Finite generation of the log canonical ring in dimension four

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To the memory of Professor Masayoshi Nagata

Abstract We treat two different topics on the log minimal model program, especially for four-dimensional log canonical pairs:

(a) finite generation of the log canonical ring in dimension four,

(b) abundance theorem for irregular fourfolds.

We obtain (a) as a direct consequence of the existence of four-dimensional log minimal models by using Fukuda's theorem on the four-dimensional log abundance conjecture. We can prove (b) only by using traditional arguments. More precisely, we prove the abundance conjecture for irregular $(n+1)$-folds on the assumption that the minimal model conjecture and the abundance conjecture hold in dimension $\leq n$.

1. Introduction

In this article, we treat two different topics on the log minimal model program, especially for four-dimensional log canonical pairs. We freely use the results on the three-dimensional log minimal model program (see [Ko], [KeMM1]). We do not always refer to the original articles since the results are scattered in various places.

1.1. Finite generation of the log canonical ring in dimension four

The following theorem is the main result of Section 3 (cf. [F5, Section 3.1]).

THEOREM 1.1 (FINITE GENERATION OF THE LOG CANONICAL RING IN DIMENSION FOUR)

Let $\pi: X \to Z$ be a proper surjective morphism from a smooth fourfold X. Let B be a boundary $\mathbb{Q}$-divisor on X such that $\operatorname{Supp} B$ is a simple normal crossing divisor on X. Then the relative log canonical ring

$$R(X/Z, K_X+B) = \bigoplus_{m\geq 0} \pi_*\mathcal{O}_X\big(\llcorner m(K_X+B)\lrcorner\big)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

Kyoto Journal of Mathematics, Vol. 50, No. 4 (2010), 671–684

DOI 10.1215/0023608X-2010-010,

Received May 6, 2009. Revised August 14, 2009. Accepted November 2, 2009.

2010 Mathematics Subject Classification: Primary 14J35; Secondary 14E30.

Author's work partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (A) 20684001 and by the Inamori Foundation.

It is easy to see that Theorem 1.1 is equivalent to Theorem 1.2.

THEOREM 1.2
Let $\pi : X \to Z$ be a proper surjective morphism from a four-dimensional log canonical pair (X, B) such that B is an effective $\mathbb{Q}$-divisor. Then the relative log canonical ring

$$R(X/Z, K_X + B) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X \left(\llcorner m(K_X + B) \lrcorner \right)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

In Section 3, we give a proof of Theorem 1.1 by using the existence theorem of four-dimensional log minimal models (cf. [B], [S2]) and Fukuda's result on the log abundance conjecture for fourfolds (see [Fk]). A key point of Fukuda's result is the abundance theorem for semi log canonical threefolds in [F1].

1.2. Abundance theorem for irregular fourfolds

In Section 4, we prove the abundance theorem for irregular $(n+1)$-folds on the assumption that the minimal model conjecture and the abundance conjecture hold in dimension $\leq n$ (see Theorem 4.5). By this result, we know that the abundance conjecture for irregular varieties is the problem for lower-dimensional varieties. Since the minimal model conjecture and the abundance conjecture hold in dimension ≤ 3, we obtain the next theorem (see Corollary 4.7).

THEOREM 1.3 (ABUNDANCE THEOREM FOR IRREGULAR FOURFOLDS)
Let X be a normal complete fourfold with only canonical singularities. Assume that K_X is nef and the irregularity $q(X)$ is not zero. Then K_X is semiample.

We also prove that there exists a good minimal model for any smooth projective irregular fourfold (see Theorem 4.8).

THEOREM 1.4 (GOOD MINIMAL MODELS OF IRREGULAR FOURFOLDS)
Let X be a smooth projective irregular fourfold. If X is not uniruled, then there exists a normal projective variety X' such that X' has only $\mathbb{Q}$-factorial terminal singularities, X' is birationally equivalent to X, and $K_{X'}$ is semiample.

We note that Sections 3 and 4 can be read independently.

We work over $\mathbb{C}$, the complex number field, throughout this article. We freely use the notation in [KMM] and [KM]. Note that we do not use $\mathbb{R}$-divisors.

2. Preliminaries

In this section, we collect basic definitions.

DEFINITION 2.1 (DIVISORS, $\mathbb{Q}$-DIVISORS)

Let X be a normal variety. For a $\mathbb{Q}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ on X such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$, where for every rational number x, $\llcorner x \lrcorner$ is the integer defined by $x - 1 < \llcorner x \lrcorner \leq x$.

