belong to \mathcal{C}_{2n} . \square

Proposition 5.6 *Let $\tilde{h} \in \mathcal{H}$ be such that $\dim(h) = 2n$. Then, for each $a \in I^{2n}$, the translate series $h(X' + a)$ belongs to \mathcal{A}_{2n} .*

Proof According to (P1), there is $\tilde{g} \in \mathcal{H}$ such that $\dim(g) = 2n$ and either $H = \tilde{h} + i\tilde{g}$ or $G = \tilde{g} + i\tilde{h}$ is holomorphic on J^{2n} . We may assume that H is holomorphic on J^{2n} . Otherwise, we replace H by G .

Suppose $a \in J^{2n}$. Clearly, the translate series $H_a := H(X + a)$ belongs to \mathcal{C}_n ; then, by Corollary 5.5, $\Re[H_a](X')$ and $\Im[H_a](X')$ belong to \mathcal{C}_{2n} . Since

$$\Re[H_a](X') = h(X'' + a) \quad \text{and} \quad \Im[H_a](X') = g(X'' + a),$$

where $X'' = (X_1, X_{n+1}, \dots, X_n, X_{2n})$, by (W1), $h(X' + a)$ and $g(X' + a)$ belong to \mathcal{C}_{2n} . On the other hand, $h(X' + a), g(X' + a) \in \mathbb{R}[[X']]$. Then $h(X' + a), g(X' + a) \in \mathcal{A}_{2n}$.

Now suppose that $a \in I^{2n}$. According to (P2), there are $0 < \varepsilon < 1$ and $\tilde{g} \in \mathcal{H}$ such that $\dim(g) = 2n$ and $h(x) = g(\varepsilon x)$ for all $x \in I^{2n}$. Since $h(X' + a) = g(\varepsilon X' + \varepsilon a)$ and $\varepsilon a \in J^{2n}$, by the above discussion and Lemma 5.2, $h(X' + a)$ belongs to \mathcal{A}_{2n} . \square

Corollary 5.7 *The structures $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{D}}$ have the same definable sets.*

Proof By compactness of I^n , the restriction of each $\tilde{f} \in L_{\mathcal{A}}$ on I^n (where $n = \dim(f)$) extends to an $\mathbb{R}_{\mathcal{H}}$ -definable analytic function on some open neighborhood of I^n . We deduce that every $\mathbb{R}_{\mathcal{A}}$ -definable set is $\mathbb{R}_{\mathcal{H}}$ -definable. Moreover, by Proposition 5.6, each $\tilde{h} \in \mathcal{H}$ is definable in $\mathbb{R}_{\mathcal{A}}$. Hence every $\mathbb{R}_{\mathcal{H}}$ -definable set is $\mathbb{R}_{\mathcal{A}}$ -definable.

From the definition, every $\mathbb{R}_{\mathcal{D}}$ -definable set is $\mathbb{R}_{\mathcal{H}}$ -definable. Now, let $\tilde{f} \in \mathcal{H}$. By (P2), there are $0 < \varepsilon < 1$ and $\tilde{g} \in \mathcal{H}$ such that $f(x) = g(\varepsilon x)$ for all $x \in I^{2n}$. Let $1 < \lambda < \frac{1}{\varepsilon}$. The subset $\lambda \cdot J^{2n}$ is an open neighborhood of I^{2n} and the analytic

function $l : \lambda.J^{2n} \rightarrow \mathbb{R}$ given by $l(x) = g(\varepsilon x)$ is $\mathbb{R}_{\mathcal{H}}$ -definable. Since $\tilde{l} = \tilde{f}$, \tilde{f} is $\mathbb{R}_{\mathcal{D}}$ -definable. Therefore, every $\mathbb{R}_{\mathcal{H}}$ -definable set is $\mathbb{R}_{\mathcal{D}}$ -definable. \square

We can now prove Theorem A.

