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TREND-FREE RUN ORDERS OF MIXED-LEVEL FRACTIONAL FACTORIAL DESIGNS[1]

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Coster and Cheng presented a generalized foldover scheme for the construction of systematic run orders of fractional factorial designs, with all factors having the same prime power number of levels, for which all the main effects components of the factors are orthogonal to a polynomial trend present in every block of the design. In this paper, we present modifications to the foldover method that allow polynomial trend-free run orders to be constructed in the following more general settings: Designs for which the number of levels of each factor is not a prime power; mixed-level factorial designs with factors at different numbers of levels; cases in which some or all two- and higher-factor interactions, not just the main effects, are required to be orthogonal to the polynomial trend.

1. Introduction. Suppose that the treatment combinations of a given fractional factorial plan are to be performed in a time (or space) sequence and that the experimenter has reason to believe that the observed yields will be influenced by a temporal (spatial) trend over the course of the experiment. In such cases, instead of the normally recommended randomized orders for the runs in each block of the design, the experimenter may prefer certain systematic run orders that improve or maximize the efficiency with which the main effects and certain multifactor interactions are estimated in the presence of this nuisance trend. In this paper, we modify the generalized foldover scheme (GFS) of Coster and Cheng (1988) to achieve optimal efficiency for this estimation problem when the trend is modeled by a (typically, low-degree) polynomial over the equally spaced run positions of the observations in each block. We define optimal efficiency to be orthogonality between the factor effects of interest and the trend effects, in terms of the usual homoscedastic linear model (see Section 2).

The principal extensions made in this paper to the results in Coster and Cheng (1988) involve the specification of sufficient conditions on the appearance of factors at nonzero levels in sequences of generators of a fractional factorial design such that two- and higher-factor interactions also achieve the trend orthogonality criterion previously applied only to the problem of main effects estimation. We further generalize the foldover approach to designs that need not have every factor with the same number of levels (mixed-level factorials) and the number of levels need not be a prime power.

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Cox (1951) introduced systematic designs for replicated variety trials with the criterion of efficient estimation of the treatment effects in the presence of a smooth polynomial trend. Other early approaches to the problem of trend elimination are discussed in Draper and Stoneman (1968), Dickinson (1974) and Cheng (1985). John (1990) found factorial arrangements for 2^n and 3^n designs for which the main effects and sometimes two-factor interactions were linear and quadratic trend free. Cheng and Jacroux (1988) and Cheng (1990) discuss an alternative and elegant approach, first introduced by Daniel and Wilcoxon (1966), to the trend elimination problem for main effects and interactions in unblocked two-level fractional factorial designs. For the 2^n series, they provide a construction technique with this approach that is essentially equivalent to the GFS generator sequence shown in Example 2 of Section 3. Bailey, Cheng and Kipnis (1992) continued this development to mixed-level factorial designs. Cheng (1990) also discusses the correspondence between the foldover method and the Daniel and Wilcoxon scheme for designs with all factors at two levels.

The primary advantage of the GFS for achieving the trend orthogonality optimality criterion is the ease with which an experimenter can try various generator sequences and quickly verify, using the sufficient conditions detailed in Theorems 1, 2 and 3, whether trend elimination for the factor effects of interest has been achieved. Except in very small designs, it is not difficult to achieve orthogonality for the main effects components. Interactions present a greater challenge, in particular when we model nonprime-leveled factors with prime-leveled pseudofactors and *require* interactions among the pseudofactors belonging to each real factor to be trend free. Examples 3 and 4 fall into this category. Numerous examples of fractional factorial designs for factors with two or three levels are available in two National Bureau of Standards publications, *Applied Mathematics Series* **48** (1957) and **54** (1959). As an illustration of the application of the GFS to many examples, for the designs in these two publications, Coster (1993) used the GFS to search among all generator sequences that produced main-effect linear trend-free orders (with minimum number of factor level changes) for a generator sequence that maximized the number of quadratic trend-free components.

In Section 2, we summarize the definition of the mixed-level fractional factorial designs to which the modified GFS is applied in Section 3 to obtain optimal, trend-free run orders. All proofs of construction results are left until the Appendix.

2. Mixed-level fractional factorial model. Let G denote a fractional factorial design involving n factors, say $a_1, \dots, a_n$, with $n_i \geq 1$ of these factors at s_i levels, for $i = 1, \dots, q$, where each s_i is a *distinct prime number*. Then $n = n_1 + \cdots + n_q$. Let the levels of any factor with s_i levels be the set of integers $\{0, 1, \dots, s_i - 1\}$, with all arithmetic on these levels being performed modulo s_i. A complete factorial design in all n factors would require $s_1^{n_1} s_2^{n_2} \cdots s_q^{n_q}$ runs or treatment combinations.

