Connes-amenability of multiplier Banach algebras

Bahman Hayati and Massoud Amini

Abstract Let B be a Banach algebra with bounded approximate identity, and let $M(B)$ be its multiplier algebra. If there exists a continuous linear injection $B^* \to M(B)$ such that, for every $b \in B$ and every $u, v \in B^*$, $\langle u, vb\rangle_B = \langle v, bu\rangle_B$, then $M(B)$ is a dual Banach algebra and the following are equivalent:

(i) B is amenable;

(ii) $M(B)$ is Connes amenable;

(iii) $M(B)$ has a normal, virtual diagonal.

1. Introduction

The concept of amenability of groups was first defined for discrete locally compact groups by John Von Neumann [Von]. Later this notion was generalized to arbitrary locally compact groups by Mahlon Day [Day]. In 1972, Barry E. Johnson [Joh2] introduced the concept of amenability for Banach algebras. He proved that a locally compact group G is amenable if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra. After the pioneering work of Johnson, several modifications of the original concept of amenability of Banach algebras are presented. One of the most important modifications was presented by Barry E. Johnson himself, in a joint paper with Richard V. Kadison and John R. Ringrose [JKR], where they introduced a notion of amenability more suitable for von Neumann algebras. It modifies the original definition in the sense that it takes into account the dual space structure of a von Neumann algebra. We note that a von Neumann algebra is a C^*-algebra that is also the dual of a Banach space. (It has a unique predual.) Due to an important contribution of Alain Connes [Con1], [Con2], Alexander Ya. Helemskii [Hel] coined the term *Connes-amenability* for this concept. Later Volker Runde [Run1] extended the notion of Connes-amenability to the more general setting of dual Banach algebras. He also showed that the notion of a normal virtual diagonal (which is the analog of virtual diagonal introduced by Johnson in [Joh2]) adapts naturally to the context of general dual Banach algebras. He showed that a dual Banach algebra with a normal virtual diagonal is Connes-amenable (with an argument

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almost verbatim to that of Edward G. Effros [Eff] for von Neuman algebras) but the converse is not true in general (see [Run2], [Run3]). Examples of dual Banach algebras (besides von Neumann algebras) include the measure algebra $M(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G. In particular, Runde showed that a locally compact group G is amenable if and only if its measure algebra $M(G)$ is Connes amenable (see [Run3]). This result sounds more interesting when compared to a deep result of H. Garth Dales, Fereidoun Ghahramani, and Alexander Ya. Helemskii [DGH] showing that $M(G)$ is amenable if and only if G is discrete and amenable. The idea of this article started with the simple observation that the measure algebra $M(G)$ is the multiplier algebra of the group algebra $L^1(G)$. It was natural to ask if, in general, amenability of a Banach algebra B (with bounded approximate identity) is related to Connes-amenability of its multiplier algebra $M(B)$. For this to make sense, we had to find suitable conditions under which $M(B)$ is a dual Banach space. These conditions were stated in a (not so well-known) article by E. O. Oshobi and John S. Pym [OP]. Although this is not explicitly stated in [OP], it is easy to see that under natural conditions on B set by the authors, the multiplier algebra $M(B)$ is a dual Banach algebra (see the paragraph after Theorem 1.0.1). In a sense, our article could be compared to that of Matthew Daws [Daw1] in which the relation between amenability of a regular Banach algebra B and Connes-amenability of its second conjugate algebra B^{**} (with Arens multiplication) is investigated. The main result of this article states that, under the natural conditions set by [OP], amenability of a Banach algebra B is equivalent to Connes-amenability of the multiplier algebra $M(B)$ as well as to the existence of a normal virtual diagonal. The proof is straightforward and gives some of the results of Runde [Run2], [Run3], and Daws [Daw1] as a special case. There is, however, a drawback. One of the conditions set by [OP] is a natural compatibility condition (see Assumption 2.1) that holds for $B = L^1(G)$ when G is compact. We were not able to prove (or disprove) that for a noncompact locally compact group G, $L^1(G)$ satisfies Assumption 2.1.

