

Toeplitz CAR flows and type I factorizations

Masaki Izumi and R. Srinivasan

Abstract Toeplitz CAR flows are a class of E_0 -semigroups including the first type III example constructed by R. T. Powers. We show that the Toeplitz CAR flows contain uncountably many mutually non-cocycle-conjugate E_0 -semigroups of type III. We also generalize the type III criterion for Toeplitz canonical anticommutation relation (CAR) flows employed by Powers (and later refined by W. Arveson), and show that Toeplitz CAR flows are always either of type I or type III.

1. Introduction

E. Wigner's famous theorem establishes that any one-parameter group of automorphisms $\{\alpha_t : t \in \mathbb{R}\}$ on $B(H)$, the algebra of all bounded operators on a separable Hilbert space H , is described by a strongly continuous one-parameter unitary group $\{U_t\}$, through the relation

$$\alpha_t(X) = \text{Ad}(U_t)(X) = U_t X U_t^*, \quad \forall X \in B(H).$$

An analogous statement of Wigner's theorem for an E_0 -semigroup, a continuous semigroup of unit-preserving endomorphisms of $B(H)$, is that the semigroup is completely determined by the set of all intertwining semigroups of isometries. That is, the E_0 -semigroup $\{\alpha_t : t \in (0, \infty)\}$ is completely described, up to cocycle conjugacy, by the set of all C_0 -semigroups of isometries $\{U_t\}$, satisfying

$$\alpha_t(X)U_t = U_t X, \quad \forall X \in B(H).$$

A subclass of E_0 -semigroups, where this analogy is indeed true, are called type I E_0 -semigroups. But due to the existence of type II and type III E_0 -semigroups in abundance, it is well known by now that such an analogy does not hold for E_0 -semigroups in general.

In [12], Powers raised the question whether such an intertwining semigroup of isometries always exists for any given E_0 -semigroup. He answered this question (see [11]) in the negative by constructing an E_0 -semigroup without any intertwining semigroup of isometries. This is the first example of what is called a

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type III E_0 -semigroup. For quite some time this was the only known example of a type III E_0 -semigroup, even though it was conjectured that there are uncountably many type III E_0 -semigroups, which are mutually non-cocycle-conjugate. In 2000, B. Tsirelson (see [17]) constructed a one-parameter family of nonisomorphic product systems of type III. Using previous results of Arveson [3], this leads to the existence of uncountably many E_0 -semigroups of type III, which are mutually non-cocycle-conjugate. Since then there has been a flurry of activity along this direction (see [5], [7], [9]).

In this article, we turn our attention to the first example of a type III E_0 -semigroup produced by Powers, which can be constructed on the type I factor obtained through the Gelfand-Naimark-Segal (GNS) construction of the CAR algebra corresponding to a (nonvacuum) quasi-free state. Although his purpose in [11] is to construct a single type III example, his construction is rather general, and it could produce several E_0 -semigroups by varying the associated quasi-free states. However, it is not at all clear whether they contain more than one cocycle conjugacy class of type III E_0 -semigroups. As is emphasized in Arveson's book (see [4, Chapter 13]), the 2-point function of Powers's quasi-free state is given by a Toeplitz operator whose symbol is a matrix-valued function with a very subtle property. Arveson clarified the role of the Toeplitz operator in Powers's construction and gave the most general form of the symbols for which the same construction works. We refer to the E_0 -semigroups obtained in this way as the *Toeplitz CAR flows*. Arveson also made a refinement of a sufficient condition obtained by Powers for the Toeplitz CAR flows to be of type III.

One of our main purposes in this article is to show that there exist uncountably many cocycle conjugacy classes of type III Toeplitz CAR flows. More precisely, we explicitly give a one-parameter family of symbols, including that of Powers, that give rise to mutually non-cocycle-conjugate type III examples. We also generalize Powers and Arveson's type III criterion mentioned above and give a necessary and sufficient condition in full generality, which solves Arveson's problem raised in [4, p. 417]. In particular, our result says that Toeplitz CAR flows are always either of type I or of type III, which is a CAR version of the same result obtained in [5] (see also [8], [9]) for product systems arising from sum systems or, equivalently, generalized canonical commutation relation (CCR) flows.

As in our previous work [9], we employ the local von Neumann algebras of an E_0 -semigroup as a classification invariant. In [9], we computed the type of the von Neumann algebras corresponding to bounded open subsets of $(0, \infty)$ for a class of generalized CCR flows. The key fact in our previous computation is that the von Neumann algebras in question always arise from quasi-free representations of the Weyl algebra. Since an analogous statement does not seem to be true in the case of Toeplitz CAR flows (even if the usual twisting operation in the duality for the CAR algebra is taken into account), we have to take an alternative approach. For this reason, we use the notion of a type I factorization, introduced by H. Araki and J. Woods [2], consisting of the local von Neumann algebras corresponding to

a countable partition of a finite interval. For each such fixed partition, whether the associated type I factorization is a complete atomic Boolean algebra of type I factors or not, is a cocycle conjugacy invariant of type III E_0 -semigroups.

2. Preliminaries

We use the following notation throughout the article.

For a family of von Neumann algebras $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ acting on the same Hilbert space H , we denote by $\bigvee_{\lambda \in \Lambda} \mathcal{M}_\lambda$ the von Neumann algebra generated by their union $\bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda$. We always denote by 1 either the identity element in a C^* -algebra or the identity operator on a Hilbert space. When we need to specify the C^* -algebra \mathfrak{A} or the Hilbert space H , we use the symbols $1_{\mathfrak{A}}$ or 1_H , respectively.

For a bounded positive operator A on a Hilbert space H , we denote by $\text{tr}(A)$ the usual trace of A , which could be infinite. For $X \in B(H)$, we denote its Hilbert-Schmidt norm by $\|X\|_{\text{HS}} = \text{tr}(X^*X)^{1/2}$.

For a tempered distribution f on \mathbb{R} , we denote by \hat{f} the Fourier transform of f with normalization

$$\hat{f}(p) = \int_{\mathbb{R}} f(x) e^{-ipx} dx, \quad f \in L^1(\mathbb{R}).$$

For an open set $O \subset \mathbb{R}$, we denote by $\mathcal{D}(O)$ the set of smooth functions on O with compact support. For a measurable set $E \subset \mathbb{R}$, we denote by $|E|$ and χ_E its Lebesgue measure and its characteristic function, respectively.

2.1. E_0 -semigroups and product systems

We briefly recall the basics of E_0 -semigroups and product systems. The reader is referred to Arveson's monograph [4] for details.

DEFINITION 2.1

Let H be a separable Hilbert space. A family of unital $*$ -endomorphisms $\alpha = \{\alpha_t\}_{t \geq 0}$ of $B(H)$ is an E_0 -semigroup if

- (i) the semigroup relation $\alpha_s \circ \alpha_t = \alpha_{s+t}$ holds for all $s, t \in (0, \infty)$ and $\alpha_0 = \text{id}$;
- (ii) the map $t \mapsto \langle \alpha_t(X)\xi, \eta \rangle$ is continuous for every fixed $X \in B(H)$, $\xi, \eta \in H$.

For an E_0 -semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ and positive t , we set

$$\mathcal{E}_\alpha(t) = \{T \in B(H); \alpha_t(X)T = TX, \forall X \in B(H)\},$$

which is a Hilbert space with the inner product $\langle T, S \rangle_{1_H} = S^*T$. The system of Hilbert spaces $\mathcal{E}_\alpha = \{\mathcal{E}_\alpha(t)\}_{t > 0}$ satisfies the following axioms of a product system.

DEFINITION 2.2

A product system of Hilbert spaces is a one-parameter family of separable complex Hilbert spaces $E = \{E(t)\}_{t > 0}$, together with unitary operators

$$U_{s,t} : E(s) \otimes E(t) \rightarrow E(s+t) \quad \text{for } s, t \in (0, \infty),$$

satisfying the following two axioms of associativity and measurability.

(i) (Associativity) For any $s_1, s_2, s_3 \in (0, \infty)$,

$$U_{s_1, s_2 + s_3}(1_{E(s_1)} \otimes U_{s_2, s_3}) = U_{s_1 + s_2, s_3}(U_{s_1, s_2} \otimes 1_{E(s_3)}).$$

(ii) (Measurability) There exists a countable set E^0 of sections

$$(0, \infty) \ni t \rightarrow h_t \in E(t)$$

such that $t \mapsto \langle h_t, h'_t \rangle$ is measurable for any two $h, h' \in E^0$, and the set $\{h_t; h \in E^0\}$ is total in $E(t)$ for each $t \in (0, \infty)$. Further, it is also assumed that the map $(s, t) \mapsto \langle U_{s,t}(h_s \otimes h_t), h'_{s+t} \rangle$ is measurable for any two $h, h' \in E^0$.

Two product systems $(\{E(t)\}, \{U_{s,t}\})$ and $(\{E'(t)\}, \{U'_{s,t}\})$ are said to be isomorphic if there exists a unitary operator $V_t : E(t) \rightarrow E'(t)$ for each $t \in (0, \infty)$ satisfying

$$V_{s+t}U_{s,t} = U'_{s,t}(V_s \otimes V_t).$$

Arveson showed that every product system is isomorphic to a product system arising from an E_0 -semigroup and that two E_0 -semigroups α and β are cocycle conjugate if and only if the corresponding product systems \mathcal{E}_α and \mathcal{E}_β are isomorphic.

For a fixed positive number a and for $0 \leq s \leq t \leq a$, we define the local von Neumann algebra $\mathcal{A}_a^E(s, t) \subset B(E(a))$ for the interval (s, t) by

$$\mathcal{A}_a^E(s, t) = U_{s, t-s, a-t}(\mathbb{C}1_{E(s)} \otimes B(E(t-s)) \otimes \mathbb{C}1_{E(a-t)})U_{s, t-s, a-t}^*,$$

where $U_{s, t-s, a-t} = U_{t, a-t}(U_{s, t-s} \otimes 1_{E(a-t)}) = U_{s, a-s}(1_{E_s} \otimes U_{t-s, a-t})$. For any open subset $O \subset [0, a]$, we set $\mathcal{A}_a^E(O) = \bigvee_{I \subset O} \mathcal{A}_a^E(I)$, where I runs over all intervals contained in O . When $a = 1$, we simply write $\mathcal{A}^E(s, t)$ for $\mathcal{A}_a^E(s, t)$. When $E = \mathcal{E}_\alpha$, we often identify $B(E(a))$ with $B(H) \cap \alpha_a(B(H))'$. When we need to distinguish them, we denote by σ the isomorphism from $B(H) \cap \alpha_a(B(H))'$ onto $\mathcal{A}_a^E(0, a)$ given by the left multiplication. By this identification, the inclusion $\mathcal{A}_a^E(s, t) \subset B(E(a))$ is identified with

$$\alpha_s(B(H) \cap \alpha_{t-s}(B(H))') \subset B(H) \cap \alpha_a(B(H))'.$$

In what follows, we often omit $U_{s,t}$ and simply write xy instead of $U_{s,t}(x \otimes y)$ if there is no possibility of confusion.

DEFINITION 2.3

A unit for a product system E is a nonzero section

$$u = \{u_t \in E_t; t > 0\},$$

such that the map $t \mapsto \langle u_t, h_t \rangle$ is measurable for any $h \in E^0$ and

$$u_s u_t = u_{s+t}, \quad \forall s, t \in (0, \infty).$$

In order to avoid possible confusion, we refer to the condition $\|x\| = 1$ for a vector $x \in E(t)$ as *normalized* instead of *unit* throughout the paper. An intertwining

C_0 -semigroup of isometries of an E_0 -semigroup α is naturally identified with a normalized unit for \mathcal{E}_α .

A product system (E_0 -semigroup) is said to be of type I if units exist for the product system and they generate the product system; that is, for any fixed $t \in (0, \infty)$, the set

$$\left\{ u_{t_1}^1 u_{t_2}^2 \cdots u_{t_n}^n; \sum_{i=1}^n t_i = t, u^i \in \mathcal{U}_E \right\}$$

is a total set in E_t , where \mathcal{U}_E is the set of all units. It is of type II if units exist but do not generate the product system. An E_0 -semigroup is said to be spatial if it is either of type I or type II. We say that a product system is of type III, or unitless, if no unit exists.

Type I product systems are further classified into type I_n , $n = 1, 2, \dots, \infty$, according to their indices n . There exists only one isomorphism class of type I_n product systems.

We recall V. Liebscher's useful criterion [10, Corollary 7.7] for isomorphic product systems in terms of the local von Neumann algebras.

