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PROPERTIES OF TESTS CONCERNING COVARIANCE MATRICES OF NORMAL DISTRIBUTIONS

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The unbiasedness and the monotonicity property of the power functions of a class of tests for the equality of covariance matrices of two p-variate normal distributions have been studied. For testing $\Sigma = I_p$ in a p-variate normal distribution with mean vector μ and covariance matrix Σ, a class of tests is proposed and their power functions and admissibility are studied.

1. Introduction and summary. The unbiasedness and the monotonicity property of the power functions of a class of tests for the equality of covariance matrices of two p-variate normal distributions have been studied. For testing $\Sigma = I_p$ in a p-variate normal distribution $N(\mu, \Sigma)$ with mean vector μ and covariance matrix Σ, a class of tests is proposed and their power functions and admissibility are studied. Some of these results are generalizations of those obtained by Das Gupta (1969) and Sugiura and Nagao (1968).

2. Tests for $\Sigma_1 = \Sigma_2$. We consider the problem of testing $\Sigma_1 = \Sigma_2$ for two p-variate normal populations $N(\mu_1, \Sigma_1)$, $N(\mu_2, \Sigma_2)$ respectively. Let S_1/N_1 and S_2/N_2 be the maximum likelihood estimates of Σ_1 and Σ_2, based on random samples of size N_1 and N_2 from the two populations, respectively.

We propose to study the power functions of the following critical regions

$$C(a, b) = \{(S_1, S_2) : |S_1 S_2^{-1}|^a / |S_1 S_2^{-1} + I|^b \leqq \lambda\}$$

where λ is a constant depending on the size α of the critical regions. For the likelihood ratio test (LRT) of this problem $a = N_1$, $b = N_1 + N_2$, and for the modified likelihood ratio test (MLRT) $a = n_1$, $b = n_1 + n_2$, where $n_i = N_i - 1$. Sugiura and Nagao (1968) proved that the MLRT is unbiased against the alternatives $\Sigma_1 \neq \Sigma_2$ and Das Gupta (1969) showed that LRT is biased when $N_1 \neq N_2$. If $a(a - b) > 0$, the result of Anderson and Das Gupta (1969) can be used to study the power functions. In the following we shall assume $0 < a < b$; in which case the critical regions $C(a, b)$ are known to be admissible (Kiefer and Schwartz (1965)).

THEOREM. (a) *The critical region $C(a, n_1 + n_2)$ is unbiased for testing $\Sigma_1 = \Sigma_2$ against alternatives $\Sigma_1 \neq \Sigma_2$ for which $(|\Sigma_1| - |\Sigma_2|)(n_1 - a) \geqq 0$.*

(b) *The critical region $C(a, b)$ is biased for testing $\Sigma_1 = \Sigma_2$ against alternatives $\Sigma_1 \neq \Sigma_2$ for which the characteristic roots of $\Sigma_1 \Sigma_2^{-1}$ lie in the interval with endpoints d and 1, where $d = a(n_1 + n_2)/bn_1$.*

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PROOF. (a) Note that

$$|S_1 S_2^{-1}|^a / |S_1 S_2^{-1} + I|^{n_1+n_2} = [|S_1|/|S_2|]^{a-n_1} [|S_1|^{n_1} |S_2|^{n_2} / |S_1 + S_2|^{n_1+n_2}] .$$

Let $\gamma_1, \cdots, \gamma_p$ be the characteristic roots of $\Sigma_1 \Sigma_2^{-1}$ and let $\Gamma = \operatorname{diag}(\gamma_1, \cdots, \gamma_p)$. Following the method of Sugiura and Nagao (1968) it can be shown that ($\bar{C}$ = complement of C)

$$P(\bar{C}(a, n_1 + n_2) \mid \Gamma = I) - P(\bar{C}(a, n_1 + n_2) \mid \Gamma) \geqq K(p, n_1, n_2, \lambda) \left\{1 - |\Gamma| \frac{(a - n_1)}{2}\right\} \int_{\bar{C}(a, n_1+n_2)} |U|^{(a-n_1-p-1)/2} \, dU$$

where K is a constant. The result (a) now follows.

(b) Consider a family of regions given by $R(a, b) = \{y : y^a (1+y)^{-b} \geqq k, y > 0\}$. These regions are either intervals or complements of intervals. When $0 < a < b$, $R(a, b)$ is a finite interval not including 0 (excluding the trivial extreme case). Consider a random variable Y such that Y/δ ($\delta > 0$) is distributed as the ratio of two independent $\chi^2_{n_1}$ and $\chi^2_{n_2}$ variates. Let $\beta(\delta) = P[Y \in R(a, b)]$. Then it can be seen by differentiation that $\beta(\delta) > \beta(1)$, if δ lies in the open interval with the endpoints 1 and d. Define Z by

$$(1) \qquad |S_1|^a |S_2|^{b-a} / |S_1 + S_2|^b = [(S_{11}^{(1)})^a (S_{11}^{(2)})^{b-a} / (S_{11}^{(1)} + S_{11}^{(2)})^b] Z$$

where $S_i = [S_{jk}^{(i)}]$, $i = 1, 2$. Suppose $\Gamma = \operatorname{diag}(\gamma_1, 1, \cdots, 1)$. Then the distribution of Z is free from γ_1 and is independent of the first factor in the right-hand side of (1). Using the argument of Das Gupta (1969), it follows that the power of the critical region $C(a, b)$ is less than its size, if γ_1 lies between 1 and d strictly.

3. Tests for $\Sigma = I_p$. We shall consider the problem of testing $\Sigma = I_p$ for $N_p(\mu, \Sigma)$. Let S/N be the maximum likelihood estimate of Σ based on a random sample of size N. Consider a class of critical regions given by

$$W(r) : |S|^{r/2} \operatorname{etr}(-S/2) \leqq \lambda$$

for $r \geqq 0$. If $r \leqq 0$ the results of Anderson and Das Gupta [1] can be used. Note that $W(r)$ can be expressed as

$$|S^*|^{n/2} \operatorname{etr}(-S^*/2) \leqq \lambda^*$$

where $S^* = (n/r)S$, $n = N - 1$. Let $\Gamma = \operatorname{diag}(\gamma_1, \cdots, \gamma_p)$, where γ_i's are the characteristic roots of Σ. It now follows from the results of Das Gupta [2] and Sugiura and Nagao [6] that (a) $P[W(r)/\Gamma]$ innreases monotonically as such γ_i deviates from r/n either in the positive direction or in the negative direction, and (b) the critical region $W(r)$ is unbiased for the alternatives $\Sigma \neq I_p$ for which $(1 - |\Sigma|)(r - n) \geqq 0$.

Using the technique of Kiefer and Schwartz [5] it can be seen that the following critical regions are unique (a.e) Bayes and hence admissible when $n > p$.

(i) $|S|^{r/2} \operatorname{etr}(-S/2) \leqq \lambda$, $1 < r < \infty$
(ii) $S^{r/2} \operatorname{etr}(-S/2) \geqq \lambda$, $-\infty < r < 0$.

This can be seen by taking $\Sigma^{-1} = cI_p + \eta\eta'$, $c > 0$ and the prior density of η as proportional to $|cI_p + \eta\eta'|^{-n/2}$. Let $r = 1/(1 - c)$. Take $0 < c < 1$ and $1 < c$ for (i) and (ii) respectively. The admissibility of this type of critical regions has yet to be studied for other values of r.

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