Proof of Theorem A By compactness of I^n , the restriction on I^n of each $\tilde{f} \in L_{\mathcal{A}}$ extends to an $\mathbb{R}_{\mathcal{H}}$ -definable analytic function on some open neighborhood of I^n . Then $\mathbb{R}_{\mathcal{D}}$ is an expansion of $\mathbb{R}_{\mathcal{A}}$ in the first-order language extending $L_{\mathcal{A}}$. By Corollary 5.7, the structures $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{D}}$ have the same definable sets. From Theorem 4.4, $(\mathbb{R}_{\mathcal{D}},^{-1})$ admits quantifier elimination. \square

6 Proof of Theorem B

Let A, B , and C be semialgebraic sets, and let $f : C \rightarrow B$ and $g : A \rightarrow C$ be strongly definable functions in \mathcal{R} . Then the function $f \circ g : A \rightarrow B$, $x \mapsto f(g(x))$ is strongly definable in \mathcal{R} . If $f_1, \dots, f_n : A \rightarrow \mathbb{R}$, then the function (f_1, \dots, f_n) is strongly definable in \mathcal{R} if and only if each f_j is strongly definable in \mathcal{R} . Furthermore, if each f_j is strongly definable in \mathcal{R} , then, for each polynomial $P \in \mathbb{R}[X_1, \dots, X_n]$, the function $P(f_1, \dots, f_n)$ is strongly definable in \mathcal{R} .

Throughout the rest of this section, we suppose that \mathcal{H} satisfies also (P3). “Strongly definable” means “strongly definable in \mathcal{R} .”

For each polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin let \mathcal{B}_{Δ} be the collection of all power series $h \in \mathbb{C}[[X]]$ which converge on Δ to a strongly definable holomorphic function $H : \Delta \rightarrow \mathbb{C}$ and all partial derivatives $\frac{\partial^{|\alpha|} H}{\partial z^{\alpha}}$ are strongly definable. Now, put $\mathcal{B}_n = \bigcup_{\Delta} \mathcal{B}_{\Delta}$, the union over all polydiscs $\Delta \subset \mathbb{C}^n$ centered at the origin. From [2], \mathcal{B}_n may easily be seen to form a local ring.

Proposition 6.1 $\langle \mathcal{B}_n : n \geq 0 \rangle$ is a convergent Weierstrass system over \mathbb{C} .

The proof of Proposition 6.1 is, modulo obvious changes, the same as that given in [2], (4.2).

Put $\mathcal{S}_n = \mathcal{B}_n \cap \mathbb{R}[[X]]$. Then $\mathcal{S} := \langle \mathcal{S}_n : n \geq 0 \rangle$ is a convergent Weierstrass system over \mathbb{R} . Let $\mathbb{R}_{\mathcal{S}}$ be the expansion of the ordered field of real numbers by each $f \in \mathcal{F}$ such that all translated series $f(X + a)$ with $a \in I^n$ belong to \mathcal{S}_n . Then we have the following theorem.

Theorem 6.2 Every definable set in $\mathbb{R}_{\mathcal{S}}$ is strongly definable (in \mathcal{R}).

The proof of Theorem 6.2 is, modulo obvious changes, the same as that given in [2], (4.4)–(4.5).

Lemma 6.3 If $f \in \mathcal{S}_p$ and $g_1, \dots, g_p \in \mathcal{S}_p$ are such that $g_1(0) = \dots = g_p(0) = 0$, then $f(g_1, \dots, g_p) \in \mathcal{S}_n$.

Proof We may assume that $g_1, \dots, g_p \in \mathcal{B}_{\Delta}$ and $f \in \mathcal{B}_{\Delta'}$, for some polydiscs $\Delta \subseteq \mathbb{R}^p$ and $\Delta' \subseteq \mathbb{R}^n$ centered at the origin. We suppose also that $(g_1, \dots, g_p)(\Delta) \subseteq \Delta'$. Clearly, $H := f(g_1, \dots, g_p)$ is strongly definable. By the claim (see below), all $\frac{\partial^{|\alpha|} H}{\partial z^{\alpha}}$ are strongly definable. Therefore, $f(g_1, \dots, g_p) \in \mathcal{S}_n$.

Claim For each $\alpha \in \mathbb{N}^n$, there are $R_1, \dots, R_k \in \mathcal{B}_{\Delta'}$ and $Q_1, \dots, Q_k \in \mathcal{B}_{\Delta}$ such that $\frac{\partial^{|\alpha|} H}{\partial z^{\alpha}} = \sum_{m=1}^k Q_m \cdot R_m(g_1, \dots, g_p)$.

