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WEAK CONVERGENCE OF TIME-SEQUENTIAL CENSORED RANK STATISTICS WITH APPLICATIONS TO SEQUENTIAL TESTING IN CLINICAL TRIALS

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A general weak convergence theory is developed for time-sequential censored rank statistics in the two-sample problem of comparing time to failure between two treatment groups, such as in the case of a clinical trial in which patients enter serially and, after being randomly allocated to one of two treatments, are followed until they fail or withdraw from the study or until the study is terminated. Applications of the theory to time-sequential tests based on these censored rank statistics are also discussed.

1. Introduction. In many clinical trials, a primary objective is to compare time to failure between two treatment groups X and Y. Typically in such a trial, patients enter the study serially and are assigned to treatment X or Y according to some random mechanism. Suppose that the trial involves $n = n' + n''$ patients with n' of them assigned to treatment X and n'' assigned to treatment Y. Let $T_i' \geq 0$ denote the entry time and $X_i > 0$ the survival time (or time to failure) after entry of the ith subject in treatment group X and let T_j'' and Y_j denote the entry time and survival time after entry of the jth subject in treatment group Y. The subjects are followed until they fail or withdraw from the study or until the study is terminated. Thus the data at calendar time t consist of $(X_i(t), \delta_i'(t))$, $i = 1, \dots, n'$, and $(Y_j(t), \delta_j''(t))$, $j = 1, \dots, n''$, where

$$\begin{aligned} &X_i(t) = X_i \wedge \xi_i' \wedge (t - T_i')^+, \qquad Y_j(t) = Y_j \wedge \xi_j'' \wedge (t - T_j'')^+, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge \text{ denotes minimum}, \\ &\delta_i'(t) = I_{\{X_i(t) = X_i\}}, \qquad \delta_j''(t) = I_{\{Y_j(t) = Y_j\}}, \end{aligned} \tag{1.1}$$

and ξ_i' (ξ_j'') denotes the withdrawal time, possibly infinite, of the ith (jth) subject in treatment group X (Y). This notation will be used throughout the sequel. We shall also let F and G denote the distribution functions of X_i and Y_j, respectively. Although the sample sizes n' and n'' may be random even though n is fixed, we shall regard them as nonrandom in the sequel since we can perform the same analysis conditional on n' and n''.

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At a given calendar time t, one can compute, on the basis of the observed data (1.1) from the two treatment groups, a rank statistic of the general form considered by Tsiatis (1982):

$$
\begin{aligned}
S_n(t) = {} & \sum_{i=1}^{n'} \delta_i'(t) Q_n(t, X_i(t)) \left\{ 1 - \frac{m'_{n,t}(X_i(t))}{m'_{n,t}(X_i(t)) + m''_{n,t}(X_i(t))} \right\} \\
& - \sum_{j=1}^{n''} \delta_j''(t) Q_n(t, Y_j(t)) \frac{m'_{n,t}(Y_j(t))}{m'_{n,t}(Y_j(t)) + m''_{n,t}(Y_j(t))},
\end{aligned}
\tag{1.2}
$$

where $Q_n(t, s)$ is some weight function satisfying certain integrability and measurability assumptions that will be specified later and

$$
m'_{n,t}(s) = \sum_{i=1}^{n'} I_{\{X_i(t) \geq s\}}, \qquad m''_{n,t}(s) = \sum_{j=1}^{n''} I_{\{Y_j(t) \geq s\}}. \tag{1.3}
$$

The case $Q_n \equiv 1$ corresponds to Mantel's (1966) logrank statistic, which is a member of the Tarone–Ware (1977) class of statistics (1.2) with

$$
Q_n(t, s) = \psi\left(n^{-1}\left[m'_{n,t}(s) + m''_{n,t}(s)\right]\right), \tag{1.4}
$$

where ψ is a nonrandom function on $[0, 1]$. Taking $\psi(x) = x$ in (1.4) gives Gehan's (1965) generalization of the Wilcoxon statistic. Letting $H_{n,t}$ denote a product-limit-type estimator of the common distribution function of H $(= F = G)$ under H_0 based on $\{X_i(t), \delta_i(t), Y_j(t), \delta_j(t)\colon i \leq n', j \leq n''\}$ [see, e.g., (1.11)], an alternative generalization of the Wilcoxon statistic proposed by Peto and Peto (1972) and Prentice (1978) is the statistic (1.2) with $Q_n(t, s) = 1 - H_{n,t}(s)$, which was subsequently extended by Harrington and Fleming (1982) to the class of weight functions

$$
Q_n(t, s) = \left(1 - H_{n,t}(s)\right)^{\rho}, \qquad \rho \geq 0. \tag{1.5}
$$

The class of rank statistics (1.2) is equivalent to Gill's (1980) class of rank statistics expressible as stochastic integrals with respect to the difference between the sample cumulative hazard functions $\Phi_{n,t}, \Gamma_{n,t}$ from the two treatment groups. Indeed, letting

$$
K_{n,t}(s) = Q_n(t, s) \frac{m'_{n,t}(s) m''_{n,t}(s)}{m'_{n,t}(s) + m''_{n,t}(s)}, \tag{1.6}
$$

we can write (1.2) as $S_n(t) = \int_0^\infty K_{n,t}(s)\, d(\Phi_{n,t}(s) - \Gamma_{n,t}(s))$, where we define

$$
N'_{n,t}(s) = \sum_{i=1}^{n'} I_{\{X_i \leq \xi_i' \wedge (t - T_i')^+ \wedge s\}}, \quad N''_{n,t}(s) = \sum_{j=1}^{n''} I_{\{Y_j \leq \xi_j'' \wedge (t - T_j'')^+ \wedge s\}}, \tag{1.7}
$$

$$
\Phi_{n,t}(s) = \int_{[0,s]} \frac{dN'_{n,t}(u)}{m'_{n,t}(u)}, \quad \Gamma_{n,t}(s) = \int_{[0,s]} \frac{dN''_{n,t}(u)}{m''_{n,t}(u)}. \tag{1.8}
$$

For $s \geq 0$, let $\mathscr{F}(s)$ denote the complete σ-field generated by

$$(1.9)\quad \begin{aligned} &I_{\{X_i \wedge \xi_i' \leq s\}}, \quad I_{\{X_i \leq \xi_i' \wedge s\}}, \quad (X_i \wedge \xi_i')I_{\{X_i \wedge \xi_i' \leq s\}}, \quad T_i', \\ &I_{\{Y_j \wedge \xi_j'' \leq s\}}, \quad I_{\{Y_j \leq \xi_j'' \wedge s\}}, \quad (Y_j \wedge \xi_j'')I_{\{Y_j \wedge \xi_j'' \leq s\}}, \quad T_j'', \qquad i \geq 1, j \geq 1. \end{aligned}$$

As in Gill's (1980) class of rank statistics, we shall assume throughout the sequel for the weight function $Q_n(t, s)$ in (1.2) that for every $t \geq 0$,

$$(1.10)\quad \{Q_n(t,s), 0 \leq s \leq t\} \text{ is a predictable process with respect to } \{\mathscr{F}(s)\}$$

[cf. pages 8 and 46 of Gill (1980)]. Thus the Tarone–Ware class of weight functions (1.4) satisfies (1.10), while the Harrington–Fleming class of weight functions (1.5) also satisfies (1.10) if $H_{n,t}$ is chosen to be a left-continuous version of the Kaplan–Meier estimator, namely,

$$(1.11)\qquad H_{n,t}(s) = 1 - \prod_{u<s}\left\{1 - \frac{\Delta N_{n,t}'(u) + \Delta N_{n,t}''(u)}{m_{n,t}'(s) + m_{n,t}''(s)}\right\},$$

where we use the convention $0/0 = 0$ and $\Delta N(s) = N(s) - N(s-)$. The predictability assumption (1.10) enables one to use martingale central limit theorems to study the asymptotic distributions of $S_n(t^*)$ for fixed t^*. In particular, a basic result in the literature is that $n^{-1/2}S_n(t^*)$ has a limiting normal distribution under the null hypothesis H_0: $F = G$ and under contiguous alternatives. This enables one to construct a simple (approximate) test of H_0 based on $S_n(t^*)$ and to compute the Pitman efficacies of the test under various local alternatives; see Gill (1980).

In the past decade there has been considerable interest in extending the asymptotic normality of these two-sample rank statistics $n^{-1/2}S_n(t^*)$, based on censored data from a fixed-duration trial terminating at a prescheduled time t^*, to weak convergence of the process of time-sequential rank statistics $\{n^{-1/2}S_n(t),\ 0 \leq t \leq t^*\}$. Such an extension would enable one to perform a sequential test as one monitors the data during the course of the trial and to terminate the study early when the two treatment groups show significant differences. Previous results in this direction only treat the case $F = G$. In particular, assuming the T_i' (and T_i'', ξ_i', ξ_i'', respectively) to be i.i.d., Tsiatis (1982) showed that $(n^{-1/2}S_n(t_1), \dots, n^{-1/2}S_n(t_k))$ has a limiting multivariate normal distribution for any k, for a large class of two-sample rank statistics. Sellke and Siegmund (1983) proved tightness and weak convergence in the case where the $S_n(t)$ are Mantel's (1966) logrank statistics, without assuming the T_i' (or T_i'', ξ_i', ξ_i'') to be i.i.d., and Slud (1984) generalized these results to weighted logrank statistics with weights not depending on t.

In Section 3 we shall establish weak convergence of $\{n^{-1/2}S_n(t),\ 0 \leq t \leq t^*\}$, both under the null hypothesis $F = G$ and under local alternatives. We also establish weak convergence of the process $\{n^{-1/2}(S_n(t) - \mu_n(t)),\ 0 \leq t \leq t^*\}$ under general alternatives (F, G), where $\mu_n(t)$ represents the drift of $S_n(t)$. A key idea in our proof of these weak convergence results is to first establish the weak convergence of a closely related two-parameter process. This key step is carried out in Section 2 by developing certain maximal inequalities for the two-parameter process. While the weak convergence proofs of Sellke and Siegmund (1983) and of Slud (1984) require the weights $Q_n(t, s)$ in the time-sequential censored rank statistics (1.2) to be the same for all t [i.e., $Q_n(t, s) = Q_n(s)$] so that $\{S_n(t),\ t \geq 0\}$ has asymptotically uncorrelated increments and can be approximated by a martingale, our new approach, which dispenses with purely martingale-based techniques by making use of certain maximal inequalities given in the Appendix, enables us to handle weights Q_n that may change with t, as is the case with the Tarone–Ware and Harrington–Fleming classes of statistics defined by (1.4) and (1.5).

Properties of the drift function $\mu_n(t)$ will be discussed in Section 4. In particular, it will be shown that $\mu_n(t)$ need not be a monotone function of t even under stochastically ordered contiguous alternatives unless an asymptotically efficient score function for those alternatives is used. This raises several basic issues concerning time-sequential tests using $S_n(t)$, which will be discussed in Section 5 in connection with applications of the weak convergence theory to construct time-sequential censored rank tests of H_0. In particular, it will be shown that because of the nonmonotonicity of the drift function $\mu_n(t)$ at stochastically ordered alternatives where the chosen score function is not asymptotically efficient (as in the case of nonproportional hazards alternatives when time-sequential logrank statistics are used), a time-sequential level-α test based on $\{S_n(t),\ t \leq t^*\}$ that terminates at or before t^* can achieve both savings in time and increase in power when compared with the fixed-duration level-α test based on $S_n(t^*)$. This remarkable feature is of particular interest in the design of clinical trials, for which both power and trial duration are important considerations.

2. Random fields, maximal inequalities and a basic weak convergence theorem. To obtain weak convergence of the normalized time-sequential censored rank statistics $n^{-1/2}(S_n(t) - \mu_n(t))$ to a limiting Gaussian process, a key step is to first establish the weak convergence of a stochastic process (random field) with two-dimensional time parameters (t, s) in the following theorem. As will be shown in Section 3, this theorem enables us to develop a unified weak convergence theory for $S_n(t)$ under the null hypothesis, local alternatives and fixed general alternatives. Not only does this theory cover cases in which the limiting Gaussian process has independent increments, like time-sequential logrank statistics under the null hypothesis previously studied by Sellke and Siegmund (1983) and Slud (1984), but it also covers cases like Gehan's statistics for which the limiting Gaussian process has correlated increments.

