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ON THE BALANCED INCOMPLETE BLOCK DESIGN FOR RANKINGS[1]

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A total of nb judges rank t objects k at a time according to n replications of a BIBD with b blocks. The Durbin statistic is commonly used in this context and is equivalent to the usual analysis of variance on the rankings. The approach considered here is to introduce the notion of compatibility so as to define distances between incomplete rankings based on metrics on the space of complete rankings. Through this device we define a class of test statistics which includes the Durbin statistic as a special case, and derive their asymptotic distributions. This analysis also yields a new interpretation of the Durbin statistic.

1. Introduction. A total of nb judges are presented k of a possible t objects to rank. Such rankings are conducted independently. The pattern of objects presented follows n replications of a basic balanced incomplete block design of b blocks of k rankings of t objects. Within each basic design every object is considered by r of b judges and each pair of objects is presented together to λ of these judges. For a balanced incomplete block design [Yates (1936)], the following must hold:

$$bk = tr, \tag{1.1}$$

$$\lambda = r(k-1)/(t-1). \tag{1.2}$$

Possible values for b, k, t, r and λ have been tabulated in Cochran and Cox (1957), pages 469–482.

The question of interest here is whether there exists a degree of agreement among the judges. The statistic commonly used in this context is due to Durbin (1951) and is a natural extension of the Friedman statistic applied to the case of a complete randomized block design. For another extension of the Friedman statistic to the case of unbalanced designs and combinations of balanced designs, see Benard and van Elteren (1953), Prentice (1979) and Skillings and Mack (1981). In this article we consider a general class of concordance statistics, develop their asymptotic distribution theory ($n \to \infty$) and compute their relative efficiencies.

We emphasize that the asymptotic results considered herein are in terms of n replications of a fixed design of b blocks under the restrictions (1.1) and

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(1.2). This point is not always made explicit in the literature, where the asymptotics are described in terms of the number of blocks increasing, which for fixed λ, t and k, can only be realized by replicating the design an arbitrarily large number of times.

2. Notation; the notion of compatibility.

Let the column vectors

$$\nu_j = \big(\nu_j(1), \dots, \nu_j(t)\big)', \qquad j = 1, 2, \dots, t!,$$

denote the $t!$ possible permutations of the integers $(1, 2, \dots, t)$. Thus $\{\nu_j\}$ is the collection of all possible complete rankings of the t objects. Occasionally we may denote a generic pair of elements of this collection by μ and η. Within the basic design of b blocks, each block represents a different pattern of k objects to be ranked. Let

$$\nu_j^{(l)} = \big(\nu_j^{(l)}(1), \dots, \nu_j^{(l)}(k)\big)', \qquad j = 1, \dots, k!,$$

represent the $k!$ possible permutations of the integers $(1, \dots, k)$, corresponding to all the possible k-partial rankings for each of the block patterns indexed by $l = 1, 2, \dots, b$. When presented with k objects according to block pattern l a judge selects a ranking from $\{\nu_j^{(l)}\}$ according to the probability vector

$$\Pi^{(l)} = \big(\pi_1^{(l)}, \dots, \pi_{k!}^{(l)}\big)'.$$

The $(bk! \times 1)$ vector of probabilities for the overall design is

$$\Pi^* = \big(\pi_1^{(1)}, \dots, \pi_{k!}^{(1)} | \pi_1^{(2)}, \dots, \pi_{k!}^{(2)} | \cdots | \pi_1^{(b)}, \dots, \pi_{k!}^{(b)}\big)'.$$

Setting $\Pi_0^* = (k!)^{-1}(1, \dots, 1 | \cdots | 1, \dots, 1)'$, the null hypothesis to be tested is

$$H_0\colon \Pi^* = \Pi_0^* \quad \text{against the alternative} \quad H_1\colon \Pi^* \neq \Pi_0^*.$$

We define the following notion of compatibility between a complete and an incomplete ranking.

DEFINITION 2.1. A complete ranking ν is said to be compatible with an incomplete ranking $\nu^{(*)}$ if the relative ranking of every pair of objects ranked in $\nu^{(*)}$ coincides with their relative ranking in ν.

In general, the number of complete rankings compatible with a specific incomplete ranking will be

$$a = t!/k!.$$

Ordering the complete rankings $\{\nu_j\}$ in some way, we may associate with every incomplete ranking $\nu^{(*)}$ a $(t! \times 1)$ compatibility vector, whose ith component is 1 or 0 according to whether ν_i is compatible to $\nu^{(*)}$ or not. The set of permutations $\nu_j^{(l)}$ corresponding to the block pattern l thus generates a $(t! \times k!)$ *matrix of compatibility* C_l, whose jth column is the compatibility vector associated with $\nu_j^{(l)}$, $j = 1, \dots, k!$. Within each block pattern a complete ranking is compatible to one and only one partial ranking, so that each row of

C_l contains exactly one element equal to 1 with the rest being 0's. In addition, each column of C_l contains $a = t!/k!$ elements equal to 1.