We proceed by induction on $|\alpha|$. Assume that $|\alpha| = 0$. Since $H = f(g_1, \dots, g_p)$, the claim holds for $|\alpha| = 0$. Now, let $\alpha \in \mathbb{N}^n$ and assume that

$$\frac{\partial^{|\alpha|} H}{\partial z^\alpha} = \sum_{m=1}^k Q_m \cdot R_m(g_1, \dots, g_p),$$

for some $k \geq 1$, $R_1, \dots, R_k \in \mathcal{B}_{\Delta'}$ and $Q_1, \dots, Q_k \in \mathcal{B}_{\Delta}$. For each $l = 1, \dots, n$, we have

$$\begin{aligned} \frac{\partial}{\partial z_l} \left(\frac{\partial^{|\alpha|} H}{\partial z^\alpha} \right) &= \sum_{m=1}^k \frac{\partial}{\partial z_l} (Q_m \cdot R_m(g_1, \dots, g_n)) \\ &= \sum_{m=1}^k \frac{\partial Q_m}{\partial z_l} R_m(g_1, \dots, g_p) + \sum_{m=1}^k \sum_{j=1}^p Q_m \frac{\partial g_j}{\partial z_l} \cdot \frac{\partial R_m}{\partial z_j}(g_1, \dots, g_p). \end{aligned}$$

Since $\frac{\partial Q_m}{\partial z_l}, Q_m \frac{\partial g_j}{\partial z_l} \in \mathcal{B}_{\Delta}$ and $\frac{\partial R_m}{\partial z_j} \in \mathcal{B}_{\Delta'}$, the claim holds for $|\alpha| + 1$. \square

Lemma 6.4 *Let $H : \Delta \rightarrow \mathbb{C}$ be a holomorphic function, where $\Delta \subset \mathbb{C}^n$ is a polydisc centered at the origin. We suppose that H is strongly definable and all partial derivatives $\frac{\partial^{|\alpha|} H}{\partial z^\alpha}$ are strongly definable. Then the Schwarz Reflection of H , H^{SR} , is strongly definable and all partial derivatives $\frac{\partial^{|\alpha|} H^{\text{SR}}}{\partial z^\alpha}$ are strongly definable.*

Proof Clearly, H^{SR} is strongly definable. Moreover, if $f \in \mathbb{C}[[X]]$, then $\frac{\partial^{|\alpha|} f^*}{\partial X^\alpha} = \left(\frac{\partial^{|\alpha|} f}{\partial X^\alpha} \right)^*$ (see Section 5 for the definition of the map $*$). It follows then that $\frac{\partial^{|\alpha|} H^{\text{SR}}}{\partial z^\alpha} = \left(\frac{\partial^{|\alpha|} H}{\partial z^\alpha} \right)^{\text{SR}}$. Therefore, all partial derivatives $\frac{\partial^{|\alpha|} H^{\text{SR}}}{\partial z^\alpha}$ are strongly definable. \square

Lemma 6.5 *Let $\tilde{h}, \tilde{g} \in \mathcal{H}$ be such that $\dim(h) = \dim(g) = 2n$ and $H = (\tilde{g} + i\tilde{h})|_{J^{2n}}$ is holomorphic, where $(\tilde{g} + i\tilde{h})|_{J^{2n}}$ is the restriction of $\tilde{g} + i\tilde{h}$ on J^{2n} . Then H is strongly definable and all partial derivatives $\frac{\partial^{|\alpha|} H}{\partial z^\alpha}$ are strongly definable.*

Proof In this proof, if $\tilde{f} \in \mathcal{H}$ and $\dim(f) = 2n$, then \hat{f} denotes the restriction of \tilde{f} on J^{2n} . Clearly, H is definable in \mathcal{R} by a quantifier-free $L_{\mathcal{R}}$ -formula. Hence H is strongly definable. By the claim (see below), all partial derivatives $\frac{\partial^{|\alpha|} H}{\partial z^\alpha}$ are definable in \mathcal{R} by quantifier-free $L_{\mathcal{R}}$ -formulas. Hence they are strongly definable.

Claim For each $\alpha \in \mathbb{N}^n$, there is a positive integer m and there are $P \in \mathbb{C}[X_1, \dots, X_m]$ and $\tilde{f}_1, \dots, \tilde{f}_m \in \mathcal{H}$ such that $\dim(f_1) = \dots = \dim(f_m) = 2n$ and $\frac{\partial^{|\alpha|} H}{\partial z^\alpha} = P(\hat{f}_1, \dots, \hat{f}_m)$.