2. Preliminaries

Let B be a Banach algebra, and let E be a Banach B-bimodule. A *derivation* from B into E is a continuous linear map $D : B \to E$ such that $D(ab) = Da \cdot b + a \cdot Db$ for all $a, b \in B$. The space of all derivations of B into E is denoted by $Z^1(B,E)$. For example, for each $x \in E$, the map $a \mapsto a \cdot x - x \cdot a$ is a derivation; these maps form the space $N^1(B,E)$ of inner derivations. Let $H^1(B,E) = Z^1(B,E)/N^1(B,E)$ be the first cohomology group of B with coefficients in E. Then B is amenable if $H^1(B,E^*) = \{0\}$, for every Banach B-bimodule E (see [Joh2], [Joh3]), where E^* is the dual Banach B-bimodule whose actions are defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \qquad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle \quad (a \in B,\ x \in E,\ x^* \in E^*).$$

A Banach algebra B is called a *dual Banach algebra* if it is dual as a Banach B-bimodule. One can see that a Banach algebra that is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous. Examples of dual Banach algebras include all von Neumann algebras, the algebra $B(E) = (E \hat{\otimes} E^*)^*$ of all bounded operators on a reflexive Banach space E, the measure algebra $M(G) = C_0(G)^*$, and the second dual B^{**} of an Arens regular Banach algebra B. Let B be a Banach algebra. We use the notation $B \hat{\otimes} B$ and $L^2(B, \mathbb{C})$, respectively, to denote the projective tensor product of B with itself and the space of bounded, bilinear forms on $B \times B$. A dual Banach B-bimodule E is called *normal* if, for each $x \in E$, the maps $a \mapsto a \cdot x$ and $b \mapsto x \cdot b$ from B into E are w^*-continuous. A dual Banach algebra B is called *Connes-amenable* if for every normal dual Banach B-bimodule E, every w^*-continuous derivation $D : B \to E$ is inner (see [Run1]). If $\Delta_B : B \hat{\otimes} B \to B$ is the diagonal operator induced by $a \otimes b \mapsto ab$, $a, b \in B$, then since the multiplication in B is separately w^*-continuous, $\Delta_B^* B_* \subset L^2_{w^*}(B; \mathbb{C})$, where B_* is a closed submodule of B^* such that $B = B_*^*$, and $L^2_{w^*}(B; \mathbb{C})$ is the set of w^*-continuous bilinear maps from $B \hat{\otimes} B$ into $\mathbb{C}$. Taking the adjoint of $\Delta_B^*|_{B_*}$, we may thus extend Δ_B to $L^2_{w^*}(B; \mathbb{C})^*$ as a B-bimodule homomorphism Δ_{w^*}. An element $M \in L^2_{w^*}(B; \mathbb{C})^*$ is called a *normal, virtual diagonal* for B if

$$a \cdot M = M \cdot a, \qquad a\Delta_{w^*} M = a \quad (a \in B).$$

Recall that a *double multiplier* τ on an algebra B consists of a pair of mappings of B into itself denoted by $b \mapsto b\tau$ and $b \mapsto \tau b$ $(b \in B)$ such that $(a\tau)b = a(\tau b)$ for all $a, b \in B$. It is easy to see that every element of B gives rise to a double multiplier via left and right multiplication. The set $M(B)$ of all double multipliers on B is an algebra (under composition of maps) with identity $1_{M(B)}$. The algebra B is called *faithful* if the only element $b \in B$ such that $abc = 0$, for all $a, c \in B$, is $b = 0$. When B is faithful, the natural map from B into $M(B)$ is injective and B is a subalgebra of $M(B)$. Also, in this case, each double multiplier τ is a left and a right multiplier. A normed algebra B with a bounded approximate identity is faithful. The theory of double multipliers is due to B. E. Johnson [Joh1] (see also [Lar]). If B is complete and faithful, then left and right multipliers are continuous linear operators (see [Joh3]). For a double multiplier μ, the norms

$$\|\mu\|_{R(B)} = \sup\{\|a\mu\|_B : \|a\|_B \le 1\}, \qquad \|\mu\|_{L(B)} = \sup\{\|\mu b\|_B : \|b\|_B \le 1\}$$

are defined. We consider $M(B)$ with the following norm:

$$\|\mu\|_{M(B)} = \max\{\|\mu\|_{R(B)}, \|\mu\|_{L(B)}\}.$$

The inclusion map $B \to M(B)$ is norm decreasing, and if B has a bounded approximate identity $(e_\alpha)_\alpha$ with bound M, then we have

$$\|b\|_B = \lim_\alpha \|e_\alpha b\|_B \le M \|b\|_{M(B)},$$

so that the B-norm is equivalent on B to the $M(B)$-norm.