THEOREM 1. *Suppose that $X_{k1}, X_{k2}, \dots, X_{kk}$ are i.i.d. nonnegative random variables with a common continuous distribution function F_k. Let $\Phi_k = -\log(1 - F_k)$. Suppose that (ξ_{ki}, T_{ki}), $i = 1, \dots, k$, are independent nonnegative random vectors that are independent of $X_{k1}, \dots, X_{kk}$ and such that*

$$\lim_{k \to \infty} k^{-1} \sum_{i=1}^{k} P\{\xi_{ki} \geq s,\ t - T_{ki} \geq s\} = b(t, s) \tag{2.1}$$

exists and is continuous for $0 \le s \le t$. *Assume that* $\lim_{k\to\infty} F_k(s) = F(s)$ *exists for all s and that F is a continuous distribution function. Define*

$$
\begin{aligned}
m_k(t,s) &= \sum_{i=1}^{k} I_{\{X_{ki} \wedge \xi_{ki} \wedge (t-T_{ki})^+ \ge s\}}, \\
N_k(t,s) &= \sum_{i=1}^{k} I_{\{X_{ki} \le \xi_{ki} \wedge (t-T_{ki})^+ \wedge s\}}, \qquad (2.2)\\
M_k(t,s) &= N_k(t,s) - \int_0^s m_k(t,u)\, d\Phi_k(u).
\end{aligned}
$$

Let $0 \le \alpha < \frac{1}{2}$ *and define*

$$
W_k(t,s) = k^{-1/2} \int_{u=0}^{s} [k/m_k(t,u)]^{\alpha} I_{\{m_k(t,u)>0\}}\, dM_k(t,u). \qquad (2.3)
$$

Then for every $\tau > 0$, W_k *converges weakly in* $D([0,\tau] \times [0,\tau])$ *as* $k \to \infty$ *to a zero-mean Gaussian process W that has continuous sample paths and covariance function*

$$
\begin{aligned}
&\operatorname{Cov}(W(t,s), W(t',s')) \\
&\quad = \int_0^{s\wedge s'} b^{-\alpha}(t,u) b^{-\alpha}(t',u) b(t\wedge t',u)(1-F(u))^{-2\alpha}\, dF(u). \qquad (2.4)
\end{aligned}
$$

We preface the proof of Theorem 1 by the following two lemmas. Lemma 1 is well known; see Gill (1980) and Sellke and Siegmund (1983). The proof of Lemma 2 is lengthy and is given in the Appendix, which also presents several basic maximal inequalities used in the proof.

LEMMA 1. *Let* $\mathscr{F}_k(s)$ *denote the complete* σ*-field generated by*

$$
I_{\{X_{ki}\wedge \xi_{ki} \le s\}}, \quad I_{\{X_{ki} \le \xi_{ki} \wedge s\}}, \quad (X_{ki} \wedge \xi_{ki}) I_{\{X_{ki}\wedge\xi_{ki} \le s\}}, \quad T_{ki}, \qquad i = 1,\dots,k. \qquad (2.5)
$$

Then for all $t \ge 0$, $\{M_k(t,s), \mathscr{F}_k(s), s \ge 0\}$ *are martingales with predictable covariation processes*

$$
\langle M_k(t,\cdot), M_k(t',\cdot)\rangle(s) = \int_0^s m_k(t\wedge t', u)\, d\Phi_k(u).
$$

LEMMA 2. *Given* $\varepsilon > 0$ *and* $\tau > 0$, *there exists a finite partition* $t_0 = 0 < \cdots < t_N = \tau$ *of* $[0,\tau]$ *such that for all large k,*

$$
P\left\{ \max_{0 \le j < N} \sup_{\substack{t_j \le t \le t_{j+1},\\ s \ge 0}} |W_k(t,s) - W_k(t_j,s)| \ge \varepsilon \right\} \le \varepsilon.
$$

PROOF OF THEOREM 1. In view of Lemma 2, it suffices to show that for every $N \ge 1$ and $0 \le t_1 < \cdots < t_N \le \tau$, $\{(W_k(t_1,s),\dots,W_k(t_N,s)), 0 \le s \le \tau\}$ converges weakly to $\{(W(t_1,s),\dots,W(t_N,s)), 0 \le s \le \tau\}$. By Lemma 1,

$\{W_k(t_j, s), \mathscr{F}_k(s), s \geq 0)$, $j = 1, \dots, N$, are martingales with predictable covariation processes $\langle W_k(t_i, \cdot), W_k(t_j, \cdot)\rangle(s)$ equal to

$$
(2.6) \qquad \begin{aligned} &k^{-1}\int_0^s [k/m_k(t_i, u)]^\alpha [k/m_k(t_j, u)]^\alpha \\ &\qquad \times I_{\{m_k(t_i, u) \geq 1,\, m_k(t_j, u) \geq 1\}} m_k(t_i \wedge t_j, u)\, d\Phi_k(u). \end{aligned}
$$

Note that for fixed t, $1 - m_k(t, \cdot)/k$ is a (left-continuous) empirical distribution function based on k independent random variables. Hence by Theorem 1.1.1 and Corollary 1.3.1 of van Zuijlen (1977), for every $0 < \beta < 1$,

$$
(2.7) \qquad P\{m_k(t, s) \leq \beta^{-1} Em_k(t, s) \text{ for all } s\} \geq 1 - 2\pi^2(1 - \beta)^{-4}\beta/3.
$$

Moreover, there exists an absolute constant C such that for all $n \geq 1$, $\lambda > 0$,

$$
(2.8) \qquad P\Big\{\sup_s |m_k(t, s) - Em_k(t, s)| \geq \lambda k^{1/2}\Big\} \leq Ce^{-2\lambda^2};
$$

see Shorack and Wellner (1986), page 797. Let $b_k(t, s) = k^{-1}\sum_{i=1}^k P\{\xi_{ki} \geq s, t - T_{ki} \geq s\}$. Since $Em_k(t, s) = k(1 - F_k(s))b_k(t, s)$ and since $b_k(t, s) \geq b_k(t', s)$ for $t \geq t'$, it follows from (2.1) that for $i \leq j$, as $k \to \infty$,

$$
(2.9) \qquad \begin{aligned} &k^{-1}\int_0^s \left[\frac{k}{Em_k(t_i, u)}\right]^\alpha \left[\frac{k}{Em_k(t_j, u)}\right]^\alpha I_{\{Em_k(t_i, u) > 0\}} Em_k(t_i, u)\, d\Phi_k(u) \\ &\qquad \to \int_0^s b^{-\alpha}(t_i, u) b^{-\alpha}(t_j, u) b(t_i, u)(1 - F(u))^{-2\alpha}\, dF(u) \\ &\qquad\qquad \text{uniformly in } s \geq 0, \end{aligned}
$$

noting that $2\alpha < 1$. Moreover, for $i \leq j$,

$$
(2.10) \qquad \begin{aligned} &k^{-1}\int I_{\{Em_k(t_i, u) > 0,\, k^{-1}Em_k(t_j, u) \leq \delta\}} [k/Em_k(t_i, u)]^\alpha \\ &\qquad \times [k/Em_k(t_j, u)]^\alpha Em_k(t_i, u)\, d\Phi_k(u) \\ &\quad \leq \int I_{\{k^{-1}Em_k(t_j, u) \leq \delta\}} [Em_k(t_i, u)/k]^{(1-2\alpha)/2} \\ &\qquad \times (1 - F_k(u))^{-1+(1-2\alpha)/2}\, dF_k(u) \to 0 \quad \text{as } \delta \downarrow 0. \end{aligned}
$$

From (2.6)–(2.10), it follows that for $i \leq j$, uniformly in $s \geq 0$,

$$
(2.11) \qquad \begin{aligned} &\langle W_k(t_i, \cdot), W(t_j, \cdot)\rangle(s) \\ &\qquad \to_P \int_0^s b^{-\alpha}(t_i, u) b^{-\alpha}(t_j, u) b(t_i, u)(1 - F(u))^{-2\alpha}\, dF(u) \\ &\qquad\qquad \text{as } k \to \infty. \end{aligned}
$$

Note that the limit in (2.11) is $\operatorname{Cov}(W(t_i, s), W(t_j, s))$, in view of (2.4). Moreover, the jump size of the process $W_k(t_i, s)$ is no larger than $k^{\alpha - 1/2}$ ($\to 0$ as

$k \to \infty$). Hence we obtain the desired weak convergence of $(W_k(t_1, s), \ldots, W_k(t_N, s))$ by a multivariate extension (via the usual Cramér–Wold device) of Rebolledo's (1980) martingale functional central limit theorem; see Gill (1980), page 17. □

The following corollary of Theorem 1 plays a basic role in establishing weak convergence of normalized censored rank statistics. Given a function f on an interval $[a, b]$, let $V_{a \le x \le b}[f(x)]$ denote the total variation of f on $[a, b]$. For a sequence of random variables U_n, we write $U_n = O_p(1)$ to denote that $\{U_n\}$ is bounded in probability, that is, $\sup_n P\{|U_n| \ge x) \to 0$ as $x \to \infty$.

COROLLARY 1. *With the same notation and assumptions as in Theorem 1, let $\{\eta_{k,t}(s), 0 \le s \le t\}$ be a predictable process with respect to $\{\mathscr{F}_k(s)\}$ for every $t \ge 0$. Let $\tau > 0$. Suppose that there exist $0 \le \alpha < \frac{1}{2}$ and a nonrandom function $\eta(t, s)$, $0 \le s \le t \le \tau$, such that $\sup_{0 \le t \le \tau}|\eta(t, 0)| < \infty$ and as $k \to \infty$,*

$$\sup_{0 \le s \le t \le \tau} |\eta_{k,t}(s) - \eta(t, s)| I_{\{k^{-1}m_k(t, s) \ge \varepsilon\}} \to_P 0 \quad \text{for every } \varepsilon > 0, \tag{2.12}$$

$$\begin{aligned} \sup_{0 \le t \le \tau} &\Big\{ V_{0 \le s \le t}\Big[\big(k^{-1}m_k(t, s)\big)^{\alpha} \eta_{k,t}(s)\Big] \\ &+ V_{0 \le s \le t}\Big[\big(1 - F(s)\big)^{\alpha} b^{\alpha}(t, s)\eta(t, s)\Big]\Big\} = O_p(1). \end{aligned} \tag{2.13}$$

Then $\{k^{-1/2}\int_0^s \eta_{k,t}(u) M_k(t, du)\colon\ 0 \le t \le \tau,\ 0 \le s \le \tau\}$ converges weakly in $D([0, \tau] \times [0, \tau])$ as $k \to \infty$ to a zero-mean Gaussian process W^* with covariance function

$$\operatorname{Cov}(W^*(t, s), W^*(t', s')) = \int_0^{s \wedge s'} \eta(t, u)\eta(t', u) b(t \wedge t', u)\, dF(u). \tag{2.14}$$

PROOF. First note that for $0 \le s \le t \le \tau$,

$$\begin{aligned} k^{-1/2} &\int_0^s \eta_{k,t}(u) M_k(t, du) \\ &= \int_0^s \big(k^{-1}m_k(t, u)\big)^{\alpha} \eta_{k,t}(u) W_k(t, du) \\ &= \big(k^{-1}m_k(t, s)\big)^{\alpha} \eta_{k,t}(s) W_k(t, s) \\ &\quad - \int_{u=0}^s W_k(t, u)\, d\Big[\big(k^{-1}m_k(t, u)\big)^{\alpha} \eta_{k,t}(u)\Big]. \end{aligned} \tag{2.15}$$

Since $\sup_{0 \le t \le \tau}|\eta(t, 0)| < \infty$, (2.12) implies that $\sup_{0 \le t \le \tau}|\eta_k(t, 0)| = O_p(1)$, so by (2.13),

$$\begin{aligned} \sup_{0 \le s \le t \le \tau} &\Big\{\big(1 - F(s)\big)^{\alpha} b^{\alpha}(t, s)|\eta(t, s)| + \big(k^{-1}m_k(t, s)\big)^{\alpha}|\eta_{k,t}(s)|\Big\} \\ &= O_p(1). \end{aligned} \tag{2.16}$$

By Theorem 1, W_k converges weakly in $D([0,\tau] \times [0,\tau])$ to a Gaussian process W with continuous sample paths. From this, together with (2.12), (2.13) and (2.16), we obtain the weak convergence of (2.15) to the Gaussian process

$$\{(1-F(s))b(t,s)\}^{\alpha}\eta(t,s)W(t,s)$$
$$-\int_{u=0}^{s} W(t,u)\, d\big[\{(1-F(u))b(t,u)\}^{\alpha}\eta(t,u)\big];$$

see Gill (1980), Theorem 2.4.3, and Shorack and Wellner (1986), Theorem 2.8.5. The covariance formula (2.14) for the limiting Gaussian process follows easily from Lemma 1. □

The following lemma provides a useful tool for checking (2.13), as will be shown in the next section.