Finally, the overall compatibility matrix for the design is defined to be

$$C = (C_1|C_2|\cdots|C_b).$$

As an example, for $t = 3$ objects presented $k = 2$ at a time, we label the $3! = 6$ complete rankings as

$$\nu_1 = (123),\quad \nu_2 = (132),\quad \nu_3 = (213),$$
$$\nu_4 = (231),\quad \nu_5 = (312),\quad \nu_6 = (321).$$

The incomplete rankings are denoted by

$$\nu_1^{(1)} = (12\_),\quad \nu_2^{(1)} = (21\_),\quad \nu_1^{(2)} = (1\_2),$$
$$\nu_2^{(2)} = (2\_1),\quad \nu_1^{(3)} = (\_12),\quad \nu_2^{(3)} = (\_21).$$

Thus ν_1, ν_2 and ν_4 are compatible with $\nu_1^{(1)}$ and ν_3, ν_5 and ν_6 are compatible with $\nu_2^{(1)}$. The corresponding matrix of compatibility C_1 for block pattern 1 is

$$C_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The overall compatibility matrix for this design is found to be

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

3. A general class of statistics and their asymptotic distributions.

For the incomplete ranking problem a general class of statistics for H_0 may be defined through the notion of a distance function over the set of incomplete rankings. This approach parallels the development in Alvo, Cabilio and Feigin (1982) in the case of complete rankings. If $d(\nu_i, \nu_j)$ is a right-invariant metric defined for pairs of complete rankings ν_i, ν_j, denote by $\Delta = (d(\nu_i, \nu_j))$ the $(t! \times t!)$ matrix of values of d.

DEFINITION 3.1. The distance between the incomplete rankings $\nu_s^{(l)}$ and $\nu_r^{(m)}$ is denoted by $d^*(\nu_s^{(l)}, \nu_r^{(m)})$ and is defined to be the average of all values $d(\nu_i, \nu_j)$ taken over all complete rankings ν_i, ν_j compatible with $\nu_s^{(l)}, \nu_r^{(m)}$, respectively.

Note that d^* is not a metric since in general the distance thus defined between an incomplete ranking and itself is greater than 0.

With this definition, the corresponding matrix of values of d^* is given by

$$\Delta^* = a^{-2} C' \Delta C. \tag{3.1}$$

Two metrics of interest related to Spearman's ρ and Kendall's τ may be defined respectively by

$$(3.2)\qquad d_S(\mu,\eta) = \frac{1}{2}\sum_{q=1}^{t}(\mu(q)-\eta(q))^2,$$

$$(3.3)\qquad d_K(\mu,\eta) = \sum_{p<q}^{t}\{1-\operatorname{sgn}(\mu(p)-\mu(q))\operatorname{sgn}(\eta(p)-\eta(q))\}.$$

For these metrics, Feigin and Alvo (1986) show that there is a matrix T and a constant c such that

$$(3.4)\qquad \Delta = cJ - T'T,$$

where J is the $(t!\times t!)$ matrix of 1's. In what follows, J will always denote a matrix of 1's, but its dimensions may differ according to the context. In drawing conclusions concerning a particular probability model Π for the complete ranking design, $\Pi'\Delta\Pi$, the expected distance between the rankings of two randomly chosen judges becomes the parameter under consideration, so that the test statistic is based on the sampling estimates of $\Pi'T'T\Pi = (T\Pi)'(T\Pi)$. It follows from (3.1) and (3.4) that

$$(3.5)\qquad \Delta^* = ca^{-2}C'JC - (a^{-1}TC)'(a^{-1}TC);$$

consequently the test statistic for testing H_0 will be based on the sampling estimates of $(T^*\Pi^*)'(T^*\Pi^*)$, where

$$T^* = a^{-1}TC.$$

Thus define $f^{(l)} = (f_1^{(l)},\dots,f_{k!}^{(l)})'$ as the vector of frequencies of rankings for block pattern l. Set the $(bk!\times 1)$ vector

$$f = \left(f_1^{(1)},\dots,f_{k!}^{(1)}|f_1^{(2)},\dots,f_{k!}^{(2)}|\cdots|f_1^{(b)},\dots,f_{k!}^{(b)}\right)',$$

and let

$$\hat\Pi^{(l)} = f^{(l)}/n,\qquad \hat\Pi^* = f/n.$$

THEOREM 3.1. *Under H_0 as $n\to\infty$,*

$$\sqrt{n}\left(\hat\Pi^* - \Pi_0^*\right) \Rightarrow_{\mathscr{L}} N(0,\Sigma_0^*),$$

where

$$\Sigma_0^* = I\otimes\Sigma_0$$

is of dimension $(bk!\times bk!)$ and

$$\Sigma_0 = (k!)^{-2}(k!I - J)$$

is of dimension $(k!\times k!)$.

PROOF. Set $X_l = \sqrt{n}(\hat\Pi^{(l)} - \Pi_0^{(l)})$, where $\Pi_0^{(l)} = (k!)^{-1}(1,1,\dots,1)'$. Under H_0, $f^{(l)}$ is a realization of a multinomial $k!$ vector, so that from the central limit theorem it follows that as $n\to\infty$,

$$X_l \Rightarrow_{\mathscr{L}} N(0,\Sigma_0).$$

On noting that $\sqrt{n}(\hat{\Pi}^* - \Pi_0^*) = (X_1'|X_2'| \cdots |X_b')'$, the theorem now follows from the independence of the X_i''s. □

A direct result of the above and the distribution of quadratic forms of normal deviates gives the following corollary.