ASSUMPTION 2.1

There is a continuous linear injection $B^* \to M(B)$ such that, for $u, v \in B^*$ and $b \in B$,

$$\langle u, vb\rangle_B = \langle v, bu\rangle_B.$$

This assumption relates the action of elements of B^* as double multipliers to their role as linear functionals on B.

THEOREM 2.1 ([OP, THEOREM 3.4])

Let B be faithful, and let it satisfy Assumption 2.1. Then, there is a Banach algebra A such that $M(B)$ is isomorphic as a Banach space with the dual A^*, and

$$\{z \in A : zA = \{0\}\} = \{z \in A : Az = \{0\}\}$$

is a closed ideal Z of A. Also, $C = A/Z$ is a Banach algebra that is algebraically a subset of both B and B^*. Moreover, C is an ideal in $M(B)$, and C^* is a complete normed algebra that can be identified with an ideal of $M(B)$. Using the identification of $M(B)$ with A^*, the w^*-closure of B is C^*.

The basic idea in [OP] behind the construction of the Banach algebra A is quite natural. For each $a, b \in B$ and $u, v \in B^*$ the authors define two elements $a \circ u$ and $v \circ' b$ of $M(B)^*$ by

$$\langle a \circ u, \tau\rangle_{M(B)} = \langle u, \tau a\rangle_B \qquad \langle v \circ' b, \tau\rangle_{M(B)} = \langle v, b\tau\rangle_B \quad (\tau \in M(B)),$$

where the indices indicate on which space the corresponding functionals are acting. Both τa and $b\tau$ belong to B since τ is a multiplier. Then A is defined by

$$A = cls_{M(B)^*}\, \mathrm{span}\{a \circ u : a \in B,\ u \in B^*\}.$$

Next, one may observe that $\|a \circ u\|_A \leq \|a\|_B \|u\|_{B^*}$. Also, for each $u, v \in B^*$ and $a \in B$, $u \circ' av = ua \circ v$. Indeed, by [OP, Theorem 3.2], B^* is an ideal of $M(B)$ and $M(B) = M(B^*)$. Now for each $\tau \in M(B)$,

$$\langle u \circ' av, \tau\rangle_{M(B)} = \langle u, av\tau\rangle_B = \langle v\tau u, a\rangle_B = \langle v, \tau ua\rangle_B = \langle ua \circ v, \tau\rangle_{M(B)}.$$

Hence we have $B^*B \circ B^* = B^* \circ' BB^*$. Finally, since both BB^* and B^*B are norm dense in B and w^*-dense in B^* (see [OP, Lemma 3.3]), A is equal to the closure of $B^*B \circ B^* = B^* \circ' BB^*$ in $M(B)^*$. In particular, for each $b \in B$ and $v \in B^*$, we get $v \circ' b \in A$. By [OP, Theorem 3.8], we observe that multiplication in $M(B)$ is separately w^*-continuous in the w^*-topology. Hence for a faithful Banach algebra B satisfying Assumption 2.1, the multiplier algebra $M(B)$ is a dual Banach algebra. Our main theorem asserts that in this case, Connes-amenability of $M(B)$ is equivalent to amenability of B.

3. Connes-amenability of $M(B)$

Throughout section, B is a Banach algebra that satisfies Assumption 2.1 and A and C are as in Theorem 2.1. Note that by [OP, Corollary 5.2], if B has a bounded right approximate identity, then algebraically, $A = C$ and $M(A) = M(B) = A^*$. In this case, B is faithful and w^*-closure of B in $M(B)$ is equal to C^*; hence B is w^*-dense in $M(B)$. In particular, by [Run1, Proposition 4.2], amenability of B implies Connes-amenability of $M(B)$. In this section we show that the converse is also true. The proof of the next lemma is straightforward and is omitted.

LEMMA 3.1

If B is faithful, then for each $b \in B$, the maps $\tau \mapsto \tau b$ and $\tau \mapsto b\tau$ of $M(B)$ into B are w^*-w-continuous.