THEOREM 2.4 (LIEBSCHER [10, COROLLARY 7.7])

Let E and F be product systems. If there is an isomorphism ρ from $B(E(1))$ onto $B(F(1))$ such that $\rho(\mathcal{A}^E(0, t)) = \mathcal{A}^F(0, t)$ for t in a dense subset of $(0, 1)$, then E and F are isomorphic.

2.2. Type I factorizations

In this subsection, we introduce a new classification invariant for type III product systems using the notion of type I factorizations introduced by Araki and Woods [2]. Throughout this subsection, every index set is assumed to be countable, and every Hilbert space is assumed to be separable.

DEFINITION 2.5

Let H be a Hilbert space. We say that a family of type I subfactors $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ of $B(H)$ is a *type I factorization* of $B(H)$ if

- (i) $\mathcal{M}_\lambda \subset \mathcal{M}'_\mu$ for any $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
- (ii) $B(H) = \bigvee_{\lambda \in \Lambda} \mathcal{M}_\lambda$.

We say that a type I factorization $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ is a *complete atomic Boolean algebra of type I factors* (abbreviated as CABATIF) if for any subset $\Gamma \subset \Lambda$, the von Neumann algebra $\bigvee_{\lambda \in \Gamma} \mathcal{M}_\lambda$ is a type I factor.

Two type I factorizations $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ of $B(H)$ and $\{\mathcal{N}_\mu\}_{\mu \in \Lambda'}$ of $B(H')$ are said to be unitarily equivalent if there exist a unitary U from H onto H' and a bijection $\sigma : \Lambda \rightarrow \Lambda'$ such that $U\mathcal{M}_\lambda U^* = \mathcal{N}_{\sigma(\lambda)}$.

EXAMPLE 2.6

Let E be a product system, and let $\{a_n\}_{n=0}^\infty$ be a strictly increasing sequence of

nonnegative numbers starting from zero and converging to $a < \infty$. Then $\{\mathcal{A}_a^E(a_n, a_{n+1})\}_{n=0}^\infty$ is a type I factorization of $B(E(a))$ because

$$B(E(a)) = \bigvee_{0 < t < a} \mathcal{A}_a^E(0, t)$$

holds (see [4, Proposition 4.2.1]). For a fixed sequence as above, the unitary equivalence class of the type I factorization $\{\mathcal{A}_a^E(a_n, a_{n+1})\}_{n=0}^\infty$ is an isomorphism invariant of the product system E . In particular, whether it is a CABATIF or not is used to distinguish concrete type III examples in Section 5. As we see now, this invariant may be useful only in the type III case.

When $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ is a type I factorization of $B(H)$, we say that a nonzero vector ξ is *factorizable* if for any λ there exists a minimal projection p_λ of \mathcal{M}_λ such that $p_\lambda \xi = \xi$.

Araki and Woods characterized a CABATIF as a type I factorization with a factorizable vector. Since we need a more precise statement, we briefly recall basics of the incomplete tensor product space (abbreviated as ITPS) now.

Let $\{(H_\lambda, \xi_\lambda)\}_{\lambda \in \Lambda}$ be a family of Hilbert spaces H_λ with normalized vectors $\xi_\lambda \in H_\lambda$. Let $\mathcal{F}(\Lambda)$ be the set of all finite subsets of Λ , which is a directed set with respect to the inclusion relation. For $F_1, F_2 \in \mathcal{F}(\Lambda)$ with $F_1 \subset F_2$, we introduce an isometric embedding V_{F_1, F_2} from $\bigotimes_{\lambda \in F_1} H_\lambda$ into $\bigotimes_{\lambda \in F_2} H_\lambda$ by

$$V_{F_1, F_2} \eta = \eta \otimes \left(\bigotimes_{\mu \in F_2 \setminus F_1} \xi_\mu \right), \quad \eta \in \bigotimes_{\lambda \in F_1} H_\lambda.$$

Then the ITPS

$$H = \bigotimes_{\lambda \in \Lambda}^{(\otimes \xi_\lambda)} H_\lambda$$

of the Hilbert spaces $\{H_\lambda\}_{\lambda \in \Lambda}$, with respect to the reference vectors $\{\xi_\lambda\}_{\lambda \in \Lambda}$, is the completion of the direct limit of the directed family $\{\bigotimes_{\lambda \in F} H_\lambda\}_{F \in \mathcal{F}(\Lambda)}$. When there is no possibility of confusion, we omit the superscript $(\otimes \xi_\lambda)$ for simplicity. We denote by $V_{F, \infty}$ the canonical embedding of $\bigotimes_{\lambda \in F} H_\lambda$ into H .

The product vector $\xi = \bigotimes_{\lambda \in \Lambda} \xi_\lambda \in H$ is understood as $V_{F, \infty} \bigotimes_{\lambda \in F} \xi_\lambda$, which does not depend on $F \in \mathcal{F}(\Lambda)$. More generally, if $\{\eta_\lambda\}_{\lambda \in \Lambda}$, $\eta_\lambda \in H_\lambda$, is a family of vectors such that $0 < \prod_{\lambda \in \Lambda} \|\eta_\lambda\| < \infty$ and if

$$\sum_{\lambda \in \Lambda} |\langle \eta_\lambda, \xi_\lambda \rangle - 1| < \infty,$$

then the net $\{V_{F, \infty} \bigotimes_{\lambda \in F} \eta_\lambda\}_{F \in \mathcal{F}(\Lambda)}$ converges in H . The product vector $\eta = \bigotimes_{\lambda \in \Lambda} \eta_\lambda$ is defined as its limit. Two product vectors $\eta = \bigotimes_{\lambda \in \Lambda} \eta_\lambda$ and $\zeta = \bigotimes_{\lambda \in \Lambda} \zeta_\lambda$ satisfy

$$\langle \eta, \zeta \rangle = \prod_{\lambda \in \Lambda} \langle \eta_\lambda, \zeta_\lambda \rangle.$$

For a subset $\Lambda_1 \subset \Lambda$, we often identify $\bigotimes_{\lambda \in \Lambda} H_\lambda$ with

$$\left(\bigotimes_{\lambda \in \Lambda_1} H_\lambda \right) \otimes \left(\bigotimes_{\mu \in \Lambda \setminus \Lambda_1} H_\mu \right)$$

in a canonical way. When Λ_1 consists of only one point λ , we set

$$\mathcal{M}_\lambda := B(H_\lambda) \otimes \mathbb{C}1_{\bigotimes_{\mu \neq \lambda} H_\mu} \subset B(H).$$

Then $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ is a CABATIF. Any type I factorization unitarily equivalent to this $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ is said to be a *tensor product factorization*. Note that there is only one tensor product factorization, up to unitary equivalence, with each constituent type I factor infinite dimensional.

One can find the following theorem in [2, Lemma 4.3, Theorem 4.1].

THEOREM 2.7 (ARAKI AND WOODS)

A type I factorization is a CABATIF if and only if it has a factorizable vector. When this condition holds, then it is a tensor product factorization.

When a product system E has a unit, then it gives a factorizable vector of the type I factorization $\{\mathcal{A}_a^E(a_n, a_{n+1})\}_{n=0}^\infty$ in Example 2.6, which is necessarily a CABATIF thanks to Theorem 2.7.

We use the following lemma in Section 4.

LEMMA 2.8

Let $H = \bigotimes_{\lambda \in \Lambda} H_\lambda$ be the ITPS of Hilbert spaces $\{H_\lambda\}_{\lambda \in \Lambda}$ with respect to reference vectors $\{\xi_\lambda\}_{\lambda \in \Lambda}$, and let

$$\rho_\lambda : B(H_\lambda) \ni X \mapsto X \otimes 1_{\bigotimes_{\mu \neq \lambda} H_\mu} \in \mathcal{M}_\lambda$$

be the canonical isomorphism. Let $R \in B(H)$ be a self-adjoint unitary such that $RM_\lambda R^ = \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$. Then there exist self-adjoint unitaries $R_\lambda \in B(H_\lambda)$ and a product vector $\eta = \bigotimes_{\lambda \in \Lambda} \eta_\lambda$ such that*

(i) *for $\forall \lambda \in \Lambda$ and $\forall X \in \mathcal{M}_\lambda$,*

$$\rho_\lambda(R_\lambda)X\rho_\lambda(R_\lambda^*) = RXR^*,$$

(ii) *$R_\lambda \eta_\lambda = \eta_\lambda$ for all $\lambda \in \Lambda$,*

(iii) *either $R\eta = \eta$ or $R\eta = -\eta$.*

Proof

Since the restriction of $\text{Ad } R$ to \mathcal{M}_λ is an automorphism of period two and \mathcal{M}_λ is a type I factor, there exist self-adjoint unitaries $R_\lambda \in B(H_\lambda)$ such that

$$\rho_\lambda(R_\lambda)X\rho_\lambda(R_\lambda^*) = RXR^*, \quad \forall X \in \mathcal{M}_\lambda.$$

By replacing R_λ with $-R_\lambda$ if necessary, we may assume $\langle R_\lambda \xi_\lambda, \xi_\lambda \rangle \geq 0$ for all $\lambda \in \Lambda$. Let $\xi = \bigotimes_{\lambda \in \Lambda} \xi_\lambda$, and let $p_\lambda \in \mathcal{M}_\lambda$ be the minimal projection satisfying $p_\lambda \xi = \xi$. Then $q_\lambda = Rp_\lambda R^*$ is a minimal projection of \mathcal{M}_λ satisfying $q_\lambda R\xi = R\xi$, and so $R\xi$ is a factorizable vector. The proof of [2, Lemma 3.2] shows that there

exist a complex number c of modulus 1 and normalized vectors $\zeta_\lambda \in H_\lambda$ such that $R\xi = c \bigotimes_{\lambda \in \Lambda} \zeta_\lambda$ and $\langle \zeta_\lambda, \xi_\lambda \rangle \geq 0$ for all $\lambda \in \Lambda$. Since $q_\lambda = \rho_\lambda(R_\lambda)p_\lambda\rho_\lambda(R_\lambda^*)$, the normalized vector ζ_λ is a scalar multiple of $R_\lambda\xi_\lambda$. Let

$$\Lambda_0 = \{\lambda \in \Lambda; \langle \zeta_\lambda, \xi_\lambda \rangle = 0\},$$

and let $\Lambda_1 = \Lambda \setminus \Lambda_0$. Then Λ_0 is a finite set. The two conditions $\langle R_\lambda\xi_\lambda, \xi_\lambda \rangle \geq 0$ and $\langle \zeta_\lambda, \xi_\lambda \rangle \geq 0$ imply that for any $\lambda \in \Lambda_1$, we actually have $R_\lambda\xi_\lambda = \zeta_\lambda$. Let Q_λ be the spectral projection of R_λ corresponding to eigenvalue 1. Then since $R_\lambda = 2Q_\lambda - 1$, we have $\langle \zeta_\lambda, \xi_\lambda \rangle = 2\langle Q_\lambda\xi_\lambda, \xi_\lambda \rangle - 1$.

For $\lambda \in \Lambda_0$, by replacing R_λ with $-R_\lambda$ if necessary, we can find a normalized vector $\eta_\lambda \in H_\lambda$ satisfying $R_\lambda\eta_\lambda = \eta_\lambda$. For $\lambda \in \Lambda_1$, we set $\eta_\lambda = Q_\lambda\xi_\lambda$. Then

$$\|\eta_\lambda\|^2 = \langle \eta_\lambda, \xi_\lambda \rangle = \frac{1 + \langle \zeta_\lambda, \xi_\lambda \rangle}{2}.$$

This shows that the product vector $\bigotimes_{\lambda \in \Lambda} \eta_\lambda \in H$ exists and $R_\lambda\eta_\lambda = \eta_\lambda$.

It remains only to show (iii). Let $e_\lambda \in B(H_\lambda)$ be the projection onto $\mathbb{C}\eta_\lambda$. Then the proof of [2, Lemma 3.2] shows that the net $\{\prod_{\lambda \in F} \rho_\lambda(e_\lambda)\}_{F \in \mathcal{F}(\Lambda)}$ strongly converges to the projection $e \in B(H)$ onto $\mathbb{C}\eta$. Since $ReR^* = e$ and R is a self-adjoint unitary, we get either $R\eta = \eta$ or $R\eta = -\eta$. \square

2.3. Quasi-free representations of the CAR algebra

We recall some of the well-known results about quasi-free representations of the algebra of canonical anticommutation relations (the CAR algebra).

