Continued fractional measure of irrationality

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Abstract   This new concept of continued fractional measure of irrationality for the real number a is introduced with the help of the classical measure of irrationality. Some relationships between this new and the classical measures are included.

1. Introduction

Following Erdős [4], we call $C(a) = \inf\{I(y); y = [a_1c_1, a_2c_2, \ldots], c_n \in \mathbb{Z}^+\}$ the continued fractional measure of irrationality for the number a while $a = [a_1, a_2, \ldots]$ is its continued fractional expansion and $I(x)$ is the measure of irrationality of the number x. In some sense the continued fractional measure of irrationality better characterizes the nature of the number a, mainly in the direction of the approximation of its partial continued fractions in average. We prove the following theorem.

THEOREM 1.1
Let $K \geq 2$ and $a_1 \geq 1$ be integers, and define a continued fraction $a = [a_1, a_2, \ldots]$ with $a_{n+1} = a_n^K + n!$ for each $n = 1, 2, \ldots$. Then $C(a) = I(a) = K + 1$.

In the same spirit, Erdős [4] defined the irrational sequences and proved that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational (see also [9]). Later Hančl, Nair, and Šustek [7] defined in a similar way the expressible set of the sequence. More information about this can be found in [8], [10], [11], and [12]. Davenport and Roth [3] proved that if $\limsup_{n\to\infty}(\sqrt{\log n}/n)\log\log q_n = \infty$ and $a_n \in \mathbb{Z}^+$ for every $n \in \mathbb{Z}^+$, then the number a is transcendental (see also [1]). Matala-aho and Merilä [16] found some measures of irrationality for the Ramanujan-type q-continued fractions. As an application of their work, let us mention the results concerning the Ramanujan-Selberg continued fractions (see [18]) and the Eisenstein continued fractions (see [5]). Certain hypergeometric functions in connection with the measure of irrationality and continued fractions were studied in Shiokawa

[17]. Further, Komatsu [15] published some results concerning the Hurwitz and Tasoev's continued fractions. By using the monodromy principle for hypergeometric functions, Huttner and Matala-aho [14] obtained measures of irrationality for certain Gauss continued fractions (see also Hata and Huttner [13]), while Bundschuh [2] worked with the special continued fractions containing a finite number of arithmetic progressions and found some estimations of the measures of irrationality for them. Throughout the whole article we consider $a = [a_1, a_2, \ldots]$ to be the continued fraction expansion such that $a_n \in \mathbb{Z}^+$ for each $n \in \mathbb{Z}^+$.

The nth partial fraction is equal to $p_n/q_n = [a_1, a_2, \ldots, a_n]$. The continued fraction expansion of the number a is infinite, so a is an irrational number. The measure of irrationality of the number a we define as

$$I(a) = -\liminf_{n \to \infty} \log_{q_n} |a - (p_n/q_n)|$$

since we know that the best approximations are directly in its partial fractions. We also use the well-known inequality for the approximation of the nth partial fraction

$$\frac{1}{q_n^2(a_{n+1}+2)} < \left|a - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2 a_{n+1}},$$

which follows, for example, from Hardy and Wright [6, (10.7.5)]. The notation $[x]$ means the integral part of the real number x. Denote $\mathbb{Z}^+$ to be the set of all positive integers. For convenience, set $\log_2 0 = 0$.

2. Main results

THEOREM 2.1
We have

$$C(a) = 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1.$$

COROLLARY 2.1
We have

$$I(a) \geq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1.$$

COROLLARY 2.2
Let $\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n = \infty$. *Then* $C(a) = I(a) = \infty$, *and thus* a *is a Liouville number.*

THEOREM 2.2
Let K *be a real number with* $K > 1$. *Assume that*

$$(2.1) \qquad 1 < R_1 = \liminf_{n\to\infty} a_n^{1/K^n} \leq \limsup_{n\to\infty} a_n^{1/K^n} = R_2,$$

where R_1 *and* R_2 *are real numbers;* R_2 *can also be infinity. Then*

$$(2.2) \qquad K + 1 \leq I(a) \leq \frac{\log_2 R_2}{\log_2 R_1}(K-1) + 2.$$

EXAMPLE 2.1

Let R_1, R_2, and K be the real numbers with $K > 1$ and $0 < R_1 \leq R_2$. Set

$$a_n = [2^{(2R_1|\sin(\log\log n)|+2R_2(1-|\sin(\log\log n)|))K^n}]$$

for each $n \in \mathbb{Z}^+$. Then $\liminf_{n\to\infty} a_n^{1/K^n} = 2^{2R_1}$, $\limsup_{n\to\infty} a_n^{1/K^n} = 2^{2R_2}$, and $I(a) = K + 1$.

