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A MEASURE OF VARIABILITY BASED ON THE HARMONIC MEAN, AND ITS USE IN APPROXIMATIONS[1]

BY MARK BROWN

The City College, CUNY

Let X be a positive random variable and assume that both $a = EX^{-1}$ and $\mu = EX$ are finite. Define $c^2 = 1 - (a\mu)^{-1}$. This quantity serves as a measure of variability for X which is reflected in the behavior of completely monotone functions of X. For g completely monotone with $g(0) < \infty$:

$$0 \le Eg(X) - g(EX) \le c^2 g(0) \quad \text{and} \quad \operatorname{Var} g(X) \le c^2 g^2(0).$$

1. Introduction. Given a random variable X and a function g, crude approximations to the mean and variance of $g(X)$ are obtainable by Taylor series arguments. The variance of X, σ^2, is a key quantity under this approach, both in approximating the bias $(Eg(X) - g(EX))$ and the variance $(\operatorname{Var}(g(X)))$. For X positive and g rapidly decreasing, the bias and variance of $g(X)$ should be relatively insensitive to the tail behavior of X, and σ^2 should therefore not play an important role. In practice, when σ^2 is very large the approximations for rapidly decreasing functions are often poor.

We take the point of view that σ^2 is not measuring that aspect of variability which is relevant to the behavior of rapidly decreasing functions of X. For this purpose, a possibly more informative measure of variability is

$$c^2 = 1 - (EXEX^{-1})^{-1}. \tag{1.1}$$

For X positive and g completely monotone, we derive

$$0 \le [Eg(X) - g(EX)]/g(0) \le c^2 \tag{1.2}$$

$$\operatorname{Var}[g(X)/g(0)] \le c^2. \tag{1.3}$$

Thus if EX^{-1} is close to $(EX)^{-1}$ (as measured by c^2) then for g completely monotone, $g(X)/g(0)$ is close to a one-point distribution at $g(EX)/g(0)$.

The quantity c^2 has an additional interpretation. For a positive random variable X with distribution F, consider a stationary renewal process on the whole real line, with interarrival time distribution F. Define T to be the length of the interval which covers $\{0\}$, and $V = T^{-1}$. Then

$$\sigma_V^2 = (EXEX^{-1} - 1)/(EX)^2 = (EX^{-1}/EX)c^2 \tag{1.4}$$

$$\sigma_V^2/EV^2 = c^2. \tag{1.5}$$

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Moreover, defining $h(x) = (EX)xg(x^{-1})$, it follows that $g(EX) = h(EV)$ and

$$Eg(X) = Eh(V). \tag{1.6}$$

Use of (1.6) and a Taylor series argument yields

$$0 \le Eg(X) - g(EX) \le (c^2/2)EX^{-1}\sup(x^3 g''(x)). \tag{1.7}$$

Inequality (1.7) is less restrictive than (1.2) in that g need only be convex with $\sup x^3 g''(x) < \infty$ (rather than completely monotone).

2. Definitions and preliminary results. A function g on $[0, \infty)$ is defined to be completely monotone if it possesses derivatives of all orders and $(-1)^n g^{(n)}(\lambda) \ge 0$ for all $\lambda > 0$. In the above $g^{(0)} = g$. Some examples are $(x + a)^{-k}$ with $a > 0$, $k > 0$, $e^{-\alpha x}$ with $\alpha > 0$, and $e^{-\lambda(1-e^{-x})}$ with $\lambda > 0$. Lemma (2.1) below is due to Bernstein. An interesting discussion and proof of Bernstein's theorem is given in Feller (1971) page 439.

LEMMA 2.1. *A function g on $[0, \infty)$ is completely monotone with $g(0) = m < \infty$ if and only if it is of the form*

$$g(x) = \int_0^\infty e^{-\lambda x}\, dH(x), \tag{2.1}$$

where H is a positive measure on $[0, \infty)$ with $H([0, \infty)) = m$.

A distribution F on $[0, \infty)$ is defined to be NWUE (new worse than used in expectation) if $\mu = EX < \infty$ and $E(X - t \mid X > t) \ge EX$ for all $t \ge 0$. This property is easily shown to be equivalent to F stochastically smaller than G, where $G(x) = \mu^{-1} \int_0^x \bar{F}(t)\, dt$ is the stationary renewal distribution corresponding to F. Lemma 2.2 below is proved in the Appendix. Lemma 2.3 presents a Laplace transform inequality.

