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ON THE ASYMPTOTIC RELATION BETWEEN *L*-ESTIMATORS AND *M*-ESTIMATORS AND THEIR ASYMPTOTIC EFFICIENCY RELATIVE TO THE CRAMÉR-RAO LOWER BOUND[1]

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This paper gives conditions under which L- and M-estimators of a location parameter have asymptotically the same distribution, conditions under which they are asymptotically equivalent and conditions under which L-estimators are asymptotically efficient relative to the Cramér-Rao lower bound. Our results differ from analogous results of Jung (1955), Bickel (1965), Chernoff, Gastwirth, Johns (1967), Jaeckel (1971) and Rivest (1978, 1982) in that we do not require that the derivative of the density of the observations is absolutely continuous nor that the function defining the M-estimator is absolutely continuous.

1. Introduction. For each $\nu = 1, 2, \cdots$, let $X_{\nu 1}, \cdots, X_{\nu n_\nu}$ be a sample from a distribution with density function $f(x - \theta_0)$, where $f(x)$ is symmetric around zero. Let $Y_{\nu 1} < \cdots < Y_{\nu n_\nu}$ be the order statistics of $X_{\nu 1}, \cdots, X_{\nu n_\nu}$, let $\hat{\theta}_{\nu n_\nu}$ be an M-estimator of θ_0 based on the function ψ and let $\hat{\mu}_{\nu n_\nu} = \sum_{i=1}^{\nu} \alpha_{\nu i} Y_{\nu i}$ be an L-estimator of θ_0.

In this paper conditions are given under which $\hat{\theta}_{\nu n_\nu}$ and $\hat{\mu}_{\nu n_\nu}$ have asymptotically the same distribution, conditions under which they are asymptotically equivalent, as well as conditions under which $\hat{\mu}_{\nu n_\nu}$ is asymptotically efficient relative to the Cramér-Rao lower bound.

Analogous results concerning the relation between $\hat{\theta}_{\nu n_\nu}$ and $\hat{\mu}_{\nu n_\nu}$, under different regularity conditions, were obtained by Bickel (1965), Jaeckel (1971) and Rivest (1978, 1982). Bickel (1965) showed that, under certain regularity conditions on f, the M-estimator based on

$$\psi(x) = \begin{cases} x & \text{if } \quad |x| \le \lambda, \\ \lambda \operatorname{sgn} x & \text{if } \quad |x| > \lambda, \end{cases}$$

has the same asymptotic distribution as an α-trimmed mean with α depending on λ and f. Jaeckel (1971) gave conditions under which $\hat{\theta}_{\nu n_\nu}$ and $\hat{\mu}_{\nu n_\nu}$ have asymptotically the same distribution, as well as conditions under which they are asymptotically equivalent. The asymptotic equivalence of $\hat{\theta}_{\nu n_\nu}$ and $\hat{\mu}_{\nu n_\nu}$ can, under less restrictive regularity conditions than those of Jaeckel (1971), be obtained as a special case of Corollary II.4 of Rivest (1978); see also Rivest (1982, page 230). Rivest (1978, 1982) needs, among other regularity conditions he uses to prove the above mentioned result, that ψ is absolutely continuous. Conditions for the asymptotic efficiency of $\hat{\mu}_{\nu n_\nu}$ relative to the Cramér-Rao lower bound have, for instance, been obtained by Jung (1955), by Chernoff, Gastwirth, Johns (1967) and by Rivest (1978, 1982); these authors suppose, among other regularity conditions, that f' is absolutely continuous. The results in this paper hold when ψ and f' are not necessarily absolutely continuous.

Section 2 contains the notation, definitions and assumptions. In Section 3 the main results are given and Section 4 contains some examples. The proofs are based on results of Shorack (1972) and on an asymptotic linearity theorem similar to the one of Jurečková (1977); this asymptotic linearity theorem is stated and proved in Section 5.

In order to simplify the notation the index ν will be omitted. Limits will be as $\nu \to \infty$ and $n_\nu \to \infty$ with ν.

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2. Notation, definitions and assumptions. Let $X_1, \cdots, X_n$ be a sample from a distribution $F(x-\theta_0)$ with density function $f(x-\theta_0)$, where $f(x)$ is assumed to satisfy

CONDITION A. (i) $f(x) = f(-x)$, $-\infty < x < \infty$; (ii) f is absolutely continuous; (iii) $\int_0^1 \varphi_F^2(u)\, du < \infty$, where

$$\begin{cases} \varphi_F(u) = -\dfrac{f'}{f}(F^{-1}(u)) \\ F^{-1}(u) = \inf\{x \mid F(x) \geq u\} \end{cases} \qquad 0 < u < 1.$$

For the M-estimator, $\hat{\theta}_n$, it will be assumed that the following conditions are satisfied. Let $\mathscr{I}$ be an open interval symmetric with respect to zero and such that $S = \{x \mid 0 < F(x) < 1\} \subseteq \mathscr{I}$. Let ψ be a function defined on $\mathscr{I}$ satisfying

CONDITION B. (i) ψ is left continuous, $\psi(x) = \psi_1(x) + \psi_2(x)$, where ψ_1 is nondecreasing, ψ_2 is non-increasing and $\int_{-\infty}^{\infty} \psi_i^2(x) f(x)\, dx < \infty$, $i = 1, 2$; (ii) $\int_{-\infty}^{\infty} \psi(x) f'(x)\, dx \neq 0$; (iii) $\psi(x) = -\psi(-x)$ for all continuity points x of ψ.

Note that A (iii) and B (i) imply that $\int_{-\infty}^{\infty} |\psi(x) f'(x)|\, dx < \infty$; further, B (ii) and B (iii) imply that ψ is not constant.

A sequence of M-estimators, $\hat{\theta}_n = \hat{\theta}_n(X_1, \cdots, X_n)$, is now defined as a sequence satisfying

$$\text{(2.1)} \qquad \text{(i) } \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i - \hat{\theta}_n) \to_{P_{\theta_0}} 0, \quad \text{(ii) } \sqrt{n}(\hat{\theta}_n - \theta_0) \text{ is bounded in } P_{\theta_0}\text{-probability.}$$

If ψ is monotone nondecreasing and continuous and $\mathscr{I}$ is the interval $(-\infty, \infty)$, then the estimator

$$\text{(2.2)} \qquad \hat{\theta}_n = \gamma \sup\{\theta \mid \textstyle\sum_{i=1}^n \psi(X_i - \theta) > 0\} + (1-\gamma)\inf\{\theta \mid \textstyle\sum_{i=1}^n \psi(X_i - \theta) < 0\},$$

where $0 \leq \gamma \leq 1$, satisfies (2.1); this follows from the fact that, in this case, $P_{\theta_0}[\sum_{i=1}^n \psi(X_i - \hat{\theta}_n) = 0] = 1$ for all n and (see e.g. Huber, 1964) $\sqrt{n}(\hat{\theta}_n - \theta_0)$ has, asymptotically, a normal distribution. Some examples of estimators satisfying (2.1) when ψ is not monotone or $\mathscr{I}$ is not the interval $(-\infty, \infty)$ will be given in Section 5.2.