We use the notation

$$G = s_1^{(n_1, p_1, r_1)} \cdots s_q^{(n_q, p_q, r_q)} \tag{2.1}$$

to denote a (possibly) blocked and/or fractional factorial design in the n factors. For each $i = 1, \ldots, q$ and the n_i factors with s_i levels, let $G_i = s_i^{(n_i, p_i, r_i)}$ denote an $s_i^{-p_i}$ fraction of the complete factorial design found by selecting a set of p_i independent defining effects, $n_i > p_i \geq 0$, involving only these n_i factors. We assume that the fraction chosen always contains the treatment combination **1** in which all the n_i factors are at level 0. Another r_i independent effects are used to block the $s_i^{n_i - p_i}$ runs into $s_i^{r_i}$ blocks each of size $s_i^{n_i - p_i - r_i}$. Then the notation of (2.1) implies that $G = G_1 \times G_2 \times \cdots \times G_q$, the product of the subdesigns G_i, that is, if run $\mathbf{g}_i \in G_i$, for $i = 1, \ldots, q$, then G contains treatment combination $\mathbf{g}_1 \mathbf{g}_2 \cdots \mathbf{g}_q$. We assume that the principal blocks of each sub-design generate the principal block of G, the block containing the run **1**. Thus, G contains $N = s_1^{n_1 - p_1} s_2^{n_2 - p_2} \cdots s_q^{n_q - p_q}$ runs blocked into $B = s_1^{r_1} s_2^{r_2} \cdots s_q^{r_q}$ blocks each of size $R = N/B$. While other methods may be used to define mixed-level fractional factorial designs, this product structure for G has, for our purposes, the two-fold advantage that (i) it is an easily applied method that is in common use and (ii) it proves compatible with the generalized foldover scheme used below to achieve the design objective of trend orthogonality.

Define $r = r_1 + \cdots + r_q$ and $p = p_1 + \cdots + p_q$. Let B_1 denote the principal block of G. Then the treatment combinations of G may be found by choosing $h = n - p - r$ independent runs in B_1 and forming all possible powers and products among them to generate B_1. There are $h_i = n_i - p_i - r_i$ *within-block* generators of B_1 contributed by each subdesign G_i, say $\mathbf{g}_{ij}$, $j = 1, \ldots, h_i$, $i = 1, \ldots, q$. Then every run in B_1 has the general form

$$\mathbf{g} = \prod_{i=1}^{q} \left[\prod_{j=1}^{h_i} \mathbf{g}_{ij}^{\xi_{ij}} \right],$$

where ξ_{ij} ranges over the set $\{0, \ldots, s_i - 1\}$ for each $i = 1, \ldots, q$. The remaining $B - 1$ blocks of G may be generated in a similar fashion using independent *between-block* runs $\mathbf{g}_1, \ldots, \mathbf{g}_r$ from distinct blocks of G.

In Coster and Cheng (1988), trend orthogonality via the GFS was developed for main effects (only) plans with all n factors having the *same* prime power number of levels, s^m, $m \geq 1$, s prime. The results of Section 3 allow us to cover not only this prime-powered case but more general mixed-level fractional factorial designs by modeling non-prime leveled factors by products of prime leveled *pseudofactors* and requiring all main effects *and* interactions among these pseudofactors to achieve the trend orthogonality condition. Theorem 2 provides the primary construction results for this purpose.

Following the development in Coster and Cheng (1988), we define below the form of the polynomial trend present in each block of G and the main effects components of each of the n factors. In these definitions, s is any one of the prime numbers $s_1, \ldots, s_q$.

DEFINITION 1. The system of orthogonal polynomials on m equally spaced points $l = 0, \dots, m-1$ is the set $\{P_{km}, k = 0, 1, 2, \dots, m-1\}$ of polynomials satisfying

$$\sum_{l=0}^{m-1} P_{km}(l) = 0 \quad \text{for all } k \geq 1, \tag{2.2}$$

$$\sum_{l=0}^{m-1} P_{km}(l) P_{k'm}(l) = 0 \quad \text{for all } k \neq k', \tag{2.3}$$

where $P_{0m}(l) = 1$ and $P_{km}(l)$ is a polynomial of degree k. We assume that each polynomial in the system is scaled so that its values are always integers.

DEFINITION 2 (Factor effects). The s coefficients of the jth main effects component of a factor, $1 \leq j \leq s-1$, are $P_{js}(l)$, $0 \leq l \leq s-1$, the values of the orthogonal polynomial of degree j on s equally spaced points.

DEFINITION 3 (Trend effects). The R values of a polynomial trend of degree j, $1 \leq j \leq R-1$, in a block of size R are $P_{jR}(l)$, $0 \leq l \leq R-1$, the values of the orthogonal polynomial of degree j on R equally spaced points.

The linear model for the N observations is

$$\mathbf{Y} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{T})(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \boldsymbol{\beta}_3')' + \boldsymbol{\varepsilon}, \tag{2.4}$$

where $\boldsymbol{\varepsilon}$ is an N-vector of zero mean, uncorrelated random errors, $\mathbf{X}_1$ is an $N \times \tau$ matrix of factor effect coefficients, $\mathbf{X}_2$ is an $N \times B$ matrix of block effect coefficients, and $\mathbf{T}$ is an $N \times k$ matrix of polynomial trend coefficients, the same in every block, of degrees $1, \dots, k$. The first R rows of $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{T})$ correspond to the R treatment combinations in the principal block B_1, the next R rows to the treatment combinations in the second block, and so on. The terms $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_3$ are the corresponding factor, block and trend parameter effects, respectively. The product design definition of G, the assumption of the same degree trend in each block, together with the requirements that (i) only one effect from each alias set is included in $\mathbf{X}_1$ and (ii) no effects confounded with blocks are included in $\mathbf{X}_1$ imply that, for the normalized columns of $\mathbf{X}$:

$$\mathbf{X}_1'\mathbf{X}_1 = \mathbf{I}_\tau, \qquad \mathbf{X}_1'\mathbf{X}_2 = \mathbf{0}, \qquad \mathbf{X}_2'\mathbf{T} = \mathbf{0}.$$

We may now define the design criterion for trend elimination.

DEFINITION 4 (Design optimality). A run order of design G is optimal for the estimation of the factor effects of interest, $\boldsymbol{\beta}_1$, in the presence of a nuisance k-degree polynomial trend in each block, if

$$\mathbf{X}_1'\mathbf{T} = \mathbf{0}. \tag{2.5}$$

If condition (2.5) is satisfied, we say that the run order of G is k-trend free.