LEMMA 3.2

If the map $\sim: L^2_{w^*}(M(B);\mathbb{C}) \to L^2(B;\mathbb{C})$ is defined by $\tilde{\psi} := \psi|_{B\times B}$ for $\psi \in L^2_{w^*}(M(B),\mathbb{C})$, then

- (i) *$\sim$ is a continuous linear map,*
- (ii) $(\Delta^*_{M(B)}(a \circ u))^\sim = a \cdot \Delta^*_B u\, (a \in B, u \in B^*)$,
- (iii) $(\psi \cdot \tau)^\sim = \tilde{\psi} \cdot \tau; (\tau \cdot \psi)^\sim = \tau \cdot \tilde{\psi}$ *for each* $\tau \in M(B)$, $\psi \in L^2_{w^*}(M(B);\mathbb{C})$.

Proof

It is straightforward to show that $\sim$ is a continuous linear map. To show (ii), let $b, c \in B$, and let ϕ be the natural map from B into $M(B)$; then

$$\begin{aligned}\left\langle \left(\Delta^*_{M(B)}(a\circ u)\right)^\sim, (b,c)\right\rangle_{B\times B} &= \left\langle \Delta^*_{M(B)}(a\circ u), \left(\phi(b),\phi(c)\right)\right\rangle_{M(B)\times M(B)}\\ &= \left\langle a\circ u, \Delta_{M(B)}\left(\phi(b)\otimes\phi(c)\right)\right\rangle_{M(B)}\\ &= \langle u, \phi(bc)a\rangle_B.\end{aligned}$$

On the other hand, since $\phi(bc) = (L_{bc}, R_{bc})$,

$$\langle a\cdot\Delta^*_B u, (b,c)\rangle_{B\times B} = \langle \Delta^*_B u, (b, ca)\rangle_{B\otimes B} = \langle u, \Delta_B(b\otimes ca)\rangle_B = \langle u, \phi(bc)a\rangle_B.$$

To prove (iii), we first note that when ϕ is the canonical map from B into $M(B)$, then for each $\tau \in M(B)$ and $b \in B$, we have $\tau\phi(b) = \phi(\tau b)$. Now let $\tau \in M(B)$ and $\psi \in L^2_{w^*}(M(B);\mathbb{C})$, and let $b, c \in B$; then

$$\begin{aligned}\langle(\psi\cdot\tau)^\sim, (b,c)\rangle_{B\times B} &= \left\langle \psi\cdot\tau, \left(\phi(b),\phi(c)\right)\right\rangle_{M(B)\times M(B)}\\ &= \left\langle \psi, \left(\phi(\tau b),\phi(c)\right)\right\rangle_{M(B)\times M(B)}\\ &= \langle\tilde{\psi}\cdot\tau, (b,c)\rangle_{B\times B}.\end{aligned}$$

Thus $(\psi\cdot\tau)^\sim = \tilde{\psi}\cdot\tau$. The second equality is proved similarly. □

THEOREM 3.1

If B has a bounded approximate identity, then the following are equivalent.

(i) *B is amenable;*
(ii) *$M(B)$ is Connes amenable;*
(iii) *$M(B)$ has a normal, virtual diagonal.*

Proof
(ii) $\Rightarrow$ (i). Let $M(B)$ be Connes amenable, let E be a pseudounital Banach B-bimodule, and let $D: B \to E^*$ be a derivation. Since B is a closed ideal of $M(B)$ with a bounded approximate identity, there exists a unique extension of D to a derivation $\tilde{D}: M(B) \to E^*$. Then $\tilde{D}$ is continuous with respect to the strict topology on $M(B)$ and the w^*-topology on E^*. We claim that E^* is a normal, dual Banach $M(B)$-bimodule and that $\tilde{D}: M(B) \to E^*$ is w^*-w^*-continuous. Let $\tau_\alpha \to 0$ in $M(B)$ in the w^*-topology. For each $x \in E$, there exist $b \in B$ and $y \in E$ such that $x = b \cdot y$, and by Lemma 3.1, we have

$$\langle \phi \cdot \tau_\alpha, x\rangle_E = \langle \phi, \tau_\alpha \cdot by\rangle_E = \langle \phi_y, \tau_\alpha b\rangle_B \to 0,$$

where ϕ_y as an element of B^*, for each $b \in B$, is defined by $\phi_y(b) = b \cdot y$. Next, let $\tau_\alpha \to 0$ in $M(B)$ in the w^*-topology. For $x \in E$, let $y \in E$ and $b \in B$ be such that $x = b \cdot y$. Then

$$\begin{aligned}\langle \tilde{D}\tau_\alpha, x\rangle_E &= \langle \tilde{D}(\tau_\alpha b) - \tau_\alpha \cdot \tilde{D}b, y\rangle_E \\ &= \langle D(\tau_\alpha b), y\rangle_E - \langle \tau_\alpha \cdot Db, y\rangle_E \to 0.\end{aligned}$$

Now, since $M(B)$ is Connes amenable, $\tilde{D}$, and so D, is an inner derivation, and therefore B is amenable.