LEMMA 2.2. *Suppose that F is NWUE with mean γ and is absolutely continuous with failure rate h. Define b to be the essential supremum of h, assumed to be finite. Then, for $x \ge 0$*

$$0 \le \bar{F}(x) - e^{-bx} \le 1 - (b\gamma)^{-1}. \tag{2.2}$$

LEMMA 2.3. *Let $\mathscr{L}$ be the Laplace transform of a probability distribution on $[0, \infty)$ with $a = \int_0^\infty \mathscr{L}(\alpha)\, d\alpha < \infty$ and $\mu = -\mathscr{L}'(0) < \infty$. Then*

$$0 \le \mathscr{L}(\alpha) - e^{-\alpha\mu} \le 1 - (a\mu)^{-1} \quad \textit{for all} \quad \alpha \ge 0. \tag{2.3}$$

PROOF. Let X be a random variable with distribution F, and $\mathscr{E}$ be exponentially distributed with mean 1 independent of X. Then $\mathscr{L}(\alpha) = \Pr(X^{-1}\mathscr{E} > \alpha)$, thus $\mathscr{L}$ is the survival function of a completely monotone distribution, i.e., a mixture of exponential distributions. Completely monotone distributions are DFR (decreasing failure rate), which in turn are NWUE (Barlow and Proschan

(1975, pages 103, 159)). Since $X^{-1}\mathscr{E}$ is DFR, its maximum failure rate is achieved at zero and equals $-\mathscr{L}'(0) = \mu = EX$. The mean of $X^{-1}\mathscr{E}$ is $a = EX^{-1} = \int_0^\infty \mathscr{L}(\alpha)\, d\alpha$. The result follows by applying Lemma 2.2 with $\bar{F} = \mathscr{L}$, $b = \mu$, and $\gamma = a$.

3. Derivation of inequalities. Consider a positive random variable X with $a = EX^{-1}$ assumed finite, as well as $\mu = EX$. Define $c^2 = 1 - (a\mu)^{-1}$. Note that $0 \le c^2 < 1$ with equality if and only if X is a constant.

THEOREM 3.1. *Let g be a completely monotone function on $[0, \infty)$ with $g(0) < \infty$. Then*

$$0 \le Eg(X) - g(\mu) \le c^2 g(0) \tag{3.1}$$

$$\mathrm{Var}(g(X)) \le c^2 g^2(0). \tag{3.2}$$

PROOF. By Lemma 2.3,

$$0 \le \mathscr{L}(\alpha) - e^{-\alpha\mu} \le c^2 \quad \text{for all} \quad \alpha \ge 0. \tag{3.3}$$

Since g is completely monotone, by Lemma 2.1 there exists a measure H on $[0, \infty)$ with $H[0, \infty) = g(0)$ and

$$g(x) = \int e^{-\alpha x}\, dH(\alpha). \tag{3.4}$$

Now

$$Eg(X) = \int\int e^{-\alpha x}\, dH(\alpha)\, dF(x) = \int \mathscr{L}(\alpha)\, dH(\alpha) \tag{3.5}$$

$$g(\mu) = \int e^{-\alpha\mu}\, dH(\alpha). \tag{3.6}$$

Since g is convex, $Eg(X) \ge g(\mu)$. Thus from (3.3) and (3.5)

$$0 \le Eg(X) - g(\mu) = \int_0^\infty (\mathscr{L}(\alpha) - e^{-\alpha\mu})\, dH(\alpha) \le c^2 g(0). \tag{3.7}$$

Since g is completely monotone so is g^2 (Feller, 1971, page 441). Applying (3.7) to g^2 we obtain

$$0 \le Eg^2(X) - g^2(\mu) \le c^2 g^2(0). \tag{3.8}$$

From (3.7) and (3.8),

$$\mathrm{Var}(g(X)) = Eg^2(X) - (Eg(X))^2 \le (g^2(\mu) + c^2 g^2(0)) - g^2(\mu) = c^2 g^2(0).$$

This concludes the proof.