Let K be a complex Hilbert space. We denote by $\mathfrak{A}(K)$ the CAR algebra over K , which is the universal C^* -algebra generated by $\{a(x); x \in K\}$, determined by the linear map $x \mapsto a(x)$ satisfying the CAR relations

$$a(x)a(y) + a(y)a(x) = 0,$$

$$a(x)a(y)^* + a(y)^*a(x) = \langle x, y \rangle 1,$$

for all $x, y \in K$. Since $\mathfrak{A}(K)$ is known to be simple, any set of operators satisfying the CAR relations generates a C^* -algebra canonically isomorphic to $\mathfrak{A}(K)$.

For any state φ of $\mathfrak{A}(K)$, there exists a unique positive contraction $A \in B(K)$ satisfying $\varphi(a(f)a(g)^*) = \langle Af, g \rangle$ for all $f, g \in K$. We call A the covariance operator (or 2-point function) of φ .

A *quasi-free state* ω_A on $\mathfrak{A}(K)$, associated with a positive contraction $A \in B(K)$, is the state whose (n, m) -point functions are determined by its 2-point function as

$$\omega_A(a(x_n) \cdots a(x_1)a(y_1)^* \cdots a(y_m)^*) = \delta_{n,m} \det(\langle Ax_i, y_j \rangle),$$

where $\det(\cdot)$ denotes the determinant of a matrix. Given a positive contraction, it is a fact that such a state always exists and is uniquely determined by the above relation. This is usually called the gauge-invariant quasi-free state (or generalized free state). Since we are dealing only with gauge-invariant quasi-free states, we just call them quasi-free states.

We denote by (H_A, π_A, Ω_A) the GNS triple associated with a quasi-free state ω_A on $\mathfrak{A}(K)$, and we set $\mathcal{M}_A := \pi_A(\mathfrak{A}(K))''$. We call π_A the *quasi-free representation* associated with A .

Recall that two representations π_1 and π_2 of a C^* -algebra \mathfrak{A} are said to be quasi-equivalent if there is a $*$ -isomorphism of von Neumann algebras

$$\theta : \pi_1(\mathfrak{A})'' \xrightarrow{\sim} \pi_2(\mathfrak{A})''$$

satisfying $\theta(\pi_1(X)) = \pi_2(X)$ for all $X \in \mathfrak{A}$. Two states are said to be quasi-equivalent if their GNS representations are quasi-equivalent.

We now summarize standard results on quasi-free states. For the proofs, the reader is referred to [4, Chapter 13], [11, Section II], and references therein.

THEOREM 2.9

Let K be a Hilbert space, let $P \in B(K)$ be a projection, and let $A, B \in B(K)$ be positive contractions.

(i) *Every quasi-free state ω_A of the CAR algebra $\mathfrak{A}(K)$ is a factor state; that is, the von Neumann algebra \mathcal{M}_A is a factor.*

(ii) *The restriction of the GNS representation π_A to $\mathfrak{A}(PK)$ is quasi-equivalent to the GNS representation π_{PAP} of $\mathfrak{A}(PK)$, where PAP is regarded as a positive contraction of PK .*

(iii) *The quasi-free state ω_A is of type I if and only if $\text{tr}(A - A^2) < \infty$.*

(iv) *The two quasi-free states ω_A and ω_B are quasi-equivalent if and only if both operators $A^{1/2} - B^{1/2}$ and $(1 - A)^{1/2} - (1 - B)^{1/2}$ are Hilbert-Schmidt.*

(v) *The two quasi-free states ω_A and ω_P are quasi-equivalent if and only if*

$$\text{tr}(P(1 - A)P + (1 - P)A(1 - P)) < \infty.$$

We frequently use the following criterion, which is more or less (v) above.

LEMMA 2.10

Let $A, B \in B(K)$ be positive contractions. We assume that ω_B is a type I state. Then the two quasi-free states ω_A and ω_B are quasi-equivalent if and only if

$$\text{tr}(B(1 - A)B + (1 - B)A(1 - B)) < \infty.$$

Proof

Let P be the spectral projection of B corresponding to the interval $[1/2, 1]$. Since ω_B is a type I state, Theorem 2.9(iii), (iv) imply that $P - B$ is a trace class operator, and ω_P and ω_B are quasi-equivalent. Thus ω_A and ω_B are quasi-equivalent if and only if ω_A and ω_P are quasi-equivalent, which is further equivalent to

$$\text{tr}(P(1 - A)P + (1 - P)A(1 - P)) < \infty,$$

thanks to Theorem 2.9(v). Now the statement follows from the fact that $P - B$ is a trace class operator. \square

Let γ be the period two automorphism of $\mathfrak{A}(K)$ determined by $\gamma(a(f)) = -a(f)$ for all $f \in K$. Since any quasi-free state ω_A is invariant under γ , the automorphism γ extends to a period two automorphism $\bar{\gamma}$ of the von Neumann algebra \mathcal{M}_A . For a $(\mathbb{Z}/2\mathbb{Z})$ -grading of $\mathfrak{A}(K)$ (resp., \mathcal{M}_A), we always refer to the one coming from γ (resp., $\bar{\gamma}$). When there is no possibility of confusion, we abuse the notation and use the same symbol γ for $\bar{\gamma}$.

Let ω_A be a type I state. Then, since every automorphism of a type I factor is inner, there exists a self-adjoint unitary $R^A \in \pi_A(\mathfrak{A})''$ satisfying $\text{Ad } R^A(X) = \gamma(X)$ for all $X \in \mathcal{M}_A$. The operator R^A is uniquely determined up to a multiple of -1 . In the same way, for every closed subspace $L \subset K$ such that the restriction of π_A to $\mathfrak{A}(L)$ is of type I, there exists a self-adjoint unitary $R_L^A \in \pi_A(\mathfrak{A}(L))''$ satisfying $\text{Ad } R_L^A(X) = \gamma(X)$ for all $X \in \pi_A(\mathfrak{A}(L))''$. For each L , we fix such an R_L^A , which itself is an even operator with respect to γ . When L_1 and L_2 are mutually orthogonal closed subspaces of K satisfying the above condition, we then have

$$R_{L_1 \oplus L_2}^A = \epsilon_{L_1, L_2} R_{L_1}^A R_{L_2}^A = \epsilon_{L_1, L_2} R_{L_2}^A R_{L_1}^A,$$

where $\epsilon_{L_1, L_2} \in \{1, -1\}$.

When ω_A is of type I, the family of operators $\{i\pi_A(a(f))R^A; f \in K\}$ also satisfies the CAR relation. We denote by π_A^t the representation of $\mathfrak{A}(K)$ determined by $\pi_A^t(a(f)) = i\pi_A(a(f))R^A$ for all $f \in K$ and call it the twisted representation associated with ω_A . Note that the two representations π_A and π_A^t coincide on the even part of $\mathfrak{A}(K)$.

LEMMA 2.11

Let ω_A be a type I quasi-free state of $\mathfrak{A}(K)$.

(i) For any subspace $L \subset K$,

$$\mathcal{M}_A \cap \pi_A(\mathfrak{A}(L))' = \pi_A^t(\mathfrak{A}(L^\perp))''.$$

(ii) Let $U = (1/\sqrt{2})(1 - iR^A) \in \mathcal{M}_A$. Then

$$U\pi_A(X)U^* = \pi_A^t(X)$$

holds for all $X \in \mathfrak{A}(K)$.

Proof

(i) Let Q be the spectral projection of A corresponding to the interval $[1/2, 1]$. Then π_A and π_Q are quasi-equivalent, and we may assume that A is a projection for the proof by replacing A with Q if necessary. Now the statement follows from the twisted duality theorem [6, Theorem 2.4].

(ii) This follows from a direct computation (or from [6, Proposition 2.3]). \square

As in [11], we also need to use a few facts about general factor states of $\mathfrak{A}(K)$.

LEMMA 2.12

Let A be the covariance operator of a state φ of $\mathfrak{A}(K)$.

- (i) If A is a projection, φ is the pure state ω_A .
- (ii) If φ is quasi-equivalent to a quasi-free state ω_B , then $A - B$ is compact.

Proof

- (i) See, for example, [1, Lemma 4.3].
- (ii) The statement follows from [4, Theorem 13.1.3]. □

2.4. Toeplitz CAR flows

Let V be an isometry of a Hilbert space K . Then we have an endomorphism ρ of $\mathfrak{A}(K)$ determined by $\rho(a(f)) = a(Vf)$ for all $f \in K$. For a positive contraction A , the composition $\pi_A \circ \rho$ gives a representation of $\mathfrak{A}(K)$, which is quasi-equivalent to π_{V^*AV} thanks to Theorem 2.9(ii). Thus if both $A^{1/2} - (V^*AV)^{1/2}$ and $(1 - A)^{1/2} - (1 - V^*AV)^{1/2}$ are Hilbert-Schmidt operators, then ρ extends to an endomorphism of the von Neumann algebra \mathcal{M}_A . In particular, if A satisfies $\text{tr}(A - A^2) < \infty$ and $\{V_t\}_{t \geq 0}$ is a strongly continuous semigroup of isometries on K satisfying the above condition for V_t in place of V , then we get an E_0 -semigroup.

In what follows, we assume that $K = L^2((0, \infty), \mathbb{C}^N)$ and that $\{S_t\}_{t \geq 0}$ is the shift semigroup

$$S_t f(x) = \begin{cases} 0, & 0 < x \leq t, \\ f(x - t), & t < x. \end{cases}$$

In his attempt to clarify Powers's construction [11] of the first example of a type III E_0 -semigroup, Arveson [4, Section 13.3] determined the most general form of a positive contraction $A \in B(K)$ satisfying $\text{tr}(A - A^2) < \infty$ and $S_t^* A S_t = A$ for all t , which we state now.

We regard K as a closed subspace of $\tilde{K} := L^2(\mathbb{R}, \mathbb{C}^N)$, and we denote by P_+ the projection from \tilde{K} onto K . We often identify $B(K)$ with $P_+ B(\tilde{K}) P_+$.

We denote by $M_N(\mathbb{C})$ the N by N matrix algebra. For $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$, we define the corresponding Fourier multiplier $C_\Phi \in B(\tilde{K})$ by

$$(C_\Phi \hat{f})(p) = \Phi(p) \hat{f}(p).$$

Then the Toeplitz operator $T_\Phi \in B(K)$ and the Hankel operator $H_\Phi \in B(K, K^\perp)$ with the symbol Φ are defined by

$$\begin{aligned} T_\Phi f &= P_+ C_\Phi f, & f \in K, \\ H_\Phi f &= (1_{\tilde{K}} - P_+) C_\Phi f, & f \in K. \end{aligned}$$

THEOREM 2.13 (ARVESON)

Let $K = L^2((0, \infty), \mathbb{C}^N)$. A positive contraction $A \in B(K)$ satisfies $\text{tr}(A - A^2) < \infty$ and $S_t^* A S_t = A$ if and only if there exists a projection $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ satisfying the following two conditions:

- (i) $A = T_\Phi$;
- (ii) the Hankel operator H_Φ is Hilbert-Schmidt.

We call the symbol Φ satisfying the condition of Theorem 2.13 *admissible*.

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be a projection. Arveson briefly mentioned in [4, p. 401], without giving a proof, that the condition Theorem 2.13(ii) holds if and only if the Fourier transform $\hat{\Phi}(x)$ (in distribution sense) restricted to $\mathbb{R} \setminus \{0\}$ is locally square integrable and

$$\sup_{\delta > 0} \int_{|x| > \delta} |x| \operatorname{tr}(|\hat{\Phi}(x)|^2) dx < \infty.$$

He also observed that any admissible symbol is necessarily quasi-continuous, though he used only the fact that H_Φ is a compact operator. Now, first we figure out the most suitable function space for the admissible symbols without using the Fourier transform, and then we give a proof to the above characterization in terms of the Fourier transform. We see the similarity between admissible symbols and logarithm of spectral density functions of off-white noises discussed in [16].

We denote by \mathbb{T} the unit circle in \mathbb{C} . Let U be the unitary from $L^2(\mathbb{R})$ onto $L^2(\mathbb{T}, \frac{dt}{2\pi})$ induced by the change of variables

$$e^{it} = -\frac{p+i}{p-i}.$$

(Since the Fourier transform $\hat{f}(p)$ of $f \in K$ has analytic continuation to the *lower* half-plane, we need a conformal transformation between the unit disk and the lower half-plane.) Let F be the unitaries associated with the Fourier transform. Then the Hankel operator H_Φ is transformed to the Hankel operator H_ϕ for \mathbb{T} by UF , where Φ and ϕ are related by $\phi(e^{it}) = \Phi(p)$ (see, e.g., [16, Section 3]). Let $\phi_{ij}(p)$ be the matrix element of $\phi(p)$. Since $\phi(e^{it})$ is a projection, the Hankel operator H_ϕ is Hilbert-Schmidt if and only if $H_{\phi_{ij}}$ and $H_{\overline{\phi_{ij}}}$ are Hilbert-Schmidt for all $i \leq j$.

