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SOME INCOMPLETE BUT BOUNDEDLY COMPLETE LOCATION FAMILIES

BY L. MATTNER

Universität Hamburg

A general result concerning noncompleteness of location families of probability measures on Euclidean space is pointed out. Examples include boundedly complete families, such as those generated by certain scale mixtures of the standard Gaussian distribution. These examples illuminate completeness criteria for location families and compare favourably in simplicity with previously known examples of incomplete boundedly complete (nonlocation) families.

1. Introduction. Completeness of a family of probability measures can occasionally be used to simplify the mathematical analysis of estimation problems [see, e.g., pages 79–81 in Lehmann (1983), Theorem 2.1 in Ghosh and Singh (1966), and the theorem in Pfanzagl (1979)]. For applications to testing problems the weaker concept of bounded completeness is just as useful [see, e.g., Lehmann (1986), pages 144 and 300].

It is known from a few examples that, in general, bounded completeness is strictly weaker than completeness. The classical example can be found in [Lehmann (1986), page 173] and, apparently, the only further examples in the statistical literature are given by Bar-Lev and Plachky (1989) in the parametric case, and by Hoeffding (1977) in the nonparametric case.

Sometimes it turns out to be rather difficult to decide whether a given family is complete or at least boundedly complete. Hence general criteria are desirable. Many of the more important and better analysed families are either exponential families or group families. Exponential families with parameter space having nonempty interior are complete. No such simple condition is known for group families in general. However, partial results are known for the special case of location families on Euclidean spaces.

The present note quickly reviews classical and recent completeness criteria for location families (Theorems 2.1 and 2.2) and points out a general result on noncompleteness of (possibly boundedly complete) location families. Examples derived from this result serve two purposes. First, they show that the seemingly technical condition (i) in Theorem 2.2 must not be omitted without substitute. Second, they enlarge the short list of noncomplete boundedly complete families mentioned above by some quite natural examples.

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2. Location families. For $\vartheta \in \mathbb{R}^d$, let δ_ϑ denote the Dirac measure located at ϑ. For any probability measure μ on the Borel sets of $\mathbb{R}^d$, we call

$$\mathscr{P} = \{\delta_\vartheta * \mu : \vartheta \in \mathbb{R}^d\}$$

the location family generated by μ. If $\mu = f\lambda^d$, that is, μ has the density f with respect to Lebesgue measure λ^d, then we also say that f generates $\mathscr{P}$. In what follows, we write tx for the inner product and $|t|$ for the Euclidean norm of vectors t and x.

Bounded completeness of $\mathscr{P}$ in the absolutely continuous case is neatly characterized in terms of the characteristic function $t \mapsto \hat{\mu}(t) := \int \exp(itx)\, d\mu(x)$ of μ.

THEOREM 2.1. *If $\mu = f\lambda^d$, then $\mathscr{P}$ is boundedly complete if and only if $\hat{\mu}(t) \neq 0$ for every $t \in \mathbb{R}^d$.*

This follows from a classical approximation theorem of Wiener [see Reiter (1968) for an exposition], as observed by Ghosh and Singh (1966).

In an attempt to find an analogous criterion for completeness, the present author obtained the following result [see Mattner (1992)].

THEOREM 2.2. *Assume that $\mu = f\lambda^d$ and that the following conditions hold*:

(i) *$f(x + y) \leq C(1 + |x|^2)^k f(y)$ for some finite C and k and all $x, y \in \mathbb{R}^d$.*
(ii) *$\hat{\mu}$ is infinitely often differentiable in $\mathbb{R}^d \setminus A$ for some finite set A.*

Then $\mathscr{P}$ is complete if and only if $\hat{\mu}(t) \neq 0$ for every $t \in \mathbb{R}^d$.

Since the above conditions (i) and (ii) look a bit technical, it may seem likely that they could be omitted without affecting the validity of the theorem. However, if condition (i) is omitted, then Theorem 2.2 becomes false. For, as the following observation makes precise, in case of light tailed densities f [which are excluded by condition (i)], completeness implies zero-freeness of $\hat{\mu}$ even in part of $\mathbb{C}^d$. This is really a stronger condition than zero-freeness of $\hat{\mu}$ on $\mathbb{R}^d$, as is seen by considering Example 2.6.

LEMMA 2.3. *If $\int \exp(-s_0 x)\, d\mu(x) < \infty$ for some $s_0 \in \mathbb{R}^d$ and $\mathscr{P}$ is complete, then $\hat{\mu}(t + is) \neq 0$ for every $t \in \mathbb{R}^d$ and every s on the line segment joining 0 and s_0.*

PROOF. Let $g(x) := \exp(\zeta x)$ for $x \in \mathbb{R}^d$ with $\zeta = -s + it$. Since $\mathscr{P}$ is complete and g $\mathscr{P}$-integrable and does not vanish [$\mathscr{P}$], we have for some $\vartheta \in \mathbb{R}^d$

$$0 \neq \int g(x + \vartheta)\, d\mu(x) = e^{\zeta\vartheta}\hat{\mu}(-i\zeta),$$

that is, $\hat{\mu}(t + is) \neq 0$. □

Using Lemma 2.3, it is easy to show that, for very light tailed probability measures, completeness of $\mathscr{P}$ is very exceptional:

THEOREM 2.4. *Let μ be a probability measure on $\mathbb{R}^d$ such that for some ε, $\delta > 0$,*

$$\int \exp(\varepsilon |x|^{1+\delta})\, d\mu(x) < \infty. \tag{1}$$

Then $\mathscr{P}$ is complete if and only if μ is Gaussian (possibly degenerate).