The above Condition B is a sufficient condition on the function ψ to show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ and $\sqrt{n}(\hat{\mu}_n - \theta_0)$ have the same asymptotic distribution; to show that $\sqrt{n}(\hat{\theta}_n - \hat{\mu}_n) \to_{P_{\theta_0}} 0$, the following, stronger Condition B* is needed.

CONDITION B*. Condition B is satisfied and

$$\int_{-\infty}^{+\infty} |\psi(x)|^r f(x)\, dx < \infty \quad \text{for some} \quad r > 2.$$

The sequence of L-estimators, $\hat{\mu}_n = \hat{\mu}_n(X_1, \cdots, X_n)$, is taken as

$$\text{(2.3)} \qquad \hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n c_{ni} Y_i + \sum_{j=1}^{\kappa} d_j Y_{[np_j]+1},$$

where $\kappa \geq 0$, $d_1, \cdots, d_\kappa$ are nonzero constants and $0 < p_1 < \cdots < p_\kappa < 1$. This estimator is a special case of the estimators considered by Shorack (1972). He gives several sets of conditions under which $\hat{\mu}_n$ is asymptotically normal. The set of conditions used in this paper is as follows (see Shorack, 1972, Assumptions 1, 2 and 4, page 413):

For each $\nu = 1, 2, \cdots$, let $J_n(t)$ be defined on $[0, 1]$ by

$$\text{(2.4)} \qquad J_n(t) = \begin{cases} c_{ni} & \dfrac{i-1}{n} < t \leq \dfrac{i}{n},\ i = 1, \cdots, n, \\ c_{n1} & t = 0. \end{cases}$$

For fixed $\beta_1, \beta_2, M > 0$ and $\delta > 0$ let

$$\begin{cases} B(t) = Mt^{-\beta_1}(1-t)^{-\beta_2} \\ \\ D(t) = Mt^{\beta_1+\delta-1/2}(1-t)^{\beta_2+\delta-1/2} \end{cases} \quad 0 < t < 1. \tag{2.5}$$

Let $J(t)$ be a fixed measurable function defined on (0, 1). It will be assumed that the following Condition C is satisfied.

CONDITION C. (i) $|F^{-1}(t)| \leq D(t)$, $|J(t)| \leq B(t)$ and, for all $\nu = 1, 2, \cdots$, $|J_n(t)| \leq B(t)$, $0 < t < 1$; (ii) Except on a set of F^{-1}-measure zero we have both J is continuous and $J_n \to J$ uniformly in some small neighbourhood of t as $\nu \to \infty$; (iii) $F^{-1}(t)$ has a derivative at $t = p_j, j = 1, \cdots, \kappa$.

Note that by Shorack (1972, Example 1, page 416–417) Condition C is satisfied if the following Condition C′ is satisfied.

CONDITION C′. (i) $\mathscr{E}|X_i|^r < \infty$ for some $r > 2$; (ii) $c_{ni} = J(t_{ni})$, $i = 1, \cdots, n$, where, for some $a > 0$, $a \min\{i/n, 1 - i/n\} \leq t_{ni} \leq 1 - a \min\{i/n, 1 - i/n\}$, $i = 1, \cdots, n$; (iii) $\max_{1\leq i\leq n}|t_{ni} - i/n| \to_{\nu\to\infty} 0$; (iv) J is continuous except at a finite number of points at which F^{-1} is continuous and

$$|J(t)| < M\{t(1-t)\}^{-1/2+1/r+\delta}, \quad 0 < t < 1$$

for some $\delta > 0$; (v) Condition C(iii) is satisfied.

It will further be assumed that the following Condition D is satisfied.

CONDITION D. (i) $\sqrt{n}(1/n \sum_{i=1}^n c_{ni} - \int_0^1 J(t)\,dt) \to 0$ as $\nu \to \infty$; (ii) $\int_0^1 J(t)\,dt + \sum_{j=1}^\kappa d_j = 1$; (iii) $c_{ni} = c_{n(n+1-i)}$, $i = 1, \cdots, n$, and, for $j = 1, \cdots, \kappa$, $d_j = d_{\kappa+1-j}$, $p_j = 1 - p_{\kappa+1-j}$.

REMARK 2.1. In Section 3 it will be shown (see Remark 3.1 after the proof of Theorem 3.1) that, if the Conditions A(i), C′(i) − (iv) and D(iii) are satisfied, then the following condition D′(i) (see Shorack, 1972, Example 1) is sufficient for D(i).

CONDITION D′(i). (a) J' exists and is continuous on (0, 1) with

$$|J'(t)| \leq M\{t(1-t)\}^{-3/2+1/r+\delta}, \quad 0 < t < 1;$$

(b) $n \max_{1\leq i\leq n}|t_{ni} - i/n| = O(1)$ as $\nu \to \infty$.

3. The main results. The main results of this paper are given in the following Theorems 3.1, 3.2 and 3.3; Theorem 3.1 gives conditions which are, if the Conditions A, B*, C and D are satisfied, necessary and sufficient for $\sqrt{n}(\hat\theta_n - \theta_0)$ and $\sqrt{n}(\hat\mu_n - \theta_0)$ to be asymptotically equivalent; Theorem 3.2 gives conditions which are, if the Conditions A, B, C and D are satisfied, sufficient for $\sqrt{n}(\hat\theta_n - \theta_0)$ and $\sqrt{n}(\hat\mu_n - \theta_0)$ to have the same asymptotic distribution; Theorem 3.3 gives conditions which are, if the Conditions A, C and D are satisfied, necessary and sufficient for $\hat\mu_n$ to be asymptotically efficient relative to the Cramér-Rao bound.

THEOREM 3.1. *If the Conditions* A, B*, C *and* D *are satisfied then* $\sqrt{n}(\hat\theta_n - \hat\mu_n) \to_{P_{\theta_0}} 0$ *if and only if, for* $p_{i-1} < t \leq p_i$, $i = 1, \cdots, \kappa + 1$,

$$K\psi(F^{-1}(t)) = \sum_{j=i}^{\kappa} \frac{d_j}{f(F^{-1}(p_j))} - \sum_{j=1}^{i-1} \frac{d_j}{f(F^{-1}(p_j))}$$

$$(3.1) \qquad + \begin{cases} -\int_{(1-t)^+}^{t} J(s)\, dF^{-1}(s) & \text{if } t > \frac{1}{2}, \\ +\int_{t}^{(1-t)^+} J(s)\, dF^{-1}(s) & \text{if } t \le \frac{1}{2}, \end{cases}$$

where K is a nonzero constant, $p_0 = 0$ *and* $p_{\kappa+1} = 1$.