It is easy to see that the Hankel operators H_h and $H_{\overline{h}}$ for $h \in L^\infty(\mathbb{T})$ are Hilbert-Schmidt if and only if h is in the Sobolev space $W_2^{1/2}(\mathbb{T})$; that is,

$$\sum_{n \in \mathbb{Z}} |n| |\hat{h}(n)|^2 < \infty,$$

where $\hat{h}(n)$ is the Fourier coefficient

$$\hat{h}(n) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) e^{-int} dt.$$

This is further equivalent to the condition that h belongs to the Besov space $B_{2,2}^{1/2}(\mathbb{T})$ because

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \frac{|h(e^{is}) - h(e^{it})|^2}{|e^{is} - e^{it}|^2} ds dt &= \int_0^{2\pi} \int_0^{2\pi} \frac{|h(e^{i(s+t)}) - h(e^{it})|^2}{|e^{is} - 1|^2} dt ds \\ &= 2\pi \int_0^{2\pi} \sum_{n \in \mathbb{Z}} \frac{|(e^{ins} - 1)\hat{h}(n)|^2}{|e^{is} - 1|^2} ds \\ &= 4\pi^2 \sum_{n \in \mathbb{Z}} |n| |\hat{h}(n)|^2. \end{aligned}$$

As was done in [16, Section 3], we can translate this condition back into that for functions on \mathbb{R} . Now we see that the Hankel operator H_Φ with a projection $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{R}^2} \frac{\operatorname{tr}(|\Phi(p) - \Phi(q)|^2)}{|p - q|^2} dp dq < \infty.$$

Although the following lemma may be found in the literature of Besov spaces, for the reader's convenience we give a proof of the first part. Parts (i) and (ii) are essentially due to Tsirelson [16, Proposition 3,6].

LEMMA 2.14

Let $\psi(p)$ be a measurable function on \mathbb{R} giving a tempered distribution, and let $0 < \mu \leq 1$. Then the following two conditions are equivalent

(1) The function ψ satisfies

$$\int_{\mathbb{R}^2} \frac{|\psi(p) - \psi(q)|^2}{|p - q|^{1+\mu}} dp dq < \infty.$$

(2) There exists a measurable function $\hat{\psi}_0(x)$ on \mathbb{R} such that

$$\int_{\mathbb{R}} |x|^\mu |\hat{\psi}_0(x)|^2 dx < \infty,$$

and $x\hat{\psi}(x) = x\hat{\psi}_0(x)$ as distributions.

Moreover,

(i) if ψ satisfies conditions (1) and (2), then

$$\int_{\mathbb{R}} |\psi(2p) - \psi(p)|^2 \frac{dp}{|p|^\mu} < \infty;$$

(ii) if ψ is an even differentiable function satisfying

$$\int_0^\infty |\psi'(p)|^2 |p|^{2-\mu} dp < \infty,$$

then ψ satisfies conditions (1) and (2).

Proof

Assume that (1) holds. Since condition (1) is written as

$$\int_{\mathbb{R}^2} \frac{|\psi(p+q) - \psi(q)|^2}{|p|^{1+\mu}} dq dp < \infty,$$

the function $q \mapsto \psi(p+q) - \psi(q)$ is square integrable for almost all $p \in \mathbb{R}$, and so is the distribution $(e^{ipx} - 1)\hat{\psi}(x)$ by the Plancherel theorem. This shows that the restriction of $\hat{\psi}$ to $\mathcal{D}(\mathbb{R} \setminus \{0\})$ is given by a locally square integrable function on $\mathbb{R} \setminus \{0\}$, say, $\hat{\psi}_0(x)$, and that for almost all $p \in \mathbb{R}$, the equation

$$(2.1) \quad (e^{ipx} - 1)\hat{\psi}(x) = (e^{ipx} - 1)\hat{\psi}_0(x)$$

holds as distributions in the variable x . In the above, we regard $\hat{\psi}_0(x)$ as a measurable function on \mathbb{R} by setting $\hat{\psi}_0(0) = 0$. Now the Plancherel formula implies

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\psi(p+q) - \psi(q)|^2}{|p|^{1+\mu}} dq dp &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(e^{ipx} - 1)\hat{\psi}_0(x)|^2}{|p|^{1+\mu}} dx dp \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin^2 \frac{px}{2} |\hat{\psi}_0(x)|^2}{|p|^{1+\mu}} dp dx \\ &= \frac{2^{1-\mu}}{\pi} \int_{\mathbb{R}} \frac{\sin^2 r}{|r|^{1+\mu}} dr \int_{\mathbb{R}} |x|^\mu |\hat{\psi}_0(x)|^2 dx. \end{aligned}$$

This implies the convergence of the integral in (2), which shows that $x\hat{\psi}_0(x)$ is a tempered distribution. Since the support of $x\hat{\psi}(x) - x\hat{\psi}_0(x)$ is contained in $\{0\}$, we have

$$x\hat{\psi}(x) - x\hat{\psi}_0(x) = \sum_{k=0}^n c_k \delta_0^{(k)}(x),$$

where $c_k \in \mathbb{C}$ and δ_0 is the Dirac mass at zero. We choose $p \neq 0$ such that (2.1) holds, and we set

$$h(x) = \begin{cases} \frac{e^{ipx} - 1}{x}, & x \neq 0, \\ ip, & x = 0. \end{cases}$$

Then

$$0 = (e^{ipx} - 1)(\hat{\psi}(x) - \hat{\psi}_0(x)) = h(x)(x\hat{\psi}(x) - x\hat{\psi}_0(x)) = \sum_{k=0}^n c_k h(x) \delta_0^{(k)}(x).$$

It is routine work to show that $c_k = 0$ for all k from this and $h(0) \neq 0$, and we get (2).

By tracing back the same computation as above, we can also show the implication from (2) to (1). \square

Summarizing our argument so far, we get the following.

THEOREM 2.15

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be a projection. Then the following three conditions are equivalent

- (1) The symbol Φ is admissible.
- (2) We have

$$\int_{\mathbb{R}^2} \frac{\text{tr}(|\Phi(p) - \Phi(q)|^2)}{|p - q|^2} dp dq < \infty.$$

- (3) There exists a $M_N(\mathbb{C})$ -valued measurable function $\hat{\Phi}_0(x)$ on \mathbb{R} such that

$$\int_{\mathbb{R}} |x| \text{tr}(|\hat{\Phi}_0(x)|^2) dx < \infty,$$

and $x\hat{\Phi}(x) = x\hat{\Phi}_0(x)$ as $M_N(\mathbb{C})$ -valued distributions.

Moreover,

(i) If Φ is admissible, then

$$\int_{\mathbb{R}} \operatorname{tr}(|\Phi(2p) - \Phi(p)|^2) \frac{dp}{|p|} < \infty.$$

(ii) If Φ is an even differentiable function satisfying

$$\int_0^{\infty} \operatorname{tr}(|\Phi'(p)|^2) p dp < \infty,$$

then Φ is admissible.

REMARK 2.16

For an admissible symbol Φ , we call $\hat{\Phi}_0$ in Theorem 2.15(3) the *regular part of $\hat{\Phi}$* . It is not clear whether $\hat{\Phi}_0$ gives a distribution on \mathbb{R} in general. However, when it is the case, for example, when $\hat{\Phi}_0 \in L^1(\mathbb{R}) \otimes M_N(\mathbb{C})$, then we have $\hat{\Phi} = \hat{\Phi}_0 + \delta_0 \otimes Q$ for some $Q \in M_N(\mathbb{C})$.

DEFINITION 2.17

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol, and let $A = T_\Phi$. We denote by $\alpha^\Phi = \{\alpha_t^\Phi\}_{t \geq 0}$ the E_0 -semigroup acting on the type I factor \mathcal{M}_A determined by

$$\alpha_t^\Phi(\pi_A(a(f))) = \pi_A(a(S_t f)), \quad \forall f \in K.$$

We call α^Φ the *Toeplitz CAR flow* associated with the symbol Φ .

For a Toeplitz CAR flow α^Φ , we simply denote $\mathcal{E}_\Phi := \mathcal{E}_{\alpha^\Phi}$ and $\mathcal{A}_a^\Phi(I) := \mathcal{A}_a^{\mathcal{E}_\Phi}(I)$.

EXAMPLE 2.18

When $\Phi \in M_N(\mathbb{C})$ is a constant projection, the corresponding Toeplitz CAR flow is nothing but the CAR flow of index N , which gives the unique cocycle conjugacy class of type I_N E_0 -semigroups.

EXAMPLE 2.19

Powers's first example of a type III E_0 -semigroup is the Toeplitz CAR flow associated with the symbol

$$\Phi(p) = \frac{1}{2} \begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix},$$

where $\theta(p) = (1 + p^2)^{-1/5}$. More generally, if $\theta(p)$ is a real differentiable function satisfying $\theta(-p) = \theta(p)$ for all $p \in \mathbb{R}$ and

$$\int_0^{\infty} |\theta'(p)|^2 p dp < \infty,$$

then Theorem 2.15 shows that the symbol Φ as above is admissible. In Section 5, we show that for $0 < \nu \leq 1/4$, the symbols Φ_ν , given by $\theta_\nu(p) = (1 + p^2)^{-\nu}$ in place of $\theta(p)$ above, give rise to mutually non-cocycle-conjugate type III E_0 -semigroups.

We summarize a few facts frequently used in this article in the next lemma. For a measurable subset $E \subset \mathbb{R}$, we set $K_E = L^2(E, \mathbb{C}^N)$. We denote by P_E the projection from \tilde{K} onto K_E . When $I \subset (0, \infty)$, we often regard P_I as an element of $B(K)$. For simplicity, we write $K_t = K_{(0,t)}$ and $P_t = P_{(0,t)}$ for $t > 0$.

LEMMA 2.20

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol, and let $\hat{\Phi}_0$ be the regular part of $\hat{\Phi}$. We set $A = T_\Phi$.

- (i) The relative commutant $\mathcal{M}_A \cap \alpha_t^\Phi(\mathcal{M}_A)'$ is $\pi_A^t(\mathfrak{A}(K_t))''$.
- (ii) Let I and J be two mutually disjoint open sets in \mathbb{R} . We assume that I and J have only finitely many connected components. Then $P_J C_\Phi P_I$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm

$$\|P_J C_\Phi P_I\|_{\text{HS}}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}} |(J+t) \cap I| \operatorname{tr}(|\hat{\Phi}_0(t)|^2) dt.$$

- (iii) Let $I \subset (0, \infty)$ be an open (finite or infinite) interval. Then the restriction of π_A to $\mathfrak{A}(K_I)$ is of type I, and the commutator $[C_\Phi, P_I]$ is Hilbert-Schmidt.

Proof

- (i) The statement follows Lemma 2.11(i).
- (ii) Let $f \in \mathcal{D}(I, \mathbb{C}^N)$ and $g \in \mathcal{D}(J, \mathbb{C}^N)$. Then

$$\langle C_\Phi f, g \rangle = \sum_{i,j=1}^N \frac{1}{2\pi} \int_{\mathbb{R}} \Phi(p)_{ij} \hat{f}_j(p) \overline{\hat{g}_i(p)} dp = \sum_{i,j=1}^N \frac{1}{2\pi} \int_{\mathbb{R}} \Phi(p)_{ij} \widehat{f_j * g_i^\#}(p) dp,$$

where $g_i^\#(x) = \overline{g_i(-x)}$. Since $\widehat{f_j * g_i^\#} \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, we get

$$\begin{aligned} \langle C_\Phi f, g \rangle &= \sum_{i,j=1}^N \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\Phi}_0(x)_{ij} f_j * g_i^\#(x) dx \\ &= \sum_{i,j=1}^N \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{\Phi}_0(y-x)_{ij} f_j(y) \overline{g_i(x)} dx dy. \end{aligned}$$

Since $\chi_J(x)\chi_I(y)\hat{\Phi}_0(y-x)$ is square integrable (as we see below), the operator $P_J C_\Phi P_I$ is Hilbert-Schmidt, and its Hilbert-Schmidt norm is

$$\begin{aligned} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi_J(x)\chi_I(y) \operatorname{tr}(|\hat{\Phi}_0(y-x)|^2) dx dy &= \frac{1}{4\pi^2} \int_{\mathbb{R}} |(J+t) \cap I| \operatorname{tr}(|\hat{\Phi}_0(t)|^2) dt \\ &< \infty, \end{aligned}$$

where we use Theorem 2.15(3).