We call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ for every j.

We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-divisors.

We call X $\mathbb{Q}$-*factorial* if and only if every Weil divisor on X is $\mathbb{Q}$-Cartier.

DEFINITION 2.2 (EXCEPTIONAL LOCUS)

For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

Let us quickly recall the definitions of singularities of pairs.

DEFINITION 2.3 (SINGULARITIES OF PAIRS)

Let X be a normal variety, and let B be an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y. We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* (resp., *klt*) if and only if $a_i \geq -1$ (resp., $a_i > -1$) for every i. We note that lc (resp., klt) is an abbreviation of *log canonical* (resp., *Kawamata log terminal*). We also note that the *discrepancy* $a(E, X, B) \in \mathbb{Q}$ can be defined for every prime divisor E *over* X.

In the above notation, if $B = 0$ and $a_i > 0$ (resp., $a_i \geq 0$) for every i, then we say that X has only *terminal* (resp., *canonical*) singularities.

DEFINITION 2.4 (DIVISORIAL LOG TERMINAL PAIR)

Let X be a normal variety, and let B be a boundary $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. If there exists a resolution $f : Y \to X$ such that

(i) both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}B)$ are simple normal crossing divisors on Y, and

(ii) $a(E, X, B) > -1$ for every exceptional divisor $E \subset Y$,

then (X, B) is called *divisorial log terminal* (*dlt* for short).

For the details of singularities of pairs, see, for example, [KM] and [F2].

DEFINITION 2.5 (CENTER, LC CENTER)

Let E be a prime divisor over X. The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X.

Let (X,B) be an lc pair. If $a(E,X,B)=-1$, $c_X(E)$ is called an *lc center* of (X,B).

The following definitions are now classical.

DEFINITION 2.6 (IITAKA'S D-DIMENSION AND NUMERICAL D-DIMENSION)
Let X be a normal complete variety, and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Assume that mD is Cartier for a positive integer m. Let

$$\Phi_{|tmD|} : X \dashrightarrow \mathbb{P}^{\dim |tmD|}$$

be rational mappings given by linear systems $|tmD|$ for positive integers t. We define *Iitaka's D-dimension*:

$$\kappa(X,D)=\begin{cases}\max_{t>0}\dim\Phi_{|tmD|}(X) & \text{if } |tmD|\neq\emptyset \text{ for some } t,\\ -\infty & \text{otherwise.}\end{cases}$$

In the case when D is nef, we can also define the *numerical D-dimension*

$$\nu(X,D)=\max\{e \mid D^e\not\equiv 0\},$$

where $\equiv$ denotes *numerical equivalence*. We note that $\nu(X,D)\geq\kappa(X,D)$ always holds.

DEFINITION 2.7 (NEF AND ABUNDANT DIVISORS)
Let X be a normal complete variety, and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that D is nef. The nef $\mathbb{Q}$-divisor D is said to be *abundant* if the equality $\kappa(X,D)=\nu(X,D)$ holds. Let $\pi: X\to Z$ be a proper surjective morphism of normal varieties, and let D be a π-nef $\mathbb{Q}$-divisor on X. Then D is said to be *π-abundant* if D_η is abundant, where $D_\eta=D|_{X_\eta}$ and X_η is the generic fiber of π.

DEFINITION 2.8 (IRREGULARITY)
Let X be a normal complete variety with only rational singularities. We put

$$q(X)=h^1(X,\mathcal{O}_X)=\dim H^1(X,\mathcal{O}_X)<\infty$$

and call it the *irregularity* of X.

Let X be as above. If $q(X)\neq 0$, then we call X *irregular*.

If X' is a normal complete variety with only rational singularities such that X' is birationally equivalent to X, then it is easy to see that $q(X)=q(X')$.

3. Log canonical ring

In this section, we prove the following theorem: Theorem 1.1.

THEOREM 3.1 (FINITE GENERATION OF THE FOUR-DIMENSIONAL LOG CANONICAL RING)
Let $\pi: X\to Z$ be a proper surjective morphism from a smooth fourfold X. Let B be a boundary $\mathbb{Q}$-divisor on X such that $\operatorname{Supp} B$ *is a simple normal crossing divisor on X. Then the relative log canonical ring*

$$R(X/Z, K_X + B) = \bigoplus_{m\geq 0} \pi_*\mathcal{O}_X\big(\llcorner m(K_X + B)\lrcorner\big)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

The next proposition is well known and a slight generalization of [K3, Theorem 7.3].