(i) $\Rightarrow$ (iii). Since B is amenable, it has a virtual diagonal $M \in (B \hat{\otimes} B)^{**} \cong L^2(B;\mathbb{C})^*$ such that $a \cdot M = M \cdot a$, $a \cdot \Delta_B^{**} M = a$ for $a \in B$. We define $\tilde{M}: L^2_{w^*}(M(B);\mathbb{C}) \to \mathbb{C}$ by $\langle \tilde{M}, \psi\rangle = \langle M, \tilde{\psi}\rangle$ for $\psi \in L^2_{w^*}(M(B);\mathbb{C}))$. Then $\tilde{M}$ is linear and $\|\tilde{M}\| \le \|M\|\|\sim\|$; hence $\tilde{M} \in L^2_{w^*}(M(B);\mathbb{C})^*$. To prove that $\tilde{M}$ is a normal, virtual diagonal for $M(B)$, it suffices to show that for each $\tau \in M(B)$, $\tilde{M} \cdot \tau = \tau \cdot \tilde{M}$ and $\tau\Delta_{w^*}\tilde{M} = \tau$. By the density of B in $M(B)$ in the strict topology, for each $\tau \in M(B)$ there exists a net $(a_\alpha)_\alpha \subseteq B$ such that $a_\alpha \to \tau$ in the strict topology. Since $L^2(B;\mathbb{C})^*$ is a pseudounital Banach B-bimodule, there exist $a \in B$ and $M' \in L^2(B;\mathbb{C})^*$ such that $M = a \cdot M'$. Thus $a_\alpha a \to \tau a$ in the norm topology, and $a_\alpha a \cdot M' \to \tau a \cdot M'$ in the w^*-topology. Next, let $b \in B$ and $M'' \in L^2(B;\mathbb{C})^*$ be such that $M = M'' \cdot b$; then since $M'' \cdot ba_\alpha \to M'' \cdot b\tau$ in the w^*-topology, we have $\tau \cdot M = M \cdot \tau$, from which, along with Lemma 3.2(iii), it follows that $\tau \cdot \tilde{M} = \tilde{M} \cdot \tau$. To complete the proof, we need to show that $\Delta_{w^*}\tilde{M}$ is $1_{M(B)}$, the unit of $M(B) = A^*$. Let $a \in B$ and $u \in B^*$; then by Lemma 3.2(ii),

$$\begin{aligned}\langle \Delta_{w^*}\tilde{M}, a \circ u\rangle_A &= \langle \tilde{M}, \Delta^*_{M(B)}(a \circ u)\rangle_{L^2_{w^*}(M(B);\mathbb{C})} \\ &= \langle M, a \cdot \Delta_B^* u\rangle_{L^2_{w^*}(B;\mathbb{C})} \\ &= \langle \Delta_B^{**}(M \cdot a), u\rangle_{B^*} \\ &= \langle u, 1_{M(B)} a\rangle_B \\ &= \langle 1_{M(B)}, a \circ u\rangle_A.\end{aligned}$$

(iii) $\Rightarrow$ (ii). This holds for any dual Banach algebra (see [Run1]). $\square$

4. Examples

EXAMPLE 4.1

Let G be a compact topological group; then $L^\infty(G) \subseteq L^1(G)$. $L^1(G)$ has bounded approximation identity. The multiplier algebra of $L^1(G)$ is equal to $M(G)$ (see [HR]). We show that $L^1(G)$ satisfies Assumption 2.1. Define $\theta : L^\infty(G) \to M(G)$ by $\psi \mapsto \psi\lambda$, where λ is the Haar measure on G. It is easy to check that θ is an injective continuous linear map (even homomorphism). Let $\psi, \varphi \in L^\infty(G)$ and $f \in L^1(G)$; then

$$\begin{aligned}
\langle \psi, \theta(\varphi) f\rangle_{L^1(G)} &= \int_G \psi(t^{-1})(\varphi\lambda * f)(t)\,dt \\
&= \int_G\int_G \psi(t^{-1})\varphi(s) f(s^{-1}t)\,ds\,dt \\
&= \int_G \varphi(s^{-1})(f * \psi\lambda)(s)\,ds \\
&= \langle \varphi, f\theta(\psi)\rangle_{L^1(G)}.
\end{aligned}$$