- (iii) Applying (ii) to I and $J = \mathbb{R} \setminus \bar{I}$, we see that $(1_{\bar{K}} - P_I)C_\Phi P_I$ is Hilbert-Schmidt. This and Theorem 2.9(ii), (iii) show the first statement. Since

$$[C_\Phi, P_I] = (1_{\bar{K}} - P_I)C_\Phi P_I - P_I C_\Phi (1_{\bar{K}} - P_I),$$

the commutator $[C_\Phi, P_I]$ is Hilbert-Schmidt. \square

3. A dichotomy theorem

Based on Powers's argument in [11], Arveson proved the following type III criterion in [4, Theorem 13.6.1].

THEOREM 3.1 (ARVESON-POWERS)

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol having the limit

$$\Phi(\infty) := \lim_{|p| \rightarrow \infty} \Phi(p).$$

If the Toeplitz CAR flow α^Φ is spatial, then

$$\int_{\mathbb{R}} \operatorname{tr}(|\Phi(p) - \Phi(\infty)|^2) dp < \infty.$$

The purpose of this section is to generalize Theorem 3.1 and to show the following dichotomy theorem, which can be considered as an analogue of [5, Theorem 39].

THEOREM 3.2

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol. Then the following conditions are equivalent

- (i) The Toeplitz CAR flow α^Φ is of type I_N .
- (ii) The Toeplitz CAR flow α^Φ is spatial.
- (iii) There exists a projection $Q \in M_N(\mathbb{C})$ satisfying

$$\int_{\mathbb{R}} \operatorname{tr}(|\Phi(p) - Q|^2) dp < \infty.$$

In particular, every Toeplitz CAR flow is either of type I or type III.

The implication from (i) to (ii) is trivial. That from (ii) to (iii) is a generalization of Theorem 3.1. Although we follow the same strategy as in the proof of Theorem 3.1, we make a significant simplification of the argument using Arveson's classification of type I product systems (see Lemma 3.5), which allows us to obtain the statement of this form. Since α^Q with a constant projection $Q \in M_N(\mathbb{C})$ is of type I_N , the implication from (iii) to (i) follows from an L^2 -perturbation theorem stated below, which can be considered as an analogue of [9, Theorem 7.4(1)].

THEOREM 3.3

Let $\Phi, \Psi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be admissible symbols. If

$$\int_{\mathbb{R}} \operatorname{tr}(|\Phi(p) - \Psi(p)|^2) dp < \infty,$$

then α^Φ and α^Ψ are cocycle conjugate.

We first give a representation-theoretical consequence of the above square integrability condition.

LEMMA 3.4

Let $\Phi, \Psi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be admissible symbols. We set $A = T_\Phi$ and $B = T_\Psi$. Then

$$\int_{\mathbb{R}} \operatorname{tr}(|\Phi(p) - \Psi(p)|^2) dp < \infty$$

if and only if for any (some) nondegenerate finite interval $I \subset (0, \infty)$, the two quasi-free states $\omega_{P_I A P_I}$ and $\omega_{P_I B P_I}$ of $\mathfrak{A}(K_I)$ are quasi-equivalent.

Proof

Thanks to Lemma 2.10, the two states $\omega_{P_I A P_I}$ and $\omega_{P_I B P_I}$ are quasi-equivalent if and only if the following quantity is finite:

$$\begin{aligned} & \operatorname{tr}(P_I C_\Phi P_I C_{1-\Psi} P_I C_\Phi P_I + P_I C_{1-\Phi} P_I C_\Psi P_I C_{1-\Phi} P_I) \\ &= \operatorname{tr}(C_{1-\Psi} (P_I C_\Phi P_I)^2 C_{1-\Psi} + C_\Psi (P_I C_{1-\Phi} P_I)^2 C_\Psi). \end{aligned}$$

Since $P_I A P_I - (P_I A P_I)^2$ and $P_I B P_I - (P_I B P_I)^2$ are trace class operators (see Theorem 2.9(iii), Lemma 2.20(iii)), we can replace $(P_I C_\Phi P_I)^2$ with $P_I C_\Phi P_I$ and $(P_I C_{1-\Phi} P_I)^2$ with $P_I C_{1-\Phi} P_I$ in the above formula, and we get

$$\begin{aligned} & \operatorname{tr}(C_{1-\Psi} P_I C_\Phi P_I C_{1-\Psi} + C_\Psi P_I C_{1-\Phi} P_I C_\Psi) \\ &= \|C_\Phi P_I C_{1-\Psi}\|_{\text{HS}}^2 + \|C_{1-\Phi} P_I C_\Psi\|_{\text{HS}}^2. \end{aligned}$$

Since the commutators $[C_{1-\Psi}, P_I]$ and $[C_\Phi, P_I]$ are Hilbert-Schmidt (see Lemma 2.20(iii)), the right-hand side is finite if and only if

$$\|C_{\Phi(1-\Psi)} P_I\|_{\text{HS}}^2 + \|C_{(1-\Phi)\Psi} P_I\|_{\text{HS}}^2$$

is finite. Proposition 13.4.1 of [4] shows that this is equal to

$$\frac{|I|}{2\pi} \int_{\mathbb{R}} \operatorname{tr}(|\Phi(p) - \Psi(p)|^2) dp,$$

and we get the statement. \square

Proof of Theorem 3.3

Assume that $\Phi - \Psi$ is square integrable. Let $A = T_\Phi$ and $B = T_\Psi$. We apply Theorem 2.4 to $E = \mathcal{E}_\Phi$ and $F = \mathcal{E}_\Psi$ and show that α^Φ and α^Ψ are cocycle conjugate.

Thanks to Lemma 3.4, the two representations π_A and π_B are quasi-equivalent when they are restricted to $\mathfrak{A}(K_1)$. This implies that there exists an isomorphism ρ_0 from $\pi_A(\mathfrak{A}(K_1))''$ onto $\pi_B(\mathfrak{A}(K_1))''$ satisfying $\rho_0(\pi_A(a(f))) = \pi_B(a(f))$ for all $f \in K_1$. Since ρ_0 preserves the grading, we may assume $\rho_0(R_{K_1}^A) = R_{K_1}^B$ by replacing $R_{K_1}^B$ with $-R_{K_1}^B$ if necessary. We may also assume $R^A = R_{K_1}^A R_{K(1,\infty)}^A$ and $R^B = R_{K_1}^B R_{K(1,\infty)}^B$.

We claim that ρ_0 extends to an isomorphism ρ_1 from $(\pi_A(\mathfrak{A}(K_1)) \cup \{R^A\})''$ onto $(\pi_B(\mathfrak{A}(K_1)) \cup \{R^B\})''$ satisfying $\rho_1(R^A) = R^B$. Indeed, since $R_{K(1,\infty)}^A$ com-

mutates with $\pi_A(\mathfrak{A}(K_1))$, we have

$$(\pi_A(\mathfrak{A}(K_1)) \cup \{R^A\})'' = \frac{1 + R_{K(1,\infty)}^A}{2} \pi_A(\mathfrak{A}(K_1))'' \oplus \frac{1 - R_{K(1,\infty)}^A}{2} \pi_A(\mathfrak{A}(K_1))''.$$

For the same reason,

$$(\pi_B(\mathfrak{A}(K_1)) \cup \{R^B\})'' = \frac{1 + R_{K(1,\infty)}^B}{2} \pi_B(\mathfrak{A}(K_1))'' \oplus \frac{1 - R_{K(1,\infty)}^B}{2} \pi_B(\mathfrak{A}(K_1))'',$$

and so ρ_0 extends to ρ_1 satisfying $\rho_1(R_{K(1,\infty)}^A) = \rho_1(R_{K(1,\infty)}^B)$. In consequence, we have $\rho_1(R^A) = R^B$.

Let ρ be the restriction of ρ_1 to $\mathcal{M}_A \cap \alpha_1^\Phi(\mathcal{M}_A)'$, which is identified with $B(\mathcal{E}_\Phi(1))$. Thanks to Lemma 2.11(i), it is generated by $\{\pi_A(a(f))R^A; f \in K_1\}$. Then the image of ρ is generated by $\{\pi_B(a(f))R^B; f \in K_1\}$, and so it is $\mathcal{M}_B \cap \alpha_1^\Psi(\mathcal{M}_B)'$, which is identified with $B(\mathcal{E}_\Psi(1))$. In the same way, we can see that ρ satisfies $\rho(\mathcal{A}^\Phi(0, s)) = \mathcal{A}^\Psi(0, s)$ for any $0 \leq s \leq 1$. Thus we get the statement from Theorem 2.4. \square

Now we start the proof of the implication (ii) \Rightarrow (iii) in Theorem 3.2. Recall that γ is the grading automorphism $\gamma(\pi_A(a(f))) = -\pi_A(a(f))$.

LEMMA 3.5

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol. If α^Φ is spatial, then there exists a unit $V = \{V_t\}_{t>0}$ for α^Φ satisfying $\gamma(V_t) = V_t$ for all $t > 0$.

Proof

Since γ commutes with α_t^Φ for all $t > 0$, it induces an automorphism of the corresponding product system \mathcal{E}_Φ . When \mathcal{E}_Φ is of type II_0 , it is easy to show the statement, and so we assume that the index of \mathcal{E}_Φ is not zero. Let E be the sub-product system of $\mathcal{E}_{\alpha^\Phi}$ generated by the units, and let β be the automorphism of E induced by γ . Then the statement follows from the following claim: For any period two automorphism β of any type I product system E , there exists a unit of E fixed by β . Note that the type I product systems are completely classified, and the action of $\text{Aut}(E)$ on the set of units \mathcal{U}_E is well known (see [3, Section 3.8]).

Let L be a Hilbert space whose dimension is the same as the index of E , and let $\mathcal{U}(L)$ be the unitary group of L . Then $\text{Aut}(E)$ is identified with $G_L = \mathbb{R} \times L \times \mathcal{U}(L)$ having the group operation

$$(\lambda, \xi, U)(\mu, \eta, V) = (\lambda + \mu + \text{Im}\langle \xi, U\eta \rangle, \xi + U\eta, UV).$$

The set \mathcal{U}_E together with the $\text{Aut}(E)$ -action on it is identified with $\mathbb{C} \times L$ with the G_L -action

$$(\lambda, \xi, U) \cdot (a, \eta) = \left(a + i\lambda - \frac{\|\xi\|^2}{2} - \langle U\eta, \xi \rangle, \xi + U\eta \right).$$

Any element $g \in G_L$ of order two is of the form $g = (0, \xi, U)$ with $U^2 = 1$ and $U\xi = -\xi$. Now we can see that $(0, (1/2)\xi) \in \mathbb{C} \times L$ is fixed by g . \square

The following lemma is a slight generalization of [11, Lemma 4.5] and [4, Lemma 13.6.5]. For later use, we show a little stronger statement than we need in this section.

LEMMA 3.6

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol, and let $A = T_\Phi$. If $V \in \mathcal{E}_\Phi(t)$ is a normalized vector satisfying $\gamma(V) = \pm V$, then there exists a pure γ -invariant state φ of $\mathfrak{A}(K_t)$ such that $V^* \pi_A(X)V = \varphi(X)1$ for any $X \in \mathfrak{A}(K_t)$.

Proof

Throughout the proof, the symbol $a^\dagger(f)$ means either $a(f)$ or $a(f)^*$. Let $f_1, f_2, \dots, f_n \in K_t$, and let $X = a^\dagger(f_1)a^\dagger(f_2) \cdots a^\dagger(f_n)$. Then for any $g \in K$, we have

$$\begin{aligned} V^* \pi_A(X)V \pi_A(a^\dagger(g)) &= V^* \pi_A(X a^\dagger(S_t g))V = (-1)^n V^* \pi_A(a^\dagger(S_t g)X)V \\ &= (-1)^n \pi_A(a^\dagger(g)) V^* \pi_A(X)V. \end{aligned}$$

If n is even, this shows that $V^* \pi_A(X)V$ is in the center $Z(\mathcal{M}_A)$ of \mathcal{M}_A , and so it is a scalar. If n is odd, the operator $R^A V^* \pi_A(X)V$ is a scalar for the same reason, and on the other hand, it is an odd operator with respect to γ . Thus $V^* \pi_A(X)V = 0$, which shows that there exists a γ -invariant state φ such that $V^* \pi_A(X)V = \varphi(X)1$ for all $X \in \mathfrak{A}(K_t)$.