Given $X > 0$ with distribution F, consider a stationary renewal process on the whole real line with interarrival time distribution F. Define T to be the length of the interval containing 0. It follows from Feller (1971) page 371 that

$$dF_T(x) = x\, dF(x)/\mu. \tag{3.9}$$

From (3.9) we see that $ET^{-1} = \mu^{-1}$ and $ET^{-2} = a\mu^{-1}$ where $a = EX^{-1}$. Defining $V = T^{-1}$ it follows that $\sigma_V^2 = (a\mu - 1)\mu^{-2} = c^2 a\mu^{-1}$, while $\sigma_V^2/EV^2 = c^2$. Also from (3.9) we see that $Eg(X) = E(\mu V g(V^{-1})) = Eh(V)$ where $h(x) = \mu x g(x^{-1})$. Note that $h(EV) = h(\mu^{-1}) = g(EX)$. Finally since

$$h(V) = h(EV) + (V - EV)h'(EV) + ((V - EV)^2/2)h''(V^*) \tag{3.10}$$

with V^* between EV and V, it follows that

$$Eg(X) \le h(EV) + (\sigma_V^2/2)\sup(h''(x)) = g(\mu) + (c^2 a\mu^{-1}/2)\sup(h''(x)). \tag{3.11}$$

But $h''(x) = \mu x^{-3} g''(x^{-1})$, and thus $\sup h''(x) = \mu \sup(x^3 g''(x))$. Thus from (3.11), for g convex with $\sup x^3 g''(x) < \infty$,

$$0 \le Eg(X) - g(\mu) \le (c^2/2)a \sup(x^3 g''(x)). \tag{3.12}$$

4. Comments and additions.

1. The relationship between X and $V = T^{-1}$ discussed in Section 3 is symmetric. A renewal process with interarrival time distribution V has a distribution of X^{-1} as the length of the interval covering zero in its stationary renewal process. Thus the dual identity to (1.5) is $\sigma_X^2/EX^2 = 1 - (EVEV^{-1})^{-1} = 1 - (ETET^{-1})^{-1}$.

2. The identity $ET^{-1} = (EX)^{-1}$ is a special case of a more general identity. If $S_n = \sum_1^n X_i$ where $X_1, \cdots, X_n$ are i.i.d. as X, then

$$E(S_n + T)^{-1} = [(n+1)\mu]^{-1}.$$

This can be seen by noting that the Laplace transform of T equals $-\mathscr{L}'(\alpha)/\mu$.

3. Lemma 3.3 is in the spirit of inequality (xi) page 422 of Brown (1983). Lemma 3.3 is more general in that it applies to the larger NWUE class, but more restrictive in that it bounds the sup norm distance between cdf's rather than the sup norm distance over all Borel sets.

4. The value of c^2 is identical for X and X^{-1}. Thus (3.1) and (3.2) hold with X^{-1} replacing X, and $a = EX^{-1}$ replacing μ in the left side of (3.1).

5. I do not know to what extent inequalities (1.2), (1.3) and (1.7) can be improved.

5. Appendix. A proof of Lemma 2.2 is now given. Note that $h(x) \le b$ implies that F is stochastically larger than $b^{-1}\mathscr{E}$ (an exponential distribution with failure rate b), and if F is NWUE then F is stochastically smaller than G, the stationary renewal distribution corresponding to F. Thus

$$e^{-bt} \le \bar{F}(t) \le \bar{G}(t) \quad \text{for all} \quad t \ge 0. \tag{5.1}$$

From (5.1)

$$\sup_t |\bar{F}(t) - e^{-bt}| \leq \sup_t |\bar{G}(t) - e^{-bt}|. \tag{5.2}$$

Define $A = \{t: \gamma^{-1}\bar{F}(t) < be^{-bt}\}$ and $D = \sup_B |G(B) - \int_B be^{-bt}\, dt|$, the sup taken over all Borel subsets of $[0, \infty)$. Since G has pdf $\gamma^{-1}\bar{F}(t)$ it follows that

$$\sup_t |\bar{G}(t) - e^{-bt}| \leq D = \int_A \left[1 - \frac{\gamma^{-1}\bar{F}(t)}{be^{-bt}}\right] be^{-bt}\, dt. \tag{5.3}$$

But F is stochastically larger than $b^{-1}\mathscr{E}$, thus

$$[1 - \gamma^{-1}\bar{F}(t)/be^{-bt}] \leq 1 - (\gamma b)^{-1}. \tag{5.4}$$

Thus

$$D \leq [1 - (\gamma b)^{-1}]\Pr(b^{-1}\mathscr{E} \in A) \leq 1 - (\gamma b)^{-1}.$$

The result now follows from (5.2), (5.3) and (5.4).

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67 TINTERN LANE
SCARSDALE, NEW YORK 10583