If $\mathbf{x}$ is any un-normalized column of $\mathbf{X}_1$ and $\mathbf{t}$ any column of $\mathbf{T}$, then we call the usual inner product $\mathbf{x}'\mathbf{t}$ the time count between $\mathbf{x}$ and $\mathbf{t}$. Criterion (2.5) states that all the time counts are zero for an optimal run order. As stated in the Introduction, our primary objective is to satisfy optimality condition (2.5) in a setting where $\mathbf{X}_1$ contains columns representing main effects plus certain two- and higher-factor interactions among factors not constrained to have the same prime power number of levels.

3. Construction of optimal run orders by the GFS. We begin by modifying the generalized foldover scheme, GFS, of Coster and Cheng (1988) for the mixed-level fractional factorial designs defined in the previous section. We then present conditions under which both main effect and interaction components of the n factors become orthogonal to the polynomial trend. This leads to a stepwise construction method for optimal run orders of G. In what follows, we may assume that G is run in a single block of size N. The usual block structure is replaced after Theorem 2 and the advantage of blocking is demonstrated by Theorem 3.

DEFINITION 5 (GFS for G). Suppose that $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$ are $n-p$ generators of G. Assume that $\mathbf{g}_j$, $j = 1, \dots, n-p$, contains at a *nonzero* level at least one factor with f_j levels, $f_j \in \{s_1, \dots, s_q\}$, and that any run $\mathbf{g}$ of the design is of the form $\mathbf{g} = \prod_{i=1}^{n-p} \mathbf{g}_i^{\xi_i}$ for a unique choice of ξ_i, $0 \le \xi_i \le f_i - 1$, $i = 1, \dots, n-p$. Let $U_0 = \mathbf{1}$. Then the run order of G produced by the GFS with respect to generator sequence $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$ and *foldover sequence* $\{f_1, \dots, f_{n-p}\}$ is given by U_{n-p} where

$$(3.1) \qquad U_j = U_{j-1}^*(\mathbf{g}_j) = \left(U_{j-1}, U_{j-1}\mathbf{g}_j, \dots, U_{j-1}\mathbf{g}_j^{f_j - 1}\right), \qquad j = 1, \dots, n-p.$$

EXAMPLE 1. Let $G = 2^{(2,1,0)}3^{(2,1,0)}$ be defined by $I = AB$ for the two factors with two levels and by $I = CD$ for those with three levels. If we choose $\mathbf{g}_1 = abc^2d$ and $f_1 = 2$ followed by $\mathbf{g}_2 = cd^2$ and $f_2 = 3$, by Definition 5, the GFS (3.1) generates $U_1 = (\mathbf{1}, abc^2d)$ and $U_2 = (U_1, U_1cd^2, U_1c^2d)$ giving run order

$$G = U_2 = (\mathbf{1}, abc^2d, cd^2, ab, c^2d, abcd^2),$$

which is not the simple product $G_1 \times G_2 = (\mathbf{1}, ab, cd^2, abcd^2, c^2d, abc^2d)$, where $G_1 = (\mathbf{1}, ab)$ and $G_2 = (\mathbf{1}, cd^2, c^2d)$.

In Example 1, since $f_1 = 2$, the highest power of $\mathbf{g}_1$ used is $f_1 - 1 = 1$. The choice of generators in Example 1 would fail if, instead, f_1 were set to 3, since $\mathbf{g}_1^2 = \mathbf{g}_2$. In this case, a different choice of $\mathbf{g}_2$ and f_2 would be required.

DEFINITION 6 (k-trend free factor effects over U_j). Let factors $a_1, \dots, a_m$, $m \ge 1$, have $t_1, \dots, t_m$ levels, respectively. Then the m-factor interaction in factors $a_1, \ldots, a_m$ is k-trend free for some $k \geq 0$ over the run order of U_j if

(a) at least one of the m factors, say a_1, occurs equally often at each of its levels over U_j and

(b) *all* $t = \prod_{l=1}^{m}(t_l - 1)$ interaction components are orthogonal to the trend polynomials $P_{0N}, \ldots, P_{kN}$.

Condition (a) of Definition 6 ensures that our definition of k-trend free interactions is in keeping with the definition for k-trend free main effects only, that is, $m = 1$, in Coster and Cheng (1988). When constructing run orders with the GFS (3.1) of Definition 5, if factor a_l with t_l levels, $l = 1, \ldots, m$, is at a nonzero level in some generator $\mathbf{g}_\nu$ of U_j for which $f_\nu = t_l$, $\nu \leq j$, then a_l occurs equally often at each of its levels over U_j. By appearing equally often at each of its levels, the main effects components of factor a_l are orthogonal to the constant polynomial P_{0N} and we say that a_l is 0-trend free. However, we do not require all m factors to meet condition (a) of Definition 6 because it is frequently possible for all t of the interaction components to be k-trend free before each one of the m factors has appeared at a nonzero level and at its foldover level in some generator. For example, if $G = 2 \times 3$ is a factorial design with factor a at three levels and factor b at 2 levels, and if $\mathbf{g}_1 = a$, $f_1 = 3$ and $\mathbf{g}_2 = b$, $f_2 = 2$, then both components of the AB-interaction are zero-trend free over U_1 and are one-trend free over $G = U_2$.

Among the foldover levels $\{f_j, j = 1, \ldots, n - p\}$ there are exactly $n_i - p_i$ appearances of the level s_i, $i = 1, \ldots, q$. If this last condition were not met, G would not be correctly generated. Note that generator $\mathbf{g}_j$ may contain, at nonzero levels, other factors with numbers of levels not equal to the foldover level f_j. This has the advantage that it easily produces run orders of G that are not simply the product of the separately ordered subdesigns G_i (see the last part of Example 1). This latter run order, while it might have the trend orthogonality properties we seek, would be considered too systematic for many practical applications.