EXAMPLE 2.2

Let R_1, R_2, and K be the real numbers with $K > 1$ and $0 < R_1 \leq R_2$. Set

$$a_n = 2^{(R_1(1+(-1)^{[\log\log n]})+R_2(1+(-1)^{1+[\log\log n]}))K^n}$$

for each $n \in \mathbb{Z}^+$. Then $\liminf_{n\to\infty} a_n^{1/K^n} = 2^{2R_1}$, $\limsup_{n\to\infty} a_n^{1/K^n} = 2^{2R_2}$, and $I(a) = (R_2/R_1)(K-1) + 2$.

REMARK 1

Examples 2.1 and 2.2 demonstrate in some sense that we cannot substantially improve upper and lower bounds for the measure of irrationality in Theorem 2.2.

COROLLARY 2.3

Let K be a real number with $K > 1$. Assume that $1 < \lim_{n\to\infty} a_n^{1/K^n} < \infty$. Then $I(a) = K + 1$ and a is a transcendental number.

EXAMPLE 2.3

Let K be a real number such that $K > 1$. Set $a_n = [2^{K^n}]$ for each $n \in \mathbb{Z}^+$. Then $I(a) = K + 1$.

3. Proofs

Theorem 1.1 is the immediate consequence of Theorem 2.1 and Corollary 2.3 as follows. First, we have

$$a_{n+1}^{1/K^{n+1}} = (a_n^K + n!)^{1/K^{n+1}} = a_n^{1/K^n}\left(1 + \frac{n!}{a_n^K}\right)^{1/K^{n+1}}$$

$$= a_1^{1/K}\prod_{k=1}^{n}\left(1 + \frac{k!}{a_k^K}\right)^{1/K^{n+1}}$$

which implies that

$$1 \leq a_1^{1/K} < a_2^{1/K^2} < \cdots < a_n^{1/K^n} < a_{n+1}^{1/K^{n+1}} < \cdots < a_1^{1/K}2^{1/K^{n+1}}.$$

Hence $1 < \lim_{n\to\infty} a_n^{1/K^n} < \infty$.

Proof of Theorem 2.1

The proof falls into two cases.

   (1) First we prove that $C(a) \geq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$. Suppose that there exists the sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $I(y) = I([a_1c_1,$

$a_2 c_2, \ldots]) < 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$. Set $A_n = a_n c_n$ for each $n \in \mathbb{Z}^+$. So there exist $Q \geq 2$ and sufficiently small δ_1 such that

$$I(y) = I([A_1, A_2, \ldots]) = Q < Q + 4\delta_1 \leq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$$

$$(3.1) \qquad\qquad\qquad\qquad \leq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 A_n} + 1.$$

From this we obtain

$$(3.2) \qquad\qquad\qquad \limsup_{n\to\infty} A_n^{1/(Q-1+3\delta_1)^n} = \infty.$$

This implies that for infinitely many N,

$$(3.3) \qquad\qquad A_{N+1}^{1/(Q-1+3\delta_1)^{N+1}} > \sup_{j=1,2,\ldots,N} A_j^{1/(Q-1+3\delta_1)^j}.$$

Otherwise, there exists n_0 such that for every $n > n_0$,

$$A_{n+1}^{1/(Q-1+3\delta_1)^{n+1}} \leq \sup_{j=1,2,\ldots,n} A_j^{1/(Q-1+3\delta_1)^j} = \sup_{j=1,2,\ldots,n-1} A_j^{1/(Q-1+3\delta_1)^j}$$

$$= \cdots = \sup_{j=1,2,\ldots,n_0} A_j^{1/(Q-1+3\delta_1)^j},$$

a contradiction with (3.2). Now from (3.3) we obtain that for infinitely many N,

$$A_{N+1} > \left(\sup_{j=1,2,\ldots,N} A_j^{1/(Q-1+3\delta_1)^j} \right)^{(Q-1+3\delta_1)^{N+1}}$$

$$\geq \left(\sup_{j=1,2,\ldots,N} A_j^{1/(Q-1+3\delta_1)^j} \right)^{(Q-2+3\delta_1)((Q-1+3\delta_1)^N+(Q-1+3\delta_1)^{N-1}+\cdots+1)}$$

$$(3.4)$$

$$\geq \left(\prod_{k=1}^{N} \left(\sup_{j=1,2,\ldots,N} A_j^{\frac{(Q-1+3\delta_1)^k}{(Q-1+3\delta_1)^j}} \right) \right)^{(Q-2+3\delta_1)}$$

$$\geq D_0 \left(\prod_{k=1}^{N} A_k \right)^{(Q-2+3\delta_1)},$$

where D_0 is a positive real number that does not depend on n. Let $[A_1, A_2, \ldots, A_k] = P_k/Q_k$ be the kth partial fraction of the number $A = [A_1, A_2, \ldots]$. This and (3.4) yield that for infinitely many N,

$$\left| A - \frac{P_N}{Q_N} \right| \leq \frac{1}{Q_N^2 A_{N+1}} \leq \frac{1}{Q_N^2 D_0 (\prod_{k=1}^{N} A_k)^{(Q-2+3\delta_1)}} \leq \frac{1}{Q_N^{Q+2\delta_1}}.$$

But this is the contradiction with (3.1), and $C(a) \geq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$ follows.