COROLLARY 3.1. *Under H_0, as $n \to \infty$,*

$$(T^*X)'(T^*X) \equiv a^{-2}(TCX)'(TCX) \Rightarrow_{\mathscr{L}} \Sigma\alpha_i z_i^2,$$

where $X = \sqrt{n}(\hat{\Pi}^* - \Pi_0^*)$, $\{z_i\}$ are independent standard normal variates and $\{\alpha_i\}$ are the eigenvalues of

$$T^*\Sigma_0^*(T^*)' = a^{-2}(TC)\Sigma_0^*(TC)'.$$

We note that the critical values of such quadratic forms may be approximated by various methods. To mention just a few, Imhof (1961) discusses a Pearson three-moment χ^2 approximation, Jensen and Solomon (1972) introduce an approximation of the Wilson–Hilferty type and Solomon and Stephens (1977) consider a four-moment fit to a power of a χ^2.

In the next section we conduct the required eigenanalysis for the statistics generated by the Spearman and Kendall metrics.

4. Asymptotic distributions for the Spearman and Kendall cases. In this section we derive the asymptotic distribution of the statistic based on the Spearman metric and show that this statistic is equivalent to the Durbin statistic. Following this, the Kendall case is considered, leading to a new test statistic. The matrices T corresponding to d_S and d_K are respectively:

$$T_S = (t_S(\nu_1), \ldots, t_S(\nu_{t!})) \tag{4.1}$$

of dimension $(t \times t!)$, where

$$t_S(\nu) = \left(\nu(1) - \frac{t+1}{2}, \ldots, \nu(t) - \frac{t+1}{2}\right)',$$

the centered rank vector, and

$$T_K = (t_K(\nu_1), \ldots, t_K(\nu_{t!})) \tag{4.2}$$

of dimension $\left(\binom{t}{2} \times t!\right)$, where the qth element of $t_K(\nu)$ is

$$\operatorname{sgn}(\nu(j) - \nu(i))$$
$$q = (i-1)(t - i/2) + (j - i), \qquad 1 \le i < j \le t,$$

so that $t_K(\nu)$ is the vector of pairwise concordances or discordances of the ranking ν with the identity rank permutation [see, e.g., Feigin and Alvo (1986).]

A property of both the matrices T_S and T_K is that their rows are orthogonal to the vector of 1's. This property is also shared by the corresponding matrices $T_S^* = T_S C$ and $T_K^* = T_K C$. This follows from the observation that the product of any row vector y with C_l results in a vector of partial sums of the elements of a partition of y. Thus $T_S C$ and $T_K C$ are orthogonal to Π_0^*, and the result of Corollary 3.1 holds equally well for $n^{-1}(T^*f)'(T^*f)$.

LEMMA 4.1. *Let R_q denote the sum of the ranks assigned to the q-th object. Then*

$$(4.3) \qquad a^{-1}(T_S Cf)'(T_S Cf) = \left(\frac{t+1}{k+1}\right)^2 \sum_{q=1}^{t} \left(R_q - nr\left(\frac{k+1}{2}\right)\right)^2,$$

$$(4.4) \qquad T_S CC' = \lambda \frac{(t+1)!}{(k+1)!} T_S.$$

PROOF. See the Appendix. □

We now conclude the eigenanalysis of the Spearman case and derive the asymptotic distribution of the test statistic.

THEOREM 4.1. *Under H_0 as $n \to \infty$,*

$$(4.5) \qquad n^{-1} G_S \equiv (a\sqrt{n})^{-2} (T_S Cf)'(T_S Cf) \Rightarrow_{\mathscr{L}} \alpha_S \chi_{t-1}^2,$$

where

$$\alpha_S = \lambda t(t+1)^2 / (12(k+1)).$$

PROOF. It is seen that

$$(k!)^2 C \Sigma_0^* C' = k! CC' - bJ.$$

Thus the eigenvalues $\{\alpha_i\}$ of Corollary 3.1 are simply the eigenvalues of

$$\frac{1}{a^2 k!} T_S CC' T_S'.$$

From Alvo, Cabilio and Feigin (1982), $T_S' T_S$ has rank $(t-1)$ and $[12/t(t+1)!] T_S' T_S$ is idempotent. Thus, making use of (4.4) of Lemma 4.1, it follows that the only nonzero eigenvalue is α_S with multiplicity $(t-1)$. Use of Corollary 3.1 yields the result. □

From (4.3) it is seen that the left-hand side of (4.5) divided by α_S is simply the Durbin statistic, which for this model is

$$(4.6) \qquad D = \frac{12}{n\lambda t(k+1)} \sum_{q=1}^{t} \left(R_q - nr\left(\frac{k+1}{2}\right)\right)^2.$$

Turning to the eigenanalysis in the Kendall case we require the following decomposition.