We now state our primary construction results. Theorem 1 is a generalization, for our mixed-level factorial design structure, of the results in Coster and Cheng (1988) that guarantee k-trend free main effects components. Essentially the same conditions must be met. Our primary result, Theorem 2, provides sufficient conditions on the generator and foldover sequences that ensure trend orthogonality for two- and (possibly) higher-factor interactions. We then recover the usual block structure and show how this is useful in Theorem 3. Before presenting these theorems, we state in Lemma 1 the essential requirement that any m-factor interaction that is k-trend free over $U_{\nu-1}$ remains k-trend free over U_ν (and hence over G). In what follows, the terminology and notation of Definition 5 apply. Proofs are in the Appendix.

LEMMA 1. *Suppose that the m-factor interaction in factors $a_1, \ldots, a_m$, $m \geq 1$, is k-trend free, $k \geq 0$, over $U_{\nu-1}$ according to Definition 6, for some $\nu \in \{2, \ldots, n - p\}$. Then this same interaction is also k-trend free over each piece $U_{\nu-1}\mathbf{g}_\nu^j$, $j = 1, \ldots, f_\nu - 1$, of U_ν and hence is k-trend free over U_ν.*

Note that some of the components of the m-factor interaction of interest may not be part of the columns of the effects matrix $\mathbf{X}_1$ of (2.4) because of aliasing or confounding in the blocking and fractionating schemes. However, our definition applies to all the components of the interaction even though some of these components are not part of the final estimation problem.

THEOREM 1. *Let a be any one of the n factors and let a have s levels, $s \in \{s_1, \dots, s_q\}$, s prime. Suppose that factor a is at a nonzero level in $(k+1)$, $k \geq 0$, of the generators for which the corresponding foldover level is s. Then all $s-1$ main effects components, as given by Definition 2, are k-trend free over G.*

Theorem 1 is, in fact, a special case of Theorem 2(ii) below and no proof of Theorem 1 is given in the Appendix. However, we have stated the sufficient conditions for main effects to be k-trend free in a separate theorem because of the assumed importance to the experimenter of achieving trend orthogonality for main effects before being concerned about interaction components.

Each factor must be at a nonzero level at its foldover level in at least one generator, that is, the main effects of all n factors are necessarily zero-trend free over G. If the conditions of Theorem 1 are met for all n factors, the resulting run order is an optimal k-trend free main effects plan. Note one limitation of the GFS for constructing k-trend free main effects run orders of an unblocked design G: if $n_i - p_i = 1$ when $s_i \geq 3$, so that there is only one use of foldover level s_i, or $n_i - p_i \leq 2$ when $s_i = 2$, then the main effects of the factor(s) with s_i levels cannot be made k-trend free for any $k \geq 1$. For example, $G = 2^2$ does not have a one-trend free main effects run order. Indeed, any design having only four distinct treatment combinations among the set of factors with two levels suffers from this same limitation. Consequently, no one-trend free run order of $G = 6^2$ can be found by the GFS using two pseudofactors each with two levels.

THEOREM 2. *Suppose that all the components of an $(m-1)$-factor interaction, $(m \geq 2)$, involving factors $a_1, \dots, a_{m-1}$ are k-trend over $U_{\nu-1}$, $2 \leq \nu \leq n-p$. Suppose that a factor a_m with s levels is at a nonzero level $\xi_{m\nu}$ in $\mathbf{g}_\nu$ and the foldover level is f_ν. Then one of the following cases may apply:*

(i) *If $a_m \notin \{a_1, \dots, a_{m-1}\}$, then the m-factor interaction in factors $a_1, \dots, a_m$ is k-trend free over U_ν if factor a_m is at level zero in generators $\mathbf{g}_1, \dots, \mathbf{g}_{\nu-1}$ of $U_{\nu-1}$.*

(ii) *if $a_m \in \{a_1, \dots, a_{m-1}\}$, say $a_m = a_1$, then the $(m-1)$-factor interaction in factors $a_1, \dots, a_{m-1}$ is $(k+1)$-trend free over U_ν if $a_2, \dots, a_{m-1}$ are at level zero in $\mathbf{g}_\nu$ and $f_\nu = s$.*

Case (i) of Theorem 2 indicates that the m-factor interaction inherits the k-trend free property of the $(m-1)$-factor interaction, provided that factor a_m has not appeared at a nonzero level in the generators of $U_{\nu-1}$. Of greater import is condition (ii), as it produces an improvement in trend robustness for the $(m-1)$-factor interaction, from k to $(k+1)$, provided that $a_m = a_1$ appears at a nonzero level in $\mathbf{g}_\nu$ while the other $(m-2)$ factors are at level zero in this generator.