PROOF. Let $\xi_1, \cdots, \xi_n$ and $U(t)$, $0 \le t \le 1$, be the uniform random variables and the Brownian bridge defined by Shorack (1972, Section A.1). Then it is sufficient (see Shorack, 1972, Section A.2) to prove the result with $X_1, \cdots, X_n$ replaced by $G^{-1}(\xi_1), \cdots, G^{-1}(\xi_n)$, where $G(x) = F(x - \theta_0)$, $-\infty < x < \infty$. Let

$$(3.2) \qquad \begin{aligned} T &= -\int_0^1 J(t)U(t)\, dG^{-1}(t) - \sum_{j=1}^{\kappa} d_j U(p_j)\left\{\frac{d}{dt} G^{-1}(t)\right\}_{t=p_j} \\ &= -\int_0^1 J(t)U(t)\, dF^{-1}(t) - \sum_{j=1}^{\kappa} \frac{d_j}{f(F^{-1}(p_j))} U(p_j), \end{aligned}$$

then it follows from Shorack (1972, Theorem 1 and Remark 2) that, if Condition C is satisfied,

$$(3.3) \qquad \sqrt{n}(\hat{\mu}_n - \eta_n) \to_{P_{\theta_0}} T,$$

where

$$(3.4) \qquad \begin{aligned} \eta_n &= \int_0^1 G^{-1}(t)J_n(t)\, dt + \sum_{j=1}^{\kappa} d_j G^{-1}(p_j) \\ &= \sum_{i=1}^{n} c_{ni} \int_{(i-1)/n}^{i/n} G^{-1}(t)\, dt + \sum_{j=1}^{\kappa} d_j G^{-1}(p_j) \\ &= \theta_0\left(\frac{1}{n}\sum_{i=1}^{n} c_{ni} + \sum_{j=1}^{\kappa} d_j\right) + \sum_{i=1}^{n} c_{ni} \int_{(i-1)/n}^{i/n} F^{-1}(t)\, dt + \sum_{j=1}^{\kappa} d_j F^{-1}(p_j). \end{aligned}$$

By the Conditions A(i), C(iii) and D(iii)

$$(3.5) \qquad \sum_{i=1}^{n} c_{ni} \int_{(i-1)/n}^{i/n} F^{-1}(t)\, dt + \sum_{j=1}^{\kappa} d_j F^{-1}(p_j) = 0,$$

and by Condition D(i)

$$(3.6) \qquad \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n} c_{ni} - \int_0^1 J(t)\, dt\right\} \to 0 \quad \text{as} \quad \nu \to \infty.$$

From (3.3), (3.4), (3.5), (3.6) and Condition D(ii) it then follows that, under the Conditions A(i), C and D,

$$(3.7) \qquad \sqrt{n}(\hat{\mu}_n - \theta_0) \to_{P_{\theta_0}} T.$$

Further note that, under the Conditions A(i), C(iii) and D(iii),

$$\frac{d_j}{f(F^{-1}(p_j))} = \frac{d_{\kappa+1-j}}{f(F^{-1}(p_{\kappa+1-j}))}, \; j = 1, \cdots, \kappa, \tag{3.8}$$

and that, under the Conditions C(ii) and D(iii), except on a set of F^{-1}-measure zero,

$$J(t) = J(1-t), \quad 0 < t < 1. \tag{3.9}$$

So if (see (3.1)), for $p_{i-1} < t \le p_i$, $i = 1, \cdots, \kappa + 1$,

$$\lambda(t) = \textstyle\sum_{j=i}^{\kappa} \dfrac{d_j}{f(F^{-1}(p_j))} - \sum_{j=1}^{i-1} \dfrac{d_j}{f(F^{-1}(p_j))} + \begin{cases} -\displaystyle\int_{(1-t)^+}^{t} J(s)\, dF^{-1}(s) & \text{if } t > \frac{1}{2} \\ +\displaystyle\int_{t}^{(1-t)^+} J(s)\, dF^{-1}(s) & \text{if } t \le \frac{1}{2}, \end{cases} \tag{3.10}$$

then

$$T = \frac{1}{2} \int_0^1 U(t)\, d\lambda(t) \tag{3.11}$$

and (3.7) becomes

$$\sqrt{n}(\hat{\mu}_n - \theta_0) \to_{P_{\theta_0}} \frac{1}{2} \int_0^1 U(t)\, d\lambda(t). \tag{3.12}$$

Now consider $\hat{\theta}_n = \hat{\theta}_n(G^{-1}(\xi_1), \cdots, G^{-1}(\xi_n))$ and note that (see (5.1.1) and (2.1)), if the Conditions A and B are satisfied, then

$$\sqrt{n}(\hat{\theta}_n - \theta_0) - \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(G^{-1}(\xi_i) - \theta_0)}{-\int_{-\infty}^{\infty} \psi(x) f'(x)\, dx} \to_{P_{\theta_0}} 0. \tag{3.13}$$

Also note that, if ψ satisfies Condition B*, then it follows from Shorack (1972, Example 1) that

$$\frac{1}{\sqrt{n}} \textstyle\sum_{i=1}^{n} \psi(G^{-1}(\xi_i) - \theta_0) - \displaystyle\int_0^1 \psi(F^{-1}(t))\, dt \to_{P_{\theta_0}} - \int_0^1 U(t)\, d\psi(F^{-1}(t)), \tag{3.14}$$

where, if Condition A(i) is also satisfied,

$$\int_0^1 \psi(F^{-1}(t))\, dt = 0. \tag{3.15}$$

From (3.13), (3.14) and (3.15) it follows that, if the Conditions A and B* are satisfied, then

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \to_{P_{\theta_0}} \frac{\int_0^1 U(t)\, d\psi(F^{-1}(t))}{\int_{-\infty}^{+\infty} \psi(x) f'(x)\, dx}. \tag{3.16}$$

From (3.12) and (3.16), it follows that, under the Conditions A, B*, C and D,

$$\sqrt{n}(\hat{\theta}_n - \hat{\mu}_n) \to_{P_{\theta_0}} \int_0^1 U(t)d\left\{\frac{\psi(F^{-1}(t))}{\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx} - \frac{1}{2}\lambda(t)\right\}. \tag{3.17}$$

Further note that

$$\int_0^1 \psi(F^{-1}(t))\,dt = \int_0^1 \lambda(t)\,dt = 0, \tag{3.18}$$

which implies that the righthandside of (3.17) is a normal random variable with mean zero and variance given by

$$\sigma^2 = \int_0^1 \left\{\frac{\psi(F^{-1}(t))}{\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx} - \frac{1}{2}\lambda(t)\right\}^2 dt, \tag{3.19}$$

which is zero if and only if

$$\frac{\psi(F^{-1}(t))}{\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx} = \frac{1}{2}\lambda(t), \quad 0 < t < 1. \tag{3.20}$$