LEMMA 3. (i) *Let f, g be nonincreasing functions on $[a,b]$ such that $g > 0$ and $0 \le f \le g^p$ for some $1 < p \le 2$. Then*

$$V_{a \le x \le b}[f(x)/g(x)] \le \frac{p}{p-1}\big[f^{(p-1)/p}(a) - f^{(p-1)/p}(b)\big]$$
$$+\frac{1}{p-1}\big[g^{p-1}(a) - g^{p-1}(b)\big].$$

(ii) *For any two functions f, g on $[a,b]$,*

$$V_{a \le x \le b}[f(x)g(x)] \le \Big(\sup_x |f(x)|\Big)V_{a \le x \le b}[g(x)]$$
$$+\Big(\sup_x |g(x)|\Big)V_{z \le x \le b}[f(x)].$$

PROOF. (ii) is obvious. To prove (i), let $a = x_1 < \cdots < x_N = b$ be a partition of $[a,b]$. For $1 \le i < N$,

$$\left|\frac{f(x_{i+1})}{g(x_{i+1})} - \frac{f(x_i)}{g(x_i)}\right| \le \frac{f(x_i) - f(x_{i+1})}{g(x_i)} + \frac{[g(x_i) - g(x_{i+1})]f(x_{i+1})}{g(x_{i+1})g(x_i)}$$
$$\le \frac{f(x_i) - f(x_{i+1})}{(f(x_i))^{1/p}} + \frac{g(x_i) - g(x_{i+1})}{(g(x_i))^{2-p}},$$

where we define $0/0 = 0$ [for the case $f(x_i) = 0$]. The desired conclusion then follows from the inequality $(x-y)/x^{\delta} \le (1-\delta)^{-1}(x^{1-\delta} - y^{1-\delta})$ for $0 < y \le x$ and $0 \le \delta < 1$, which in turn follows from the well-known inequality $\delta x + (1-\delta)y \ge x^{\delta}y^{1-\delta}$. □

3. Weak convergence of time-sequential censored rank statistics. In this section we apply Corollary 1 and Lemma 3 to analyze the two-sample time-sequential rank statistics $S_n(t)$, defined in (1.2). More generally, instead of $S_n(t)$, we shall consider

$$S_n(t,s) = \sum_{i=1,\, X_i(t)\le s}^{n'} \delta_i'(t) Q_n(t, X_i(t)) \left\{1 - \frac{m'_{n,t}(X_i(t))}{m'_{n,t}(X_i(t)) + m''_{n,t}(X_i(t))}\right\}$$

$$(3.1) \qquad - \sum_{j=1,\, Y_j(t)\le s}^{n''} \delta_j''(t) Q_n(t, Y_j(t)) \frac{m'_{n,t}(Y_j(t))}{m'_{n,t}(Y_j(t)) + m''_{n,t}(Y_j(t))}$$

$$= \int_0^s Q_n(t,u)\left[m'_{n,t}(u) + m''_{n,t}(u)\right]^{-1} \times \left\{m''_{n,t}(u)\, dN'_{n,t}(u) - m'_{n,t}(u)\, dN''_{n,t}(u)\right\},$$

where $m'_{n,t}, m''_{n,t}$ are defined in (1.3) and $N'_{n,t}, N''_{n,t}$ are defined in (1.7). Note that

$$(3.2) \qquad S_n(t,t) = S_n(t,\infty) = (1.2).$$

Let $\Phi = -\log(1-F)$ and $\Gamma = -\log(1-G)$ denote the cumulative hazard functions of the X_i and Y_j, respectively, and let

$$(3.3) \qquad \begin{aligned} M'_{n,t}(s) &= N'_{n,t}(s) - \int_0^s m'_{n,t}(u)\, d\Phi(u), \\ M''_{n,t}(s) &= N''_{n,t}(s) - \int_0^s m''_{n,t}(u)\, d\Gamma(u), \end{aligned}$$

$$(3.4) \qquad \mu_n(t,s) = \int_0^s Q_n(t,u) \frac{m'_{n,t}(u) m''_{n,t}(u)}{m'_{n,t}(u) + m''_{n,t}(u)} (d\Phi(u) - d\Gamma(u)).$$

From (3.1), (3.3) and (3.4), we obtain the stochastic integral representation

$$(3.5) \qquad \begin{aligned} &n^{-1/2}(S_n(t,s) - \mu_n(t,s)) \\ &\quad = n^{-1/2} \int_0^s Q_{n,t}(u)\left[m'_{n,t}(u) + m''_{n,t}(u)\right]^{-1} \\ &\qquad \times \left\{m''_{n,t}(u)\, dM'_{n,t}(u) - m'_{n,t}(u)\, dM''_{n,t}(u)\right\}. \end{aligned}$$

As in Corollary 1, we use the notation $V_{a\le x\le b}[f(x)]$ to denote the total variation of a function f on the interval $[a,b]$ and assume that there exist $0 \le \alpha < \frac{1}{2}$ and a nonrandom function $q(t,s)$ such that $\sup_{0\le t\le\tau}|q(t,0)| < \infty$ and as $n \to \infty$,

$$(3.6) \qquad \sup_{0\le s\le t\le\tau} |Q_n(t,s) - q(t,s)| I_{\{n^{-1}m'_{n,t}(s) + n^{-1}m''_{n,t}(s)\ge\varepsilon\}} \to_P 0 \quad \text{for every } \varepsilon > 0,$$

$$(3.7) \qquad \begin{aligned} \sup_{0\le t\le\tau} \Big\{ &V_{0\le s\le t}\left[\left(n^{-1}m'_{n,t}(s) + n^{-1}m''_{n,t}(s)\right)^{\alpha} Q_n(t,s)\right] \\ &+ V_{0\le s\le t}\left[\{(1-F(s))b'(t,s) + \right. \\ &\left.\quad (1-G(s))b''(t,s)\}^{\alpha} q(t,s)\right]\Big\} = O_p(1). \end{aligned}$$

The following theorem establishes weak convergence of $n^{-1/2}S_n(t,s)$ under local alternatives that are characterized by (3.11) and (3.12) and weak convergence of $n^{-1/2}(S_n(t,s)-\mu_n(t,s))$ under fixed (F,G). Note that for the special case of the null hypothesis $F=G$, $\mu_n(t,s)=0$ by (3.4). Assuming the existence of the limits

$$b'(t,s)=\lim_{m\to\infty} m^{-1}\sum_{i=1}^{m} P\{\xi_i'\geq s,\, t-T_i'\geq s\},$$
$$b''(t,s)=\lim_{m\to\infty} m^{-1}\sum_{j=1}^{m} P\{\xi_j''\geq s,\, t-T_j''\geq s\}, \tag{3.8}$$

$$n'/n\to\gamma \quad \text{as } n(=n'+n'')\to\infty \text{ with } 0<\gamma<1, \tag{3.9}$$

the theorem expresses the covariance of the limiting Gaussian process in terms of the function

$$\begin{aligned} h_{\Phi,\Gamma}(t,t^*;s) = \int_0^s & q(t,u)q(t^*,u)\{m'(t,u)+m''(t,u)\}^{-1} \\ & \times\{m'(t^*,u)+m''(t^*,u)\}^{-1} \\ & \times\{m'(t\wedge t^*,u)m''(t,u)m''(t^*,u)\,d\Phi(u) \\ & \quad +m''(t\wedge t^*,u)m'(t,u)m'(t^*,u)\,d\Gamma(u)\}, \end{aligned} \tag{3.10}$$
$$m'(t,u)=\gamma(1-F(u))b'(t,u),$$
$$m''(t,u)=(1-\gamma)(1-G(u))b''(t,u).$$

THEOREM 2. *Suppose that* $(X_i,\xi_i',T_i'),(Y_j,\xi_j'',T_j'')$, $i=1,2,\dots$, $j=1,2,\dots$, *are independent random vectors with nonnegative components such that* (ξ_i',T_i') *is independent of* X_i *and* (ξ_j'',T_j'') *is independent of* Y_j *for all i and j. Assume the existence of the limits* (3.8) *and* (3.9), *in which* $b'(t,s)$ *and* $b''(t,s)$ *are also assumed to be continuous for* $0\leq s\leq t$. *Define* $S_n(t,s)$ *by* (3.1) *and assume that the weight function* Q_n *in* (3.1) *satisfies* (1.10) *for every* $t\geq 0$. *Suppose that* X_i *are i.i.d. with a continuous cumulative hazard function* Φ *and* Y_j *are i.i.d. with a continuous cumulative hazard function* Γ. *Let* $\tau\geq 0$.

(i) *For fixed* Φ, *suppose that as* $n\to\infty$, $\Gamma\to\Phi$ *such that*

$$n^{1/2}\{(d\Gamma/d\Phi)(s)-1\}\to g(s) \tag{3.11}$$

uniformly in $s\in I$ *and* $\sup_{s\in I}|g(s)|<\infty$ *for all closed subintervals* I *of* $\{s\in[0,\tau]\colon F(s)<1\}$. *Assume that as* $n\to\infty$,

$$\int_0^\tau |d\Gamma/d\Phi-1|\,d\Phi = O(n^{-1/2}), \tag{3.12}$$

and that the weight function Q_n *satisfies* (3.6) *and* (3.7) *for some* $0\leq\alpha<\frac{1}{2}$ *and nonrandom function* q *with* $\sup_{0\leq t\leq\tau}|q(t,0)|<\infty$. *Then as* $n\to\infty$, $\{n^{-1/2}S_n(t,s),\ 0\leq t\leq\tau,\ 0\leq s\leq\tau\}$ *converges weakly in* $D([0,\tau]\times[0,\tau])$ *to a Gaussian process* Z^* *with* $\mathrm{Cov}(Z^*(t,s),Z^*(t',s'))=h_{\Phi,\Phi}(t,t';\,s\wedge s')$, *where*

$h_{\Phi,\Gamma}$ *is given in* (3.10) *and*

$$EZ^*(t,s) = -\gamma(1-\gamma)\int_0^s q(t,u)g(u)b'(t,u)b''(t,u) \times\left[\gamma b'(t,u) + (1-\gamma)b''(t,u)\right]^{-1} dF(u). \tag{3.13}$$

(ii) *For fixed* Φ *and* Γ, *suppose that there exist* $0 \le \alpha < \frac{1}{2}$ *and a nonrandom function* q *with* $\sup_{0\le t\le\tau}|q(t,0)| < \infty$ *such that* (3.6) *and* (3.7) *hold as* $n\to\infty$. *Then as* $n\to\infty$, $\{n^{-1/2}(S_n(t,s)-\mu_n(t,s)),\ 0\le t\le\tau,\ 0\le s\le\tau\}$ *converges weakly in* $D([0,\tau]\times[0,\tau])$ *to a zero-mean Gaussian process* Z *with* $\operatorname{Cov}(Z(t,s),Z(t',s')) = h_{\Phi,\Gamma}(t,t';\ s\wedge s')$, *where* $h_{\Phi,\Gamma}$ *is given in* (3.10).

PROOF. Let $U_n(t,s) = n^{-1/2}(S_n(t,s)-\mu_n(t,s))$, which has the stochastic integral representation (3.5). Let $U(t,s)$ be a zero-mean Gaussian process with covariance equal to that of Z^* for part (i) of the theorem, or equal to that of Z for part (ii) of the theorem. Let $\mathscr{F}(s)$ denote the complete σ-field generated by (1.9). As in Lemma 1, $\{M'_{n,t}(s),\mathscr{F}(s),\ s\ge 0\}$ and $\{M''_{n,t}(s),\mathscr{F}(s),\ s\ge 0\}$ are martingales with predictable covariation processes $\langle M'_{n,t}, M''_{n,t^*}\rangle(s) = 0$,

$$\langle M'_{n,t}, M'_{n,t^*}\rangle(s) = \int_0^s m'_{n,t\wedge t^*}(u)\,d\Phi(u),$$

$$\langle M''_{n,t}, M''_{n,t^*}\rangle(s) = \int_0^s m''_{n,t\wedge t^*}(u)\,d\Gamma(u);$$

see Sellke and Siegmund (1983) and Slud (1984). Hence, in view of (3.5), an argument similar to that used in the proof of Theorem 1 can be used to show that as $n\to\infty$, under the setting of part (i) or (ii) of the theorem, $\{(U_n(t_1,s),\dots,U_n(t_M,s)),\ 0\le s\le s_0\}$ converges weakly to $\{(U(t_1,s),\dots,U(t_M,s)),\ 0\le s\le s_0\}$, for every $M\ge 1$ and $0\le t_1<\cdots<t_M\le\tau$ and for every $s_0\in[0,\tau]$ such that

$$(1-F(s_0))b'(t_1,s_0) + (1-G(s_0))b''(t_1,s_0) > 0. \tag{3.14}$$

Since $b'(t,s_0)$ and $b''(t,s_0)$ are nondecreasing in t, the assumption (3.14) implies that

$$\liminf_{n\to\infty}\left\{\inf_{s\le s_0,\,1\le j\le M}\left[n^{-1}Em'_{n,t_j}(s) + n^{-1}Em''_{n,t_j}(s)\right]\right\} > 0,$$

so we can avoid arguments analogous to (2.7) and (2.10) in the proof of Theorem 1. By (3.4),

$$n^{-1/2}\mu_n(t,s) = -\int_0^s \frac{(n^{-1}m'_{n,t}(u))(n^{-1}m''_{n,t}(u))}{n^{-1}m'_{n,t}(u) + n^{-1}m''_{n,t}(u)} \times\left[n^{1/2}\left(\frac{d\Gamma}{d\Phi}(u) - 1\right)\right]Q_n(t,u)\,d\Phi(u). \tag{3.15}$$

Hence $\sup_{s\le s_0,\ 1\le j\le M}|n^{-1/2}\mu_n(t_j,s) - EZ^*(t_j,s)| \to_P 0$ under the assumptions of part (i) of the theorem.