LEMMA 4.2. *There exist a matrix A and unique constants θ_1, θ_2 such that*

$$T_K CC' = \theta_1 T_K + \theta_2 A, \tag{4.7}$$

where

$$AT_K' = I \tag{4.8}$$

and

$$\theta_1 = \frac{\lambda a}{(t-2)(k+1)}(t^2 - k - 2), \qquad \theta_2 = \frac{-\lambda a^2 k!}{3}\left(\frac{t+1}{k+1}\right)\left(\frac{t-k}{t-2}\right).$$

PROOF. See the Appendix. □

We may now complete the eigenanalysis in the Kendall case. Let

$$c_K = t(t-1)/2, \qquad \lambda_1 = 2(t+1)!/(3t(t-1)), \qquad \lambda_2 = 2t!/(3t(t-1)).$$

THEOREM 4.2. *Under H_0 as $n \to \infty$,*

$$n^{-1}G_K \equiv (a\sqrt{n})^{-2}(T_K Cf)'(T_K Cf) \Rightarrow_{\mathscr{L}} \alpha_1 \chi^2_{(t-1)} + \alpha_2 \chi^2_{\binom{t-1}{2}}, \tag{4.9}$$

where

$$\alpha_1 = \frac{\lambda(t+1)^2}{3(k+1)}, \qquad \alpha_2 = \frac{\lambda(t(k-1)-2)}{3(t-2)(k+1)} \tag{4.10}$$

and the χ^2 variates are independent.

PROOF. From Alvo, Cabilio and Feigin (1982), $(c_K)^{-1}T_K T_K'$ has two distinct eigenvalues λ_1 and λ_2 with multiplicities $(t-1)$ and $\binom{t-1}{2}$, respectively. From Lemma 4.2, equation (4.7), it follows that

$$T_K CC' T_K' = \theta_1 T_K T_K' + \theta_2 I.$$

Thus the eigenvalues of $(a^2 k!)^{-1} T_K CC' T_K'$ are

$$\alpha_1 = (\lambda_1 c_K \theta_1 + \theta_2)/(a^2 k!) \quad \text{with multiplicity } (t-1)$$

and

$$\alpha_2 = (\lambda_2 c_K \theta_1 + \theta_2)/(a^2 k!) \quad \text{with multiplicity } \binom{t-1}{2}.$$

A simplification yields the forms claimed by the theorem. The proof is now completed through the use of Corollary 3.1 and the arguments in the initial part of the proof of Theorem 4.1, which remain valid in this case. □

In order to compare the Bahadur efficiencies of the two test statistics under the same design, we consider the approximate Bahadur slope at an alternative Π^*. Proceeding as in Alvo, Cabilio and Feigin (1982) or Alvo and Cabilio

(1984), these slopes are found to be

$$a_S(\Pi^*) = (\Pi^* - \Pi_0^*)'C'T_S'T_SC(\Pi^* - \Pi_0^*)/(a^2 b\alpha_S), \tag{4.11}$$

$$a_K(\Pi^*) = (\Pi^* - \Pi_0^*)'C'T_K'T_KC(\Pi^* - \Pi_0^*)/(a^2 b\alpha_1). \tag{4.12}$$

Since $C\Pi_0^* = ab\Pi_0$ and $C\Pi^* = ab\Pi$, where Π_0 and Π are $t!$ dimensional probability vectors with $\Pi_0 = (t!)^{-1}\mathbf{1}$, then

$$a(\Pi^*) = b(\Pi - \Pi_0)'T'T(\Pi - \Pi_0)(\alpha)^{-1} \tag{4.13}$$

for both (4.11) and (4.12). From Alvo, Cabilio and Feigin (1982),

$$T_K'T_K = \frac{4}{t}T_S'T_S + A,, \tag{4.14}$$

where $(T_S'T_S)A = 0$ and A is positive semidefinite. Application of (4.14) to (4.13) gives

$$a_K(\Pi^*) \geq a_S(\Pi^*),$$

so that on the basis of this criterion the test based on the Kendall metric is the superior of the two.

5. Computational formulas and an example. The statistic G_S in (4.5) is easily calculated from (4.3). Based on the proof of Lemma 4.2, a computational form for G_K may also be obtained. We recall that the t objects are labeled $1, \dots, t$. For each pair of objects labeled (q_1, q_2), $q_1 < q_2$, the incomplete ranking μ_j^* of judge j is assigned a score $a_j(q_1, q_2)$, where

$$a_j(q_1, q_2) = \begin{cases} \operatorname{sgn}\big(\mu_j^*(q_2) - \mu_j^*(q_1)\big), & \text{if judge } j \text{ ranks both } q_1 \text{ and } q_2, \\ \big(1 - 2\mu_j^*(q_1)/(k+1)\big), & \text{if judge } j \text{ ranks only } q_1, \\ \big(2\mu_j^*(q_2)/(k+1) - 1\big), & \text{if judge } j \text{ ranks only } q_2, \\ 0, & \text{otherwise.} \end{cases}$$

This score is a measure of the level of agreement of the relative ranking of the object pair by judge j as compared to the complete ranking $\nu_1 = (1, 2, 3, \dots, t)$. This measure is 1 or -1 when there is perfect agreement or disagreement, respectively, 0 when there is no basis for comparison and a fraction between $-(k-1)/(k+1)$ and $(k-1)/(k+1)$ if only one object is ranked. The actual value of this fraction is the expected level of agreement adjusted by multiplication by $(k-1)/(k+1)$.