(2) Now we prove that $C(a) \leq 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$. To prove this, we find for every sufficiently small positive real number δ_2 the sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $I(y) = I([a_1 c_1, a_2 c_2, \ldots]) < 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1 + 2\delta_2$. Set $S = 2^{\limsup_{n\to\infty}(1/n)\log_2\log_2 a_n} + 1$. From this we obtain that there exists n_0 such that for each $n > n_0$ we have

$$a_n < 2^{(S-1+\delta_2)^n}.$$

Now we take the sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $c_1 = c_2 = \cdots = c_{n_0} = 1$ and for every $n > n_0$

(3.5) $$2^{(S-1+\delta_2)^n} \leq A_n = a_n c_n \leq 2^{(S-1+\delta_2)^n+1}$$

where we set $A_n = a_n c_n$ for each $n \in \mathbb{Z}^+$. From (3.5) we obtain that there exists a positive real number D_1 which does not depend on n and such that for every positive integer n

$$D_1 \prod_{k=1}^{n} 2^{(S-1+\delta_2)^k} \leq \prod_{k=1}^{n} A_k.$$

Hence

(3.6) $$D_2 2^{(S-1+\delta_2)^{n+1}/(S-2+\delta_2)} \leq \prod_{k=1}^{n} A_k$$

where D_2 is a suitable positive real constant which does not depend on n. Let $[A_1, A_2, \ldots, A_k] = \frac{P_k}{Q_k}$ be the kth partial fraction of the number $A = [A_1, A_2, \ldots]$. Inequalities (3.5) and (3.6) yield that for every sufficiently large positive integer n,

$$\left| A - \frac{P_n}{Q_n} \right| \geq \frac{1}{Q_n^2(A_{n+1}+2)} \geq \frac{1}{Q_n^2 8\left((1/D_2)\prod_{k=1}^{n} A_k\right)^{(S-2+\delta_2)}} \geq \frac{1}{Q_n^{S+2\delta_2}}.$$

From this and the fact that partial continued fractions are the best approximations we obtain that $I(A) \leq S + 2\delta_2$. □

Proof of Theorem 2.2

From (2.1) we obtain that for every sufficiently small δ_3 there exists n_0 such that for each $n > n_0$ we have

$$1 < R_1 - \delta_3 \leq a_n^{1/K^n} \leq R_2 + \delta_3.$$

Hence

(3.7) $$2^{K^n \log_2(R_1-\delta_3)} \leq a_n \leq 2^{K^n \log_2(R_2+\delta_3)}.$$

It implies that there exists a positive real number D_3 such that for all sufficiently large positive integers n we have

(3.8) $$D_3 2^{\log(R_1-\delta_3)/(K-1)K^{n+1}} \leq \prod_{k=1}^{n} a_k.$$

Now the proof falls into two cases.

(1) First we prove that $I(a) \geq K + 1$. From (3.7) we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log_2 \log_2 a_n \geq \log_2 K.$$

Then this and Corollary 2.1 imply that $I(a) \geq K + 1$.

(2) Now we prove that $I(a) \leq (\log_2 R_2)/(\log_2 R_1)(K - 1) + 2$. To prove this we estimate the partial continued fractions of the number a. By the way, we assume that $R_2 < \infty$ since the case $R_2 = \infty$ is trivial. From (3.7) and (3.8) we

obtain that for every sufficiently large positive integer n we have

$$\left| a - \frac{p_n}{q_n} \right| \geq \frac{1}{q_n^2(a_{n+1}+2)} \geq \frac{1}{q_n^2 4\left((1/D_3)\prod_{k=1}^n a_k\right)^{((K-1)\log_2(R_2+\delta_3))/\log_2(R_1-\delta_3)}}$$

$$\geq \frac{1}{q_n^{(((K-1)\log_2(R_2+2\delta_3))/\log_2(R_1-\delta_3))+2}}.$$

This and the fact that partial continued fractions are the best approximations yield $I(a) \leq \left(((K-1)\log_2(R_2+2\delta_3))/\log_2(R_1-\delta_3)\right) + 2$. But this holds for all sufficiently small δ_3. Thus $I(a) \leq (\log_2 R_2/\log_2 R_1)(K-1) + 2$ and (2.2) follows.