It remains to show tightness of $\{U_n(t, s)\colon s, t \in [0, \tau],\ n \geq 1\}$. Take $\delta > 0$ such that $\alpha + \delta < \frac{1}{2}$. Let

$$\eta'_{n,t}(u) = Q_n(t,u)\big[m'_{n,t}(u) + m''_{n,t}(u)\big]^{-1} m''_{n,t}(u)$$
$$= n^{-1} \frac{m''_{n,t}(u)}{\big[n^{-1}m'_{n,t}(u) + n^{-1}m''_{n,t}(u)\big]^{1+\alpha}}$$
$$\times \big[n^{-1}m'_{n,t}(u) + n^{-1}m''_{n,t}(u)\big]^{\alpha} Q_n(t,u).$$

By first applying Lemma 3(i) with $f = (n^{-1}m''_{n,t})(n^{-1}m'_{n,t})^{\alpha+\delta}$, $g = [n^{-1}m'_{n,t} + n^{-1}m''_{n,t}]^{1+\alpha}$ and $p = (1+\alpha+\delta)/(1+\alpha)$ and then applying Lemma 3(ii) and (3.7), it can be shown that

$$\sup_{0 \leq t \leq \tau} V_{0 \leq s \leq t}\Big[\big(n^{-1}m'_{n,t}(u)\big)^{\alpha+\delta} \eta'_{n,t}(u)\Big] = O_p(1).$$

Hence we can apply Corollary 1 to obtain the tightness of

$$n^{-1/2}\int_0^s \eta'_{n,t}(u)\, dM'_{n,t}(u).$$

The same argument can be used to show tightness of the other component $n^{-1/2}\int_0^s \eta''_{n,t}(u)\, dM''_{n,t}(u)$ in the stochastic integral representation (3.5) of $U_n(t, s)$. □

In view of (3.2), we obtain from Theorem 2 the following corollary.

COROLLARY 2. *Under the assumptions of Theorem 2, define the censored rank statistics $S_n(t)$ by (1.2). Then as $n \to \infty$, $\{n^{-1/2}(S_n(t) - \mu_n(t)),\ 0 \leq t \leq \tau\}$ converges weakly in $D[0,\tau]$ to a zero-mean Gaussian process $\{Z_{\Phi,\Gamma}(t),\ 0 \leq t \leq \tau\}$ with covariance $\mathrm{Cov}(Z_{\Phi,\Gamma}(t), Z_{\Phi,\Gamma}(t')) = h_{\Phi,\Gamma}(t, t';\ t \wedge t')$, for fixed (Φ, Γ), where $h_{\Phi,\Gamma}$ is defined in (3.10) and $\mu_n(t) = \mu_n(t,t)$ is defined in (3.4). Moreover, for fixed Φ, as $n \to \infty$ and $\Gamma \to \Phi$ such that (3.11) and (3.12) hold, $\{n^{-1/2}S_n(t),\ 0 \leq t \leq \tau\}$ converges weakly in $D[0,\tau]$ to the Gaussian process $\{Z_{\Phi,\Phi}(t) + \mu(t),\ 0 \leq t \leq \tau\}$, where*

$$\mu(t) = -\gamma(1-\gamma)\int_0^t q(t,u)g(u)b'(t,u)b''(t,u) \times\big[\gamma b'(t,u) + (1-\gamma)b''(t,u)\big]^{-1} dF(u). \tag{3.16}$$

The limiting Gaussian process $Z_{\Phi,\Phi}$ has independent increments if the limit function $q(t,s)$ in (3.6) satisfies the condition

$$q(t_1,s)q(t_2,s) = q^2(t_1 \wedge t_2, s), \qquad 0 \leq s \leq t_1 \wedge t_2,\ t_1, t_2 \in [0,\tau]. \tag{3.17}$$

In fact, in this case, $\operatorname{Cov}(Z_{\Phi,\Phi}(t), Z_{\Phi,\Phi}(t^*))$ *is equal to*

$$\gamma(1-\gamma)\int_0^{t\wedge t^*} q^2(t\wedge t^*, u)b'(t\wedge t^*, u)b''(t\wedge t^*, u)$$
$$\times\left[\gamma b'(t\wedge t^*, u) + (1-\gamma)b''(t\wedge t^*, u)\right]^{-1} dF(u).$$

Condition (3.17) is clearly satisfied if $q(t, s)$ does not depend on t, or more generally, if $q(t, s)$ is of the form

$$q(t,s) = Q(s)I_{\{s\le s(t)\}}, \quad \text{where } s(t) \ge 0 \text{ is nondecreasing in } t. \tag{3.18}$$

The following examples show that the Tarone–Ware and Harrington–Fleming classes of weight functions $Q_n(t, s)$ satisfy the assumptions of Theorem 2 and therefore, in particular, the theorem is applicable to the commonly used logrank, Gehan and Peto–Prentice statistics. Moreover, condition (3.18) is shown to be satisfied in Examples 2 and 3 below.

EXAMPLE 1. Consider the Tarone–Ware class of weight functions defined by (1.4), in which we assume ψ: $[0, 1] \to (-\infty, \infty)$ to be continuous on $(0, 1]$ and such that $\sup_{0\le x\le 1} V_{0\le x\le 1}[x^\alpha \psi(x)] < \infty$ for some $0 \le \alpha < \frac{1}{2}$. Letting

$$q(t,s) = \psi\left[\gamma(1-F(s))b'(t,s) + (1-\gamma)(1-G(s))b''(t,s)\right], \tag{3.19}$$

note that (3.6) and (3.7) are satisfied under assumptions (3.8) and (3.9) of Theorem 2.

EXAMPLE 2. Define the left-continuous Kaplan–Meier curve $H_{n,t}(s)$ of the combined sample $\{X_i(t), \delta_i'(t), Y_j(t), \delta_j''(t): i = 1, \dots, n', j = 1, \dots, n''\}$ at time t by (1.11) and let ψ: $[0, 1] \to (-\infty, \infty)$ be continuous on $[0, 1)$ and such that

$$V_{0\le x\le 1}\left[(1-x)^\beta \psi(x)\right] < \infty \quad \text{for some } 0 \le \beta < \tfrac{1}{2}. \tag{3.20}$$

Let $Q_n(t, s) = \psi(H_{n,t}(s))$. For the Harrington–Fleming (1982) class of statistics (1.5), $\psi(x) = (1 - x)^\rho$ with $\rho \ge 0$ and (3.20) is obviously satisfied. In the context of Theorem 2, for the case $G = F$ or the case $G \to F$, define $q(t, s) = \psi(F(s))$, which does not depend on t and therefore satisfies (3.18) with $s(t) = \tau$, and it will be shown below that conditions (3.6) and (3.7) are satisfied in either case. For the case $F \ne G$, it will be shown that (3.6) and (3.7) still hold with

$$q(t,s) = \psi(H(t,s)), \tag{3.21}$$

where $1 - H(t, s)$ is equal to

$$\exp\left\{-\int_0^s \frac{\gamma b'(t,u)\,dF(u) + (1-\gamma)b''(t,u)\,dG(u)}{\gamma b'(t,u)(1-F(u)) + (1-\gamma)b''(t,u)(1-G(u))}\right\}.$$

To check condition (3.7), take $0 < \alpha < \beta$ such that $\alpha + \beta < \frac{1}{2}$ and apply Lemma 3 and (3.20) to

$$V_{0 \le s \le t}\left[\frac{\left(n^{-1}m'_{n,t}(s) + n^{-1}m''_{n,t}(s)\right)^{\alpha+\beta}}{\left(1 - H_{n,t}(s)\right)^{\beta}} \times \left(1 - H_{n,t}(s)\right)^{\beta}\psi\left(H_{n,t}(s)\right)\right], \tag{3.22}$$

noting that $1 - H_{n,t}(s) \ge n^{-1}[m'_{n,t}(s) + m''_{n,t}(s)]$ and using a similar representation and argument for the other (nonrandom) term of (3.7). To check condition (3.6), first use (1.3), (1.7), (3.8), (3.9) and (2.8) to show that for every $\delta > 0$,

$$P\left\{\sup_{0 \le s \le t \le \tau}\left[\left|n^{-1}m'_{n,t}(s) - \gamma b'(t,s)(1 - F(s))\right| + \left|n^{-1}m''_{n,t}(s) - (1-\gamma)b''(t,s)(1 - G(s))\right| + \left|n^{-1}N'_{n,t}(s) - \gamma\int_0^s b'(t,u)\,dF(u)\right| + \left|n^{-1}N''_{n,t}(s) - (1-\gamma)\int_0^s b''(t,u)\,dG(u)\right|\right] \ge \delta\right\} \to 0$$

as $n \to \infty$. Using the identity $\int_0^s A_n\,dB_n - \int_0^s A\,dB = \int_0^s (A_n - A)\,dB + \int_0^s A_n\,d(B_n - B)$ and applying integration by parts to $\int_0^s A_n\,d(B_n - B)$, it then follows that for every $\varepsilon > 0$ and $\delta > 0$,

$$P\left\{\sup_{0 \le s \le t \le \tau}\left|(1 - H_{n,t}(s)) - (1 - H(t,s))\right| \times I_{\{m'_{n,t}(s) + m''_{n,t}(s) \ge \varepsilon n\}} \ge \delta\right\} \to 0 \quad \text{as } n \to \infty, \tag{3.23}$$

in view of (3.21) and the integral representation

$$\log(1 - H_{n,t}(s)) = -\int_{u<s}\left\{\left[m'_{n,t}(u) + m''_{n,t}(u)\right]^{-1} + O\left(\left[m'_{n,t}(u) + m''_{n,t}(u)\right]^{-2}\right)\right\} d\left(N'_{n,t}(u) + N''_{n,t}(u)\right).$$

EXAMPLE 3. Let $\psi\colon [0,1] \to (-\infty,\infty)$ be continuous on $[0,1)$ such that (3.20) holds. Let $0 < \rho < 1$. Under assumptions (3.8) and (3.9) of Theorem 2, define

$$Q_n(t,s) = \psi\left(\gamma F_{n,t}(s) + (1-\gamma)G_{n,t}(s)\right) I_{\{m'_{n,t}(s) \ge (n')^{\rho},\ m''_{n,t}(s) \ge (n'')^{\rho}\}}, \tag{3.24}$$

where

$$F_{n,t}(s) = 1 - \prod_{u<s}\left(1 - \Delta N'_{n,t}(u)/N'_{n,t}(u)\right),$$
$$G_{n,t}(s) = 1 - \prod_{u<s}\left(1 - \Delta N''_{n,t}(u)/N''_{n,t}(u)\right)$$

are the left-continuous Kaplan–Meier estimates of F and G, respectively, Let

$$q(t,s) = \psi(\gamma F(s) + (1-\gamma)G(s))I_{\{s \le s(t)\}},$$
$$(3.25) \qquad s(t) = \inf\{u: (1-F(u))b'(t,u) = 0\} \wedge \inf\{u: (1-G(u))b''(t,u) = 0\},$$

putting $G = F$ for the setting of Theorem 2(i). Then conditions (3.6) and (3.7) are satisfied, as can be shown by an argument similar to (3.22) and (3.23). Note that the limit function (3.25) is of the form (3.18).

4. Asymptotic drift and covariance function of the rank statistics $S_n(t)$. Consider the two-sample time-sequential rank statistics $S_n(t)$ defined in (1.2). By Corollary 2, under the null hypothesis H_0: $F = G$ and the assumptions of Theorem 2, $\{n^{-1/2}S_n(t),\ 0 \le t \le \tau\}$ converges weakly in $D[0,\tau]$ to a zero-mean Gaussian process $\{Z(t),\ 0 \le t \le \tau\}$ with covariance function

$$(4.1) \qquad \begin{aligned} &\mathrm{Cov}(Z(t), Z(t^*)) \\ &\quad = \gamma(1-\gamma)\int_0^{t \wedge t^*} q(t,u)q(t^*,u)b'(t \wedge t^*, u)b''(t \wedge t^*, u) \\ &\qquad \times [\gamma b'(t \wedge t^*, u) + (1-\gamma)b''(t \wedge t^*, u)]^{-1}\, dF(u). \end{aligned}$$

In particular, for the Tarone–Ware class of statistics with weight functions given by (1.4), in which we assume ψ: $[0,1] \to (-\infty, \infty)$ to be continuous on $(0,1]$ and such that $V_{0 \le x \le 1}[x^\alpha \psi(x)] < \infty$ for some $0 \le \alpha < \frac{1}{2}$, the function q in (4.1) is given by

$$(4.2) \qquad q(t,u) = \psi((1-F(s))[\gamma b'(t,s) + (1-\gamma)b''(t,s)]).$$

The Gaussian process $Z(t)$ with covariance function (4.1)–(4.2) does not have independent increments unless ψ = constant (as in Mantel's logrank statistic) or unless $\gamma b'(t,s) + (1-\gamma)b''(t,s)$ does not depend on t (as in the case of simultaneous entry with $T_i' = T_j'' = 0$).