The sum over all rankings of the scores for the pair (q_1, q_2) is

$$\frac{1}{a}(T_K Cf)_q = \sum_{j=1}^{nb} a_j(q_1, q_2) \tag{5.1}$$

the qth element of $(1/a)(T_K Cf)$, where q is the index of the pair (q_1, q_2), and thus

$$G_K = \sum_{q_1 < q_2} \left(\sum_{j=1}^{nb} a_j(q_1, q_2) \right)^2. \tag{5.2}$$

This form is reminiscent of the one derived by Hays (1960) for the complete ranking situation, and in fact reduces to it in that case.

The distance between μ_i^* and μ_j^*, the incomplete rankings of judges i and j, respectively, is given by (3.5) as

$$d^*(\mu_i^*, \mu_j^*) = c - \frac{1}{a^2}[C(\mu_i^*)]'T'TC(\mu_j^*), \tag{5.3}$$

where $C(\mu^*)$ is the compatibility vector of μ^*; $c = c_S = (t(t^2 - 1)/12)$ in the Spearman case and $c = c_K = t(t-1)/2$ in the Kendall case. Let

$$\omega_j(s) = \mu_j^*(s)\delta(s,j) + \left(\frac{k+1}{2}\right)(1 - \delta(s,j)), \tag{5.4}$$

where

$$\delta(s,j) = \begin{cases} 1, & \text{if judge } j \text{ ranks object } s, \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Lemma 4.1 shows that

$$\left(\frac{t+1}{k+1}\omega_j(s) - \frac{t+1}{2}\right)$$

is the sth element of $(T_S C(\mu_j^*)/a)$, so that in the Spearman case a simplified form of this distance is

$$d_S^*(\mu_i^*, \mu_j^*) = \frac{t(t+1)(2t+1)}{6} - \left(\frac{t+1}{k+1}\right)^2 \sum_{s=1}^{t} \omega_i(s)\omega_j(s). \tag{5.5}$$

In the Kendall case the distance becomes

$$d_K^*(\mu_i^*, \mu_j^*) = \frac{t(t-1)}{2} - \sum_{q_1 < q_2} a_i(q_1, q_2) a_j(q_1, q_2). \tag{5.6}$$

The computational formula for G_K is illustrated in the following example due to Durbin (1951). Here $t = 7$, $k = 3$, $r = 3$, $\lambda = 1$, $b = 7$, $n = 1$.

	Object Numbers						
Rankers	1	2	3	4	5	6	7
1	2	1	—	3	—	—	—
2	—	2	3	—	1	—	—
3	—	—	1	2	—	3	—
4	—	—	—	1	3	—	2
5	1	—	—	—	2	3	—
6	—	1	—	—	—	2	3
7	1	—	2	—	—	—	3

The matrix of scores $a_j(q_1, q_2)$ is thus

Object Pairs

12	13	14	15	16	17	23	24	25	26	27	34	35	36	37	45	46	47	56	57	67
b	0	a	0	0	0	c	a	c	c	c	c	—	—	—	d	d	d	—	—	—
0	c	—	d	—	—	a	0	b	0	0	d	b	d	d	d	—	—	c	c	—
—	d	0	—	c	—	d	0	—	c	—	a	c	a	c	0	a	0	c	—	c
—	—	d	c	—	0	—	d	c	—	0	d	c	—	0	a	c	a	d	b	0
c	c	c	a	a	c	—	—	0	c	—	—	0	c	—	0	c	—	a	0	d
d	—	—	—	0	c	c	c	c	a	a	—	—	0	c	—	0	c	0	c	a
c	a	c	c	c	a	0	—	—	—	c	0	0	0	a	—	—	c	—	c	c

where $a = 1$, $b = -1$, $c = 0.5$, $d = -0.5$.

Summing each column gives $\sum_{j=1}^{7} a_j(q_1, q_2)$ for each pair $q_1 < q_2$. Thus

$$G_K = \sum_{q_1 < q_2} \left(\sum_{j=1}^{7} a_j(q_1, q_2) \right) = 39.5.$$

Since for this model $n = 1$, one would not expect that the asymptotic distributions would provide accurate approximations to the exact null distributions. For this data set the observed value of Durbin's D is $48/7$. Whereas the maximum possible value of D for this design is 12, the upper 5% value of a χ_6^2 is 12.592, so that we could never reject the null hypothesis at this level. This example was included in a simulation study by Fawcett and Salter (1987) which indicated that when $n = 1$ the χ^2 distribution is not a very accurate approximation to the null distribution of the Durbin statistic. A similar situation holds in this case for the G_K statistic. Indeed the asymptotic distribution is $5.333\chi_6^2 + 0.2\chi_{15}^2$, and its upper 5% critical value exceeds 5.333 multiplied by (upper 5% critical value of a χ_6^2) = 67.153, whereas the maximum possible value of G_K is 66.5.