EXAMPLE 2. Suppose that the design is $G = s^{(n,0,0)}$, a complete factorial design in n factors each with s levels, where s is a prime number and $n \geq 4$. Then, for the following two cases:

(i) n is *even*, and we choose generators

$$\mathbf{g}_i = a_1 \cdots a_{i-1}a_{i+1} \cdots a_n, \qquad i = 1,\dots,n-1,$$
$$g_n = a_1 \cdots a_{n-2}a_{n-1}^{[q(s-1)(\mathrm{mod}\ s)]};$$

(ii) n is *odd*, and we choose generators

$$\mathbf{g}_1 = a_2 \cdots a_n, \qquad \mathbf{g}_{n-1} = a_1 \cdots a_{n-3}a_{n-2}^{[q(s-1)(\mathrm{mod}\ s)]}, \qquad \mathbf{g}_n = a_n,$$
$$\mathbf{g}_i = a_1 \cdots a_{i-1}a_{i+1}a_{n-1}, \qquad i = 2,\dots,n-2.$$

The resulting run order of G has all $n(s-1)$ main effects components and all $n(n-1)(s-1)^2/2$ two-factor interaction components one-trend free. Note that the integer q in the generator expressions above is any choice of $1 \leq q \leq s-1$ such that we obtain a complete set of n independent generators. It is sufficient to choose $q \neq (n-2)$ (mod s). Note that each factor is at a nonzero level in at least two generators, making main effects at least one-trend free. For each pair of distinct factors a_i, a_j, $i \neq j$, at least one generator has a_i at a nonzero level while a_j is at level zero and the converse holds in some other generator. Hence, by Theorem 2(ii), all two-factor interactions are one-trend free.

For example, the design $G = 2^{(4,0,0)}$ presented in Coster and Cheng (1988) would now use generator sequence $\{bcd, acd, abd, abc\}$, while design $G = 2^{(5,0,0)}$ would have generator sequence $\{bcde, acd, abd, abc, e\}$. In a similar fashion, design $G = 3^{(4,0,0)}$ is generated by the sequence $\{bcd, acd, abd, abc^2\}$, except that the foldover level is now 3, not 2, at every stage. For this last case, the value $q = 1$ was used for $\mathbf{g}_4$ since $(4-2) = 2$ (mod 3) cannot be used.

Our construction objectives are now apparent. To achieve main effects orthogonality to a kth degree trend, we seek a generator foldover sequence that has each factor appearing at a nonzero level and at its foldover level in $(k+1)$ generators. Interactions involving two factors are k-trend free if one factor is at a nonzero level while the other is at level zero in $(k+1)$ generators with the appropriate foldover levels, or if case (i) applies to the two factors of interest. We may similarly proceed to conditions for higher factor interactions to be trend free. When attempting to meet the trend orthogonality conditions for main effects, if the foldover level for a generator is f_ν, we would like to have as many as possible of the factors with f_ν levels appearing at a nonzero level in $\mathbf{g}_\nu$. Conversely, trend orthogonality for two-factor interactions requires these factors to be isolated from one another in $\mathbf{g}_\nu$, that is, one at level zero and the other at a nonzero level in this generator. Clearly, these conditions compete with each other and in some cases run orders with, say, all main effects and two-factor interaction components k-trend free cannot be generated by the GFS.

We now replace the usual block structure of G defined by (2.1) and state a theorem that exploits the block structure and, in particular, the assumption that the same degree trend is present in every block. Recall that the principal block uses $h = n - p - r$ generators. Let $\mathbf{g}_\nu$, $h + 1 \leq \nu \leq n - p$, be a between block generator with foldover level f_ν. Then, with the GFS construction method, the following conditions hold (see the Appendix for the proof).

THEOREM 3 (Blocked designs). *Suppose that factor a_1 with $t_1 = f_\nu$ levels is at a nonzero level in $\mathbf{g}_\nu$. Then*:

(i) *all $t_1 - 1$ main effects components of a_1 are orthogonal to all the trend columns of matrix* $\mathbf{T}$ *of model* (2.4);

(ii) *if factors $a_2, \dots, a_m$ are at level zero in $\mathbf{g}_\nu$, then the m-factor interaction in $a_1, \dots, a_m$ is orthogonal to all the trend columns of matrix* $\mathbf{T}$ *of model* (2.4).

The approach of Bailey, Cheng and Kipnis (1992) and the GFS ideas presented here provide the experimenter with alternative (although closely related) methodologies and some flexibility in constructing trend-free run orders. One possible advantage to the use of pseudofactors and the GFS might be as follows. A factor with a nonprime number of levels frequently contains subsets of levels that are of special interest to the experimenter, for example, treatments versus controls. In small fractional factorial designs, it may not be possible to have all effects (say main effects and two-factor effects) trend free. Instead, the experimenter might use pseudofactors to describe the effects of special interest, and arrange for these to be trend-free, while sacrificing trend robustness for the other components associated with the overall effect. For example, suppose factor A at 4 levels represents two controls and two test treatments, in the order $(Control_1, Control_2, Treatment_1, Treatment_2)$. Code the levels of A by two two-level pseudofactors B_1 and B_2, with their four treatment combinations being $(11, 22, 21, 12)$, corresponding to the order of the levels of A given above. Then the B_1B_2 interaction contrast is equivalent to the contrast comparing average treatment effect to average control effect, usually a comparison of primary interest. If not all components of A can be made trend free, the experimenter might use a trend free order for the B_1B_2 interaction, giving lower priority to making the main effects of either pseudofactor trend free. If instead the factor A had six levels, then the second pseudofactor, B_1, would have three levels. Assuming that the B_1B_2 interaction again represents contrasts of special interest, the experimenter must look for generators and corresponding foldover levels to make the interaction components trend resistant. For example, $\mathbf{g}_1 = b_1$, $f_1 = 2$ and $\mathbf{g}_2 = b_2$, $f_2 = 3$ makes both interaction components one-trend-free, but the main effects components are only zero-trend-free.

We end this paper with two examples. Example 3 shows how interactions among pseudofactors are made trend free, resulting in trend free main effects components for the real factors. Example 4 combines both pseudofactors and blocking for a mixed-level, blocked factorial design.

EXAMPLE 3. Consider designs for n factors each with a proper prime-power number of levels. For our purposes, we will restrict attention to the prime powers 4, 8 and 9. (The prime powers 16, 25, 27 and so on seem unreasonably large for most practical factorial designs in at least $n \geq 2$ factors, especially complete factorial designs.)