That (3.20) is equivalent to (3.1) can be seen as follows. First note that

$$0 \neq \int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx = -\int_0^1 \psi(F^{-1}(t))\varphi_F(t)\,dt, \tag{3.21}$$

so that, in order to prove that $K \neq 0$, it is sufficient to prove that

$$\int_0^1 \lambda(t)\varphi_F(t)\,dt = -2. \tag{3.22}$$

This can be seen as follows. First note that

$$\begin{aligned}\int_{p_{i-1}}^{p_i}\varphi_F(t)\,dt &= -\int_{p_{i-1}}^{p_i}\frac{f'}{f}(F^{-1}(t))\,dt = -\int_{F^{-1}(p_{i-1})}^{F^{-1}(p_i)} f'(y)\,dy \\ &= f(F^{-1}(p_{i-1})) - f(F^{-1}(p_i)),\end{aligned} \tag{3.23}$$

which implies that

$$\begin{aligned}\sum_{i=1}^{K}\int_{p_{i-1}}^{p_i}\varphi_F(t)\,dt\sum_{j=i}^{K}\frac{d_j}{f(F^{-1}(p_j))} \\ &= \sum_{i=1}^{K}\{f(F^{-1}(p_{i-1})) - f(F^{-1}(p_i))\}\sum_{j=i}^{K}\frac{d_j}{f(F^{-1}(p_j))} \\ &= \sum_{j=1}^{K}\frac{d_j}{f(F^{-1}(p_j))}\sum_{i=1}^{j}\{f(F^{-1}(p_{i-1})) - f(F^{-1}(p_i))\} \\ &= \sum_{j=1}^{K}\frac{d_j}{f(F^{-1}(p_j))}\{f(F^{-1}(p_0)) - f(F^{-1}(p_j))\} \\ &= -\sum_{j=1}^{K} d_j.\end{aligned} \tag{3.24}$$

In the same way it can be shown that

$$\sum_{i=2}^{\kappa+1} \int_{p_{i-1}}^{p_i} \varphi_F(t)\,dt \sum_{j=1}^{i-1} \frac{d_j}{f(F^{-1}(p_j))} = -\sum_{j=1}^{\kappa} d_j. \tag{3.25}$$

Further, by using Condition A(i),

$$\begin{aligned} \int_0^1 \varphi_F(t)\,dt \int_{1-t}^{t} J(s)\,dF^{-1}(s) &= \int_0^1 J(s)\,dF^{-1}(s)\left[-\int_0^s \frac{f'}{f}(F^{-1}(t))\,dt + \int_{1-s}^1 \frac{f'}{f}(F^{-1}(t))\,dt\right] \\ &= -2\int_0^1 J(s)f(F^{-1}(s))\,dF^{-1}(s) = -2\int_0^1 J(s)\,ds. \end{aligned} \tag{3.26}$$

From (3.24), (3.25), (3.26) and Condition D(ii) it follows that (3.22) holds. □

REMARK 3.1. If the Conditions C′(i)–(iv) and D′(i) are satisfied, then (see Shorack, 1972, Example 1)

$$\sqrt{n}\left(\int_0^1 G^{-1}(t)J_n(t)\,dt - \int_0^1 G^{-1}(t)J(t)\,dt\right) \to 0 \quad \text{as} \quad \nu \to \infty, \tag{3.27}$$

which, if the Conditions A(i) and D(iii) are also satisfied, is equivalent to the Condition D(i).

REMARK 3.2. For the special case when $\kappa = 0$ the above result can, under less restrictive regularity conditions, be obtained from Rivest (1978, Corollary II.4; 1982, Remark 8, page 230). Jurečková (1982) gives, without stating regularity conditions, the relation (3.1) for the asymptotic equivalence between L- and M-estimators for the special case when $\kappa = 0$ as well as for the special case when $J(t) = 0$, $0 \le t \le 1$.

THEOREM 3.2. *If the Conditions* A, B, C *and* D *are satisfied then* $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ *and* $\sqrt{n}(\hat{\mu}_n - \theta_0)$ *have, when* (3.1) *holds, the same asymptotic distribution.*

PROOF. First note that, by (3.7), (3.11) and (3.18), $\sqrt{n}\,(\hat{\mu}_n - \theta_0)$ is, under the Conditions A(i), C and D, asymptotically normal with mean zero and variance equal to

$$\frac{1}{4}\int_0^1 \lambda^2(t)\,dt. \tag{3.28}$$

Further, by (3.13), $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is, under the Conditions A and B, asymptotically normal with mean zero and variance equal to

$$\frac{\int_{-\infty}^{+\infty} \psi^2(x)f(x)\,dx}{\left\{\int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx\right\}^2} = \frac{\int_0^1 \psi^2(F^{-1}(t))\,dt}{\left\{\int_{+\infty}^{+\infty} \psi(x)f'(x)\,dx\right\}^2}. \tag{3.29}$$

The result then follows from the fact that (3.1) is equivalent to (3.20) and that (3.20) implies that (3.28) equals (3.29). □

THEOREM 3.3. *If the Conditions* A, C *and* D *are satisfied then the variance of the asymptotic distribution of* $\sqrt{n}\,(\hat{\mu}_n - \theta_0)$ *equals the Cramér-Rao lower bound if and only if,* *for almost all* $t \in (0, 1)$, *for* $p_{i-1} < t \le p_i$, $i = 1, \cdots, \kappa + 1$

$$\frac{-2\varphi_F(t)}{\int_0^1 \varphi_F^2(t)\,dt} = \sum_{j=i}^{\kappa} \frac{d_j}{f(F^{-1}(p_j))} - \sum_{j=1}^{i-1} \frac{d_j}{f(F^{-1}(p_j))}$$

$$+ \begin{cases} -\int_{(1-t)+}^{t} J(s)\,dF^{-1}(s) & \text{if } t > \frac{1}{2} \\ +\int_{t}^{(1-t)+} J(s)\,dF^{-1}(s) & \text{if } t \le \frac{1}{2} \end{cases} \tag{3.30}$$

PROOF. Under the Conditions A(i), C and D, $\sqrt{n}(\hat{\mu}_n - \theta_0)$ is asymptotically normal with mean zero and variance

$$\sigma^2 = \frac{1}{4}\int_0^1 \lambda^2(t)\,dt. \tag{3.31}$$

Further (see (3.22))

$$\int_0^1 \lambda(t)\varphi_F(t)\,dt = -2 \tag{3.32}$$

and from (3.32) one obtains

$$1 = \frac{1}{4}\left\{\int_0^1 \lambda(t)\varphi_F(t)\,dt\right\}^2 \le \frac{1}{4}\int_0^1 \lambda^2(t)\,dt \int_0^1 \varphi_F^2(t)\,dt \tag{3.33}$$

or

$$\sigma^2 = \frac{1}{4}\int_0^1 \lambda^2(t)\,dt \ge \frac{1}{\int_0^1 \varphi_F^2(t)\,dt}, \tag{3.34}$$

with equality if and only if, for some $a \ne 0$,

$$\lambda(t) = a\varphi_F(t) \quad \text{for almost all} \quad t \in (0, 1). \tag{3.35}$$

Further, (3.32) implies that

$$a = \frac{-2}{\int_0^1 \varphi_F^2(t)\,dt}.$$

□

REMARK 3.3. For the special case when $\kappa = 0$, the fact that the variance of the asymptotic distribution of $\sqrt{n}(\hat{\mu}_n - \theta_0)$ equals the Cramér-Rao lower bound when (3.30) is satisfied has, under different regularity conditions than ours, e.g. been proved by Jung (1955), Chernoff, Gastwirth, Johns (1967) and by Rivest (1978, 1982).