We proceed by induction on $|\alpha|$. For $|\alpha| = 0$, we have

$$H = (\tilde{g} + i\tilde{h})|_{J^{2n}} = \hat{g} + i\hat{h}.$$

Hence the claim holds for $|\alpha| = 0$. Let $\alpha \in \mathbb{N}^n$. We suppose that the claim holds for α . We have

$$\frac{\partial^{|\alpha|} H}{\partial z^\alpha} = P(\hat{f}_1, \dots, \hat{f}_m)$$

of the required form. Then, for each $l = 1, \dots, n$,

$$\frac{\partial}{\partial z_l} \left(\frac{\partial^{|\alpha|} H}{\partial z^\alpha} \right) = \frac{\partial}{\partial z_l} (P(\hat{f}_1, \dots, \hat{f}_m)).$$

Put $P = R + iT$, with $R, T \in \mathbb{R}[X_1, \dots, X_m]$. Say $F := (\widehat{f}_1, \dots, \widehat{f}_m)$. Then

$$\begin{aligned} \frac{\partial}{\partial z_l}(P(F)) &= \frac{1}{2} \left(\frac{\partial}{\partial x_l}(P(F)) - i \frac{\partial}{\partial y_l}(P(F)) \right) \\ &= \frac{1}{2} \left(\frac{\partial}{\partial x_l}(R(F)) + \frac{\partial}{\partial y_l}(T(F)) + i \left(\frac{\partial}{\partial x_l}(R(F)) - \frac{\partial}{\partial y_l}(T(F)) \right) \right). \end{aligned}$$

Let S be either R or T , and let t_l be either x_l or y_l . We have

$$\begin{aligned} \frac{\partial}{\partial t_l}(S(F)) &= \sum_{j=1}^m \frac{\partial \widehat{f}_j}{\partial t_l} \cdot \frac{\partial S}{\partial X_j}(F) \\ &= \sum_{j=1}^m \frac{\partial \widehat{f}_j}{\partial t_l} \cdot \frac{\partial S}{\partial X_j}(F). \end{aligned}$$

By (P3), all $\frac{\partial \widehat{f}_j}{\partial t_l}$ belong to \mathcal{H} , and clearly $\dim\left(\frac{\partial f_j}{\partial t_l}\right) = 2n$. Therefore, the claim holds for $|\alpha| + 1$. \square

Proposition 6.6 *Let $\widetilde{h} \in \mathcal{H}$ be such that $\dim(h) = 2n$. Then for each $a \in I^{2n}$ the translate series $h(X' + a)$ belongs to \mathcal{S}_{2n} .*

Proof According to (P1), there is $\widetilde{g} \in \mathcal{H}$ such that $\dim(g) = 2n$ and either $H = \widetilde{h} + i\widetilde{g}$ or $G = \widetilde{g} + i\widetilde{h}$ is holomorphic on J^{2n} . We may assume that H is holomorphic on J^{2n} . Otherwise, we replace H by G .

Suppose $a \in J^{2n}$. By Lemma 6.5, the translate series $H_a := H(X + a)$ belongs to \mathcal{B}_n . By Proposition 6.1 and Lemmas 6.3–6.4, Corollary 5.5 also holds for the system $\langle \mathcal{B}_n : n \geq 0 \rangle$. Then $\Re[H_a](X')$ and $\Im[H_a](X')$ belong to \mathcal{B}_{2n} . Since $\Re[H_a](X') = h(X'' + a)$ and $\Im[H_a](X') = g(X'' + a)$, where $X'' = (X_1, X_{n+1}, \dots, X_n, X_{2n})$, by (W1), $h(X' + a)$ and $g(X' + a)$ belong to \mathcal{B}_{2n} . On the other hand, $h(X' + a), g(X' + a) \in \mathbb{R}[[X']]$. Hence $h(X' + a), g(X' + a) \in \mathcal{S}_{2n}$.

Now suppose that $a \in I^{2n}$. According to (P2), there are $0 < \varepsilon < 1$ and $\widetilde{g} \in \mathcal{H}$ such that $\dim(g) = 2n$ and $h(x) = g(\varepsilon x)$ for all $x \in I^{2n}$. Since $h(X' + a) = g(\varepsilon X' + \varepsilon a)$ and $\varepsilon a \in J^{2n}$, by the above discussion and Lemma 6.3, $h(X' + a)$ belongs to \mathcal{S}_{2n} . \square

Proof of Theorem B By Proposition 6.6, each $\widetilde{h} \in \mathcal{H}$ is definable in $\mathbb{R}_{\mathcal{S}}$. Hence every \mathcal{R} -definable set is $\mathbb{R}_{\mathcal{S}}$ -definable. Therefore, by Theorem 6.2, the structure \mathcal{R} is strongly model complete. \square

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