PROPOSITION 3.2
Let (X, B) be a proper log canonical fourfold such that $K_X + B$ is nef and $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is abundant; that is, $\kappa(X, K_X + B) = \nu(X, K_X + B)$.

Proof
See, for example, [Fk, Proposition 3.1]. We note that we need the three-dimensional log minimal model program and log abundance theorem here (see [Ko], [KeMM1], [KeMM2]). □

Let us recall Fukuda's result [Fk]. We generalize this in Theorem 3.10.

THEOREM 3.3 (CF. [Fk, THEOREM 1.5])
Let (X, B) be a proper dlt fourfold. Assume that $K_X + B$ is nef and that $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is semiample.

Proof
By Proposition 3.2, $\kappa(X, K_X + B) = \nu(X, K_X + B)$. We put $S = \llcorner B\lrcorner$ and $K_S + B_S = (K_X + B)|_S$. Then the pair (S, B_S) is semidivisorial log terminal and $K_S + B_S$ is semiample by [F1, Theorem 0.1]. Finally, by [F3, Corollary 6.7], we obtain that $K_X + B$ is semiample. □

REMARK 3.4
The proof of [Fk, Proposition 3.3] depends on [K3, Theorem 5.1]. It requires [K3, Theorem 4.3], whose proof contains a nontrivial gap (see [F2, Remark 3.10.3], [F6]). So we adopted [F3, Corollary 6.7] in the proof of Theorem 3.3.

In this section, we adopt Birkar's definition of the log minimal model (see [B, Definition 2.4]), which is slightly different from [KM, Definition 3.50] (see Remark 3.6 and Example 3.7).

DEFINITION 3.5 (CF. [B, DEFINITION 2.4])
Let (X, B) be a log canonical pair over Z. A *log minimal model* $(Y/Z, B_Y + E)$ of $(X/Z, B)$ consists of a birational map $\phi : X \dashrightarrow Y/Z$, $B_Y = \phi_*B$, and $E = \sum_j E_j$, where E_j is a prime divisor on Y, ϕ^{-1}-exceptional for every j, and satisfies the following conditions:

(1) Y is $\mathbb{Q}$-factorial, and $(Y, B_Y + E)$ is dlt;

(2) $K_Y + B_Y + E$ is nef over Z; and
(3) for every prime divisor D on X which is exceptional over Y, we have

$$a(D, X, B) < a(D, Y, B_Y + E),$$

where $a(D, X, B)$ (resp., $a(D, Y, B_Y + E)$) denotes the discrepancy of D with respect to (X, B) (resp., $(Y, B_Y + E)$).

REMARK 3.6
In [KM, Definition 3.50], it is required that ϕ^{-1} have no exceptional divisors.

EXAMPLE 3.7
Let $X = \mathbb{P}^2$, and let D_X be the complement of the big torus. Then $K_X + D_X$ is dlt and $K_X + D_X \sim 0$. Let $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, and let D_Y be the complement of the big torus. Then (Y, D_Y) is a log minimal model of (X, D_X) in the sense of Definition 3.5. Of course, $K_Y + D_Y$ is dlt and $K_Y + D_Y \sim 0$. On the other hand, (Y, D_Y) is not a log minimal model of (X, D_X) in the sense of [KM, Definition 3.50].

We prepare the following two easy lemmas.

LEMMA 3.8
We use the notation in Definition 3.5. Then we have

$$a(\nu, X, B) \leq a(\nu, Y, B_Y + E)$$

for every divisor ν over X. Thus, we obtain

$$R(X/Z, K_X + B) \simeq R(Y/Z, K_Y + B_Y + E).$$

Proof
It is an easy consequence of the negativity lemma (see, e.g., [KM, Proposition 3.51, Theorem 3.52]). □

LEMMA 3.9
Let $\pi : X \to Z$ be a projective surjective morphism between projective varieties. Assume that (X, B) is log canonical and H is an ample Cartier divisor on Z. Let R be a $(K_X + B)$-negative extremal ray of $\overline{NE}(X)$ such that

$$R \cdot \big(K_X + B + (2 \dim X + 1)\pi^* H\big) < 0.$$

Then R is a $(K_X + B)$-negative extremal ray of

$$\overline{NE}(X/Z) = \big\{ z \in \overline{NE}(X) \bigm| z \cdot \pi^* H = 0 \big\} \subset \overline{NE}(X).$$

In particular, if $K_X + B$ is π-nef, then $K_X + B + (2 \dim X + 1)\pi^* H$ is nef.