In the particular case when $\psi(x)$ satisfies

$$\psi(F^{-1}(t)) = -\varphi_F(t), \quad 0 < t < 1, \tag{3.36}$$

the estimator $\hat{\theta}_n = \hat{\theta}_n(F)$ is a maximum likelihood estimator in the sense that (see (2.1)) it is an "asymptotic solution" to the likelihood equation for which $\sqrt{n}\{\hat{\theta}_n(F) - \theta_0\}$ is bounded in P_{θ_0} – probability; further, under the Conditions A and B, $\sqrt{n}\{\hat{\theta}_n(F) - \theta_0\}$ is asymptotically normal with mean zero and variance $\{\int_0^1 \varphi_F^2(u)\,du\}^{-1}$. From Theorem 3.1 it follows that, if the Conditions A, B*, C and D are satisfied, then

$$\sqrt{n}\{\hat{\theta}_n(F) - \hat{\mu}_n\} \to_{P_{\theta_0}} 0$$

if and only if (3.30) is satisfied.

Now consider the particular case when

$$f(x) > 0 \quad \text{for all} \quad x \in S = \{x \mid 0 < F(x) < 1\}; \tag{3.37}$$

then $F^{-1}(t)$ is absolutely continuous on the interval $[t_1, t_2]$ if $0 < t_1 < t_2 < 1$; so the right hand side of (3.1) has a finite number of discontinuities which occur at $p_1, \cdots, p_\kappa$ and is, when $\kappa = 0$, absolutely continuous on $[t_1, t_2]$ if $0 < t_1 < t_2 < 1$. This implies that, if (3.1) and (3.37) are satisfied, then ψ has a finite number of discontinuities which occur at $F^{-1}(p_1)$, $\cdots, F^{-1}(p_\kappa)$, ψ is absolutely continuous when $\kappa = 0$ and (3.1) is equivalent to

$$\begin{aligned} &K\psi'(x) = -2J(F(x)) \quad \text{almost everywhere} \\ &K[\psi(\{F^{-1}(p_i)\}^+) - \psi(F^{-1}(p_i))] = \frac{-2d_i}{f(F^{-1}(p_i))}, \quad i = 1, \cdots, \kappa. \end{aligned} \tag{3.38}$$

Further, if (3.36) is satisfied then (3.38) is equivalent to

$$\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = -J(F(x)) \int_0^1 \varphi_F^2(t)\,dt \quad \text{almost everywhere}$$

$$f'(\{F^{-1}(p_i)\}^+) - f'(\{F^{-1}(p_i)\}^-) = -d_i \int_0^1 \varphi_F^2(t)\,dt, \quad i = 1, \cdots, \kappa, \tag{3.39}$$

so that, in this case, $f'(x)$ has a finite number of discontinuities which occur at $F^{-1}(p_1)$, $\cdots, F^{-1}(p_\kappa)$ and f' is absolutely continuous when $\kappa = 0$.

REMARK 3.4. If $\int_0^1 |J(t)|\,dt < \infty$ and Condition A is satisfied then (3.30) implies D(ii); this follows from the fact that (see the proof of (3.22))

$$\int_0^1 \lambda(t)\varphi_F(t)\,dt = -2\left\{\int_0^1 J(t)\,dt + \textstyle\sum_{j=1}^{\kappa} d_j\right\}. \tag{3.40}$$

Further, if (3.37) is also satisfied, then (see (3.39))

$$\begin{aligned} 1 &= \int_0^1 J(t)\,dt + \textstyle\sum_{j=1}^{\kappa} d_j \\ &= \frac{-1}{\int_0^1 \varphi_F^2(t)\,dt}\left\{\int_{-\infty}^{+\infty}\left\{\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right)\right\} f(x)\,dx \right. \\ &\qquad \left. + \textstyle\sum_{j=1}^{\kappa}[f'(\{F^{-1}(p_j)\}^+) - f'(\{F^{-1}(p_j)\}^-)]\right\}, \end{aligned} \tag{3.41}$$

which is equivalent to

$$\begin{aligned} &\int_{-\infty}^{+\infty}\left\{\frac{d^2}{dx^2}\log f(x)\right\} f(x)\,dx + \textstyle\sum_{j=1}^{\kappa}\{f'(\{F^{-1}(p_j)\}^+) - f'(\{F^{-1}(p_j)\}^-)\} \\ &\qquad = -\int_{-\infty}^{+\infty}\left\{\frac{f'(x)}{f(x)}\right\}^2 f(x)\,dx. \end{aligned} \tag{3.42}$$

4. Examples.

EXAMPLE 1. Let, for $p > 1$,

$$f(x) = \begin{cases} \frac{p+1}{2}(1+x)^p, & -1 \le x \le 0, \\ \frac{p+1}{2}(1-x)^p, & 0 < x \le 1. \end{cases} \tag{4.1}$$

Then

$$\frac{f'(x)}{f(x)} = \begin{cases} \frac{p}{1+x}, & -1 < x < 0, \\ \frac{-p}{1-x}, & 0 < x \le 1, \end{cases} \tag{4.2}$$

$$\int_0^1 \varphi_F^2(t)\, dt = \frac{p^2(p+1)}{p-1} \tag{4.3}$$

and it is easy to see that Condition A and (3.37) are satisfied. The maximum likelihood estimator $\hat{\theta}_n(F)$ is obtained by taking $\psi(x)$ equal to the left continuous version of $f'(x)/f(x)$, that is

$$\psi(x) = \begin{cases} \frac{p}{1+x}, & -1 < x \le 0 \\ \frac{-p}{1-x}, & 0 < x \le 1, \end{cases} \tag{4.4}$$

which satisfies Condition B* with $r - 2 \le p - 1$. The estimator $\hat{\theta}_n(F)$ is, by Theorem 3.1, asymptotically equivalent to a linear combination of the order statistics $\hat{\mu}_n$ if and only if (3.39) is satisfied, that is if and only if $\kappa = 1$, $p_1 = ½$, $d_1 = (p-1)/p$, and, for almost all $t \in (0, 1)$,

$$J(t) = \begin{cases} \frac{p-1}{p(p+1)}(2t)^{-2/(p+1)}, & 0 < t \le \frac{1}{2}, \\ \frac{p-1}{p(p+1)}(2(1-t))^{-2/(p+1)}, & \frac{1}{2} \le t < 1, \end{cases} \tag{4.5}$$

provided the Conditions C and D are satisfied. To verify these conditions, first define $J(t)$ on the whole interval (0, 1) by (4.5). Then note that Condition C(i) on F^{-1} and J is equivalent to

$$\begin{aligned} &\text{(i)} \quad 1 - (2t)^{1/(p+1)} \le Mt^{\beta+\delta-1/2}(1-t)^{\beta+\delta-1/2}, \quad 0 < t \le \frac{1}{2}, \\ &\text{(ii)} \quad \frac{p-1}{p(p+1)}(2t)^{-2/(p+1)} \le Mt^{-\beta}(1-t)^{-\beta}, \quad 0 < t \le \frac{1}{2}, \end{aligned} \tag{4.6}$$

for some β and some $M > 0$, $\delta > 0$. For the second part of Condition (4.6) to be satisfied it is necessary that $\beta \ge 2/(p+1)$ and for the first part to be satisfied it is necessary that $\beta + \delta - ½ \le 0$. This implies that

$$0 < \delta \le \frac{1}{2} - \beta \le \frac{1}{2} - \frac{2}{p+1} = \frac{p-3}{2(p+1)} \tag{4.7}$$

is necessary for (4.6); further (4.7) implies that $p > 3$ is necessary for (4.6).