It only remains to show that φ is pure. Recall that the twisted representation π_A^t is defined by $\pi_A^t(a(f)) = i\pi_A(a(f))R^A$ and that $\mathcal{M}_A \cap \alpha_t^\Phi(\mathcal{M}_A)' = \pi_A^t(\mathfrak{A}(K_t))''$. We denote by π the irreducible representation of $\mathfrak{A}(K_t)$ on $\mathcal{E}_\Phi(t)$ given by $\pi(X) = \sigma(\pi_A^t(X))$ on $\mathcal{E}_\Phi(t)$, where $\sigma(Y)$ denotes the left multiplication of Y . Then the pure state of $\mathfrak{A}(K_t)$ given by $X \mapsto \langle \pi(X)V, V \rangle = V^* \pi_A^t(X)V$ coincides with φ because both φ and this state are γ -invariant, and π_A and π_A^t coincide on the even part of $\mathfrak{A}(K_t)$. \square

Proof of (ii) \Rightarrow (iii) in Theorem 3.2

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol, and let $A = T_\Phi$. Assume that α^Φ is spatial. Then Lemma 3.5 shows that there exists a normalized unit $V = \{V_t\}_{t \geq 0}$ satisfying $\gamma(V_t) = V_t$ for all t . Let φ be the state of $\mathfrak{A}(K_1)$ defined by $\varphi(X) = \langle \pi_A(X)V_1\Omega_A, V_1\Omega_A \rangle$ for $X \in \mathfrak{A}(K_1)$, and let $B \in B(K_1)$ be the covariance operator for φ . Then Lemma 3.6 shows that $V_t^* \pi_A(a(f))V_t = 0$ for any $f \in K_t$. We claim that there exists a positive contraction $Q \in L^\infty((0, 1)) \otimes M_N(\mathbb{C})$ such that B is the multiplication operator of Q . To prove the claim, it suffices to show that B commutes with P_t for all $0 < t < 1$. Indeed, if $f \in K_t$ and $g \in K_{(t, 1)}$, then

$$V_1^* \pi_A(a(f)a(g)^*)V_1 = V_{1-t}^* V_t^* \pi_A(a(f))V_t \pi_A(a(S_t^* g)^*)V_{1-t} = 0.$$

Thus we get $P_{(t, 1)} B P_t = 0$, and the claim is shown.

Note that φ is quasi-equivalent to $\omega_{P_1 A P_1}$. We claim that B is a projection. Let $\mathbb{K}(K_1)$ be the set of compact operators of K_1 , and let $q: B(K_1) \rightarrow B(K_1)/\mathbb{K}(K_1)$ be the quotient map. Then, thanks to Lemma 2.12(ii), we have

$q(P_1 A P_1) = q(B)$. Since $\omega_{P_1 A P_1}$ is a type I state, we have $q(P_1 A P_1)^2 = q(P_1 A P_1)$, and so $B - B^2$ is a compact operator. This is possible only if $Q(x)$ is a projection for almost every $x \in (0, 1)$, and so B is a projection.

Since B is a projection, Lemma 2.12(i) implies $\varphi = \omega_B$. Since $\omega_{P_1 A P_1}$ and ω_B are quasi-equivalent, Theorem 2.9(v) implies

$$\begin{aligned} & \|C_\Phi(P_1 - B)\|_{\text{HS}}^2 + \|C_{1-\Phi}B\|_{\text{HS}}^2 \\ &= \text{tr}((P_1 - B)C_\Phi(P_1 - B) + BC_{1-\Phi}B) < \infty. \end{aligned}$$

A computation similar to that in [4, Proposition 13.4.1] shows that the left-hand side is

$$\frac{1}{2\pi} \int_0^1 \int_{\mathbb{R}} \text{tr}(|\Phi(p) - Q(x)|^2) dp dx.$$

Thus the integral

$$\int_{\mathbb{R}} \text{tr}(|\Phi(p) - Q(x)|^2) dp$$

is finite for almost every $x \in (0, 1)$, and the proof is finished. \square

EXAMPLE 3.7

Let $\theta(p)$ be a real smooth function satisfying $\theta(-p) = \theta(p)$ for all $p \in \mathbb{R}$ and $\theta(p) = \log(\log|p|)$ (or $\theta(p) = \log^\alpha|p|$ with $0 < \alpha < 1/2$) for large $|p|$. Then Φ associated with θ in Example 2.19 is an admissible symbol without having limit at infinity. While Theorem 3.1 does not apply to such Φ , now we know from Theorem 3.2 that the Toeplitz CAR flow α^Φ is of type III.

4. Type I factorizations associated with Toeplitz CAR flows

Thanks to Theorem 3.2, we have a complete understanding of spatial Toeplitz CAR flows now. The purpose of this section is to calculate the invariant we introduced in Subsection 2.2 in the case of type III Toeplitz CAR flows.

THEOREM 4.1

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol, and let $\{a_n\}_{n=0}^\infty$ be a strictly increasing sequence of nonnegative numbers such that $a_0 = 0$, and it converges to a finite number a . Let $I_n = (a_n, a_{n+1})$ and $O = \bigcup_{n=0}^\infty I_{2n}$.

(i) If

$$\sum_{n=0}^{\infty} \|(1_{\tilde{K}} - P_{I_n})C_\Phi P_{I_n}\|_{\text{HS}}^2 < \infty,$$

then $\{\mathcal{A}_a^\Phi(I_n)\}_{n=1}^\infty$ is a CABATIF.

(ii) If $\{\mathcal{A}_a^\Phi(I_n)\}_{n=0}^\infty$ is a CABATIF, then $\|(1_{\tilde{K}} - P_O)C_\Phi P_O\|_{\text{HS}}^2 < \infty$.

We prepare a few facts used in the proof of (i) first.

LEMMA 4.2

Let H be a Hilbert space, and let $P, Q \in B(H)$ be projections. Then

$$\|(1 - P)QP\|_{\text{HS}} = \|(1 - Q)PQ\|_{\text{HS}}.$$

Proof

There is a decomposition of H into closed subspaces (each subspace could possibly be $\{0\}$)

$$H = H_1 \oplus H_2 \oplus H_3 \oplus H_4 \otimes \mathbb{C}^2 \oplus H_5$$

such that the two projections are expressed as

$$P = 1_{H_1} \oplus 1_{H_2} \oplus 0 \oplus \begin{pmatrix} 1_{H_4} & 0 \\ 0 & 0 \end{pmatrix} \oplus 0,$$

$$Q = 1_{H_1} \oplus 0 \oplus 1_{H_3} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0,$$

where c and s are nonsingular positive contractions satisfying $c^2 + s^2 = 1_{H_4}$ (see [15, p. 308]). Then we have

$$\begin{aligned} \|(1 - P)QP\|_{\text{HS}}^2 &= \left\| \begin{pmatrix} 0 & 0 \\ cs & 0 \end{pmatrix} \right\|_{\text{HS}}^2 = \text{tr}(c^2 s^2), \\ \|(1 - Q)PQ\|_{\text{HS}}^2 &= \left\| \begin{pmatrix} s^2 & -cs \\ -cs & c^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \right\|_{\text{HS}}^2 \\ &= \left\| \begin{pmatrix} c^2 s^2 & cs^3 \\ -c^3 s & -c^2 s^2 \end{pmatrix} \right\|_{\text{HS}}^2 = \text{tr}(2c^4 s^4 + c^2 s^6 + c^6 s^2) \\ &= \text{tr}(c^2 s^2 (c^2 + s^2)^2) = \text{tr}(c^2 s^2). \end{aligned}$$

□

LEMMA 4.3

Let the notation be as in Theorem 4.1, and let $A = T_{\Phi}$. We set

$$B = \sum_{n=0}^{\infty} P_{I_n} A P_{I_n} + P_{(a, \infty)} A P_{(a, \infty)}.$$

Then the following conditions are equivalent

- (i) The assumption of Theorem 4.1(i) holds.
- (ii) The quasi-free state ω_B is of type I.
- (iii) The two quasi-free states ω_A and ω_B are quasi-equivalent.

Proof

Theorem 2.9(iii) and Lemma 2.20(iii) imply that (i) and (ii) are equivalent. We show the equivalence of (i) and (iii). We set $I_{-1} = (a, \infty)$. Since ω_A is of type I,

Lemma 2.10 shows that ω_A and ω_B are quasi-equivalent if and only if the following quantity is finite:

$$\begin{aligned} & \operatorname{tr}(A(1_K - B)A + (1_K - A)B(1_K - A)) \\ &= \sum_{n=-1}^{\infty} \operatorname{tr}(P_+ C_{\Phi} P_{I_n} C_{1-\Phi} P_{I_n} C_{\Phi} P_+ + P_+ C_{1-\Phi} P_{I_n} C_{\Phi} P_{I_n} C_{1-\Phi} P_+) \\ &= \sum_{n=-1}^{\infty} (\|C_{1-\Phi} P_{I_n} C_{\Phi} P_+\|_{\text{HS}}^2 + \|C_{\Phi} P_{I_n} C_{1-\Phi} P_+\|_{\text{HS}}^2). \end{aligned}$$

Note that since

$$\begin{aligned} & \sum_{n=-1}^{\infty} (\|C_{1-\Phi} P_{I_n} C_{\Phi} (1_{\tilde{K}} - P_+)\|_{\text{HS}}^2 + \|C_{\Phi} P_{I_n} C_{1-\Phi} (1_{\tilde{K}} - P_+)\|_{\text{HS}}^2) \\ & \leq \sum_{n=-1}^{\infty} \operatorname{tr}((1_{\tilde{K}} - P_+) C_{\Phi} P_{I_n} C_{\Phi} (1_{\tilde{K}} - P_+) \\ & \quad + (1_{\tilde{K}} - P_+) C_{1-\Phi} P_{I_n} C_{1-\Phi} (1_{\tilde{K}} - P_+)) \\ & = \|P_+ C_{\Phi} (1_{\tilde{K}} - P_+)\|_{\text{HS}}^2 + \|P_+ C_{1-\Phi} (1_{\tilde{K}} - P_+)\|_{\text{HS}}^2 \\ & = 2\|P_+ C_{\Phi} (1_{\tilde{K}} - P_+)\|_{\text{HS}}^2 < \infty, \end{aligned}$$

the above quantity is finite if and only if

$$\sum_{n=-1}^{\infty} \|C_{1-\Phi} P_{I_n} C_{\Phi}\|_{\text{HS}}^2 < \infty.$$

Thanks to Lemma 4.2, this is equivalent to

$$\sum_{n=-1}^{\infty} \|(1_{\tilde{K}} - P_{I_n}) C_{\Phi} P_{I_n}\|_{\text{HS}}^2 < \infty.$$

Since $\|(1_{\tilde{K}} - P_{I_{-1}}) C_{\Phi} P_{I_{-1}}\|_{\text{HS}}^2 < \infty$, we conclude that (i) is equivalent to (iii). \square

Proof of Theorem 4.1(i)

Assume that the assumption of Theorem 4.1(i) holds. It suffices to show that for any strictly increasing sequence of nonnegative integers $\{n_m\}_{m=0}^{\infty}$, the von Neumann algebra $\mathcal{A}_a^{\Phi}(E) := \bigvee_{m=0}^{\infty} \mathcal{A}_a^{\Phi}(I_{n_m})$ is a type I factor, where $E = \bigcup_{m=0}^{\infty} I_{n_m}$. Note that $\mathcal{A}_a^{\Phi}(E)$ is always a factor (see [9, Remark 8.2]). We may assume $n_0 = 0$ without loss of generality. Identifying $B(\mathcal{E}^{\Phi}(a))$ with $\mathcal{M}_A \cap \alpha_a^{\Phi}(\mathcal{M}_A)'$, we see that it suffices to show that the factor

$$\bigvee_{m=0}^{\infty} \alpha_{a_{n_m}}^{\Phi}(\mathcal{M}_A \cap \alpha_{a_{n_m+1}-a_{n_m}}^{\Phi}(\mathcal{M}_A)')$$

is of type I. Recall that we have $\mathcal{M}_A \cap \alpha_t^{\Phi}(\mathcal{M}_A)' = \pi_A^t(\mathfrak{A}(K_t))''$, where $\pi_A^t(a(f)) = i\pi_A(a(f))R^A$. Since

$$\alpha_t^{\Phi}(R^A) = \pm R_{K(t,\infty)}^A = \pm \epsilon_{K_t, K(t,\infty)} R^A R_{K_t}^A,$$

we get

$$\begin{aligned} & \alpha_{a_{n_m}}^{\Phi} (\mathcal{M}_A \cap \alpha_{a_{n_m+1}-a_{n_m}}^{\Phi} (\mathcal{M}_A)') \\ &= \{ \pi_A^t(a(f)) R_{K_{a_{n_m}}}^A, R_{K_{a_{n_m}}}^A \pi_A^t(a(f))^*; f \in K_{I_{n_m}} \}'' . \end{aligned}$$

Thanks to Lemma 2.11(ii), it suffices to show that the factor

$$\bigvee_{m=0}^{\infty} \{ \pi_A(a(f)) R_{K_{a_{n_m}}}^A, R_{K_{a_{n_m}}}^A \pi_A(a(f))^*; f \in K_{I_{n_m}} \}''$$

is of type I.