Suppose that we modify the Tarone–Ware weight function (1.4) by

$$(4.3) \qquad Q_n(t,s) = \psi(H_{n,t}(s)), \quad \text{where } H_{n,t}(s) \text{ is defined (1.11)},$$

as in Example 2, or by (3.24), as in Example 3. Then the function q in (4.1) is given by $q(t,u) = \psi(F(u))$ for the case (4.3), and by (3.25) with $G = F$ for the case (3.24). In either case, (4.1) becomes

$$(4.4) \qquad \begin{aligned} &\mathrm{Cov}(Z(t), Z(t^*)) = h(t \wedge t^*), \\ &h(t) = \gamma(1-\gamma)\int_0^t \psi^2(F(u))b'(t,u)b''(t,u) \\ &\qquad \times [\gamma b'(t,u) + (1-\gamma)b''(t,u)]^{-1}\, dF(u). \end{aligned}$$

Hence, for the normalized rank statistics $n^{-1/2}S_n(t)$ with weight functions (4.3) or (3.24), the limiting Gaussian process $Z(t)$ under H_0: $F = G$ has independent increments. We next consider the limiting drift $\mu(t)$ of $n^{-1/2}S_n(t)$ under contiguous alternatives.

Suppose that $F = F_0$ and $G = F_\theta$, where $\{F_\theta, \ -\varepsilon \le \theta \le \varepsilon\}$ is a family of continuous distribution functions whose cumulative hazard functions $\Phi_\theta = -\log(1 - F_\theta)$ satisfy

$$\int_0^\infty |d\Phi_\theta/d\Phi_0 - 1|\, d\Phi_0 = O(\theta) \quad \text{as } \theta \to 0,$$

$$\theta^{-1}\{(d\Phi_\theta/d\Phi_0)(s) - 1\} \to g(s) \tag{4.5}$$

uniformly in $s \in I$ as $\theta \to 0$ and $\sup_{s \in I}|g(s)| < \infty$ for all closed subintervals I of $\{s\colon F_0(s) < 1\}$. Define the weight function Q_n by (4.3) or by (3.24). Take $c \neq 0$. Under the assumptions of Theorem 2, we obtain from Corollary 2 that $\{n^{-1/2}S_n(t), 0 \le t \le \tau\}$ converges weakly under $H_n\colon (F, G) = (F_0, F_{c/\sqrt{n}})$ to the Gaussian process $\{Z(t) + \mu(t), 0 \le t \le \tau\}$, where $Z(t)$ has mean 0 and covariance (4.4) and

$$\begin{aligned}\mu(t) = -c\gamma(1-\gamma)\int_0^t &\psi(F(u))g(u)b'(t,u)b''(t,u)\\ &\times[\gamma b'(t,u) + (1-\gamma)b''(t,u)]^{-1}\,dF(u).\end{aligned} \tag{4.6}$$

From (4.4) and (4.6), it follows by the Schwarz inequality that

$$\begin{aligned}\frac{\mu^2(t)}{\operatorname{Var}(Z(t))} \le c^2\gamma(1-\gamma)\int_0^t &g^2(u)b'(t,u)b''(t,u)\\ &\times[\gamma b'(t,u) + (1-\gamma)b''(t,u)]^{-1}\,dF(u)\end{aligned} \tag{4.7}$$

and that equality is attained in (4.7) if $\psi \circ F$ is a scalar multiple of g. In particular, for the choice

$$\psi(x) = -cg(F^{-1}(x)), \tag{4.8}$$

which is the asymptotically optimal score function for two-sample rank statistics based on complete (uncensored) data, we have equality in (4.7) and

$$\begin{aligned}\mu(t) &= \operatorname{Var} Z(t)\\ &= \gamma(1-\gamma)c^2\int_0^t g^2(u)b'(t,u)b''(t,u)\\ &\qquad\times[\gamma b'(t,u) + (1-\gamma)b''(t,u)]^{-1}\,dF(u).\end{aligned} \tag{4.9}$$

Summarizing, for the time-sequential rank statistics (1.2) with weight functions $Q_n(t, s)$ of the form (4.3) or (3.24), the limiting Gaussian process has independent increments under the null hypothesis and under contiguous alternatives. The variance function of the limiting Gaussian process is $h(t)$ given by (4.4), while the drift of the limiting Gaussian process under these alternatives is the function $\mu(t)$ given by (4.6). Moreover, $\mu(t)$ is a linear function (scalar multiple) of $h(t)$ if the score function ψ is chosen to be a scalar multiple of the asymptotically optimal score function $g \circ F^{-1}$ associated with these local alternatives via (4.5). However, for other choices of ψ, $\mu(t)$ given by (4.6) need not even be a monotone function of t although $h(t)$ given by (4.4) is always nondecreasing in t, as will be shown in a subsequent example.

We next consider the asymptotic properties of these time-sequential rank statistics under general fixed alternatives (F, G) with cumulative hazard functions Φ, Γ. Defining the random drift $\mu_n(t) = \mu_n(t, t)$ by (3.4) and assuming the setting of Theorem 2(ii), it can be shown that

$$\begin{aligned} n^{-1}\mu_n(t) &\to_P \gamma(1-\gamma) \\ &\times \int_0^{s(t)} \frac{(1-F(u))(1-G(u))b'(t,u)b''(t,u)}{\gamma(1-F(u))b'(t,u) + (1-\gamma)(1-G(u))b''(t,u)} \\ &\qquad \times q(t,u)(d\Phi(u) - d\Gamma(u)), \end{aligned} \tag{4.10}$$

the convergence being uniform in $0 \le t \le \tau$, where $s(t)$ is defined in (3.25). By Corollary 2, $\{n^{-1/2}(S_n(t) - \mu_n(t)),\ 0 \le t \le \tau\}$ converges weakly in $D[0, \tau]$ to a zero-mean Gaussian process $\{Z_{\Phi,\Gamma}(t),\ 0 \le t \le \tau\}$ which usually does not have independent increments when $F \ne G$.

The following simple example in the case of logrank statistics shows that the limit of $n^{-1}\mu_n(t)$ in (4.10) or the limiting drift $\mu(t)$ in (4.6) may not be a monotone function of t even for stochastically ordered alternatives. Let F_0 be the exponential distribution with constant hazard rate $\lambda > 0$, and define for $\theta > 0$,

$$1 - F_\theta(x) = \begin{cases} \exp\{-(1-\theta)\lambda x\}, & 0 \le x \le 1, \\ \exp\{3\theta\lambda/2 - (1+\theta/2)\lambda x\}, & 1 < x \le 3, \\ \exp(-\lambda x), & x > 3. \end{cases} \tag{4.11}$$

Let $F = F_0$ and $G = F_\theta$. Clearly $\{F_\theta,\ \theta \ge 0\}$ is stochastically ordered; in fact, $F_\theta \ge F_{\theta'}$ for $\theta \le \theta'$. For $\theta > 0$, the hazard rate of F_θ is $\lambda_\theta(x) = (1-\theta)\lambda(<\lambda)$ if $0 \le x \le 1$, $\lambda_\theta(x) = (1+\theta/2)\lambda(>\lambda)$ if $1 < x \le 3$ and $\lambda_\theta(x) = \lambda$ for $x > 3$. Therefore the function g in (4.5) is given by

$$g(x) = \begin{cases} -1, & 0 \le x \le 1, \\ \frac{1}{2}, & 1 < x \le 3, \\ 0, & x > 3. \end{cases} \tag{4.12}$$

Hence, for the time-sequential logrank statistics [with $Q_n(t, u) \equiv 1$], the limiting drift $\mu(t)$, given by (4.6) for contiguous alternatives $\theta = cn^{-1/2} > 0$, is increasing for $0 < t < 1$, decreasing for $1 < t < 3$ and constant for $t \ge 3$, under the assumption that $b'(t, u)b''(t, u) > 0$ for all $0 \le u < t$. Moreover, under this assumption, the limit of $n^{-1}\mu_n(t)$ in (4.10) for a fixed alternative in the family (4.11) is again increasing for $0 < t < 1$, decreasing for $1 < t < 3$ and constant for $t \ge 3$. This shows that a level-α test of H_0: $F = G$ based on the logrank statistic $S_n(t_1)$ with $1 < t_1 < 3$ can have higher power at the stochastically ordered alternatives $(F, G) = (F_0, F_\theta)$ with $\theta > 0$ than that based on

$S_n(t_2)$ evaluated at a later time $t_2 > t_1$, providing therefore both savings in time and increase in power.

5. Applications to sequential testing problems and concluding remarks. Assuming a Lehmann (proportional hazards) family of the form $1 - G(s) = (1 - F(s))^{1-\theta}$, Jones and Whitehead (1979) considered the use of time-sequential logrank statistics $S_n(t)$ [given by (1.2) with $Q_n \equiv 1$] to test sequentially over time the one-sided null hypothesis H_0': $\theta \leq 0$. They suggested plotting $S_n(t)$ versus $\tilde{V}_n(t)$, where $\tilde{V}_n(t)$ is Mantel's (1966) estimate of the variance of $S_n(t)$ under $F = G$:

$$\tilde{V}_n(t) = \int_0^t m'_{n,t}(s) m''_{n,t}(s) \times [m'_{n,t}(s) + m''_{n,t}(s)]^{-2} d(N'_{n,t}(s) + N''_{n,t}(s)). \tag{5.1}$$

They argued heuristically that $\{(\tilde{V}_n(t), S_n(t)),\ t \geq 0\}$ should behave approximately like $\{(v, W(v)),\ v \geq 0\}$, where $W(v)$ is the standard Wiener process under $\theta = 0$, and is a Wiener process with drift under alternatives θ close to 0. Using this Wiener process approximation, one can therefore simply replace $(v, W(v))$ in a sequential test for the drift of a Wiener process by $(\tilde{V}_n(t), S_n(t))$ to construct a corresponding time-sequential logrank test for H_0'. In particular, Jones and Whitehead considered the case where the sequential test based on $(v, W(v))$ is a sequential probability ratio test. Earlier, Armitage (1975), Chapter 7, suggested using similar approximations to construct repeated significance tests of H_0: $\theta = 0$ based on time-sequential logrank statistics. The Sellke–Siegmund (1983) weak convergence theory for time-sequential logrank statistics provides a rigorous asymptotic justification of the heuristics of Jones and Whitehead under H_0: $\theta = 0$. Moreover, in view of (4.9), the heuristic argument that $\{(\tilde{V}_n(t), S_n(t)),\ t \geq 0\}$ should also behave like a Wiener process $\{(v, W(v)),\ v \geq 0)$ with $EW(v)/v =$ constant, under contiguous proportional hazards alternatives, is also valid asymptotically since the function g in (4.5) reduces to a constant and $Q_n \equiv 1$ for the logrank statistics.

Instead of using logrank statistics, one can use other rank statistics (1.2) with weight functions $Q_n(t, s)$ of the form (3.24) or (4.3) to construct repeated significance or other sequential tests of H_0: $F = G$ or H_0': $F \leq G$. As shown in Section 4, the limiting Gaussian process for these time-sequential rank statistics has independent increments under the null hypothesis and under contiguous alternatives. An extension of (5.1) to estimate the variance of $S_n(t)$ under $F = G$ is

$$\tilde{V}_n(t) = \int_0^t Q_n^2(t, s) m'_{n,t}(s) m''_{n,t}(s) \times [m'_{n,t}(s) + m''_{n,t}(s)]^{-2} d(N'_{n,t}(s) + N''_{n,t}(s)); \tag{5.2}$$

see Gill (1980), where an alternative estimator

$$\hat{V}_n(t) = \int_0^t Q_n^2(t,s)[m'_{n,t}(s) + m''_{n,t}(s)]^{-2} \times \left\{ (m''_{n,t}(s))^2 dN'_{n,t}(s) + (m'_{n,t}(s))^2 dN''_{n,t}(s) \right\} \tag{5.3}$$

is also discussed. As pointed out by Gill (1980), pages 103 and 104, there is no clear choice between $\tilde{V}_n(t)$ and $\hat{V}_n(t)$ since one estimator has a smaller limiting value (when normalized by n) than the other for some (F, G) pairs but the situation is reversed for other (F, G) pairs. We propose to use

$$V_n(t) = \{\hat{V}_n(t) + \tilde{V}_n(t)\}/2 \tag{5.4}$$

as a comparison between these two choices. For Q_n of the form (3.24), it follows from (3.25) that

$$n^{-1}V_n(t) \to_P \int_0^{s(t)} \psi^2(\gamma F(s) + (1-\gamma)G(s)) m'(t,s) m''(t,s) \times [m'(t,s) + m''(t,s)]^{-1} d(\Phi(s) + \Gamma(s))/2, \tag{5.5}$$

where m' and m'' are defined in (3.10), γ is given in (3.9) and $s(t)$ is given in (3.25). Since $s(t)$, $m'(t,s)$ and $m''(t,s)$ are nondecreasing in t, the limit function in (5.5) is also nondecreasing in t, which is an obvious advantage in view of the independent increments property of the limiting Gaussian process under $F = G$, noting also that the limit functions

$$n^{-1}\tilde{V}_n(t) \to_P \int_0^{s(t)} \psi^2(\gamma F(s) + (1-\gamma)G(s)) m'(t,s) m''(t,s) \times \frac{m'(t,s)\, d\Phi(s) + m''(t,s)\, d\Gamma(s)}{\{m'(t,s) + m''(t,s)\}^2},$$

$$n^{-1}\hat{V}_n(t) \to_P \int_0^{s(t)} \psi^2(\gamma F(s) + (1-\gamma)G(s)) m'(t,s) m''(t,s) \times \frac{m''(t,s)\, d\Phi(s) + m'(t,s)\, d\Gamma(s)}{\{m'(t,s) + m''(t,s)\}^2},$$

however, need not be nondecreasing in t unless $\Phi = \Gamma$.