It is important to note that a parallel situation occurs in the complete block design, where the statistics based on the Spearman and Kendall metrics have asymptotic distributions (as $n \to \infty$) which are of the same form as those considered here. The Spearman case gives rise to the Friedman statistic, and various approximations to its null distribution have been proposed and are considered in Kendall (1975) and Quade (1972). Further approximations have been studied extensively in Iman and Davenport (1980). Similarly, approximations to the null distribution of the average Kendall statistic are proposed and studied in Ehrenberg (1952), Hays (1960), Quade (1972) and Alvo and Cabilio (1984). Various such approximations appear to be quite accurate in the situations where the χ^2 asymptotic distributions are inappropriate, that is, when n, the number of complete blocks, is not large compared to the number of treatments. The inappropriateness of χ^2 approximations in such cases is further emphasized by the fact that one can show that for n fixed the distributions of these statistics are asymptotically normal (as $t \to \infty$).

Due to the relationship between the structures of the complete and incomplete designs, one would expect that similar results hold for the tests considered here.

In the case of the Durbin statistic, various approximations have been proposed. Kendall (1975) suggests approximating the null distribution of

$$F = \frac{\left(\dfrac{\lambda(t+1)}{(k+1)}\right)W}{1-W} \tag{5.7}$$

by an F-distribution with degrees of freedom

$$p = \frac{tr\left(1 - \dfrac{k+1}{\lambda(t+1)}\right)}{\left(\dfrac{tr}{t-1} - \dfrac{k}{k-1}\right)} - \frac{2(k+1)}{\lambda(t+1)}, \qquad q = 2\left(\frac{\lambda(t+1)}{(k+1)} - 1\right)p. \tag{5.8}$$

The concordance statistic W is defined by

$$W = \frac{(k+1)}{n\lambda(t^2-1)}D$$

and achieves its maximum value of 1 when all the incomplete rankings share one compatible complete ranking.

Clearly the statistic in (5.7) is unbounded. For this example the degrees of freedom are $p = q = 4.25$, and the upper 5% value of this F-distribution is 5.97.

Another F-distribution approximation may be derived by conducting the usual analysis of variance on the ranks. In this case, the null distribution of

$$F = \frac{(b(k-1)-(t-1))D}{(t-1)(b(k-1)-D)} \tag{5.9}$$

is approximated by an F-distribution with degrees of freedom

$$p = t - 1, \qquad q = b(k-1) - t + 1. \tag{5.10}$$

In the example, the maximum value of (5.9) is 4, whereas the upper 5% value of an F-distribution with $p = 6$, $q = 8$ is 3.58.

Although it is possible for either of these F tests to reject H_0 at the 5% level, the accuracy of these and other approximations must await further study.

6. Summary. Through the notion of compatibility between a complete and incomplete ranking, we have been able to extend distance-based methods in the analysis of rankings to the balanced incomplete block design. Such an approach not only leads to a connection between Spearman distance and the Durbin statistic, but allows for the definition of other concordance statistics through metrics which are appropriate to the investigator's needs. Various examples of such metrics may be found in Diaconis (1988).

The asymptotic distributions of statistics in this class are investigated for the case that n, the number of replications of the model, becomes arbitrarily large. Such asymptotic distributions turn out to be the distributions of weighted sums of squares of independent standard normal variates whose weights are the eigenvalues of transformed covariance matrices.

The required eigenanalysis was carried out both for the Spearman and the Kendall metric. In both cases the asymptotic distributions have forms similar to those found in the complete block situation. Consistent with what has been observed in the complete case, for n very small the asymptotic distributions do not appear to be accurate approximations to the upper tail probabilities of the exact null distributions.

The asymptotic behaviour as $b \to \infty$ with n fixed has yet to be addressed, but if the complete case is any guide the asymptotics will be quite different from those derived here.

We are presently engaged in a study of such asymptotic behaviour, as well as an examination and comparison of approximating distributions.

APPENDIX

This section contains the proofs of Lemmas 4.1 and 4.2. For what follows we recall that $c_{ij}^{(l)}$, the (i,j)th element of C_l, is either 1 or 0 depending on whether ν_i is or is not compatible with $\nu_j^{(l)}$.

PROOF OF LEMMA 4.1. Denote the $(t \times t!)$ matrix of rank vectors $\{\nu_j\}$ by T_V. The (i,j)th element of $T_V C_l = \sum_m \nu_m(i)$, where the sum is taken over all ν_m which are compatible with $\nu_j^{(j)}$. Define the indicator

$$\delta(q,l) = \begin{cases} 1, & \text{if } q\text{th object is ranked within the block pattern } l, \\ 0, & \text{otherwise.} \end{cases}$$

We first show that the (q,j) element of $T_V C_l$ is

$$\text{(A.1)} \quad (T_V C_l)_{qj} = \frac{(t+1)!}{(k+1)!}\left[\delta(q,l)\nu_j^{(l)}(i(q,l)) + (1-\delta(q,l))\left(\frac{k+1}{2}\right)\right],$$

where $i(q,l)$ is the label of the qth object in the incomplete pattern l.