To use the GFS construction method of Section 3 and the results developed there, we first define pseudofactors, each with a prime number of levels, to represent each real factor. If each factor a_i has s^m levels, s prime, $m > 1$, let m pseudofactors each with s levels be a_{ij}, $j = 1, \dots, m$. Then *all* main effects and interactions among the m pseudofactors are equivalent to the $s^m - 1$ main effects components of the real factor. Thus, a factor with four levels requires two pseudofactors each at two levels, a factor with nine levels is represented by two pseudofactors with three levels, and so on. Then, we have the following construction results:

(a) For $n \geq 2$ factors each at $t = 4$ or $t = 9$ levels, there is a run order of $G = t^n$ with all main effects components one-trend free. This follows from the fact that for the $2n \geq 4$ pseudofactors with two or three levels, respectively, Example 2 above produces a run order with all main effects and two-factor interactions one-trend free, which is more than sufficient to make the main effects of the real factors one-trend free.

(b) For $n \geq 2$ factors each at $t = 8$ levels, there is a run order of $G = t^n$ with all main effects components one-trend free. In this case, we have the additional requirement that the three factor interaction between each set of pseudofactors (a_{i1}, a_{i2}, a_{i3}) also be one-trend free. A sequence of generators having the required properties is

$$\mathbf{g}_1 = \prod_{i=1}^{n} a_{i1}, \qquad \mathbf{g}_2 = \prod_{i=1}^{n} a_{i2}, \qquad \mathbf{g}_3 = \prod_{i=1}^{n} a_{i3},$$

$$\mathbf{g}_4 = \left[\prod_{i=1}^{n-1} \prod_{j=1}^{3} a_{ij}\right] a_{n1} a_{n2}, \qquad \mathbf{g}_5 = a_{11} a_{n3}, \dots,$$

where the remaining $(n - 5)$ generators may be anything to complete the design. Note that the two- and three-factor interactions among each set of three pseudofactors are one-trend free over U_3, by Theorem 2(i) and (ii), while $\mathbf{g}_4$ and $\mathbf{g}_5$ simply meet the main effects requirement that each pseudofactor appear at least twice in the sequence.

EXAMPLE 4. Suppose G is a $2 \times 3 \times 4 \times 6$ factorial blocked into six blocks of size 24. Let factor a be at two levels; factor b be at three levels; factor c be at four levels; and factor d be at six levels. Let factor c be represented by two pseudofactors, c_1 and c_2, each at two levels. Let factor d be represented by two pseudofactors, d_1 at two levels and d_2 at three levels. Suppose that the effects confounded with blocks are $AC_1C_2D_1$, part of the ACD interaction, and BD_2. Then, if four within-block generators and two between-block generators and their foldover levels are:

$$\mathbf{g}_1 = ac_1,\ f_1 = 2, \qquad \mathbf{g}_2 = ac_2,\ f_2 = 2, \qquad \mathbf{g}_3 = ac_1c_2d_1,\ f_3 = 2,$$
$$\mathbf{g}_4 = bd_2^2,\ f_4 = 3, \qquad \mathbf{g}_5 = c_1c_2d_1,\ f_5 = 2, \qquad \mathbf{g}_6 = bd_2,\ f_6 = 3,$$

the resulting run-order is one-trend free in the four real factors. Note that $\mathbf{g}_1$ and $\mathbf{g}_2$ make the C_1C_2 interaction and the A main effect one-trend free. When $\mathbf{g}_3$ is used, the C_1 and C_2 main effects become one-trend free also, making real factor C main effects one-trend free over U_3. The D_1D_2 interaction is one-trend free over U_4. Because the foldover level can be 3 for only one within-block generator, the main effects of the three-leveled factors b and d_2 can only be made one-trend free "between blocks" after use of $\mathbf{g}_6$. The main effect of d_1 becomes one-trend free over U_5. Because of the blocking in this example, by Theorem 3, many of the two- and three-factor interactions among the six pseudofactors are also one-trend free.

APPENDIX

The proofs of Lemma 1 and Theorems 2 and 3 are presented here.

Proof of Lemma 1. Without loss of generality, let the m factors be $a_1, \dots, a_m$, with a_l having t_l levels, $l = 1, \dots, m$. Let f denote the number of runs in $U_{\nu-1}$. Let $\eta_l \in \{0, \dots, t_l - 1\}$ be the possible levels of factor a_l. In generator $\mathbf{g}_\nu$, let factor a_l be at level ξ_l, for some $\xi_l \in \{0, \dots, t_l - 1\}$.

Let the treatment combination $(\eta_1, \dots, \eta_m)$ occur $c(\eta_1, \dots, \eta_m)$ times over $U_{\nu-1}$ in run positions $i_{(\eta_1, \dots, \eta_m)j}$, $j = 1, \dots, c(\eta_1, \dots, \eta_m)$. By the statement of the lemma, the m-factor interaction is k-trend free over $U_{\nu-1}$. This means that, for the trend of degree u, $0 \le u \le k$, and for *every* component $(q_1, \dots, q_m)$ of the interaction, $1 \le q_l \le t_l - 1$,

$$0 = \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left[W_{uN}(\eta_1, \dots, \eta_m) \prod_{l=1}^{m} P_{q_l t_l}(\eta_l) \right], \tag{A.1}$$

where

$$W_{uN}(\eta_1, \dots, \eta_m) = \sum_{j=1}^{c(\eta_1, \dots, \eta_m)} P_{uN}\big(i_{(\eta_1, \dots, \eta_m)j}\big) \tag{A.2}$$

is the contribution to the time count, from the trend of degree u with any component of the m-factor interaction, made by the treatment combination $(\eta_1, \dots, \eta_m)$ over the run positions of $U_{\nu-1}$. If $c(\eta_1, \dots, \eta_m) = 0$, set $W_{uN}(\eta_1, \dots, \eta_m) = 0$.