The independent increments property of the limiting Gaussian process under H_0: $F = G$ for the time-sequential censored rank statistics (1.2) with weight functions Q_n of the form (3.24) or (4.3) is basic to the relatively simple Wiener process approximations for calculating type I error probabilities of the associated sequential tests [cf. Siegmund (1985), Chapters 3–5]. If the limiting Gaussian process has correlated increments, as is the case with the Tarone–Ware weight functions (1.4) with nonconstant ψ, the type I errors of the corresponding sequential tests are much harder to evaluate by the Gaussian process approximation; see Slud and Wei's (1982) sequential test using Gehan's statistics.

In summary, we have established in Corollary 2 of Section 3 weak convergence to Gaussian processes for time-sequential two-sample censored rank statistics, under the null hypothesis $F = G$ and contiguous alternatives and also under fixed general alternatives. The examples in that section show the wide scope of applicability of the results, which are further specialized in Section 4 to the Tarone–Ware and Harrington–Fleming classes of statistics. Conditions (3.17) and (3.18) characterize the settings in which the limiting Gaussian process has independent increments under the null hypothesis and contiguous alternatives and are satisfied by censored rank statistics $S_n(t)$ with weight functions Q_n of the form (3.24) or (4.3). This independent increments property of the limiting Gaussian process enables us to construct sequential tests based on $S_n(t)$ by using relatively simple Wiener process approximations, and (5.4) provides an estimate of the asymptotic variance of $S_n(t)$ under $F = G$ for the implementation of these tests. In Section 4 it is shown, however, that the asymptotic drift of $S_n(t)$ may level off or even decrease with increasing t under stochastically ordered alternatives for which the weight function Q_n associated with $S_n(t)$ is not asymptotically optimal, such as the nonproportional hazards alternatives (4.11) in the case of logrank statistics. Since the asymptotic variance of $S_n(t)$ continues to increase with t because of the asymptotically uncorrelated increments property under the null hypothesis and contiguous alternatives, the efficacy of $S_n(t)$ may actually decrease with increasing t, therefore allowing sequential tests to achieve both savings in sample size and increase in power over fixed-duration tests. Simulation studies illustrating this remarkable phenomenon and details of the implementation and performance of various time-sequential rank tests of H_0: $F = G$ and of the one-sided hypothesis H_0': $F \leq G$ will be presented elsewhere.

APPENDIX: PROOF OF LEMMA 2

To prove Lemma 2, we shall make use of some basic maximal inequalities that are given in the following two lemmas. First, Lemma 4 states the continuous-parameter version, due to Lenglart, Lepingle and Pratelli (1980), of the Burkholder–Gundy–Davis inequality for martingales and also an exponential inequality [cf. Shorack and Wellner (1986), page 899] for continuous-parameter martingales. Next, we extend the Longnecker–Serfling inequality to a form that is applicable to our setting in Lemma 5, which can be proved by an induction argument similar to that used in the proof of Lemma 2 of Longnecker and Serfling (1977).

LEMMA 4. *Let $\{Z(s), \mathscr{G}(s),\ s \geq 0\}$ be a martingale with right-continuous sample paths that have left-hand limits. Take any $a > 0$.*

(i) *For any $q > 1$, there exists a universal constant C_q, depending only on q, such that*

$$E\Big(\sup_{s \leq a} |Z(s)|^q\Big) \leq C_q\Big\{E[\langle Z\rangle(a)]^{q/2} + E\Big[\sup_{s \leq a} |\Delta Z(s)|^q\Big]\Big\},$$

where $\langle Z\rangle(s)$ *denotes the predictable variation process of the martingale* $\{Z(s)\}$ *and* $\Delta Z(s) = Z(s) - Z(s-)$.

(ii) *Suppose that* $\sup_{s\le a}|\Delta Z(s)| \le c$. *Then for any* $\lambda > 0$ *and* $\tau > 0$,

$$P\Big\{\sup_{s\le a}|Z(s)| \ge \lambda, \langle Z\rangle(a) \le \tau\Big\} \le 2\exp\big\{-\tfrac{1}{2}(\lambda^2/\tau)\psi(\lambda c/\tau)\big\},$$

where $\psi(x) = 2x^{-2}\{(1+x)[\log(1+x) - 1] + 1\}$.

LEMMA 5. *Let* $q > 0$ *and* $\gamma > 1$. *Let* $\{Z_n, n \ge 1\}$ *be a sequence of random variables defined in the same probability space and let* $\{g_n\}$ *be a sequence of nonnegative integrable functions on a measure space* $(\mathscr{X}, \mathscr{B}, \mu)$. *Suppose that for every fixed* $x \in \mathscr{X}$, $g_n(x)$ *is nondecreasing in* $n \le N$ *and that*

$$\text{(A.1)} \quad E|Z_i - Z_j|^q \le \Big\{\int_{\mathscr{X}}[g_i(x) - g_j(x)]\,d\mu(x)\Big\}^\gamma \quad \textit{for all } 1 \le j \le i \le N.$$

Then there exists a universal constant $C_{q,\gamma}$ *depending only on* q *and* γ *such that*

$$\text{(A.2)} \qquad E\Big(\sup_{n\le N}|Z_n - Z_1|^q\Big) \le C_{q,\gamma}\Big\{\int_{\mathscr{X}}[g_N(x) - g_1(x)]\,d\mu(x)\Big\}^\gamma.$$

We now proceed to prove Lemma 2. The first step in the proof is to replace the half-line $[0,\infty]$ by $n(k) = O(k^\nu)$ points to $t_{k,0} < \cdots < t_{k,n(k)}$ which may depend on the entry times $T_{k1},\dots,T_{kk}$. This is the content of the following lemma.

LEMMA 6. *With the same notation and assumptions as in Theorem* 1, *let* ν *be an integer greater than or equal to* 7. *Let* $\theta_0^k = 0$ *and define* θ_j^k *inductively by*

$$\text{(A.3)} \qquad F_k\big(\theta_j^k\big) - F_k\big(\theta_{j-1}^k\big) = k^{-(\nu-1)} \quad \textit{for } 1 \le j < k^{\nu-1},$$

recalling that F_k *is a continuous distribution function on* $[0,\infty)$. *Let*

$$\text{(A.4)} \quad \mathscr{T}_k = \{0\} \cup \{\infty\} \cup \big\{\theta_j^k + T_{ki}: i = 1,\dots,k;\ j = 0,\dots,k^{\nu-1} - 1\big\},$$

and order the distinct elements of $\mathscr{T}_k$ *as* $0 = t_{k,0} < \cdots < t_{k,n(k)} = \infty$. *Then* $n(k) \le k^\nu + 2$, *and for every* $\eta > 0$ *such that* $\frac{1}{2} > \alpha + \eta$,

$$P\Big\{\sup_{1\le n\le n(k)}\ \sup_{\substack{t_{k,n-1}\le t\le t_{k,n},\\ s\ge 0}} |W_k(t,s) - W_k(t_{k,n-1},s)| \ge k^{-1/2+\alpha+\eta}\Big\} = O(k^{-\nu+6}).$$

PROOF. In view of (A.4), given any $i \in \{1,\dots,k\}$ and $n \in \{1,\dots,n(k)\}$, we can choose $j = j(i,n) \in \{0,\dots,k^{\nu-1} - 1\}$ such that

$$\text{(A.5)} \quad [t_{k,n-1}, t_{k,n}] \subset \big[\theta_{j(i,n)}^k + T_{ki}, \theta_{j(i,n)+1}^k + T_{ki}\big], \qquad \big(\theta_{k^{\nu-1}}^k = \infty\big).$$

Since $N_k(t,s) = \sum_1^k I_{\{X_{ki} \le \xi_{ki} \wedge (t-T_{ki})^+ \wedge s\}}$, it follows that for fixed $n \le n(k)$,

$$\begin{aligned}
&\{N_k(t_{k,n},\infty) - N_k(t_{k,n-1},\infty) \ge 2\} \\
&\quad \subset \{X_{ki} \le \xi_{ki} \text{ and } t_{k,n-1} - T_{ki} < X_{ki} \le t_{k,n} - T_{ki} \text{ for at least two } i\text{'s}\} \\
&\quad \subset \left\{X_{ki} \in \left[\theta^k_{j(i,n)}, \theta^k_{j(i,n)+1}\right] \text{ for at least two } i\text{'s}\right\}, \qquad [\text{by (A.5)}].
\end{aligned}$$

Therefore by (A.3),

$$P\{N_k(t_{k,n},\infty) - N_k(t_{k,n-1},\infty) \ge 2\} \le \binom{k}{2} k^{-2(\nu-1)} \le k^{-2\nu+4}. \tag{A.6}$$

Since $m_k(t,s) = \sum_1^k I_{\{X_{ki} \wedge \xi_{ki} \wedge (t-T_{ki})^+ \ge s\}}$, it follows that for fixed $n \le n(k)$,

$$\begin{aligned}
&\left\{\int_0^\infty I_{\{m_k(t_{k,n-1},u) \ne m_k(t_{k,n},u)\}} N_k(t_{k,n},du) \ge 2\right\} \\
&\quad \subset \{m_k(t_{k,n-1}, X_{kh}) \ne m_k(t_{k,n}, X_{kh}) \text{ for at least two } h\text{'s}\} \\
&\quad \subset \big\{\exists\, h_1 \ne h_2 \in \{1,\dots,k\} \text{ and } i_1, i_2 \in \{1,\dots,k\} \text{ such that} \\
&\qquad t_{k,n-1} - T_{k,i_1} < X_{k,h_1} \le t_{k,n} - T_{k,i_1} \text{ and} \\
&\qquad t_{k,n-1} - T_{k,i_2} < X_{k,h_2} \le t_{k,n} - T_{k,i_2}\big\}.
\end{aligned}$$

Hence by (A.5) and (A.3),

$$\begin{aligned}
P\left\{\int_0^\infty I_{\{m_k(t_{k,n-1},u) \ne m_k(t_{k,n},u)\}} N_k(t_{k,n-1},du) \ge 2\right\} &\le \binom{k}{2} k^2 k^{-2(\nu-1)} \\
&\le k^{-2\nu+6}.
\end{aligned} \tag{A.7}$$

From (2.2) and (2.3), it follows that for $t_{k,n-1} \le t \le t_{k,n}$,

$$\begin{aligned}
&k^{1/2-\alpha} |W_k(t,s) - W_k(t_{k,n-1},s)| \\
&\quad = \left| \int_0^s m_k^{-\alpha}(t,u) I_{\{m_k(t,u)\ge 1\}} M_k(t,du) \right. \\
&\qquad \left. - \int_0^s m_k^{-\alpha}(t_{k,n-1},u) I_{\{m_k(t_{k,n-1},u)\ge 1\}} M_k(t_{k,n-1},du) \right| \\
&\quad \le [N_k(t_{k,n},\infty) - N_k(t_{k,n-1},\infty)] \\
&\qquad + \int_0^\infty I_{\{m_k(t_{k,n-1},u) \ne m_k(t_{k,n},u)\}} N_k(t_{k,n},du) \\
&\qquad + \int_0^s [m_k^{1-\alpha}(t_{k,n},u) - m_k^{1-\alpha}(t_{k,n-1},u)]\, d\Phi_k(u).
\end{aligned} \tag{A.8}$$

Since the function $x - x^{1-\alpha}$ is increasing in $x \ge 1$,

$$\begin{aligned}
&\int_0^s [m_k^{1-\alpha}(t_{k,n},u) - m_k^{1-\alpha}(t_{k,n-1},u)]\, d\Phi_k(u) \\
&\quad \le \int_0^s [m_k(t_{k,n},u) - m_k(t_{k,n-1},u)]\, d\Phi_k(u).
\end{aligned} \tag{A.9}$$

Let $Z_{k,n}(s) = M_k(t_{k,n}, s) - M_k(t_{k,n-1}, s)$ and let $\mathscr{F}_k(s)$ denote the complete σ-field generated by (2.5). Then $\{Z_{k,n}(s), \mathscr{F}_k(s), s \geq 0\}$ is a martingale with $\sup_s|\Delta Z_{k,n}(s)| \leq 1$ a.s. and therefore by the exponential inequality in Lemma 4(ii),

$$P\left(\bigcup_{i=1}^{\infty}\left\{\sup_s |Z_{k,n}(s)| \geq ik^{2\eta/3}, \tfrac{1}{2}ik^{\eta} \leq \langle Z_{k,n}\rangle(\infty) \leq ik^{\eta}\right\}\right) \leq 2\sum_{i=1}^{\infty} \exp\left(-\tfrac{1}{3}ik^{\eta/3}\right) \tag{A.10}$$

for all large k. Since $\langle Z_{k,n}\rangle(s) = \int_0^s [m_k(t_{k,n}, u) - m_k(t_{k,n-1}, u)]\, d\Phi_k(u)$ and $Z_{k,n}(s) = N_k(t_{k,n}, s) - N_k(t_{k,n-1}, s) - \langle Z_{k,n}\rangle(s)$, we have for $k^{\eta/3} \geq 4$,

$$\begin{aligned} &\left\{N_k(t_{k,n}, \infty) - N_k(t_{k,n-1}, \infty) \leq 1,\right. \\ &\quad \left.\int_0^{\infty} [m_k(t_{k,n}, u) - m_k(t_{k,n-1}, u)]\, d\Phi_k(u) \geq \tfrac{1}{2}k^{\eta}\right\} \\ &\quad \subset \bigcup_{i=1}^{\infty}\left\{\sup_s |Z_{k,n}(s)| \geq ik^{2\eta/3}, \tfrac{1}{2}ik^{\eta} \leq \langle Z_{k,n}\rangle(\infty) \leq ik^{\eta}\right\}. \end{aligned} \tag{A.11}$$

From (A.6)–(A.11), it follows that for all large k,

$$\begin{aligned} &\sup_{n \leq n(k)} P\left\{\sup_{t_{k,n-1} \leq t \leq t_{k,n}} k^{1/2-\alpha}|W_k(t,s) - W_k(t_{k,n-1}, s)| \geq 4 + \tfrac{1}{2}k^{\eta}\right\} \\ &\quad \leq k^{-2\nu+4} + k^{-2\nu+6} + 2\sum_{i=1}^{\infty}\exp\left(-\tfrac{1}{3}ik^{\eta/3}\right). \end{aligned}$$

Since $n(k) = O(k^{\nu})$, the desired conclusion follows. □

The next step in the proof of Lemma 2 is to make use of Lemma 6 and the exponential inequality in Lemma 4(ii) to show that the interval $[0, \tau]$ can be replaced further by nonrandom partition points τ_i^k defined in the following.