<div style="text-align:right">□</div>

Corollaries 2.1 and 2.2 are immediate consequences of Theorem 2.1. Corollary 2.3 is an immediate consequence of Theorem 2.2. Example 2.3 is an immediate consequence of Corollary 2.3.

Proof of Example 2.1

(1) First, we prove that $I(a) \geq K+1$. From the fact that the sequence $\{|\sin(\log_2\log_2 k)|\}_{k=1}^\infty$ is dense in $[0,1]$, we obtain $\liminf_{n\to\infty} a_n^{1/K^n} = 2^{2R_1}$ and $\limsup_{n\to\infty} a_n^{1/K^n} = 2^{2R_2}$. This and Theorem 2.2 yield $I(a) \geq K+1$.

(2) Let ε be sufficiently small, and let n be sufficiently large. From the mean value theorem we obtain that for every $k, j \in \{[n/2], \ldots, n, n+1\}$,

(3.9)
$$\left| |\sin(\log_2\log_2 k)| - |\sin(\log_2\log_2 j)| \right|$$
$$\leq |\sin(\log_2\log_2 k) - \sin(\log_2\log_2 j)|$$
$$= \left| (k-j)\cos(\log_2\log_2 \zeta)\frac{1}{\zeta\log_2 \zeta} \right| \leq \frac{3}{\log_2 n},$$

where $\zeta \in [[n/2], n+1]$. The definition of the sequence $\{a_k\}_{k=1}^\infty$ and (3.9) yield that for every $k \in \{[n/2], \ldots, n, n+1\}$

$$2^{(2R_2+(2R_1-2R_2)(|\sin(\log_2\log_2 n)|+(3/\log_2 n)))K^k}$$

$$\leq a_k$$
$$\leq 2^{(2R_2+(2R_1-2R_2)(|\sin(\log_2\log_2 n)|-(3/\log_2 n)))K^k};$$

hence

(3.10)
$$S_n^{(1-\varepsilon)K^k} \leq a_k \leq S_n^{(1+\varepsilon)K^k},$$

where

$$S_n = 2^{2R_2+(2R_1-2R_2)|\sin(\log_2\log_2 n)|}.$$

Now we prove that $I(a) \leq K+1$. To prove this we find the lower bound for $\prod_{k=1}^n a_k$. Inequality (3.10) implies that

$$\prod_{k=1}^n a_k \geq \prod_{k=[n/2]}^n a_k \geq \prod_{k=[n/2]}^n S_n^{(1-\varepsilon)K^k} \geq S_n^{((1-2\varepsilon)/(K-1))K^{n+1}}.$$

This and (3.10) yield

$$\left| a - \frac{p_n}{q_n} \right| \geq \frac{1}{q_n^2(a_{n+1}+2)} \geq \frac{1}{q_n^2 4\left(\prod_{k=1}^n a_k\right)^{((K-1)(1+\varepsilon))/(1-2\varepsilon)}}$$

$$\geq \frac{1}{q_n^{((K-1)(1+2\varepsilon))/(1-2\varepsilon)+2}}.$$ □

Proof of Example 2.2

(1) We have either $a_k = 2^{2R_1 K^k}$ or $a_k = 2^{2R_2 K^k}$. From this we obtain $\liminf_{n\to\infty} a_n^{1/K^n} = 2^{2R_1}$ and $\limsup_{n\to\infty} a_n^{1/K^n} = 2^{2R_2}$. This and Theorem 2.2 yield $I(a) \leq R_2/R_1(K-1) + 2$.

(2) Now we prove that $I(a) \geq (R_2/R_1)(K-1) + 2$. Let s be a sufficiently large positive integer and ε let be a sufficiently small positive real number. Set $n + 1 = 2^{2^{2s+1}}$. Then $a_{n+1} = 2^{2R_2 K^{n+1}}$ and $a_k = 2^{2R_1 K^k}$ for all $k = (n+1)/2, \ldots, n$. From this we obtain

$$\prod_{k=1}^n a_k \leq \prod_{k=1}^{(n-1)/2} 2^{2R_2 K^k} \prod_{k=(n+1)/2}^n 2^{2R_1 K^k} \leq 2^{((2R_1(1+\varepsilon))/(K-1))K^{n+1}}.$$

This yields

$$\left| a - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2 a_{n+1}} \leq \frac{1}{q_n^2 \left(\prod_{k=1}^n a_k\right)^{(R_2(K-1))/(R_1(1+\varepsilon))}} \leq \frac{1}{q_n^{2+(R_2(K-1))/(R_1(1+2\varepsilon))}}.$$

This implies that $I(a) \geq R_2/R_1(K-1) + 2$. □

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