COROLLARY 2.5. *If μ has compact support, then $\mathscr{P}$ is complete if and only if μ is a Dirac measure.*

PROOF OF THEOREM 2.4. The if part is, in case that μ is nondegenerate (i.e., has a Gaussian density with respect to λ^d), contained in the completeness theorem for exponential families mentioned in the Introduction. The obvious reduction of the degenerate cases (μ concentrated on some strict subspace of $\mathbb{R}^d$) to nondegenerate cases in lower dimensions is left to the reader.

To prove the only if part, assume that $\mathscr{P}$ is complete and let X be a random vector with distribution μ. Fix any $t \in \mathbb{R}^d$ with $|t| = 1$ and consider the location family $\mathscr{Q}$ generated by the distribution ν of tX.

$\mathscr{Q}$ is complete, since it is the family of distributions of the random variables $t(X + \vartheta)$ with $\vartheta \in \mathbb{R}^d$, that is, $\mathscr{Q}$ is induced by the complete family $\mathscr{P}$ via a statistic. Since $|t| = 1$, (1) clearly continues to hold when ν replaces μ. By an application of Young's inequality $ab \le (1/p)a^p + (1/q)b^q$ (a, $b \ge 0$, $1/p + 1/q = 1$) with $p = 1 + \delta$, we get

$$\begin{aligned} |\hat{\nu}(\zeta)| &\le \int \exp\big((\varepsilon^{1/p}|x|)(\varepsilon^{-1/p}|\zeta|)\big)\, d\nu(x) \\ &\le A \exp(B|\zeta|^q), \qquad \zeta \in \mathbb{C}^d \end{aligned}$$

for some finite constants A, B.

Thus $\hat{\nu}$ is an entire characteristic function of finite order without zeros (the latter in view of Lemma 2.3). By a theorem of Marcinkiewicz [see, e.g., Linnik and Ostrovskiĭ (1977), page 42, Corollary], ν must be Gaussian.

Since t was arbitrary, we conclude that μ is Gaussian. □

EXAMPLE 2.6. The following measures μ or densities f generate incomplete but boundedly complete location families $\mathscr{P}$ on $\mathbb{R}$.

(a) $f(x) = (1/6)(x^2 - 3)^2 \cdot (1/\sqrt{2\pi})e^{-x^2/2}$.

(b) Nontrivial finite scale mixtures of the standard Gaussian $N(0,1)$-distribution, for example,

$$\mu = \tfrac{1}{2} \cdot N(0,1) + \tfrac{1}{2} \cdot N(0,2).$$

(c) $f(x) = 1/2(1 - |x|)_+ + (1/2)(1/\alpha)(1 - |\alpha x|)_+$ with $\alpha > 0$ irrational.

(d) $\mu(\{-1\}) = \mu(\{1\}) = 1/6$, $\mu(\{0\}) = 2/3$.

PROOF. In each case noncompleteness of $\mathscr{P}$ obviously follows from Theorem 2.4. To verify bounded completeness, we apply Theorem 2.1. The characteristic functions are given by

$$\text{(a)} \qquad \hat{\mu}(t) = \frac{1}{6}\left(\left(\frac{1}{i}\frac{d}{dt}\right)^2 - 3\right)^2 e^{-t^2/2} = \frac{1}{6}(t^4 + 6)e^{-t^2/2},$$

$$\text{(b)} \qquad \hat{\mu}(t) = \tfrac{1}{2}e^{-t^2/2} + \tfrac{1}{2}e^{-2t^2},$$

$$\text{(c)} \qquad \hat{\mu}(t) = \frac{1}{2}\frac{\sin^2(t/2)}{(t/2)^2} + \frac{1}{2}\frac{\sin^2(t/2\alpha)}{(t/2\alpha)^2}$$

[see Feller (1971), page 502],

$$\text{(d)} \qquad \hat{\mu}(t) = \tfrac{2}{3} + \tfrac{1}{3}\cos t,$$

respectively. In each case $\hat{\mu}$ is zero-free on the real line [in case (c) observe that $\hat{\mu}(t) = 0$ would imply that $t/2\pi$ and $(t/2\pi)(1/\alpha)$ were nonzero integers, contradicting the irrationality of α]. □

REMARK. Evidently, Example 2.6(b) may be extended to nondiscrete mixing measures with compact support. On the other hand, Student's t-densities as well as the symmetric stable laws are examples of scale mixtures of $N(0,1)$-distributions (see the following Note) for which completeness of the generated location families has been proved [Mattner (1992), Theorems 3.1 and 3.2].

NOTE. The scale mixture properties stated above are well known and follow from the possibility of representing the corresponding characteristic functions as scale mixtures of the standard Gaussian characteristic function. In case of the t-densities see Mattner (1992) for an explicit formula. In case of the symmetric stable laws with characteristic function $\hat{\mu}(t) = e^{-|t|^\alpha}$ for some $\alpha \in (0,2)$ apply Bernstein's theorem to the completely monotone function $\lambda \mapsto \varphi(\lambda) = e^{-\lambda^{\alpha/2}}$ [Feller (1971), page 439 and Criterion 2 on page 441]. [This argument is essentially due to Bochner (1937)].

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UNIVERSITÄT HAMBURG
INSTITUT FÜR MATHEMATISCHE STOCHASTIK
BUNDESSTRASSE 55
D-20146 HAMBURG
GERMANY