LEMMA 7. *With the same notation and assumptions as Theorem* 1, *let* $b_k(t,s) = k^{-1}\sum_1^k P\{\xi_{ki} \geq s, t - T_{ki} \geq s\}$. *Let* $0 < \gamma < 1 - 2\alpha$, $0 < \delta < \gamma/2$. *Let* $\tau_0^k = 0$ *and define*

$$\begin{aligned} \tau_{i+1}^k = \inf\Big\{t \in (\tau_i^k, \tau]: &\int_0^{\infty}\left[b_k(t,s) - b_k(\tau_i^k, s)\right]^{1-2\alpha} \\ &\times (1 - F_k(s))^{-2\alpha}\, dF_k(s) \geq k^{-\gamma}\Big\}, \end{aligned} \tag{A.12}$$

$\inf\varnothing = \tau$, *noting that* $b_k(t,s)$ *is nondecreasing and right-continuous in* t. *Then*

$$P\left\{\sup_i \sup_{\tau_i^k \leq t < \tau_{i+1}^k,\, s \geq 0} \left|W_k(t,s) - W_k(\tau_i^k, s)\right| \geq k^{-\delta}\right\} \to 0 \quad \textit{as } k \to \infty.$$

PROOF. From (A.12), it follows that there are $O(k^\gamma)$ partition points τ_i^k of $[0, \tau]$. In view of Lemma 6, it suffices to show that

$$P\left\{\sup_i \sup_{t \in \mathscr{T}_k,\, \tau_i^k \le t < \tau_{i+1}^k} \left(\sup_{s \ge 0} \left|W_k(t,s) - W_k(\tau_i^k, s)\right|\right) \ge k^{-\delta}\right\} \to 0 \tag{A.13}$$

as $k \to \infty$,

where $\mathscr{T}_k$ is defined in (A.4). For $\tau_i^k \le t < \tau_{i+1}^k$, by the definition (A.12) of τ_{i+1}^k,

$$\int_0^\infty \left[b_k(t,s) - b_k(\tau_i^k, s)\right]^{1-2\alpha} (1 - F_k(s))^{-2\alpha}\, dF_k(s) < k^{-\gamma}. \tag{A.14}$$

Moreover, $\{W_k(t,s) - W_k(\tau_i^k, s), \mathscr{F}_k(s), s \ge 0\}$ is a martingale with predictable variation process

$$\begin{aligned}
&\langle W_k(t,\cdot) - W_k(\tau_i^k, \cdot)\rangle(s) \\
&= k^{2\alpha-1}\left\{\int_0^s m_k^{-2\alpha}(t,u) I_{\{m_k(t,u)>0\}}\left[m_k(t,u) - m_k(\tau_i^k, u)\right] d\Phi_k(u)\right. \\
&\qquad + \int_0^s \left[m_k^{-\alpha}(t,u) I_{\{m_k(t,u)>0\}}\right. \\
&\qquad\qquad \left.\left. - m_k^{-\alpha}(\tau_i^k, u) I_{\{m_k(\tau_i^k,u)>0\}}\right]^2 m_k(\tau_i^k, u)\, d\Phi_k(u)\right\} \\
&\le 2\int_0^s \left[m_k(t,u)/k - m_k(\tau_i^k, u)/k\right]^{1-2\alpha} \\
&\qquad \times I_{\{m_k(t,u)\ge 1\}} (1 - F_k(u))^{-1}\, dF_k(u),
\end{aligned}$$

using the inequalities $0 < 2\alpha < 1$ and $1 - x^\alpha \le 1 - x \le 1$ for $0 \le x \le 1$. Since $m_k(t,u) \ge 1 \Rightarrow k^{-1}\Sigma_1^k I_{\{X_{kj} \ge u\}} \ge k^{-1}$, it then follows that for $\tau_i^k \le t < \tau_{i+1}^k$,

$$\begin{aligned}
\langle W_k(t,\cdot) - W_k(\tau_i^k, \cdot)\rangle(\infty) &\le 2\int_0^\infty \{[m_k(\tau_{i+1}^k -, u) - m_k(\tau_i^k, u)]/k\}^{1-2\alpha} \\
&\qquad \times I_{\{k^{-1}\Sigma_1^k I_{\{X_{kj}\ge u\}} \ge k^{-1}\}} (1 - F_k(u))^{-1}\, dF_k(u).
\end{aligned} \tag{A.15}$$

We next make use of (A.14) and (A.15) to show that for every $0 < \beta < 1$,

$$\begin{aligned}
&P\left\{\sup_i \sup_{\tau_i^k \le t < \tau_{i+1}^t} \langle W_k(t,\cdot) - W_k(\tau_i^k, \cdot)\rangle(\infty) \ge 6k^{-\gamma}\right\} \\
&\quad \le P\left\{k^{-1}\sum_{j=1}^k I_{\{X_{kj}\ge u\}} > \beta^{-1}(1 - F_k(u)) \text{ for some } u \ge 0\right\} \\
&\qquad + O(k^\gamma \exp(-k^\theta)) \quad \text{as } k \to \infty
\end{aligned} \tag{A.16}$$

for some $\theta > 0$. Take $0 < \rho < 1$ such that $(1 - \rho)(1 - 2\alpha) > \gamma$. On the event

$$E_k = \bigcap_i \left\{ m_k(\tau_{i+1}^k - ,s) - m_k(\tau_i^k, s) \right.$$
$$\left. \leq 2\max\left(k^\rho, E\left[m_k(\tau_{i+1}^k - ,s) - m_k(\tau_i^k, s)\right]\right) \text{ for all } s \geq 0 \right\}$$
$$\cap \left\{ k^{-1} \sum_{j=1}^k I_{\{X_{kj} \geq s\}} \leq \beta^{-1}(1 - F_k(s)) \text{ for all } s \geq 0 \right\},$$

since $E[m_k(\tau_{i+1}^k - ,s) - m_k(\tau_i^k, s)]/k = (1 - F_k(s))[b_k(\tau_{i+1}^k - ,s) - b_k(\tau_i^k, s)]$, we obtain from (A.14) that

$$\int_0^\infty \left\{ \left[m_k(\tau_{i+1}^k - , s) - m_k(\tau_i^k, s) \right] / k \right\}^{1-2\alpha}$$
$$\times I_{\{k^{-1}\sum_{j=1}^k I_{\{X_{kj} \geq s\}} \geq k^{-1}\}} (1 - F_k(s))^{-1} \, dF_k(s)$$
$$\leq 2k^{-\gamma} + 2(k^{\rho-1})^{1-2\alpha}$$
$$\times \int_{E[m_k(\tau_{i+1}^k - , s) - m(\tau_i^k, s)] \leq k^\rho} I_{\{1 - F_k(s) \geq \beta k^{-1}\}} (1 - F_k(s))^{-1} \, dF_k(s)$$
$$\leq 2k^{-\gamma} + 2k^{-(1-\rho)(1-2\alpha)} \log(k/\beta) < 3k^{-\gamma}$$

for all large k. Therefore, in view of (A.15), the left-hand side of (A.16) is majorized by $P(E_k^c)$ for all large k. Since there are $O(k^\gamma)$ partition points τ_i^k of $[0, \tau]$, Lemma 8 below shows that $P(E_k^c)$ is majorized by the right-hand side of (A.16).

To prove that (A.13) holds, we shall assume without loss of generality that $(\gamma/2 >)\delta > \gamma - (\frac{1}{2} - \alpha)$. By the exponential inequality in Lemma 4(ii), for all large k, say $k \geq k_0$, we have for every i and for all $t \in \mathscr{T}_k$ with $t \geq \tau_i^k$,

$$P\left[\sup_{s \geq 0} \left| W_k(t, s) - W_k(\tau_i^k, s) \right| \geq k^{-\delta}, \right.$$
$$\left. \langle W_k(t, \cdot) - W_k(\tau_i^k, \cdot) \rangle(\infty) \leq 6k^{-\gamma} T \right| T_{k1}, \dots, T_{kk} \Big]$$
$$\leq 2\exp(-k^{\gamma - 2\delta}/13).$$

Since $\mathscr{T}_k$ has $O(k^\nu)$ elements while there are $O(k^\gamma)$ partition points τ_i^k, it then follows that

$$P\left(\bigcup_i \bigcup_{t \in \mathscr{T}_k : \tau_i^k \leq t < \tau_{i+1}^k} \left\{ \sup_{s \geq 0} \left| W_k(t, s) - W_k(\tau_i^k, s) \right| \geq k^{-\delta}, \right.\right.$$
$$\text{(A.17)} \qquad \left.\left. \langle W_k(t, \cdot) - W_k(\tau_i^k, \cdot) \rangle(\infty) \leq 6k^{-\gamma} \right\} \right)$$
$$= O\left(k^{\gamma+\nu} \exp(-k^{\gamma - 2\delta}/13)\right).$$

From (A.16) and (A.17) together with Theorem 1.1.1 of van Zuijlen (1977), the desired conclusion (A.13) follows by letting $\beta \to 0$. □

The following lemma, which has been used in the proof of Lemma 7 and will also be used later in the proof of Lemma 2, follows from Corollary 1.3 and Theorem 2.1 of Alexander (1985), noting that for fixed $t_1 \le t_2$,

$$m_k(t_2, s) - m_k(t_1, s) = \sum_1^k I_{\{X_{ki} \wedge \xi_{ki} \ge s,\, t_2 - s \ge T_{ki} > t_1 - s\}}.$$

LEMMA 8. *With the same notation and assumptions as in Theorem* 1, *given any* $0 < \rho < 1$ *and* $\eta > 0$, *there exists* $\theta > 0$ *such that as* $k \to \infty$,

$$\sup_{0 \le t_1 \le t_2 \le \tau} P\left\{ \sup_{s:\, E[m_k(t_2, s) - m_k(t_1, s)] \ge k^\rho} \left| \frac{m_k(t_2, s) - m_k(t_1, s)}{Em_k(t_2, s) - Em_k(t_1, s)} - 1 \right| \ge \eta \right\}$$
$$= O(\exp(-k^\theta)),$$

$$\sup_{0 \le t_1 \le t_2 \le \tau} P\left\{ \sup_{s:\, E[m_k(t_2, s) - m_k(t_1, s)] \le k^\rho} [m_k(t_2, s) - m_k(t_1, s)] \ge 2k^\rho \right\}$$
$$= O(\exp(-k^\theta)),$$

$$\sup_{t \ge 0} P\left\{ \sup_{s:\, Em_k(t, s) \ge k^\rho} \left| \frac{m_k(t, s)}{Em_k(t, s)} - 1 \right| \ge \eta \right\} = O(\exp(-k^\theta)).$$

PROOF OF LEMMA 2. Let Λ_k denote the set of all partition points τ_i^k of $[0, \tau]$ introduced in Lemma 7. Then Λ_k has $O(k^\gamma)$ elements. Take any $0 < \rho < 1$ and define

$$\tilde{W}_k(t, s) = k^{\alpha - 1/2} \int_0^s m_k^{-\alpha}(t, u) I_{\{m_k(t, u) \ge 1,\, Em_k(t, u) \ge k^\rho\}} M_k(t, du).$$