To this end assume without loss of generality that the partial ranking $\nu_j^{(l)}(i) = i$, $i = 1,\dots,k$. If the qth object is ranked and is given rank i by $\nu_j^{(l)}$, then the number of complete rankings compatible with $\nu_j^{(l)}$ which assign rank s to this object is

$$\begin{cases} \binom{s-1}{i-1}\binom{t-s}{k-i}(t-k)!, & \text{if } 1 \le i \le s,\ 0 \le k-i \le t-s, \\ 0, & \text{otherwise.} \end{cases}$$

On using a variation of a well-known combinatorial identity [see Riordan (1968), page 10], it is seen that the sum of the ranks for the qth object is

$$(t-k)! \sum_{s=i}^{t-k+i} s\binom{s-1}{i-1}\binom{t-s}{k-i} = i\binom{t+1}{t-k}(t-k)! = i\frac{(t+1)!}{(k+1)!}.$$

On the other hand, if the qth object is unranked in block pattern l, then among the complete rankings compatible to $\nu_j^{(l)}$, the rank s is awarded with frequency

$$\binom{t-1}{t-k-1}(t-k-1)! \quad \text{for } s = 1, 2, \ldots, t.$$

The sum of the ranks for sucha an object is thus

$$\sum_{s=1}^{t} s\binom{t-1}{t-k-1}(t-k-1)! = \left(\frac{k+1}{2}\right)\frac{(t+1)!}{(k+1)!}.$$

Thus the qth element of $T_S Cf$ is given by

$$\begin{aligned}
(T_S Cf)_q &= \left[T_V Cf - \left(\frac{t+1}{2}\right)JCf\right]_q \\
&= \left[\frac{(t+1)!}{(k+1)!}\left(R_q + n(b-r)\left(\frac{k+1}{2}\right)\right) - \frac{t+1}{2}anb\right] \\
&= \frac{(t+1)!}{(k+1)!}\left[R_q - nr\frac{(k+1)}{2}\right],
\end{aligned}$$

which gives the result (4.3).

Turning to (4.4), let $V_{(l)} = (k+1)!/(t+1)! T_V C_l$ and denote its elements by $\{v_{qj}^{(l)}\}$, where

$$v_{qj}^{(l)} = \delta(q,l)\nu_j^{(l)}(i(q,l)) + (1-\delta(q,l))\left(\frac{k+1}{2}\right), \tag{A.2}$$
$$q = 1, \ldots, t,\ j = 1, \ldots, k!,$$

and let

$$V = (V_1 | V_2 | \cdots | V_b).$$

Then $(VC')_{q1}$, the $(q,1)$th element of VC' is given by

$$\begin{aligned}
\sum_l \sum_j v_{qj}^{(l)} c_{1j}^{(l)} &= \sum_l \sum_j \nu_j^{(l)}(i(q,l))\delta(q,l)c_{1j}^{(l)} \\
&\quad + \sum_l \sum_j \left(\frac{k+1}{2}\right)(1-\delta(q,l))c_{1j}^{(l)}.
\end{aligned} \tag{A.3}$$

The first term on the right-hand side of (A.3) is the sum of the ranks awarded to the qth object over all the different incomplete ranking patterns which rank that object and are compatible with the complete ranking $\nu_1 \equiv (1, 2, \ldots, t)$. This may also be interpreted as the sum of the ranks assigned to the qth object in one replication of the incomplete design when all judges agree with the complete ranking ν_1. From Kendall (1975) this is simply $r + (q-1)\lambda$.

Thus

$$(VC')_{q1} = r + (q-1)\lambda + (b-r)\left(\frac{k+1}{2}\right) \tag{A.4}$$

and hence

$$(T_V CC')_{q1} = \frac{(t+1)!}{(k+1)!}\left\{\left(\frac{k+1}{2}\right)b + \lambda\left(q - \frac{t+1}{2}\right)\right\}.$$

A similar expression holds for every element of $T_V CC'$, and (4.4) now follows on recalling that $T_S = T_V - ((t+1)/2)J$ and noting that

$$\frac{t+1}{2}JCC' = \frac{(t+1)!}{(k+1)!}\frac{k+1}{2}bJ. \qquad \square$$

PROOF OF LEMMA 4.2. Let $S_i(p,q) = \operatorname{sgn}(\nu_i(p) - \nu_i(q))$ and $\eta_{ij} = (CC')_{ij}$. If constants θ_1, θ_2 are to satisfy (4.7) under conditions (4.8), then

$$T_K CC' T_K' = \theta_1 T_K T_K' + \theta_2 I.$$

Thus

$$\text{(A.5)} \qquad \sum_{i,j} S_i(p,q) S_j(p,q) \eta_{ij} = \theta_1 \sum_i S_i^2(p,q) + \theta_2 \quad \text{for all } p \neq q$$

and

$$\text{(A.6)} \qquad \sum_{i,j} S_i(p,q) S_j(s,q) \eta_{ij} = \theta_1 \sum_i S_i(p,q) S_i(s,q), \qquad s \neq p \neq q.$$

Summing (A.6) over s, using the fact that

$$\sum_{s=1}^{t} S_j(s,q) = -2\nu_j(q) + (t+1), \qquad s \neq q,$$

we obtain

$$\text{(A.7)} \qquad \begin{aligned} &-\sum_{i,j} S_i(p,q)\{2\nu_j(q) - (t+1)\}\eta_{ij} - \sum_{ij} S_i(p,q) S_j(p,q) \eta_{ij} \\ &\quad = -\theta_1 \sum_i S_i(p,q)\{2\nu_i(q) - (t+1)\} - \theta_1 \sum_i S_i^2(p,q). \end{aligned}$$