Proof
If (X, B) is klt, then it is obvious by Kawamata's bound of the length of extremal rays (see [K4]). When (X, B) is lc, it is sufficient to use [F5, Section 3.1.3] or [F4, Section 18]. □

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1
We can assume that the fiber of π is connected. First, if $\kappa(X_\eta, K_{X_\eta} + B_\eta) = -\infty$, where η is the generic point of Z, then the statement is trivial. We note that the statement is obvious when Z is a point and $\kappa(X, K_X + B) = 0$. So we can assume that $\kappa(X_\eta, K_{X_\eta} + B_\eta) \geq 0$ and that $\kappa(X, K_X + B) \geq 1$ when Z is a point. Since the problem is local, we can assume that Z is affine. By compactifying Z and X and taking a resolution of X, we can assume that X and Z are projective and that $\operatorname{Supp} B$ is a simple normal crossing divisor. By the assumption, we can find an effective $\mathbb{Q}$-divisor M on X such that $K_X + B \sim_{\mathbb{Q},\pi} M$; that is, there exists a $\mathbb{Q}$-divisor N on Z such that $K_X + B \sim_{\mathbb{Q}} M + \pi^* N$. We take a log minimal model of (X, B) over Z by using the arguments in [B, Section 3]. Then we obtain a projective surjective morphism $\pi_Y : Y \to Z$ such that $\bigl(Y/Z, B_Y + \sum_j E_j\bigr)$ is a log minimal model of $(X/Z, B)$, where B_Y is the pushforward of B on Y by $\phi : X \dashrightarrow Y$ and E_j is exceptional over X and is a prime divisor on Y for every j. Let A be a sufficiently ample general Cartier divisor on Z. Then $(Y, B_Y + E + \pi_Y^* A)$, where $E = \sum_j E_j$, is a log minimal model of $(X, B + \pi^* A)$ by Lemma 3.9. Since $\kappa(Y, K_Y + B_Y + E + \pi_Y^* A) \geq 1$, $K_Y + B_Y + E + \pi_Y^* A$ is semiample by Theorem 3.3. In particular,

$$K_Y + B_Y + E = K_Y + B_Y + E + \pi_Y^* A - \pi_Y^* A$$

is π_Y-semiample. Thus,

$$R(Y/Z, K_Y + B_Y + E) = \bigoplus_{m \geq 0} \pi_{Y*} \mathcal{O}_Y \bigl(\llcorner m(K_Y + B_Y + E) \lrcorner\bigr)$$

is a finitely generated $\mathcal{O}_Z$-algebra. Therefore,

$$R(X/Z, K_X + B) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X \bigl(\llcorner m(K_X + B) \lrcorner\bigr)$$

is a finitely generated $\mathcal{O}_Z$-algebra by Lemma 3.8. We finish the proof. □

The final theorem in this section is a generalization of Fukuda's theorem (see Theorem 3.3).

THEOREM 3.10 (A SPECIAL CASE OF THE LOG ABUNDANCE THEOREM)
Let $\pi : X \to Z$ be a proper surjective morphism from a four-dimensional log canonical pair (X, B) such that B is an effective $\mathbb{Q}$-divisor and $K_X + B$ is π-nef. When Z is a point, we further assume that $\kappa(X, K_X + B) > 0$. Then $K_X + B$ is π-semiample.

Proof
Without loss of generality, we can assume that π has connected fibers. By Proposition 3.2 and the log abundance theorem in dimension ≤ 3, $K_{X_\eta} + B_\eta$ is nef and abundant, where η is the generic point of Z. By Theorem 1.2, $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(\llcorner m(K_X+B)\lrcorner)$ is a finitely generated $\mathcal{O}_Z$-algebra. Therefore, K_X+B is π-semiample by Lemma 3.12. □

The next lemma is well known. We leave the proof as an exercise for the reader.

LEMMA 3.11 (CF. [L, THEOREM 2.3.15])
Let $\pi : X \to Z$ be a projective surjective morphism from a smooth variety X to a normal variety Z, and let M be a π-nef and π-big Cartier divisor on X. Then $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(mM)$ is a finitely generated $\mathcal{O}_Z$-algebra if and only if M is π-semiample.