It will now be shown that, if $p > 5$, then the Conditions C′ and D are satisfied with the choice

$$t_{ni} = \frac{i}{n+1}, \quad i = 1, \cdots, n. \tag{4.8}$$

This implies that the Condition C is satisfied with

$$J_n(t) = \begin{cases} J\left(\frac{i}{n+1}\right) & \frac{i-1}{n} < t \le \frac{i}{n}, \quad i = 1, \cdots, n, \\ J\left(\frac{1}{n+1}\right) & t = 0. \end{cases} \tag{4.9}$$

In order to show that the Conditions C′ and D are satisfied if $p > 5$ and the t_{ni} are given by (4.8) first note that C′(i), C′(iii), C′(v) and D(iii) are obviously satisfied; that D(ii) is satisfied follows from Remark 3.4. Further, by choosing $a \in (0, 1)$, it is easily seen that C′(ii) is satisfied for all $n \ge a/(1 - a)$. For C′(iv) note that $J(t)$ is continuous for $t \in (0, 1)$ and satisfies

$$|J(t)| \le M\{t(1-t)\}^{-1/2+1/r+\delta}, \quad 0 < t < 1, \tag{4.10}$$

when r, δ, M and p satisfy

$$\text{(i)}\ \frac{1}{2} - \frac{1}{r} - \delta \ge \frac{2}{p+1}, \quad \text{(ii)}\ M \ge \frac{p-1}{p(p+1)} 2^{-2/(p+1)}, \tag{4.11}$$

where (i) is equivalent to $\delta \le (p-3)/2(p+1) - 1/r$ and where, because $p > 5$, $r \le p + 1$ can be chosen such that $(p-3)/2(p+1) - 1/r$ is positive. Now consider Condition D(i); because Condition C′ is satisfied, it is (see Remark 3.1) sufficient to show that D′(i) is satisfied. For this, first note that D′(i)(b) is obviously satisfied. However, J' does not satisfy D′(i)(a) because $J'(t)$ is not continuous at $t = ½$. But, on the interval (0, ½] (on the interval [½, 1)), the left continuous (right continuous) version of $J'(t)$ does satisfy the Condition D′(i)(a) with $\delta \le (p-3)/2(p+1) - 1/r$ and it can easily be seen from Shorack's (1972) proof that these conditions on J' are sufficient for (3.27) and thus for D(i).

Example 2. Let

$$f(x) = \frac{2}{(4+|x|)^2}, \ -\infty < x < \infty. \tag{4.12}$$

Then

$$\frac{f'(x)}{f(x)} = \frac{-2}{4+|x|} \operatorname{sgn}(x), \tag{4.13}$$

$$\int_0^1 \varphi_F^2(t)\, dt = \frac{1}{12} \tag{4.14}$$

and it is easily seen that Condition A and (3.37) are satisfied. The maximum likelihood estimator $\hat{\theta}_n(F)$ is obtained by taking $\psi(x)$ equal to the left continuous version of $f'(x)/f(x)$, that is

$$\psi(x) = \begin{cases} \frac{2}{4-x}, & x \le 0 \\ \frac{-2}{4+x}, & x > 0. \end{cases} \tag{4.15}$$

This function ψ is not monotone, but can be written as the sum of a nondecreasing function $\psi_1(x)$ and a nonincreasing function $\psi_2(x)$ for instance by taking

$$\psi_1(x) = \begin{cases} \psi(x), & x \le 0, \\ \psi(x) + 1, & x > 0, \end{cases}$$
$$\psi_2(x) = \begin{cases} 0, & x \le 0, \\ -1, & x > 0, \end{cases} \tag{4.16}$$

and it is easily seen that with this choice of ψ_1 and ψ_2 the condition $\int_{-\infty}^{+\infty} \psi_i^2(x)f(x)\,dx < \infty$, $i = 1, 2$ is satisfied. Further B* is satisfied because

$$\int_{-\infty}^{+\infty} |\psi(x)|^r f(x)\,dx < \infty \quad \text{for all} \quad r \geq 0.$$

The estimator $\hat{\theta}_n(F)$ is then, by Theorem 3.1, asymptotically equivalent to a linear combination of the order statistics if and only if (3.39) holds, that is if and only if $\kappa = 1$, $p = ½$, and

$$\text{(4.17)} \qquad \text{(i) } d_1 = \frac{3}{2}, \text{ (ii) } J(t) = \begin{cases} -6t^2 & t \leq \frac{1}{2}, \\ -6(1-t)^2 & t \geq \frac{1}{2}, \end{cases}$$

for almost all $t \in (0, 1)$, provided the Conditions C and D are satisfied. Note that in this case the Condition C′ is not satisfied because $\mathscr{E}|X_i|^r$ does not exist for $r \geq 1$. Concerning the Conditions C and D, first let $J(t)$ be defined by (4.17.2) for all $t \in (0, 1)$. The Conditions C(iii) and D(ii) are satisfied. Now for $i = 1, \cdots, n$ choose

$$\text{(4.18)} \qquad c_{ni} = \begin{cases} -6\left(\dfrac{i-1}{n}\right)^2, & i \leq \dfrac{n+1}{2}, \\ -6\left(\dfrac{n-i}{n}\right)^2, & i > \dfrac{n+1}{2}, \end{cases}$$

then D(iii) is satisfied. Further, (4.18) implies that

$$\text{(4.19)} \qquad J_n(t) = \begin{cases} -6\left\{\dfrac{[nt]^+ - 1}{n}\right\}^2, & t \leq \dfrac{1}{2} \\ -6\left\{\dfrac{n - [nt]^+}{n}\right\}^2, & t > \dfrac{1}{2}, \end{cases}$$

where $[nt]^+$ is the smallest integer $\geq nt$. It is easily seen that, because $|J(t)|$ is increasing on $[0, ½]$ and decreasing on $[½, 1]$,

$$\text{(4.20)} \qquad |J_n(t)| \leq |J(t)|, \qquad 0 \leq t \leq 1.$$

Further note that

$$\text{(4.21)} \qquad |J(t) - J_n(t)| \leq \frac{6(n-1)}{n^2}, \quad 0 \leq t \leq 1,$$

which proves that C(ii) and D(i) are satisfied. Finally, Condition C(i) is out, because of (4.20), equivalent to

$$\text{(4.22)} \quad \text{(i) } \frac{2-4t}{t} \leq M\{t(1-t)\}^{\beta+\delta-1/2}, \quad \text{(ii) } 6t^2 \leq M\{t(1-t)\}^{-\beta}, \quad 0 \leq t \leq \frac{1}{2},$$

for some β and some $M > 0$, $\delta > 0$. It is easily seen that (4.22) is satisfied with $\beta = -2$, $\delta = 1$ and $M \geq 3/2$.