Note that for every t, $\{W_k(t, s) - \tilde{W}_k(t, s), \mathscr{F}_k(s),\, s \ge 0\}$ is a martingale. Let $0 < \lambda < (1 - \rho)(1 - 2\alpha)$. From Lemma 8 it follows that

$$P\left\{ \sup_{t \in \Lambda_k} \langle W_k(t, \cdot) - \tilde{W}_k(t, \cdot) \rangle(\infty) \ge k^{-\lambda} \right\}$$
$$\le P\left\{ \sup_{t \in \Lambda_k} \int_0^\infty [m_k(t, u)/k]^{1 - 2\alpha} I_{\{1 \le m_k(t, u) \le 2k^\rho\}} \right.$$
$$\text{(A.18)} \qquad \left. \times (1 - F_k(u))^{-1} dF_k(u) \ge k^{-\lambda} \right\} + O(k^\gamma e^{-k^\theta})$$
$$\le P\left\{ k^{-1} \sum_1^k I_{\{X_{ki} \ge u\}} \ge \beta^{-1}(1 - F_k(u)) \text{ for some } u \ge 0 \right\} + O(k^\gamma e^{-k^\theta})$$
$$\text{as } k \to \infty$$

for every $0 < \beta < 1$. To see the last inequality above, note that $m_k(t, u)/k \le k^{-1}\sum_1^k I_{\{X_{ki} \ge u\}}$ and that $k^{-(1-\rho)(1-2\alpha)} = o(k^{-\lambda})$. Hence, making use of Theorem 1.1.1 of van Zuijlen (1977) and Lemma 4(ii), we obtain from (A.18) by letting $\beta \to 0$ that for every $0 < \delta < \lambda/2$,

$$\text{(A.19)} \qquad P\left\{\sup_{t \in \Lambda_k} \sup_{s \ge 0} \left| W_k(t, s) - \tilde{W}_k(t, s) \right| \ge k^{-\delta}\right\} \to 0 \quad \text{as } k \to \infty.$$

Choose positive numbers q, p such that

$$\text{(A.20)} \qquad q(1/2 - \alpha) > 1, \qquad p\{q(1/2 - \alpha) - 1\} > 1 + q.$$

Let $t_0 = 0$ and define t_j inductively by

$$\text{(A.21)} \quad t_{j+1} = \inf\left\{t > t_j \colon \int_0^\infty \left[b(t, s) - b(t_j, s)\right]\left(1 - F(s)\right)^{-2\alpha} dF(s) = \varepsilon^p\right\}.$$

There are finitely many, say $N - 1$, such points in the interval $(0, \tau)$. Redefine t_N by $t_N = \tau$. As $k \to \infty$, $b_k(t, s) \to b(t, s)$ by (2.1), and the convergence is uniform in $t \in [0, \tau]$ and $s \ge 0$ since $b_k(t, s)$ is monotone in t and in s and since b is continuous; moreover, $F_k(u) \to F(u)$ uniformly in u, and F_k, F are monotone and continuous. Therefore, by (A.21), for large k,

$$\text{(A.22)} \quad \max_{0 \le j < N} \int_0^\infty \left[b_k(t_{j+1}, s) - b_k(t_j, s)\right]\left(1 - F_k(s)\right)^{-2\alpha} dF_k(s) \le 2\varepsilon^p.$$

Moreover, in view of the continuity of b and the uniform convergence of b_k to b, it follows from (A.12) that as $k \to \infty$,

$$\text{(A.23)} \quad \sup_i \int_0^\infty \left[b_k(\tau_{i+1}^k, s) - b_k(\tau_i^k, s)\right]\left(1 - F_k(s)\right)^{-2\alpha} dF_k(s) \to 0.$$

To prove that the conclusion of Lemma 2 holds for the partition points t_j defined by (A.21), it suffices to show, in view of Lemma 7, that for all large k,

$$P\left\{\max_{0 \le j < N} \sup_{t \in \Lambda_k,\, t_j \le t \le t_{j+1}} \left(\sup_{s \ge 0} \left| W_k(t, s) - W_k\left(\tau_{i_k(j)}^k, s\right) \right|\right) \ge \varepsilon/2\right\} \le \varepsilon,$$

where $i_k(j) = \max\{i \colon \tau_i^k \le t_j\}$, recalling that Λ_k consists of $O(k^\gamma)$ partition points $\tau_0^k < \tau_1^k < \cdots$. By (A.19), it further suffices to show that for all large k,

$$\text{(A.24)} \quad P\left\{\max_{0 \le j < N}\left[\sup_{s \ge 0} \max_{i_k(j) \le i < i_k(j+1)} \left| \tilde{W}_k(\tau_i^k, s) - \tilde{W}_k\left(\tau_{i_k(j)}^k, s\right) \right| \right] \ge \varepsilon/3\right\} \le \varepsilon.$$

Fix $j = 0, \dots, N - 1$. Since $\{\tilde{W}_k(\tau_i^k, s), \mathscr{F}_k(s),\ s \ge 0)$ is a martingale, $\{\max_{i_k(j) \le i < i_k(j+1)} |\tilde{W}_k(\tau_i^k, s) - \tilde{W}_k(\tau_{i_k(j)}^k, s)|, \mathscr{F}_k(s),\ s \ge 0)$ is a nonnegative submartingale, and therefore

$$E\left(\sup_{0\le s\le a}\max_{i_k(j)\le i<i_k(j+1)}\left|\tilde W_k(\tau_i^k,s)-\tilde W_k(\tau_{i_k(j)}^k,s)\right|^q\right)$$
$$\le\left(\frac{q}{q-1}\right)^q E\left(\max_{i_k(j)\le i<i_k(j+1)}\left|\tilde W_k(\tau_i^k,a)-\tilde W_k(\tau_{i_k(j)}^k,a)\right|^q\right), \tag{A.25}$$

by Doob's (1953) inequality, where we take any $a \ge \tau^k_{i_k(j+1)}$, noting that

$$\tilde W_k(t,s)=\tilde W_k(t,a)\quad\text{for all } s\ge a \text{ and } t\le \tau^k_{i_k(j+1)}. \tag{A.26}$$

Recalling that $0<\gamma<1-2\alpha$ in (A.12), let

$$A_k=\bigcup_{r<i}\Big\{m_k(\tau_i^k,u)-m_k(\tau_r^k,u)\ge\max\Big(k^{(1-\gamma/(1-2\alpha))/2},$$
$$2E\big[m_k(\tau_i^k,u)-m_k(\tau_r^k,u)\big]\Big)\text{ for some } u\Big\}.$$

It follows from Lemma 8 that

$$P(A_k)=O\big(k^{2\gamma}\exp(-k^\theta)\big)\quad\text{as } k\to\infty. \tag{A.27}$$

Let $i_k(j)\le r<i<i_k(j+1)$. A similar argument as in (A.15) shows that

$$\langle\tilde W_k(\tau_i^k,\cdot)-\tilde W_k(\tau_r^k,\cdot)\rangle(a)$$
$$\le 2\int_0^a\big[m_k(\tau_i^k,u)/k-m_k(\tau_r^k,u)/k\big]^{1-2\alpha} \tag{A.28}$$
$$\times I_{\{Em_k(\tau_i^k,u)\ge k^\rho\}}(1-F_k(u))^{-1}\,dF_k(u).$$

Since $\{Em_k(\tau_i^k,u)\ge k^\rho\}\subset\{1-F_k(u)\ge k^{-1+\rho}\}$, (A.28) implies that

$$\langle\tilde W_k(\tau_i^k,\cdot)-\tilde W_k(\tau_r^k,\cdot)\rangle(a)\le 2\log k. \tag{A.29}$$

Moreover, on A_k^c, since either $m_k(\tau_i^k,u)-m_k(\tau_r^k,u)\le k^{(1-\gamma/(1-2\alpha))/2}$ or $m_k(\tau_i^k,u)-m_k(\tau_r^k,u)\le 2E[m_k(\tau_i^k,u)-m_k(\tau_r^k,u)]$, (A.28) implies that for all large k,

$$\langle\tilde W_k(\tau_i^k,\cdot)-\tilde W_k(\tau_r^k,\cdot)\rangle(a)$$
$$\le 2k^{-(1-2\alpha+\gamma)/2}\log k$$
$$+4\int_0^\infty\big[b_k(\tau_i^k,u)-b_k(\tau_r^k,u)\big]^{1-2\alpha}(1-F_k(u))^{-2\alpha}\,dF_k(u) \tag{A.30}$$
$$\le 5\int_0^\infty\big[b_k(\tau_i^k,u)-b_k(\tau_r^k,u)\big]^{1-2\alpha}(1-F_k(u))^{-2\alpha}\,dF_k(u)\quad\text{on } A_k^c,$$

where the last inequality follows from $(1 - 2\alpha + \gamma)/2 > \gamma$ and

$$\int_0^\infty \left[b_k(\tau_i^k, u) - b_k(\tau_r^k, u)\right]^{1-2\alpha}(1 - F_k(u))^{-2\alpha}\, dF_k(u) \geq k^{-\gamma} \quad \text{by (A.12)}. \tag{A.31}$$

Combining (A.29) and (A.30), we have

$$\begin{aligned} E\left[\langle \tilde{W}_k(\tau_i^k, \cdot) - \tilde{W}_k(\tau_r^k, \cdot)\rangle(a)\right]^{q/2} &\leq (2\log k)^{q/2} P(A_k) \\ &\quad + \left\{5\int_0^\infty \left[b_k(\tau_i^k, u) - b_k(\tau_r^k, u)\right]^{1-2\alpha}(1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{q/2} \\ &\leq \left\{6\int_0^\infty \left[b_k(\tau_i^k, u) - b_k(\tau_r^k, u)\right]^{1-2\alpha}(1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{q/2} \end{aligned} \tag{A.32}$$

for all large k, by (A.27) and (A.31). Since the absolute value of the jump size of the process $\{\tilde{W}_k(\tau_i^k, s) - \tilde{W}_k(\tau_r^k, s),\ s \geq 0)$ is no larger than $k^{\alpha - 1/2} = o(k^{-\gamma/2})$, it then follows from Lemma 4(i) together with (A.31) and (A.32) that there exists a constant $C > 0$ such that for all $i_k(j) \leq r < i \leq i_k(j+1)$,

$$\begin{aligned} E\left|\tilde{W}_k(\tau_i^k, a) - \tilde{W}_k(\tau_r, a)\right|^q &\leq C\left\{\int_0^\infty \left[b_k(\tau_i^k, u) - b_k(\tau_r^k, u)\right]^{1-2\alpha}(1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{q/2} \\ &\leq C\left\{\int_0^\infty \left[b_k(\tau_i^k, u) - b_k(\tau_r^k, u)\right](1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{(1-2\alpha)q/2} \\ &\quad \times \left\{\int_0^\infty (1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{\alpha q}, \end{aligned}$$

where the last inequality follows from the bound $\int_0^\infty g^{1-2\alpha}\, d\mu/\mu[0,\infty) \leq \{\int_0^\infty g\, d\mu/\mu[0,\infty)\}^{1-2\alpha}$ for any nonnegative function g and finite measure μ on $[0,\infty)$. Since $q(\frac{1}{2} - \alpha) > 1$, we then obtain by Lemma 5 that there exists $C^* > 0$ such that for all large k,

$$\begin{aligned} E\left(\max_{i_k(j) \leq i < i_k(j+1)} \left|\tilde{W}_k(\tau_i^k, a) - \tilde{W}_k(\tau_{i_k(j)}^k, a)\right|^q\right) &\leq C^*\left\{\int_0^\infty \left[b_k(\tau_{i_k(j+1)}^k, u) - b_k(\tau_{i_k(j)}^k, u)\right]\right. \\ &\qquad \left.\times (1 - F_k(u))^{-2\alpha}\, dF_k(u)\right\}^{q(1/2-\alpha)} \end{aligned} \tag{A.33}$$

From (A.25), (A.26), (A.33) and (A.22) together with the Markov inequality, it follows that for all large k,

$$P\left\{\max_{0\le j<N}\left[\sup_{s\ge 0}\max_{i_k(j)\le i<i_k(j+1)}\left|\tilde{W}_k(\tau_i^k,s)-\tilde{W}_k\left(\tau_{i_k(j)}^k,s\right)\right|\right]\ge \varepsilon/3\right\}$$

$$\le\left(\frac{3}{\varepsilon}\right)^q\left(\frac{q}{q-1}\right)^q C^*\sum_{j=0}^{N-1}(2\varepsilon^p)^{q(1/2-\alpha)-1}$$

$$\times\int_0^\infty\left[b_k\left(\tau_{i_k(j+1)}^k,u\right)-b_k\left(\tau_{i_k(j)}^k,u\right)\right](1-F_k(u))^{-2\alpha}\,dF_k(u)$$

$$\le \overline{C}\varepsilon^{p\{q(1/2-\alpha)-1\}-q}\int_0^\infty b_k(\tau,u)(1-F_k(u))^{-2\alpha}\,dF_k(u),$$

where $\overline{C}=C^*(6q)^q/(q-1)^q$. Since $p\{q(1/2-\alpha)-1\}-q>1$ by (A.20) and since $b_k(\tau,u)\le 1$, the desired conclusion (A.24) follows, provided that ε is sufficiently small. □

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