By [KMM, Proposition 6-1-3], we can reduce Lemma 3.12 to Lemma 3.11.

LEMMA 3.12
Let $\pi : X \to Z$ be a proper surjective morphism between normal varieties, and let M be a π-nef and π-abundant Cartier divisor on X. Then $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(mM)$ is a finitely generated $\mathcal{O}_Z$-algebra if and only if M is π-semiample.

3.1. Appendix
In this appendix, we explicitly state the results in dimension ≤ 3 because we can find no good references for the relative statements (cf. [Ft], [KeMM1], [KeMM2]).

THEOREM 3.13
Let $\pi : X \to Z$ be a proper surjective morphism between normal varieties. Assume that (X, B) is log canonical with $\dim X \leq 3$ and that B is an effective $\mathbb{Q}$-divisor. Then

$$\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X\big(\llcorner m(K_X+B)\lrcorner\big)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

Proof
When Z is a point, this theorem is well known (cf. [Ft], [KeMM1], [KeMM2]). So we assume that $\dim Z \geq 1$. By the arguments in the proof of Theorem 1.1, we can prove that $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(\llcorner m(K_X+B)\lrcorner)$ is a finitely generated $\mathcal{O}_Z$-algebra. □

THEOREM 3.14
Let $\pi : X \to Z$ be a proper surjective morphism such that (X, B) is log canonical with $\dim X \leq 3$. Assume that K_X+B is π-nef and that B is an effective $\mathbb{Q}$-divisor. Then K_X+B is π-semiample.

Proof
When Z is a point, this theorem is well known (cf. [Ft], [KeMM1], [KeMM2]). So we assume that $\dim Z \geq 1$. Without loss of generality, we can assume that π has connected fibers. It is well known that $K_X + B$ is π-nef and π-abundant by the log abundance theorem in dimension ≤ 2. By Theorem 3.13 and Lemma 3.12, $K_X + B$ is π-semiample. □

We close this appendix with a remark.

REMARK 3.15
Let $\pi : X \to Z$ be a proper surjective morphism between normal varieties. Assume that (X, B) is klt and that B is an effective $\mathbb{Q}$-divisor. Then

$$\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X \big(\llcorner m(K_X + B) \lrcorner \big)$$

is a finitely generated $\mathcal{O}_Z$-algebra by [BCHM]. Therefore, by Lemma 3.12, $K_X + B$ is π-semiample if and only if $K_X + B$ is π-nef and π-abundant by Lemma 3.12.

Of course, we know that $K_X + B$ is π-semiample if and only if $K_X + B$ is π-nef and π-abundant without appealing to [BCHM] (see, e.g., [F6]). It is known as Kawamata's theorem (see [K3, Theorem 6.1]).

4. Abundance theorem for irregular varieties

In this section, we treat the abundance conjecture for irregular varieties. Let us recall the following minimal model conjecture.

CONJECTURE 4.1 (MINIMAL MODEL CONJECTURE)
Let X be a smooth projective variety. Assume that K_X is pseudoeffective. Then there exists a normal projective variety X' which satisfies the following conditions.

(i) *X' is birationally equivalent to X.*
(ii) *X' has only $\mathbb{Q}$-factorial terminal singularities.*
(iii) *$K_{X'}$ is nef.*

We call X' a minimal model of X.

In Conjecture 4.1, if $K_{X'}$ is semiample, X' is usually called a *good minimal model* of X.

CONJECTURE 4.2 (ABUNDANCE CONJECTURE)
Let X be a projective variety with only canonical singularities. Assume that K_X is nef. Then K_X is semiample. In particular, $\kappa(X) = \kappa(X, K_X)$ is nonnegative.

We know that Conjectures 4.1 and 4.2 hold in dimension ≤ 3 (cf. [KMM], [Ko]).

REMARK 4.3

In Conjecture 4.1, by [BCHM], we can replace (ii) with the following slightly weaker condition: (ii′) X' has at most canonical singularities. Similarly, we can assume that X has only $\mathbb{Q}$-factorial terminal singularities in Conjecture 4.2.

REMARK 4.4

Let X be a smooth projective variety. Then X is uniruled if and only if K_X is not pseudoeffective by [BDPP].

The next theorem is the main theorem of this section.