Let

$$\mathcal{N}_m := \bigvee_{k=0}^m \{ \pi_A(a(f)) R_{K_{a_{n_k}}}^A, R_{K_{a_{n_k}}}^A \pi_A(a(f))^*; f \in K_{I_{n_k}} \}'' ,$$

and let $J_m = \bigcup_{k=1}^m (a_{n_{k-1}+1}, a_{n_k})$. Since

$$R_{K_{a_{n_m}}}^A = \pm R_{K_{J_m}}^A \prod_{k=0}^{m-1} R_{K_{I_{n_k}}}^A$$

and

$$R_{I_{n_k}}^A \in \{ \pi_A(a(f)) R_{K_{J_k}}^A, R_{K_{J_k}}^A \pi_A(a(f))^*; f \in K_{I_{n_k}} \}'' ,$$

we can show

$$\mathcal{N}_m = \bigvee_{k=0}^m \{ \pi_A(a(f)) R_{K_{J_k}}^A, R_{K_{J_k}}^A \pi_A(a(f))^*; f \in K_{I_{n_k}} \}''$$

by induction, where we use the convention $R_{K_{J_0}}^A = 1$. Thus to prove the statement, it suffices to show that the factor

$$\bigvee_{m=0}^{\infty} \{ \pi_A(a(f)) R_{K_{J_m}}^A, R_{K_{J_m}}^A \pi_A(a(f))^*; f \in K_{I_{n_m}} \}''$$

is of type I.

Let B be as in Lemma 4.3. Since π_A and π_B are quasi-equivalent, there exists an isomorphism θ from \mathcal{M}_A onto \mathcal{M}_B satisfying $\theta(\pi_A(f)) = \theta(\pi_B(a(f)))$ for any $f \in K$. Since θ preserves the grading, we may assume $\theta(R_{K_I}^A) = R_{K_I}^B$ for any interval $I \subset (0, \infty)$. Thus to prove the statement, it suffices to show that the factor

$$\mathcal{N} := \bigvee_{m=0}^{\infty} \{ \pi_B(a(f)) R_{K_{J_m}}^B, R_{K_{J_m}}^B \pi_B(a(f))^*; f \in K_{I_{n_m}} \}''$$

is of type I.

Since J_m is disjoint from E , the self-adjoint unitary $R_{K_{J_m}}^B$ commutes with any $\pi_B(a(f))$ with $f \in K_E$. Thus \mathcal{N} is generated by the factor representation π of $\mathfrak{A}(K_E)$ determined by $\pi(a(f)) = \pi_B(a(f)) R_{J_m}^B$ for $f \in K_{I_{n_m}}$. Let ω be the state of $\mathfrak{A}(K_E)$ defined by $\omega(X) := \langle \pi(X) \Omega_B, \Omega_B \rangle$ for $X \in \mathfrak{A}(K_E)$. Since π is a factor representation, the GNS representation of ω is quasi-equivalent to π .

We claim that ω coincides with $\omega_{P_E B P_E}$. Let $X_i \in \mathfrak{A}(K_{I_{n_i}})$, $i = 0, 1, \dots, m$, be of the form

$$X_i = a^\dagger(f_1^i) a^\dagger(f_2^i) \cdots a^\dagger(f_{l_i}^i)$$

with $f_j^i \in K_{I_{n_i}}$, where $a^\dagger(f)$ means either $a(f)$ or $a(f)^*$. Then we have $\pi(X_i) = \pi_B(X_i) R_{K_{J_i}}^B$, and

$$\omega(X_1 X_2 \cdots X_m) = \langle \pi_B(X_1 X_2 \cdots X_m) Y \Omega_B, \Omega_B \rangle,$$

where Y is an element in the even part of $\pi_B(\mathfrak{A}(K_E^\perp))''$. Since B commutes with P_{I_n} for any n , if one of l_1, l_2, \dots, l_m is odd, then approximating Y by polynomials of $\pi_B(a^\dagger(f))$ with $f \in K_E^\perp$, we see that the right-hand side is zero. (Consider the contributing 2-point functions.) When l_1, l_2, \dots, l_m are all even, we have

$$\omega(X_1 X_2 \cdots X_m) = \langle \pi_B(X_1 X_2 \cdots X_m) \Omega_B, \Omega_B \rangle = \omega_B(X_1 X_2 \cdots X_m),$$

which shows $\omega = \omega_{P_E B P_E}$. Thus to prove the statement, it suffices to show that $\omega_{P_E B P_E}$ is of type I.

Since P_E commutes with B , we get

$$\mathrm{tr}(P_E B P_E - (P_E B P_E)^2) = \mathrm{tr}(P_E (B - B^2)) \leq \mathrm{tr}(B - B^2).$$

Now the statement follows from Theorem 2.9(iii) and Lemma 4.3. \square

We proceed to the proof of Theorem 4.1(ii).

LEMMA 4.4

Let L_n , $n = 0, 1, \dots$, be Hilbert spaces, and let $L = \bigoplus_{n=0}^{\infty} L_n$. Assume that φ is a γ -invariant state of $\mathfrak{A}(L)$ satisfying the following two conditions

- (i) For any natural number n and $X_i \in \mathfrak{A}(L_i)$, $i = 0, 1, \dots, n$,

$$\varphi(X_1 X_2 \cdots X_n) = \varphi(X_1) \varphi(X_2) \cdots \varphi(X_n).$$

- (ii) The restriction φ_n of φ to $\mathfrak{A}(L_n)$ is a pure state for any n .

Then φ is a pure state.

Proof

Let (H_n, π_n, Ω_n) be the GNS triple of φ_n , and let $H = \bigotimes_{n=0}^{\infty} (\otimes^{\Omega_n}) H_n$ be the ITPS of the Hilbert spaces $\{H_n\}_{n=0}^{\infty}$ with respect to the reference vectors $\{\Omega_n\}_{n=0}^{\infty}$. We set $\Omega = \bigotimes_{n=0}^{\infty} \Omega_n$. Since φ_n is a γ -invariant state of $\mathfrak{A}(L_n)$, there exists a self-adjoint unitary $R_n \in B(H_n)$ satisfying $R_n \pi_n(X) \Omega_n = \pi_n(\gamma(X))$ for all $X \in \mathfrak{A}(L_n)$. We introduce a representation π of $\mathfrak{A}(L)$ on H by setting $\pi(a(f))$ for $f \in L_n$ as

$$\pi(a(f)) = \begin{cases} \pi_0(a(f)) \otimes 1_{\bigotimes_{k=1}^{\infty} H_k}, & n = 0, \\ R_0 \otimes R_1 \otimes \cdots \otimes R_{n-1} \otimes \pi_n(a(f)) \otimes 1_{\bigotimes_{k=n+1}^{\infty} H_k}, & n > 0. \end{cases}$$

Then π is irreducible, and the pure state ψ of $\mathfrak{A}(L)$ defined by $\psi(X) = \langle \pi(X) \Omega, \Omega \rangle$ satisfies conditions (i) and (ii). Moreover, the restriction of ψ to $\mathfrak{A}(L_n)$ coincides

with φ_n . Since $\{\varphi_n\}_{n=0}^\infty$ and condition (i) uniquely determine φ , we conclude that $\varphi = \psi$ and it is a pure state. \square

LEMMA 4.5

If the assumption of Theorem 4.1(ii) holds, then there exist normalized vectors $V \in \mathcal{E}_\Phi(a)$, $V_n \in \mathcal{E}_\Phi(a_{n+1} - a_n)$, and $W_n \in \mathcal{E}_\Phi(a - a_{n+1})$, $n = 0, 1, 2, \dots$, such that V is factorized as $V = V_0 V_1 V_2 \cdots V_n W_n$, and $\gamma(V_n) = \pm V_n$, $\gamma(W_n) = \pm W_n$ for any nonnegative integer n .

Proof

Assume that $\{\mathcal{A}_a^\Phi(I_n)\}_{n=0}^\infty$ is a CABATIF. Then thanks to Theorem 2.7, there exists a sequence of Hilbert spaces with normalized vectors $\{(H_n, \xi_n)\}_{n=0}^\infty$ and unitary U from ITPS $H := \bigotimes_{n=0}^\infty \otimes^{\xi_n} H_n$ onto $\mathcal{E}_\Phi(a)$ such that $U\mathcal{M}_n U^* = \mathcal{A}_a^\Phi(I_n)$, where

$$\mathcal{M}_n = B(H_n) \otimes \mathbb{C}1_{\bigotimes_{m \neq n} H_m}.$$

For $X \in \mathcal{M}_A \cap \alpha_a^\Phi(\mathcal{M}_A)'$, we denote by $\sigma(X) \in B(\mathcal{E}_\Phi(a))$ the corresponding left multiplication operator. We claim that for any $0 < t < a$, there exists $\epsilon_t \in \{1, -1\}$ such that $\gamma(X) = \epsilon_t \sigma(R_{K_t}^A)X$ for any $X \in \mathcal{E}^\Phi(t)$. Indeed, let ϵ_t be the constant determined by $\alpha_t^\Phi(R^A) = \epsilon_t R_{K_t}^A R^A$. Then

$$\gamma(X) = R^A X R^{A*} = R^A \alpha_t^\Phi(R^A)^* X = \epsilon_t R_{K_t}^A X,$$

which shows the claim.

The claim (or Lemma 2.11) implies that for any $X \in \mathcal{M}_A \cap \alpha_a^\Phi(\mathcal{M}_A)'$, we have $R_{K_a}^A X R_{K_a}^{A*} = \gamma(X)$. Thus $\sigma(R_{K_a}^A)$ is a self-adjoint unitary satisfying

$$\sigma(R_{K_a}^A) \mathcal{A}_a^\Phi(I_n) \sigma(R_{K_a}^A)^* = \mathcal{A}_a^\Phi(I_n).$$

For the same reason, the operator $\sigma(R_{K_{I_n}}^A)$ is a self-adjoint unitary in $\mathcal{A}_a^\Phi(I_n)$ satisfying

$$(4.1) \quad \sigma(R_{K_a}^A) X \sigma(R_{K_a}^A)^* = \sigma(R_{K_{I_n}}^A) X \sigma(R_{K_{I_n}}^A)^*, \quad \forall X \in \mathcal{A}_a^E(I_n).$$

Applying Lemma 2.8 to the self-adjoint unitary $R = U^* \sigma(R_{K_a}^A) U \in B(H)$, we get a product vector $\eta = \bigotimes_{n=1}^\infty \eta_n \in H$ and self-adjoint unitaries $R_n \in B(H_n)$ satisfying the three conditions in the conclusion of Lemma 2.8. We may assume $\|\eta_n\| = 1$ by normalizing each η_n . We set $V := U\eta$. Then we have $\gamma(V) = \epsilon_a \sigma(R_{K_a}^A) V = \pm V$.

Let $e_n \in \mathcal{M}_n$ be the minimal projection satisfying $e_n \eta = \eta$ for all n , and set $f_n = U e_n U^*$, which is a minimal projection of $\mathcal{A}_a^\Phi(I_n)$. Then we have $f_n V = V$ for all n . For each n , we can choose a normalized vector $V_n \in \mathcal{E}_\Phi(a_{n+1} - a_n)$ so that for any $X \in \mathcal{E}_\Phi(a_n)$ and $Y \in \mathcal{E}_\Phi(a - a_{n+1})$, we have $f_n(X V_n Y) = X V_n Y$. Since the self-adjoint unitary $U \rho_n(R_n) U^* \in \mathcal{A}_a^\Phi(I_n)$ satisfies the same equation as (4.1) in place of $\sigma(R_{K_{I_n}}^A)$, we have either $U \rho_n(R_n) U^* = \sigma(R_{K_{I_n}}^A)$ or $U R_n U^* = -\sigma(R_{K_{I_n}}^A)$. Thus $\rho_n(R_n) e_n \rho_n(R_n)^* = e_n$ implies $\sigma(R_{K_{I_n}}^A) f_n \sigma(R_{K_{I_n}}^A)^* = f_n$. Since f_n is a minimal projection of $\mathcal{A}_a^\Phi(I_n)$ and $\sigma(R_{K_{I_n}}^A) \in \mathcal{A}_a^\Phi(I_n)$ is a self-adjoint

unitary, this shows that XV_nY is an eigenvector of $\sigma(R_{K_{I_n}}^A)$ and $R_{K_{I_n}}^A XV_nY = \pm XV_nY$. On the other hand, since $\alpha_{a_n}^\Phi(R_{a_{n+1}-a_n}^A) = \pm R_{I_n}^A$ and

$$R_{K_{I_n}}^A XV_nY = X(\alpha_{a_n}^{\Phi^{-1}}(R_{I_n}^A)V_n)Y,$$

we see that V_n is an eigenvector of $\sigma(R_{a_{n+1}-a_n}^A)$. Thus we get $\gamma(V_n) = \pm V_n$. Letting $W_n = (V_1V_2 \cdots V_n)^*V$, we finish the proof. \square

Proof of Theorem 4.1(ii)

Since $\|(1_{\tilde{K}} - P_O)C_\Phi P_O\|_{\text{HS}}^2 = \text{tr}(P_O A P_O - (P_O A P_O)^2)$, it suffices to show that the restriction of π_A to $\mathfrak{A}(K_O)$ is a type I representation thanks to Theorem 2.9(ii), (iii).