Note that the L-estimator

$$\text{(4.23)} \qquad \hat{\mu}_n = \frac{1}{n} \textstyle\sum_{i=1}^n c_{ni} Y_i + d_1 Y_{[n/2]+1},$$

defined by (4.17) and (4.18), does not have a variance for any n; this follows from the fact that $\mathscr{E}Y_2^2$ and $\mathscr{E}Y_{n-1}^2$ do not exist and that c_{n2} and c_{nn-1} are non-zero. An L-estimator with the same asymptotic properties as the one defined by (4.17) and (4.18) but with a finite variance for all n, can be obtained by replacing c_{n2} and c_{nn-1} by zero.

5. Appendix

5.1. *The asymptotic linearity of* $n^{-1/2}\sum_{i=1}^n \psi(X_i - \theta)$. In this section the following theorem will be proved.

THEOREM 5.1.1. *If the Conditions* A, B(i) *and* B(iii) *are satisfied then, for every* $C > 0$,

$$\sup_{\sqrt{n}|\theta-\theta_0|\le C}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i-\theta) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i-\theta_0) - \sqrt{n}(\theta-\theta_0)\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx\right| \to_{P_{\theta_0}} 0. \tag{5.1.1}$$

PROOF. The following proof is analogous to the proof given by Jurečková of her asymptotic linearity theorem (1977, Theorem 4.1).

Without loss of generality it can be supposed that ψ is monotone nondecreasing and that $\theta_0 = 0$. Further note that, with $\theta_0 = 0$, (5.1.1) is equivalent to

$$\sup_{|\theta|\le C}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi\left(X_i - \frac{\theta}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i) - \theta\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx\right| \to_{P_{\theta_0}} 0. \tag{5.1.2}$$

Because ψ is defined on the interval $\mathscr{I}$ and $S = \{x \mid 0 < F(x) < 1\} \subseteq \mathscr{I}$, $\sum_{i=1}^n \psi(X_i)$ is defined with probability 1. Further, let $\mathscr{I}$ be the interval $(-b, b)$, then, for $\sqrt{n} \ge C/b$, $\sum_{i=1}^n \psi(X_i - \theta/\sqrt{n})$ is defined for all $\theta \in [-C, C]$ if and only if

$$\frac{C}{\sqrt{n}} - b \le Y_1 < Y_n \le b - \frac{C}{\sqrt{n}}, \tag{5.1.3}$$

where, for each $C > 0$, the probability of the event (5.1.3) tends to 1 as $\nu \to \infty$.

Now note that for fixed θ,

$$\sum_{i=1}^n \log\frac{f\left(X_i + \frac{\theta}{\sqrt{n}}\right)}{f(X_i)} + \frac{\theta}{\sqrt{n}}\sum_{i=1}^n \varphi_F(F(|X_i|))\operatorname{sgn} X_i + \frac{1}{2}\theta^2\int_0^1 \varphi_F^2(u)\,du \to_{P_0} 0; \tag{5.1.4}$$

cf. the proof of Theorem VI 2.5, Hájek and Šidák (1967). Further

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \varphi_F(F(|X_i|))\operatorname{sgn} X_i = \frac{1}{\sqrt{n}}\sum_{i=1}^n \varphi_F(F(X_i)) \tag{5.1.5}$$

and (5.1.4) and (5.1.5) imply that

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i),\ \sum_{i=1}^n \log\frac{f\left(X_i + \frac{\theta}{\sqrt{n}}\right)}{f(X_i)}\right) \tag{5.1.6}$$

has asymptotically the same distribution as

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(X_i),\ -\frac{\theta}{\sqrt{n}}\sum_{i=1}^n \varphi_F(F(X_i)) - \frac{1}{2}\theta^2\int_0^1 \varphi_F^2(u)\,du\right). \tag{5.1.7}$$

From the Conditions A(iii), B(i) and B(iii) it follows that the asymptotic distribution of

(5.1.7) is two dimensional normal with means $\mu_1 = 0$, $\mu_2 = -\frac{1}{2}\theta^2 \int_0^1 \varphi_F^2(u)\,du$, variances

$$\sigma_1^2 = \int_{-\infty}^{+\infty} \psi^2(x)f(x)\,dx, \quad \sigma_2^2 = \theta^2 \int_0^1 \varphi_F^2(u)\,du$$

and with covariance

$$-\theta \int_0^1 \psi(F^{-1}(u))\varphi_F(u)\,du = \theta \int_{\infty}^{\infty} \psi(x)f'(x)\,dx.$$

Using LeCam's third lemma (see Hájek and Šidák, 1967, page 208) it follows that, for fixed θ,

$$\text{(5.1.8)} \qquad \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi\left(X_i - \frac{\theta}{\sqrt{n}}\right) \to_{\mathscr{L}} N\left(\theta \int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx, \int_{-\infty}^{+\infty} \psi^2(x)f(x)\,dx\right).$$

Further (see the proof of Theorem 4.1 of Jurečková, 1977), (5.1.8) implies that, for fixed θ,

$$\text{(5.1.9)} \qquad \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi\left(X_i - \frac{\theta}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i) - \theta \int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx \to_{P_0} 0$$

and the uniformity in θ, for $|\theta| \le C$, then follows, as in Jurečková's proof, from the fact that $\sum_{i=1}^n \psi(X_i - \theta)$ is, with probability 1, monotone in θ. □

5.2. *Some examples of estimators satisfying* (2.1). In this section it will be supposed that the conditions A and B are satisfied.

First consider the case when ψ is monotone nondecreasing. Then the estimator (2.2) is defined when $\sum_{i=1}^n \psi(X_i - \theta)$ takes, on its interval of definition, i.e. on the interval $(Y_n - b, Y_1 + b)$, both positive and negative values. The probability that this is the case is 1 when $b = \infty$ and tends to 1 as $\nu \to \infty$ when $b < \infty$; this can be seen as follows. Let δ be a positive number and let $C(\delta) > 0$ and $n_1(\delta)$ an integer be such that

$$\text{(5.2.1)} \qquad P_{\theta_0}(\sqrt{n}|\tilde{\theta}_n - \theta_0| \le C(\delta)) \ge 1 - \frac{\delta}{3} \quad \text{for} \quad n > n_1(\delta),$$

where $\tilde{\theta}_n$ is the solution (in θ) to

$$\text{(5.2.2)} \qquad \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i - \theta) + \sqrt{n}(\theta - \theta_0) \int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx = 0;$$

the existence of $C(\delta)$ and $n_1(\delta)$ follows from the Conditions B(i) and B(ii). Further let $n_2(\delta)$ be an integer such that

$$\text{(5.2.3)} \qquad \frac{2C(\delta)}{\sqrt{n}} < b \quad \text{for} \quad n > n_2(\delta),$$

then, for $n > n_2(\delta)$, $\sum_{i=1}^n \psi(X_i - \theta)$ is defined for all

$$\theta \in \left[\theta_0 - \frac{2C(\delta)}{\sqrt{n}}, \theta_0 + \frac{2C(\delta)}{\sqrt{n}}\right]$$

if and only if

$$\text{(5.2.4)} \qquad -b + \theta_0 + \frac{2C(\delta)}{\sqrt{n}} < Y_1 < Y_n < b + \theta_0 - \frac{2C(\delta)}{\sqrt{n}}.$$