Since G is compact, $L^1(G)$ is amenable (see [Joh2]). So by Theorem 3.3, $M(G)$ is Connes amenable and has a normal virtual diagonal, which is known as [Run1, Proposition 5.2].

EXAMPLE 4.2

Let G be a discrete group, and let $A(G)$, $B(G)$, and $VN(G)$ be the Fourier algebra, Fourier-Stieltjes algebra, and the group von Neumann algebra of G (see [Eym]). Let us show that $A(G)$ satisfies Assumption 2.1. We know that $A(G)^* \simeq VN(G)$ (see [Eym, Theorem 3.10]). Let us show the pairing of $T \in VN(G)$ with $u \in A(G)$ with $\langle T, u\rangle$. There is a left action of $VN(G)$ on $A(G)$ defined by $\langle S, Tu\rangle = \langle \check{T}S, u\rangle$, where $\langle \check{T}, u\rangle = \langle T, \check{u}\rangle$ and $\check{u}(x) = u(x^{-1})$ for $x \in G, u \in A(G)$, and $S, T \in VN(G)$. Define $\theta : VN(G) \to M(A(G))$ by $\theta(T)(u) = (T\delta_e)u$, where e is the identity of G and $\delta_e = \delta_e * \delta_e \in A(G)$ (see [Eym, Proposition 3.4]). Note that since $A(G)$ is a commutative Banach algebra, here the left and right multiplier are the same. By [Eym, Proposition 3.17.1], we have

$$\|(T\delta_e)u\| \le \|T\delta_e\| \cdot \|u\| \le \|T\| \cdot \|\delta_e\| \cdot \|u\| \le \|T\| \cdot \|u\|$$

for each $u \in A(G)$ and $T \in VN(G)$; hence θ is continuous. Now each $T \in VN(G)$ could be written as $T = \sum_{x\in G} \alpha_x \rho(x)$, where $\alpha_x \in \mathbb{C}$ and $\rho : G \to B(\ell^2(G))$ is the right regular representation. The sum is convergent in strong operator topology. Since $\delta_e \in A(G) \cap \ell^2(G)$, we have $T\delta_e = T(\delta_e) \in A(G) \cap \ell^2(G)$ (see [Eym, Proposition 3.17.3]), where T on the right-hand side is considered as an operator on $\ell^2(G)$. In particular, if $T\delta_e = 0$, then

$$\sum_{x\in G} \alpha_x \rho(x)(\delta_e) = \sum_{x\in G} \alpha_x \delta_x = 0$$

in $\ell^2(G)$. Hence $T=0$ in $B(\ell^2(G))$ or, equivalently, in $VN(G)$. Therefore θ is also one-to-one. Finally, if $S,T\in VN(G)$ and $u\in A(G)$, we want to show that $\langle S,(T\delta_e)u\rangle=\langle T,(S\delta_e)u\rangle$. First, let us assume that $T=\rho(x)$ and $S=\rho(y)$ for some $x,y\in G$. Then

$$\big\langle\rho(y),\big(\rho(x)\delta_e\big)u\big\rangle=\langle\rho(y),\delta_x u\rangle=\delta_x(y)u(y)=\delta_y(x)u(x)=\big\langle\rho(y),\big(\rho(x)\delta_e\big)u\big\rangle.$$

Next, if $T=\sum_x\alpha_x\rho(x)$ and $S=\sum_y\beta_y\rho(y)$ are finite linear combinations, then by linearity, we get

$$\begin{aligned}\langle S,(T\delta_e)u\rangle&=\sum_{x,y}\alpha_x\beta_y\big\langle\rho(y),\big(\rho(x)\delta_e\big)u\big\rangle=\sum_{x,y}\alpha_x\beta_y\big\langle\rho(x),\big(\rho(y)\delta_e\big)u\big\rangle\\&=\langle T,(S\delta_e)u\rangle.\end{aligned}$$

Finally, taking limits in strong operator topology, we get the desired equality for arbitrary $S,T\in VN(G)$. It follows that when G is discrete and amenable, then Connes-amenability of $B(G)$ is equivalent to amenability of $A(G)$. But the latter is equivalent to G having an abelian subgroup of finite index (see [FR]). Hence for a discrete amenable group, $B(G)$ is Connes-amenable if and only if G is abelian by finite. This was proved in [Run4] through different methods. The same result is proved in [Uyg], without G a priori supposed to be amenable.