Let $L_n = K_{a_{2n+1}-a_{2n}}$, and let $L = \bigoplus_{n=0}^{\infty} L_n$. We denote by π the representation of $\mathfrak{A}(L)$ on H_A determined by

$$\pi(a(f)) = \pi_A(a(S_{a_{2n}}f)) = \alpha_{a_{2n}}^\Phi(\pi_A(a(f))), \quad f \in L_n.$$

Since $\pi(\mathfrak{A}(L)) = \pi_A(\mathfrak{A}(K_O))$, it suffices to show that π is a type I representation. Let V , V_n , and W_n be the normalized vectors obtained in Lemma 4.5. We set $\varphi(X) = \langle \pi(X)V\Omega_A, V\Omega_A \rangle$ for $X \in \mathfrak{A}(L)$. Then φ is a state of $\mathfrak{A}(L)$ whose GNS representation is quasi-equivalent to π . We show that φ is pure using Lemma 4.4.

Lemma 3.6 shows that there exists a γ -invariant pure state φ_n of $\mathfrak{A}(L_n)$ satisfying $V_n^* \pi_A(X)V_n = \varphi_n(X)1$ for all $X \in \mathfrak{A}(L_n)$. Let $X_i \in \mathfrak{A}(L_i)$, $i = 1, 2, \dots, n$. Then

$$\begin{aligned} & V^* \pi_A(X_0 X_1 \cdots X_n) V \\ &= W_{2n}^* V_{2n}^* \cdots V_1^* V_0^* \pi_A(X_0) \alpha_{a_2}^\Phi(\pi_A(X_1)) \cdots \alpha_{a_{2n}}^\Phi(\pi_A(X_n)) V_0 V_1 \cdots V_{2n} W_{2n} \\ &= W_{2n}^* V_{2n}^* \cdots V_1^* V_0^* \pi_A(X_0) V_0 V_1 \pi_A(X_1) \cdots \alpha_{a_{2n}-a_2}^\Phi(\pi_A(X_n)) V_2 \cdots V_{2n} W_{2n} \\ &= \varphi_0(X_0) W_{2n}^* V_{2n}^* \cdots V_2^* \pi_A(X_1) \cdots \alpha_{a_{2n}-a_2}^\Phi(\pi_A(X_n)) V_2 \cdots V_{2n} W_{2n} \\ &= \varphi_0(X_0) \varphi_1(X_1) W_{2n}^* V_{2n}^* \cdots V_4^* \pi_A(X_2) \cdots \alpha_{a_{2n}-a_4}^\Phi(\pi_A(X_n)) V_4 \cdots V_{2n} W_{2n} \\ &= \cdots = \varphi_0(X_0) \varphi_1(X_1) \cdots \varphi_n(X_n). \end{aligned}$$

Thus Lemma 4.4 shows that φ is a pure state, and in consequence, π is a type I representation. \square

In order to apply Theorem 4.1 to concrete examples, we state the assumptions of Theorem 4.1 in terms of the regular part $\hat{\Phi}_0$ of the Fourier transform $\hat{\Phi}$.

LEMMA 4.6

Let the notation be as in Theorem 4.1. Assume that Φ is an even function.

(i) *The assumption of Theorem 4.1(i) holds if and only if*

$$\int_0^\infty \sum_{n=0}^{\infty} \min\{x, |I_n|\} \text{tr}(|\hat{\Phi}_0(x)|^2) dx < \infty.$$

(ii) *The assumption of Theorem 4.1(ii) holds if and only if*

$$\int_0^\infty |O \ominus (O + x)| \operatorname{tr}(|\hat{\Phi}_0(x)|^2) dx < \infty,$$

where $O \ominus (O + x)$ is the symmetric difference of O and O translated by x .

Proof

(i) The statement follows from $|(I_n \setminus (I_n + t))| = \min\{|t|, |I_n|\}$ and Lemma 2.20(ii).

(ii) We set $J_{-1} := (-\infty, 0)$, $J_0 := (a, \infty)$, and $J_n = I_{2n-1}$ for $n \in \mathbb{N}$. Then

$$\|(1_{\tilde{K}} - P_O)C_\Phi P_O\|_{\text{HS}}^2 = \sum_{m=-1}^{\infty} \sum_{n=0}^{\infty} \|P_{J_m} C_\Phi P_{I_{2n}}\|_{\text{HS}}^2.$$

The statement follows from this and Lemma 2.20(ii). \square

Lemma 4.2 implies $\|(1_{\tilde{K}} - P_E)C_\Phi P_E\|_{\text{HS}}^2 = \|C_{1-\Phi} P_E C_\Phi\|_{\text{HS}}^2$. Thus by using Fourier transform, we can also get the following criteria, though we do not use them in this article.

LEMMA 4.7

Let the notation be as in Theorem 4.1.

(i) *The assumption of Theorem 4.1(i) holds if and only if*

$$\int_{\mathbb{R}^2} \frac{\operatorname{tr}(|\Phi(p) - \Phi(q)|^2)}{|p - q|^2} \sum_{n=0}^{\infty} \sin^2 \frac{|I_n|(p - q)}{2} dp dq < \infty.$$

(ii) *The assumption of Theorem 4.1(ii) holds if and only if*

$$\int_{\mathbb{R}^2} \operatorname{tr}(|\Phi(p) - \Phi(q)|^2) |\chi_O(p - q)|^2 dp dq < \infty.$$

5. Examples

Applying Theorem 4.1 to concrete sequences, we get the following theorem, which provides us with a computable invariant for type III Toeplitz CAR flows.

THEOREM 5.1

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be an admissible symbol satisfying $\Phi(p) = \Phi(-p)$ for all $p \in \mathbb{R}$, and let $0 < \mu < 1$. We set $a_0 = 0$,

$$a_n = \sum_{k=1}^n \frac{1}{k^{1/(1-\mu)}}, \quad n \in \mathbb{N},$$

and $a = \lim_{n \rightarrow \infty} a_n$. Then the following three conditions are equivalent

- (1) *The type I factorization $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ is a CABATIF.*
- (2) *We have*

$$\int_0^\infty \int_0^\infty \frac{\operatorname{tr}(|\Phi(p) - \Phi(q)|^2)}{|p - q|^{1+\mu}} dp dq < \infty.$$

(3) We have

$$\int_0^\infty x^\mu \operatorname{tr}(|\hat{\Phi}_0(x)|^2) dx < \infty.$$

Moreover,

(i) If $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ is a CABATIF, then

$$\int_0^\infty \operatorname{tr}(|\Phi(2p) - \Phi(p)|^2) \frac{dp}{p^\mu} < \infty.$$

(ii) If Φ is differentiable and

$$\int_0^\infty \operatorname{tr}(|\Phi'(p)|^2) p^{2-\mu} dp < \infty,$$

then $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ is a CABATIF.

Proof

The statement follows from Lemma 2.14, Lemma 4.6, and Lemma 5.2 applied to $h(x) = x^{\mu-1}$. \square

The following lemma is more or less [9, Lemma 8.6].

LEMMA 5.2

Let $h(x)$ be a nonnegative strictly decreasing continuous function on $(0, \infty)$ satisfying $\lim_{x \rightarrow +0} h(x) = \infty$, $\lim_{x \rightarrow \infty} h(x) = 0$, and

$$\int_0^1 h(x) dx < \infty.$$

We set $a_0 = 0$,

$$a_n = \sum_{k=1}^n h^{-1}(k), \quad n \in \mathbb{N},$$

$I_n = (a_n, a_{n+1})$, and $O = \bigcup_{n=0}^\infty I_{2n}$. Then the sequence $\{a_n\}_{n=0}^\infty$ converges, and

$$x(h(x) - 1) \leq |O \ominus (O + x)| \leq 2 \sum_{n=0}^\infty \min\{x, |I_n|\} \leq 2 \int_0^x h(t) dt, \quad \forall x > 0.$$

Proof

Note that we have

$$\sum_{k=n+1}^\infty h^{-1}(k) \leq \int_0^{h^{-1}(n+1)} h(t) dt - nh^{-1}(n+1),$$

and in particular, the sequence $\{a_n\}_{n=0}^\infty$ converges. Since $\min\{x, |I_n|\} = |I_n \setminus (I_n \pm x)|$, the middle inequality follows from the definition of O .

For fixed $x > 0$, we take the unique nonnegative integer n satisfying $h^{-1}(n+1) < x \leq h^{-1}(n)$ (or, equivalently, $n \leq h(x) < n+1$). Then

$$\begin{aligned} \sum_{k=0}^{\infty} \min\{x, |I_k|\} &= \sum_{k=0}^{n-1} x + \sum_{k=n}^{\infty} |I_k| = nx + \sum_{k=n}^{\infty} h^{-1}(k+1) \\ &\leq \int_0^{h^{-1}(n+1)} h(t) dt + n(x - h^{-1}(n+1)) \\ &\leq \int_0^x h(t) dt. \end{aligned}$$

When n is even, counting only contribution from $\{I_{2k}\}_{k=0}^{(n-2)/2}$, we get

$$|(U+x) \setminus U| \geq \frac{n}{2}x.$$

In a similar way, we get

$$|U \setminus (U+x)| \geq \frac{n}{2}x,$$

and so

$$|U \ominus (U+x)| \geq nx \geq (h(x) - 1)x.$$

When n is odd, we have $|(U+x) \setminus U| \geq ((n+1)/2)x$ and $|U \setminus (U+x)| \geq ((n+1)/2)x$ in a similar way, which shows $|U \ominus (U+x)| \geq xh(x)$. \square

Now we apply Theorem 5.1 to concrete examples.

THEOREM 5.3

For $\nu > 0$, let $\theta_\nu(p) = (1+p^2)^{-\nu}$, and let

$$\Phi_\nu(p) = \frac{1}{2} \begin{pmatrix} 1 & e^{i\theta_\nu(p)} \\ e^{-i\theta_\nu(p)} & 1 \end{pmatrix}.$$

Then Φ_ν is admissible. Let $\alpha^\nu := \alpha^{\Phi_\nu}$ be the corresponding Toeplitz CAR flow.

- (i) If $\nu > 1/4$, then α^ν is of type I_2 .
- (ii) If $0 < \nu \leq 1/4$, then α^ν is of type III.
- (iii) If $0 < \nu_1 < \nu_2 \leq 1/4$, then α^{ν_1} and α^{ν_2} are not cocycle conjugate.

Proof

The fact that Φ_ν is admissible follows from Theorem 2.15(ii). Conditions (i) and (ii) follow from Theorem 3.2. To show (iii), we choose μ in the interval $(1-4\nu_2, 1-4\nu_1)$, which satisfies $0 < \mu < 1$. Applying Theorem 5.1(i), (ii) to this μ and $\Phi = \Phi_{\nu_i}$, $i = 1, 2$, we see that $\{\mathcal{A}_a^{\Phi_{\nu_2}}(a_n, a_{n+1})\}_{n=0}^{\infty}$ is a CABATIF, while $\{\mathcal{A}_a^{\Phi_{\nu_1}}(a_n, a_{n+1})\}_{n=0}^{\infty}$ is not. Therefore α^{ν_1} and α^{ν_2} are not cocycle conjugate. \square

REMARK 5.4

Let Φ be as in Example 3.7, and let μ and $\{a_n\}_{n=0}^{\infty}$ be as in Theorem 5.1.

Then Theorem 5.1(i) implies that $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ is not a CABATIF for any $0 < \mu < 1$. This shows that α^Φ is not cocycle conjugate to α^ν for any ν .

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Izumi: Department of Mathematics, Graduate School of Science, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan; izumi@math.kyoto-u.ac.jp

Srinivasan: Chennai Mathematical Institute, Siruseri 603103, India; vasanth@cmi.ac.in