Adding (A.5) and (A.7) and summing over $p(\neq q)$ gives

$$\text{(A.8)} \qquad \begin{aligned} &4\sum_{i,j}\left(\nu_i(q) - \frac{t+1}{2}\right)\left(\nu_j(q) - \frac{t+1}{2}\right)\eta_{ij} \\ &\quad = \theta_2(t-1) + 4\theta_1 \sum_i \left(\nu_i(q) - \frac{t+1}{2}\right)^2. \end{aligned}$$

The left-hand side of (A.8) is 4 times the qth diagonal element of $T_S CC' T_S'$, which by (4.4) is equal to

$$\lambda \frac{(t+1)!}{(k+1)!} \sum_i \left(\nu_i(q) - \frac{t+1}{2}\right)^2.$$

Thus (A.8) becomes

$$\lambda \frac{(t+1)!}{(k+1)!} = \theta_1 + \frac{3\theta_2}{(t+1)!}. \tag{A.9}$$

On the other hand, (A.5) must hold in particular for $p = 2$, $q = 1$, so that θ_1, θ_2 must also satisfy

$$(T_K CC'T_K')_{11} = \sum_{i,j} S_i(2,1) S_j(2,1) \eta_{ij} = \theta_1 t! + \theta_2. \tag{A.10}$$

In order to compute the left-hand side of (A.10), we enumerate the possible values of the elements $(T_K C)_{1m} = \sum_{i=1}^{t!} S_i(2,1) c_{im}$, $m = 1, 2, \dots, bk!$, and the frequencies with which these values occur. Recall that $c_{im} = 1$ whenever ν_i is compatible with the incomplete ranking indexed by m. Such a partial ranking may rank both, or one, or none of the objects 1 and 2. This leads us to consider four cases.

CASE 1. The incomplete ranking indexed by m ranks both objects 1 and 2.

If this incomplete ranking is denoted by $\nu_j^{(l)}$, then because of compatibility

$$S_i(2,1) = \operatorname{sgn}\left(\nu_j^{(l)}(2) - \nu_j^{(l)}(1)\right)$$

and since there are $a = t!/k!$ compatible complete rankings, then

$$(T_k C)_{1m} = \operatorname{sgn}\left(\nu_j^{(l)}(2) - \nu_j^{(l)}(1)\right) \frac{t!}{k!}$$

for all $\lambda k!$ such cases.

CASE 2. The incomplete ranking indexed by m ranks neither object 1 nor 2.

This case can only occur if $t - 2 \geq k$. Given any compatible complete ranking for this case, the ranking produced by permuting the ranks given to objects 1 and 2 is also compatible. Thus $(T_K C)_{1m} = 0$.

CASE 3. The incomplete ranking indexed by m ranks object 1 but not 2.

We consider incomplete rankings which award rank $j = 1, 2, \dots, k$ to object 1, with the remaining $(k - 1)$ ranks in natural order. The number of complete rankings ν for which $\nu(1) = i$ and $\nu(2) > i$ [and thus $S(2,1) = 1$] is given by

$$\begin{aligned} &\frac{(t-i)!}{1!(k-j)!((t-i)-(k-j)-1)!} \binom{i-1}{j-1} (t-k-1)! \\ &\quad = ((t-i)-(k-j)) \binom{t-i}{k-j} \binom{i-1}{j-1}. \end{aligned} \tag{A.11}$$

Similarly, the number of ν for which $\nu(1) = i$ and $\nu(2) < i$ [and thus

$S(2,1) = -1]$ is given by

$$\text{(A.12)}\qquad \begin{aligned} &\frac{(i-1)!}{1!(j-1)!(i-j-1)!}\binom{t-i}{k-j}(t-k-1)! \\ &= (i-j)\binom{i-1}{j-1}\binom{t-i}{k-j}. \end{aligned}$$

Both (A.11) and (A.12) are valid only if $0 \le i - j \le t - k$. Thus $(T_K C)_{1m}$ for this case is the sum over all i such that $j \le i \le t - k + j$ of the difference (A.11) minus (A.12). Another variation of the combinatorial identity used in the proof of Lemma 4.1 [see Riordan (1968), page 9] and some simplification shows that this sum is equal to $(k + 1 - 2j)(t!/(k + 1)!)$. The total number of such terms is $(r - \lambda)(k - 1)!$ for each $j = 1, 2, \dots, k$.

CASE 4. The incomplete ranking indexed by m ranks object 2 but not 1.

By analogy to Case 3 typical terms have the value $-(k + 1 - 2j)(t!/(k + 1)!)$ for $j = 1, 2, \dots, k$.

To conclude the proof, we use the above information to write

$$(T_K CC'T_K')_{11} = \lambda k!\left(\frac{t!}{k!}\right)^2 + 2(r-\lambda)(k-1)!\sum_{j=1}^{k}\left\{\frac{t!}{(k+1)!}(k+1-2j)\right\}^2,$$

which on simplification becomes

$$\text{(A.13)}\qquad (T_K CC'T_K')_{11} = \lambda a^2 k!\left(1 + \frac{2}{3}\frac{(t-k)}{(k+1)}\right).$$

The lemma now follows on using (A.13) in solving for θ_1 and θ_2 in (A.9) and (A.10). □

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