Now consider the f run positions in $U_{\nu-1}\mathbf{g}_\nu$. Over these next f runs, the contribution to the time count from the trend of degree u made by those run positions containing treatment combination $(\eta_1, \dots, \eta_m)$ in $U_{\nu-1}$ is, by (A.2),

$$\sum_{j=1}^{c(\eta_1, \dots, \eta_m)} P_{uN}\big(i_{(\eta_1, \dots, \eta_m)j} + f\big)$$

$$= \sum_{j_1=0}^{u} \left[b_{j_1}(f) \sum_{j=1}^{c(\eta_1, \dots, \eta_m)} P_{j_1 N}\big(i_{(\eta_1, \dots, \eta_m)j}\big) \right] \tag{A.3}$$

$$= \sum_{j_1=0}^{u} b_{j_1}(f) W_{j_1 N}(\eta_1, \dots, \eta_m).$$

In the run positions of $U_{\nu-1}\mathbf{g}_\nu$, the treatment combination $(\eta_1, \dots, \eta_m)$ is replaced by $(\eta_1 + \xi_1, \dots, \eta_m + \xi_m)$, where it is understood that the addition $(\eta_j + \xi_j)$ is performed modulo t_j, $j = 1, \dots, m$. Hence, the value of the $(q_1, \dots, q_m)$-component of the m-factor interaction at such a run position becomes

$$\prod_{l=1}^{m} P_{q_l t_l}(\eta_l + \xi_l). \tag{A.4}$$

As was done with the trend polynomials $P_{uN}(\cdot)$ in (A.3), for each $l = 1, \dots, m$, we may express the term in the product of expression (A.4) as

$$P_{q_l t_l}(\eta_l + \xi_l) = \sum_{i_l=0}^{t_l-1} d_{i_l}(\xi_l) P_{i_l t_l}(\eta_l), \tag{A.5}$$

where the coefficients d_{i_l} depend on ξ_l but not on η_l. However, by Definition 2, summing the left-hand side of (A.5) over $\eta_l = 0, \dots, t_l - 1$ yields 0, as does each term on the right-hand side of (A.5) except the first, constant term involving $P_{0t_l}(\eta_l)$, where the sum is $d_0(\xi_l)t_l$. Hence, we must have that $d_0(\xi_l) = 0$ for $l = 1, \dots, m$. Then the time count T_u from the trend of degree u with the $(q_1, \dots, q_m)$ component of the interaction over the f run positions of $U_{\nu-1}\mathbf{g}_\nu$ is, using (A.3) and (A.5),

$$T_u = \sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} \left\{ \left[\sum_{j_1=0}^{u} b_{j_1}(f) W_{j_1 N}(\eta_1, \dots, \eta_m) \right] \prod_{l=1}^{m} P_{q_l t_l}(\eta_l + \xi_l) \right\}$$

$$= \sum_{j_1=0}^{u} \sum_{i_1=1}^{t_1-1} \cdots \sum_{i_m=1}^{t_m-1} \left\{ b_{j_1}(f) \prod_{l=1}^{m} d_{i_l}(\xi_l) \right.$$

$$\left. \times \left[\sum_{\eta_1=0}^{t_1-1} \cdots \sum_{\eta_m=0}^{t_m-1} W_{j_1 N}(\eta_1, \dots, \eta_m) \prod_{l=1}^{m} P_{i_l t_l}(\eta_l) \right] \right\}$$

$$= 0,$$

because, by (A.1), the inner summations within the brackets "[$\cdots$]" are all equal to 0.

Hence, every component of the m-factor interaction is k-trend free over $U_{\nu-1}\mathbf{g}_\nu$, and by the same argument is k-trend free over each piece $U_{\nu-1}\mathbf{g}_\nu^w$,

$w = 1, \ldots, f_\nu - 1$ with ξ_l replaced by $w\xi_l$ in the expressions above. So this interaction remains k-trend free over G. □

PROOF OF THEOREM 2. Before establishing the results, we introduce a slightly more convenient notation. Let the level of factor a_l be ξ_{lj} in generator $\mathbf{g}_j$, $l = 1, \ldots, m$ and $j = 1, \ldots, \nu$, and the foldover levels be f_j, as usual. For index $i_j \in \{0, \ldots, f_j - 1\}$, $j = 1, \ldots, \nu$, the run position in U_ν given by a fixed choice of $(i_1, \ldots, i_\nu)$ may be expressed as

$$\sum_{j=1}^{\nu} \left(\prod_{l_1=1}^{j-1} f_{l_1} \right) i_j = f i_\nu + d, \qquad f = \prod_{l_1=1}^{\nu-1} f_{l_1}, \qquad d = \sum_{j=1}^{\nu-1} \left(\prod_{l_1=1}^{j-1} f_{l_1} \right) i_j,$$

where d depends on $i_1, \ldots, i_{\nu-1}$ but not on i_ν. The level of factor a_l in this run position is of the form

$$\sum_{j=1}^{\nu} \xi_{lj} i_j = \xi_l^* + i_\nu \xi_{l\nu},$$

where ξ_l^* is also independent of i_ν. Arithmetic in the above expression is understood to be carried out modulo t_l. Then the assumptions of the theorem together with Lemma 1 imply that, for any $0 \le u \le k$ and all $1 \le q_l \le t_l - 1$, $l = 1, \ldots, m$,

$$0 = \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{\nu-1}=0}^{f_{\nu-1}-1} \left[P_{uN}(f i_\nu + d) \prod_{l=1}^{m-1} P_{q_l t_l}(\xi_l^* + i_\nu \xi_{l\nu}) \right], \tag{A.6}$$

for any $i_\nu = 0, \ldots, f_\nu - 1$.