THEOREM 4.5 (ABUNDANCE THEOREM FOR IRREGULAR $(N+1)$-FOLDS)

Assume that Conjectures 4.1 and 4.2 hold in dimension $\leq n$*. Let* X *be a normal complete* $(n+1)$*-fold with only canonical singularities. If* K_X *is nef and* $q(X)\neq 0$*, then* K_X *is semiample.*

Proof

Let $\pi:\overline{X}\to X$ be a resolution, and let $\alpha:\overline{X}\to A=\mathrm{Alb}(\overline{X})$ be the Albanese mapping. By the assumption, we have $\dim A\geq 1$. Since X has only rational singularities, $\beta=\alpha\circ\pi^{-1}:X\to A$ is a morphism (cf. [R, Proposition 2.3], [BS, Lemma 2.4.1]).

CLAIM 1

We have $\kappa(X,K_X)=\kappa(\overline{X},K_{\overline{X}})\geq 0$.

Proof

Let $f:\overline{X}\to S$ be the Stein factorization of α, and let F be a general fiber of f. Then, by [K2, Corollary 1.2], we have

$$\kappa(\overline{X},K_{\overline{X}})\geq\kappa(F,K_F)+\kappa(\overline{S},K_{\overline{S}}),$$

where $\overline{S}$ is a resolution of S. We note that $\kappa(\overline{S},K_{\overline{S}})\geq 0$ because $S\to\beta(X)\subset A$ is generically finite (see, e.g., [U, Theorem 6.10, Lemma 10.1]). We also note that $\kappa(F,K_F)=\kappa(G,K_G)\geq 0$ since $\dim G\leq n$, G has only canonical singularities, and K_G is nef, where $G=\pi(F)$. Here, we used Conjectures 4.1 and 4.2 in dimension $\dim G=\dim F\leq n$. Therefore, we obtain $\kappa(\overline{X},K_{\overline{X}})\geq 0$. □

CLAIM 2

If $\kappa(X,K_X)=0$*, then* $\nu(X,K_X)=0$.

Proof

By Kawamata's theorem (see [K1, Theorem 1]), β is surjective and $\beta_*\mathcal{O}_X\simeq\mathcal{O}_A$. Let G be a general fiber of β. Then $\kappa(G,K_G)=0$ by

$$0=\kappa(X,K_X)\geq\kappa(G,K_G)+\kappa(A,K_A)=\kappa(G,K_G)$$

as in Claim 1, and $\kappa(G, K_G) \geq 0$ by Conjecture 4.2 in $\dim G \leq n$ since K_G is nef. We note that X and G have only canonical singularities. By Remark 3.15 and the assumption: Conjecture 4.2 in dimension $\leq n$, K_X is β-semiample. Therefore, $\beta : X \to A$ can be written as

$$\beta : X \xrightarrow{f} S \xrightarrow{g} A,$$

where $K_X \sim_{\mathbb{Q}} f^*D$ for some g-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on S, $g : S \to A$ is a birational morphism, and S is a normal variety. Since $\kappa(X, K_X) = 0$, we obtain $\kappa(S, D) = 0$. So it is sufficient to prove that $D \sim_{\mathbb{Q}} 0$. By [A, Theorem 0.2], we can write $D \sim_{\mathbb{Q}} K_S + \Delta_S$ such that (S, Δ_S) is klt. In particular, Δ_S is effective. By Lemma 4.6, we obtain that g is an isomorphism. Therefore, $D \sim_{\mathbb{Q}} 0$ since $\kappa(S, D) = 0$ and $S = A$ is an Abelian variety. □

By Claims 1 and 2, $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$. In this case, we obtain $\kappa(X, K_X) = \nu(X, K_X)$ by Kawamata's argument and the assumption: Conjectures 4.1 and 4.2 in dimension $\leq n$ (see the proof of [K3, Theorem 7.3]). Therefore, K_X is semiample by Remark 3.15. □

We used the following lemma in the proof of Claim 2.

LEMMA 4.6

Let $g : S \to A$ be a projective birational morphism from a klt pair (S, Δ_S) to an Abelian variety A. Assume that $K_S + \Delta_S$ is g-nef and $\kappa(S, K_S + \Delta_S) = 0$. Then g is an isomorphism.