Now let $n_3(\delta)$ be an integer with $n_3(\delta) \geq n_2(\delta)$ and such that (see (5.2.4))

$$P_{\theta_0}\left(-b+\theta_0+\frac{2C(\delta)}{\sqrt{n}}<Y_1<Y_n<b+\theta_0-\frac{2C(\delta)}{\sqrt{n}}\right)\geq 1-\frac{\delta}{3} \quad \text{for} \quad n>n_3(\delta); \tag{5.2.5}$$

the existence of $n_3(\delta)$ follows from the fact that $\{x \mid 0<F(x)<1\} \subseteq \mathscr{I}$. Further let ε satisfy

$$0<\varepsilon<C(\delta)\left|\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx\right| \tag{5.2.6}$$

and let $n(\varepsilon, \delta)$ be an integer such that, by Theorem 5.1.1,

$$P_{\theta_0}\left\{\sup_{\sqrt{n}|\theta-\theta_0|\leq 2C(\delta)}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n\psi(X_i-\theta)-\frac{1}{\sqrt{n}}\sum_{i=1}^n\psi(X_i-\theta_0)\right.\right.$$
$$\left.\left.-\sqrt{n}(\theta-\theta_0)\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx\right|<\varepsilon\right\}\geq 1-\frac{\delta}{3} \quad \text{for} \quad n>n(\varepsilon,\delta). \tag{5.2.7}$$

Now note that if (i) $\sqrt{n}|\tilde{\theta}_n-\theta_0| \leq C(\delta)$, (ii) $\sum_{i=1}^n\psi(X_i-\theta)$ is defined for all $\theta \in [\theta_0 - 2C(\delta)/\sqrt{n}, \theta_0+2C(\delta)/\sqrt{n}]$, (iii) $0<\varepsilon<C(\delta)|\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx|$, (iv) $n>n_2(\delta)$, and (v)

$$\sup_{\sqrt{n}|\theta-\theta_0|\leq 2C(\delta)}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n\psi(X_i-\theta)\right.$$
$$\left.-\frac{1}{\sqrt{n}}\sum_{i=1}^n\psi(X_i-\theta_0)-\sqrt{n}(\theta-\theta_0)\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx\right|<\varepsilon \tag{5.2.8}$$

then $\sum_{i=1}^n\psi(X_i-\theta)$ takes both positive and negative values on $[\theta_0 - 2C(\delta)/\sqrt{n}, \theta_0 + 2C(\delta)/\sqrt{n}]$. The fact that the probability of the intersection of the events (i), (ii) and (v) is at least $1-\delta$ for $n>\max(n_1(\delta), n_3(\delta), n(\varepsilon,\delta))$, then implies that the probability that the estimator (2.2) is defined tends to 1 as $\nu \to \infty$. Further (i)-(v) also imply

$$\left|\sqrt{n}(\hat{\theta}_n-\theta_0)-\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n\psi(X_i-\theta_0)}{\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx}\right|\leq\varepsilon, \quad \frac{1}{\sqrt{n}}\left|\sum_{i=1}^n\psi(X_i-\hat{\theta}_n)\right|\leq 2\varepsilon, \tag{5.2.9}$$

where $\hat{\theta}_n$ is the estimator (2.2). This shows that any estimator defined by (2.2) whenever it exists satisfies (2.1).

Now consider the case when ψ is not necessarily monotone and suppose that $\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx$ is known. Then a linearized estimator, analogous to the linearized signed rank estimators of Kraft and van Eeden (1970, 1972) can be used. Let $\hat{\theta}_{1n}$ be a location invariant estimator of θ_0 such that $\sqrt{n}\,|\hat{\theta}_{1n}-\theta_0|$ is bounded in probability and let

$$\hat{\theta}_n=\hat{\theta}_{1n}+\frac{\sum_{i=1}^n\psi(X_i-\hat{\theta}_{1n})}{n\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx} \tag{5.2.10}$$

whenever $\sum_{i=1}^n\psi(X_i-\hat{\theta}_{1n})$ is defined. By a reasoning similar to the one given above for the estimator (2.2), it can be shown that the probability that the estimator (5.2.10) is defined tends to 1 as $\nu \to \infty$. That the estimator (5.2.10) satisfies (2.1) follows from Theorem 5.1.1; for a proof see Kraft and van Eeden (1970, 1972).

If $\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx$ is unknown then an estimator with the same asymptotic properties as (5.2.10) can be obtained by replacing, in (5.2.10), $\int_{-\infty}^{+\infty}\psi(x)f'(x)\,dx$ by a consistent estimator. Such an estimator of $\int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx$ can, for instance, be obtained as follows. Let $A > 0$ be a given constant, then by (5.1.1)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} - \frac{A}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i - \theta_0) - \sqrt{n}\left(\hat{\theta}_{1n} - \theta_0 - \frac{A}{\sqrt{n}}\right) \int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx \to_{P_{\theta_0}} 0,$$

(5.2.11)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} + \frac{A}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i - \theta_0) - \sqrt{n}\left(\hat{\theta}_{1n} - \theta_0 + \frac{A}{\sqrt{n}}\right) \int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx \to_{P_{\theta_0}} 0,$$

which implies that

$$\frac{\sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} + \frac{A}{\sqrt{n}}\right) - \sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} - \frac{A}{\sqrt{n}}\right)}{2\sqrt{n}A} \tag{5.2.12}$$

is a consistent estimator of $\int_{-\infty}^{+\infty} \psi(x)f'(x)\,dx$. The resulting estimator of θ_0

$$\hat{\theta}_{1n} + \frac{2A \sum_{i=1}^n \psi(X_i - \hat{\theta}_{1n})}{\sqrt{n}\left[\sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} + \frac{A}{\sqrt{n}}\right) - \sum_{i=1}^n \psi\left(X_i - \hat{\theta}_{1n} - \frac{A}{\sqrt{n}}\right)\right]}$$

is analogous to the linearized signed rank estimators proposed by Kraft and van Eeden (1970, page 272).

Finally note that (5.1.1) implies that an estimator $\hat{\theta}_n$ satisfying (2.1) satisfies (3.13).

REMARK 5.1. Another estimator satisfying (2.1) is (see Kraft and van Eeden, 1970, 1972) obtained by taking, among the roots to $\sum_{i=1}^n \psi(X_i - \theta) = 0$, the one that is closest to a linearized estimator.

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