EXAMPLE 4.3

Let $\mathcal{X}$ be a reflexive Banach space with the approximation property. The norm closure of the set of finite-rank operators on $\mathcal{X}$ is denoted by $\mathcal{A}(\mathcal{X})$. Since $\mathcal{X}$ has the approximation property, $\mathcal{A}(\mathcal{X})$ is equal to $\mathcal{K}(\mathcal{X})$, the algebra of compact operators on $\mathcal{X}$, and has bounded approximate identity (see [DU, p. 242], [CLM, p. 86]). Let $B=\mathcal{A}(\mathcal{X})=\mathcal{X}^*\check{\otimes}\mathcal{X}$ (see [CLM, p. 63]), then $B^*=\mathcal{X}\hat{\otimes}\mathcal{X}^*=\mathcal{N}(\mathcal{X}^*)$, where $\mathcal{N}(\mathcal{X}^*)$ is the set of nuclear operators on $\mathcal{X}^*$. We use the canonical isomorphism $\mathcal{X}\hat{\otimes}\mathcal{X}^*\cong\mathcal{X}^*\hat{\otimes}\mathcal{X}$ to identify B^* with $\mathcal{N}(\mathcal{X})=\mathcal{X}^*\hat{\otimes}\mathcal{X}$; then the identity map on the algebraic tensor product $\mathcal{X}\otimes\mathcal{X}^*$ extends to a contraction $\theta: B^*\to B$. Let us observe that B satisfies Assumption 2.1. Each operator $u\in B^*=\mathcal{N}(\mathcal{X})$ acts as a linear functional on B via $\langle u,x\otimes x^*\rangle_B=\langle x^*,ux\rangle_{\mathcal{X}}$, $x\in\mathcal{X},x^*\in\mathcal{X}^*$. Let $x\in\mathcal{X}$ and $x^*\in\mathcal{X}^*$; then for $u\in B^*$, we have

$$\theta(u)(x\otimes x^*)=ux\otimes x^*,\qquad (x\otimes x^*)\theta(u)=x\otimes u^*x^*.$$

Hence for $u,v\in B^*$, $x\in\mathcal{X}$, and $x^*\in\mathcal{X}^*$,

$$\begin{aligned}\langle u,\theta(v)(x\otimes x^*)\rangle_B&=\langle u,vx\otimes x^*\rangle_B=\langle x^*,uvx\rangle_{\mathcal{X}}\\&=\langle u^*x^*,vx\rangle_{\mathcal{X}}=\langle v,x\otimes u^*x^*\rangle_B\\&=\langle v,(x\otimes x^*)\theta(u)\rangle_B.\end{aligned}$$

From Theorem 3.3, it thus follows that the following are equivalent: $\mathcal{A}(\mathcal{X})$ is amenable, $\mathcal{B}(\mathcal{X})$ is Connes amenable, and $\mathcal{B}(\mathcal{X})$ has a normal virtual diagonal. This was shown in [Run4] by using a different method. Examples of Banach spaces $\mathcal{X}$ for which $\mathcal{A}(\mathcal{X})$ is amenable can be found in [GJW]. In particular, this happens when $\mathcal{X}$ has property $(\mathbb{A})$ (see [GJW, Theorem 4.2]).

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Hayati: Department of Mathematics, Malayer University, Malayer, Hamedan, Iran; hayati@malayeru.ac.ir

Amini: Department of Mathematics, Tarbiat Modares University, Tehran 14115-175, Iran; mamini@modares.ac.ir; and Institut Penyelidikan Matematik Universiti Putra Malaysia, UPM 43400 Serdang, Selangor Darul Ehsan, Malaysia; massoud@putra.upm.edu.my