Proof

By replacing S with its small projective $\mathbb{Q}$-factorialization (cf. [BCHM]), we can assume that S is $\mathbb{Q}$-factorial. We note that $K_S = E$, where E is effective and $\operatorname{Supp} E = \operatorname{Exc}(g)$ since A is an Abelian variety. If $B = g_*\Delta_S \neq 0$, then $g^*B \leq m(K_S + \Delta_S)$ for some $m > 0$. In this case,

$$1 \leq \kappa(A, B) = \kappa(S, g^*B) \leq \kappa(S, K_S + \Delta_S) = 0.$$

It is a contradiction. Therefore, $B = 0$. This means that Δ_S is g-exceptional. Thus, $K_S + \Delta_S$ is effective, g-exceptional, and $\operatorname{Exc}(g) = \operatorname{Supp}(K_S + \Delta_S)$. By the assumption, $K_S + \Delta_S$ is g-nef. So g is an isomorphism by the negativity lemma. □

As a special case of Theorem 4.5, we obtain the abundance theorem for irregular fourfolds.

COROLLARY 4.7 (ABUNDANCE THEOREM FOR IRREGULAR FOURFOLDS)

Let X be a normal complete fourfold with only canonical singularities. Assume that K_X is nef and the irregularity $q(X)$ is not zero. Then K_X is semiample.

Proof
It is obvious by Theorem 4.5 because Conjectures 4.1 and 4.2 hold in dimension $\leq$ 3 (cf. [Ko], [KM]). $\square$

We close this section with the following theorem.

THEOREM 4.8 (GOOD MINIMAL MODELS OF IRREGULAR FOURFOLDS)
Let X be a smooth projective irregular fourfold. If X is not uniruled, then X has a good minimal model. More precisely, there exists a normal projective variety X' such that X' has only $\mathbb{Q}$-factorial terminal singularities, X' is birationally equivalent to X, and $K_{X'}$ is semiample.

Proof
We run the minimal model program. Then we obtain a minimal model X' of X since K_X is pseudoeffective by the assumption. Here we use the existence and the termination of four-dimensional terminal flips (cf. [KMM, Theorem 5-1-15], [S1], [HM, Corollary 5.1.2]). We note that $q(X') = h^1(X', \mathcal{O}_{X'}) = q(X) \neq 0$. Therefore, by Theorem 4.5, we obtain that $K_{X'}$ is semiample. $\square$

4.1. Appendix

In this appendix, we give a remark on the abundance conjecture for fourfolds for the reader's convenience.

CONJECTURE 4.9 (ABUNDANCE CONJECTURE FOR FOURFOLDS)
Let X be a complete fourfold with only canonical singularities. If K_X is nef, then K_X is semiample.

This conjecture is still open. By Corollary 4.7 and Kawamata's argument (cf. [K3, Theorem 7.3]), we can reduce Conjecture 4.9 to the following two problems.

PROBLEM 4.10
Let X be a smooth projective fourfold. If X is not uniruled and $q(X) = 0$, then $\kappa(X) \geq 0$.

PROBLEM 4.11
Let X be a projective fourfold with only $\mathbb{Q}$-factorial terminal singularities. If K_X is nef, $q(X) = 0$, and $\kappa(X, K_X) = 0$, then K_X is numerically trivial; equivalently, $K_X \sim_{\mathbb{Q}} 0$.

We explain the reduction argument closely. Let X be a complete fourfold with only canonical singularities such that K_X is nef. If $q(X) \neq 0$, then K_X is semiample by Corollary 4.7. So from now on, we can assume that $q(X) = 0$. By taking a resolution of X and running the minimal model program (cf. [KMM, Theorem 5-1-15], [S1], [HM, Corollary 5.1.2]), there exists a projective variety X'

such that $K_{X'}$ is nef and X' has only $\mathbb{Q}$-factorial terminal singularities. Let

$$X \xleftarrow{f} W \xrightarrow{g} X'$$

be a common resolution. Then $f^*K_X = g^*K_{X'}$ by the negativity lemma. Therefore, we can replace X with X' to prove Conjecture 4.9. If we solve Problem 4.10, then we obtain $\kappa(X, K_X) \geq 0$ since X has only terminal singularities. Furthermore, if we solve Problem 4.11, then we can prove that $\nu(X, K_X) > 0$ implies $\kappa(X, K_X) > 0$. By the proof of [K3, Theorem 7.3], we obtain $\nu(X, K_X) = \kappa(X, K_X)$ (cf. Proposition 3.2). Thus, K_X is semiample (cf. Remark 3.15).

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