PROOF OR PART (i). The time count over U_ν between the trend of degree k and any component of the m-factor interaction is expression (A.6) summed also over i_ν, except that the product at the end now runs from $l = 1$ to $l = m$ and includes a term of the form $P_{q_m t_m}(\xi_m^* + i_\nu \xi_{m\nu})$. By the assumptions of part (i), $\xi_m^* = 0$, independent of $i_1, \ldots, i_{\nu-1}$, so this additional term in the product may be taken outside the summations over $i_1, \ldots, i_{\nu-1}$ and by expression (A.6) the result is 0 for each value of i_ν. □

Note that part (i) allows a factor that has not yet appeared in any previous generator to inherit, in its interactions with those other factors already used in the earlier generators, any trend free properties existing among this first set of factors.

PROOF OF PART (ii). The time count over U_ν between the trend of degree $(k + 1)$ and any component of the $(m - 1)$-factor interaction is of the form

$$T(k+1, \nu) = \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_\nu=0}^{f_\nu-1} P_{k+1,N}(f i_\nu + d) \prod_{l=0}^{m-1} P_{q_l t_l}(\xi_l^* + i_\nu \xi_{l\nu}). \tag{A.7}$$

By the assumptions of part (ii), $\xi_{1\nu} = \xi_{m\nu} \neq 0$, $\xi_{l\nu} = 0$, $l = 2, \ldots, m - 1$ and

$f_\nu = s$. If we express the trend polynomial in (A.7) in the form

$$P_{k+1,N}(fi_\nu + d) = \sum_{u=0}^{k+1} \alpha_u(fi_\nu) P_{uN}(d), \tag{A.8}$$

where $\alpha_{k+1}(fi_\nu) = 1$ independent of i_ν, we may write (A.7) as

$$\begin{aligned} T(k+1,\nu) &= \sum_{u=0}^{k+1} \sum_{i_\nu=0}^{s-1} \alpha_u(fi_\nu) \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{\nu-1}=0}^{f_{\nu-1}-1} \\ &\quad\times\left\{\left[P_{uN}(d) \prod_{l=2}^{m-1} P_{q_l t_l}(\xi_l^*)\right] P_{q_m t_m}(\xi_m^* + i_\nu \xi_{m\nu})\right\}. \end{aligned} \tag{A.9}$$

For $u = 0, \dots, k$, the inner summations over $i_1, \dots, i_{\nu-1}$ yield 0 for any value of i_ν by the assumptions of the theorem and Lemma 1. Then, since $\alpha_{k+1}(fi_\nu) = 1$, expression (A.9) reduces to

$$\begin{aligned} T(k+1,\nu) &= \sum_{i_1=0}^{f_1-1} \cdots \sum_{i_{\nu-1}=0}^{f_{\nu-1}-1} \left\{ P_{k+1,N}(d) \prod_{l=2}^{m-1} P_{q_l t_l}(\xi_l^*) \right. \\ &\qquad \left. \times\left[\sum_{i_\nu=0}^{s-1} P_{q_m t_m}(\xi_m^* + i_\nu \xi_{m\nu})\right]\right\}. \end{aligned}$$

Finally, the inner summation over i_ν always yields 0 by (2.2) of Definition 1. Hence, the $(m-1)$-factor interaction is $(k+1)$-trend free over U_ν. □

Proof of Theorem 3. Let factor a_1 be at level $\xi \neq 0$ in $\mathbf{g}_\nu$.

Proof of part (i). When $U_{\nu-1}$ is folded over with respect to generator $\mathbf{g}_\nu$ at foldover level $f_\nu = t_1$, each block of size R of $U_{\nu-1}$ generates $f_{\nu-1} - 1$ new blocks of the same size. If factor a_1 is at level ξ_j in run position j, $j = 1, \dots, R$, in any block of $U_{\nu-1}$, then a_1 is at level $\xi_j + i_\nu \xi$ in some block of U_ν generated from this starting block. Since $\xi_j + i_\nu \xi$ takes each possible level of a_1 exactly once as i_ν runs from 0 to $f_\nu - 1$, and since the trend polynomial of any degree k is the same in every such block in run position j, the contribution to the time count from starting position j for any main effects component q_1 of a_1 is given by

$$\sum_{i_\nu=0}^{f_\nu-1} P_{q_1 t_1}(\xi_j + i_\nu \xi) P_{kR}(j) \tag{A.10}$$

which is 0 for every j by (2.2) of Definition 1. Hence, a_1 is k-trend free over G for any $0 \le k \le R-1$. □

Proof of part (ii). For starting position $1 \le j \le R$ of any fixed block of $U_{\nu-1}$ and trend polynomial of degree $0 \le k \le R-1$, since $a_2, \dots, a_m$ are at level zero in $\mathbf{g}_\nu$, the product of $P_{kR}(j)$ and any component of the $(m-1)$-factor interaction in factors $a_2, \dots, a_m$ is constant for this same position j in all

$f_\nu - 1$ blocks generated from the current starting block of interest. So the time count contributed by position j is the same as (A.10) except for the addition of a constant product, that does not depend on i_ν, involving any component of the $(m - 1)$-factor interaction. The resulting sum is again 0 for every j and hence the m-factor interaction of $a_1, \dots, a_m$ is k-trend free over G. □

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